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Oscillating waves and optimal smoothing effect for one-dimensional nonlinear scalar conservation laws

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Abstract

Lions, Perthame, Tadmor conjectured in 1994 an optimal smoothing effect for entropy solutions of nonlinear scalar conservations laws ([19]). In this short paper we will restrict our attention to the simpler one-dimensional case. First, supercritical geometric optics lead to sequences of C^∞ solutions uniformly bounded in the Sobolev space conjectured. Second we give continuous solutions which belong exactly to the suitable Sobolev space. In order to do so we give two new definitions of nonlinear flux and we introduce fractional BV spaces.

MSC: Primary: 35L65, 35B65; Secondary: 35B10, 35B40, 35C20.

Keywords: conservation laws, nonlinear flux, geometric optics, oscillations, Sobolev spaces, smoothing effect, fractional BV spaces.

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1 Introduction and nonlinear flux definitions

We focus on oscillating smooth solutions for one-dimensional scalar conservations laws:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \qquad u(0,x) = u_0(x), \qquad t > 0,\, x \in \mathbb{R}. \tag{1}$$

The aim of this paper is to build solutions related to the maximal regularity or the uniform Sobolev bounds conjectured in [19] for entropy solutions. In the one-dimensional case, piecewise smooth solutions with the maximal regularity are obtained in [12] for power-law fluxes. We seek supercritical geometric optics expansions and some special oscillating solutions. Our constructions are valid for all C^∞ flux and show that one cannot expect a better smoothing effect.

The more complex multidimensional case is dealt with in [17, 4]. For recent other approaches we refer the reader to [9, 7, 11, 8, 15, 14]. Recall that the first famous BV smoothing effect for uniformly convex flux was given by the Oleinik one-sided Lipschitz condition in the 1950s (see for instance the books [10, 18]). For solutions with bounded entropy production, the smoothing effect is weaker than for entropy

solutions ([12, 14]).

Let us give various definitions of nonlinear flux from [19, 1, 17, 2]. Throughout the paper, K denotes a compact real interval.

Definition 1 [Lions-Perthame-Tadmor nonlinear flux,[19]]
$f \in C^1(K, \mathbb{R})$ is said to be a nonlinear flux on K with degeneracy α if there exists a constant $C > 0$ such that for all $\delta > 0$,

$$\sup_{\tau^2+\xi^2=1} \left(measure\{v \in K,\ |\tau + \xi\, f'(v)| < \delta\}\right) \leq C\delta^\alpha. \tag{2}$$

In [19], the authors proved a smoothing effect for entropy solutions in some Sobolev space. They obtained uniform Sobolev bounds with respect to L^∞ bounds of initial data. Moreover, they conjectured a better smoothing effect :

$$u_0 \in L^\infty(\mathbb{R}) \Rightarrow u(t,.) \in W^{s,1}_{loc}(\mathbb{R}_x), \quad \text{for all } s < \alpha \text{ and for all } t > 0 \tag{3}$$

where the parameter α is defined in Definition 1. They proved a weaker smoothing effect which was improved in [21]. The conjecture (3) is still an open problem.

In [17] was given another definition related to the derivatives of the flux. It generalizes a notion of nonlinear flux arising in geometric optics ([5]). The next one-dimensional definition of smooth nonlinear flux is simpler than in the multidimensional case ([1, 17]).

Definition 2 [Smooth nonlinear flux, [17]]
$f \in C^\infty(K, \mathbb{R})$ is said to be a nonlinear flux on K with degeneracy d if

$$d = \max_{u \in K} \left(\min \left\{ k \geq 1,\ \frac{d^{1+k} f}{du^{1+k}}(u) \neq 0 \right\} \right) < +\infty. \tag{4}$$

For the Burgers equation or for uniformly convex flux, the degeneracy is $d = 1$. That is the minimal possible value. For the cubic flux $f(u) = u^3$ on $K = [-1, 1]$, the degeneracy is $d = 2$. The cubic flux is "less" nonlinear than the quadratic flux. Notice that, with this definition, a linear flux is not nonlinear: $d = +\infty$ with the natural convention $\min(\emptyset) = +\infty$.

This definition is equivalent to Definition 1 for C^∞ flux with $\alpha = \dfrac{1}{d}$ ([1, 17]). Therefore the Lions-Perthame-Tadmor parameter α is for

smooth flux the inverse of an integer.

The conjectured smoothing effect (3) is proved for the first time in fractional BV spaces for the class of nonlinear (degenerate) convex fluxes ([2]).

Definition 3 [Nonlinear degenerate convex flux, [3, 2]]
Let f belong to $C^1(I,\mathbb{R})$ where I is an interval of $\mathbb{R}$. We say that the degeneracy of f on I is at least p if the continuous derivative $a(u) = f'(u)$ satisfies:

$$0 < \inf_{I\times I} \frac{|a(u)-a(v)|}{|u-v|^p} \tag{5}$$

The lowest real number p, if there exists, is called the degeneracy of f on I. If there is no p such that (5) is satisfied, we set $p = +\infty$.
Let $f \in C^2(I)$. We say that a real number $y \in I$ is a degeneracy point of f on I if $f''(y) = 0$ (i.e. y is a critical point of a).

For instance, if f is the power-law flux on $[-1,1]$: $f(u) = |u|^{1+\alpha}$ where $\alpha > 0$, then the degeneracy is $p = \max(1,\alpha)$, ([3, 2]).

Remark 1 *Definition 3 implies the convexity (or the concavity) of the flux f.*

Indeed, by definition there exists $C > 0$ such that $|f'(u) - f'(v)| \geq C|u-v|^p$. Hence the difference $f'(u) - f'(v)$ never vanishes for $u \neq v$. Since the flux is continuous, it has got a constant sign for $u > v$, which implies the monotonicity of f' and then the convexity (or the concavity) of the flux.

Remark 2 *Definition 3 is less general than Definition 1. Nevertheless, if f satisfies (5) then it also satisfies (2) with $\alpha = \dfrac{1}{p}$, and also (4) with $d = p$ when f is smooth.*

The paper is organized as follows. The sequence given in Section 2 is exactly uniformly bounded in the Sobolev space conjectured in [19]. Furthermore, this sequence is unbounded in all smoother Sobolev spaces. In Section 3, we build solutions with the suitable regularity (3).

2 Supercritical geometric optics

We give a sequence of high frequency waves with small amplitude exactly uniformly bounded in the Sobolev space conjectured in [19]. The construction uses a WKB expansion ([5, 20]).

Theorem 4 *Let $f \in C^\infty(K, \mathbb{R})$ be a nonlinear flux with degeneracy d defined by (4). There exists a constant state $\underline{u} \in K$ such that for any smooth periodic function U_0 satisfying for all $0 < \varepsilon \leq 1$, for all $x \in \mathbb{R}$, $u_0^\varepsilon(x) = \underline{u} + \varepsilon U_0\left(\frac{x}{\varepsilon^d}\right) \in K$, the following properties hold:*

1. *there exists a positive time T such that the entropy solution u^ε of equation (1) with $u_0 = u_0^\varepsilon$ is smooth on $[0, T] \times \mathbb{R}$ for all $0 < \varepsilon \leq 1$,*

2. *the sequence (u^ε) is uniformly bounded in $W^{s,1}_{loc}([0, T] \times \mathbb{R})$ for $s = \alpha = \dfrac{1}{d}$ and unbounded for $s > \alpha$ when $U_0' \neq 0$ a.e.*

The key point is to construct a sequence of very high frequency waves near the state $\underline{u}$ where the maximum in (4) is reached. Next we compute the optimal Sobolev bounds uniformly with respect to ε on the WKB expansion:

$$u^\varepsilon(t, x) = \underline{u} + \varepsilon\, U\left(t, \frac{\varphi(t, x)}{\varepsilon^d}\right) + \varepsilon\, r_\varepsilon(t, x).$$

To estimate the remainder in Sobolev norms, we build a smooth sequence of solutions. It is quite surprising to have such smooth sequence on uniform time strip $[0, T]$. Indeed, it is a sequence of solutions with no entropy production, without shock. But for any higher frequency, the life span T_ε of u_ε as a continuous solution goes towards 0 and oscillations are canceled ([17]). Thus the construction is optimal.

Remark 3 *The uniform life span of the smooth sequence (u^ε) is at least*

$$T \sim \frac{1}{\sup_\theta \left|\dfrac{d\, U_0}{d\, \theta}\right|},$$

as one can see in [17]. So we can build such smooth sequence for any large time T and any non constant initial periodic profile U_0 small enough in C^1. But we cannot take $T = +\infty$ since shocks always occur when U_0 is not constant.

Remark 4 *For C^∞ flux, the parameter α in Definition 1 is always the inverse of an integer. To get supercritical geometric optics expansions for all $\alpha \in]0,1]$ and not only $\alpha \in \left\{\frac{1}{n},\ n \in \mathbb{N}^*\right\}$, we shall consider power-law flux $f(u) = |u|^{1+p}$, where $p = \frac{1}{\alpha} \in [1, +\infty[$, as in [12]. In this case, $\underline{u} = 0$ and the sequence is simply $u^\varepsilon(t,x) = \varepsilon U\left(t, \frac{x}{\varepsilon^p}\right)$, the exact entropy solution of (1) and $U(t,\theta)$ is the entropy solution of $\partial_t U + \partial_\theta |U|^{1+p} = 0$, $U(0,\theta) = U_0(\theta)$.*

Proof: We give a sketch of the proof (see [17] for more details).

- Existence of $\underline{u}$: the map $u \to \min\{k \geq 1,\ f^{(1+k)}(u) \neq 0\}$ is upper semi-continuous, so it achieves its maximum on the compact K.

- WKB expansion ([13, 16, 5, 17]): we plug the ansatz

$$u^\varepsilon(t,x) = \underline{u} + \varepsilon U_\varepsilon\left(t, \frac{\varphi(t,x)}{\varepsilon^d}\right)$$

  into (1). Notice that the exact profile U_ε depends on ε.
  Set $\lambda = f'(\underline{u})$ and $b = \frac{f^{(1+d)}(\underline{u})}{(1+d)!} \neq 0$. After simplification, the Taylor expansion of the flux $f(\underline{u} + \varepsilon U_\varepsilon) = f(\underline{u}) + \varepsilon\,\lambda\, U_\varepsilon + \varepsilon^{1+d}\, b\, U_\varepsilon^{1+d} - \varepsilon^{2+d}\, R_\varepsilon(U_\varepsilon)$ gives an equation for the exact profile U_ε and the phase φ:

$$\frac{\partial U_\varepsilon}{\partial t} + b\frac{\partial U_\varepsilon^{1+d}}{\partial \theta} = \varepsilon\frac{\partial R_\varepsilon(U_\varepsilon)}{\partial \theta}, \quad U_\varepsilon(0,\theta) = U_0(\theta), \quad \varphi(t,x) = x - \lambda t. \tag{6}$$

  The profile, which does not depend on ε, is

$$\frac{\partial U}{\partial t} + b\frac{\partial U^{1+d}}{\partial \theta} = 0, \qquad U(0,\theta) = U_0(\theta). \tag{7}$$

- Existence of smooth solutions for a time $T > 0$ independent of ε: it is a consequence of the method of characteristics. Indeed, the characteristics of equation (6) are a small perturbation of characteristics of equation (7).

- Approximation in $C^1([0,T]\times\mathbb{R})$: it comes again from the method of characteristics since $\varepsilon R_\varepsilon \to 0$.
  Notice that the expansion is valid in L^1_{loc} after shock waves ([5]). But it is not enough to estimate the Sobolev norms.

- Sobolev estimates: roughly speaking, the order of growth of the s fractional derivative $\dfrac{d^s}{dx^s}U_0\left(\frac{x}{\varepsilon^d}\right)$ is ε^{-sd}. For the profile U, this estimate is propagated along the characteristics on $[0,T]$. We have the same estimate for U_ε since U_ε is near U in C^1. Then we get the Sobolev bounds for u_ε.

□

3 Oscillating solutions

In this section we give exact continuous solutions with the Sobolev regularity conjectured in [19]. Indeed, we choose a suitable initial data such that the regularity is not spoiled by the nonlinearity of the flux for a positive time T. Furthermore, the conjectured smoothing effect is proved for the first time in fractional BV spaces ([2]) for the degenerate convex class of nonlinear flux given by Definition 3. The next theorem shows the optimality of this smoothing effect. The optimality was also given in [12] in Besov spaces framework. Let us introduce the BV^s spaces.

Definition 5 (Fractional BV spaces)
Let I be a non empty interval of $\mathbb{R}$. A partition σ of the interval I is a finite ordered subset: $\sigma = \{x_0, x_1, \cdots, x_n\} \subset I$, $x_0 < x_1 < \cdots < x_n$. We denote by $S(I)$ the set of all partitions of I. Let s belong to $]0,1]$ and $p = \dfrac{1}{s} \geq 1$. The s-total variation of a real function u on I is

$$TV^s u\{I\} \quad = \quad \sup_{\sigma \in S(I)} \sum_{k=1}^{n} |u(x_k) - u(x_{k-1})|^p \, .$$

$BV^s(I)$ is the space of real functions u such that $TV^s u\{I\} < +\infty$.

BV^s spaces are introduced in [2] for applications to conservation laws. These spaces measure the regularity of regulated functions: $BV = BV^1 \subset BV^s \subset L^\infty$. Indeed, $BV^s(K)$ is very close to the Sobolev space $W^{s,1/s}(K)$ ([2]):

- $BV^s(K) \subset W^{s-\eta,1/s}(K)$ for all $0 < \eta < s$.

- $BV^s(K) \neq W^{s,1/s}(K)$

We now give continuous functions which have the BV^s regularity.

Proposition 1 (A continuous BV^s function [3])
Let $0 < s < 1$, $0 < \eta < 1-s$ and let $g = g_{s,\eta}$ be the real function defined on $[0,1]$ by $g(0) = 0$ and for all $x \in]0,1]$:

$$g(x) = x^b \cos\left(\frac{\pi}{x^c}\right), \quad \textit{where } b = s + \frac{s^2}{\eta} \textit{ and } c = \frac{s}{\eta}.$$

The function g belongs to $BV^s([0,1]) \cap C^0([0,1])$ but not to $BV^{s+\eta}([0,1])$.

Notice that such example do not provide a function which belongs to BV^s but not to $\bigcup_{\eta>0} BV^{s+\eta}$. ***Proof:*** The extrema of g are achieved on $x_k = k^{-1/c}$. Let $p = \dfrac{1}{s} > 1$, $q \leq p$ and

$$V_q = \sum_{k=1}^{+\infty} |g(x_{k+1}) - g(x_k)|^q.$$

Since $qb/c = q(s+\eta)$, the asymptotic behavior $|g(x_{k+1}) - g(x_k)|^q \sim 2^q k^{-qb/c}$ when $k \to +\infty$ yields $V_q = +\infty$ when $q = 1/(s+\eta)$ and $V_p < +\infty$. First this implies $g \notin BV^{s+\eta}$. Second, for such oscillating function with diminishing amplitudes, we choose the optimal infinite partition to compute the s-total variation (see Proposition 2.3. p. 6 in [2]). Then g belongs to BV^s. □

We are now able to find oscillating initial data with the critical Sobolev exponent propagated by the nonlinear conservation law (1).

Theorem 6 *Assume $f \in C^\infty(K, \mathbb{R})$ be nonlinear in the sense of Definition 2. We denote by d its degeneracy and $s = \dfrac{1}{d}$. For any $\eta > 0$ and any time $T > 0$ there exists a solution $u \in C^0([0,T] \times \mathbb{R}, \mathbb{R})$ such that for all $t \in [0\,,\,T]$*

$$u(t,\cdot) \in BV^s(\mathbb{R},\mathbb{R}) \textit{ and } u(t,\cdot) \notin BV^{s+\eta}(\mathbb{R},\mathbb{R}).$$

The idea follows the K-S Cheng construction ([6]) with the function g given in Proposition 1.

Proof: Let $\underline{u} \in K$ a point where the maximum of degeneracy of f is achieved. We also suppose that $\underline{u} \in \overset{\circ}{K}$ (the proof of Theorem 6 is

quite similar if $\underline{u} \in \partial K$).
We define the initial condition u_0 by :

$$\begin{cases} u_0(x) = \underline{u} & \text{if } x < 0 \\ u_0(x) = \underline{u} + \delta g(x) & \text{if } 0 \leq x \leq 1 \\ u_0(x) = \underline{u} - \delta & \text{if } 1 < x \end{cases},$$

where $\delta > 0$ is chosen such that for all $x \in [0,1]$, $\underline{u} + \delta g(x) \in K$. Notice that for all $x \in [0,1]$, $-1 \leq g(x) \leq 1$ and $g(1) = -1$.

Then, following the method of characteristics, we define the function $u(t,x)$ by :

$$\begin{cases} u(t,x) = 0 & \text{if } x < 0 \\ u(t,x) = \underline{u} + \delta g(y) & \text{if } x = y + ta(\underline{u} + \delta g(y)), \quad 0 \leq y \leq 1 \\ u(t,x) = \underline{u} - \delta & \text{if } 1 + ta(\underline{u} - \delta) < x \end{cases}.$$

$u_0 \in BV^s([0\,,\,1])$ and $u_0 \notin BV^{s+\eta}([0\,,\,1])$. Let be $t > 0$ and for all y,

$$\theta_t(y) = y + ta(\underline{u} + \delta g(y)).$$

Considering the change of variable $y = x - a(\underline{u})t$, we can assume without loss of generality that $f'(\underline{u}) = a(\underline{u}) = 0$. Since $f \in C^\infty(K, \mathbb{R})$, we derive from a Taylor expansion that

$$a(u) = \frac{1}{d!}\left(a^{(d)}(\underline{u})(u - \underline{u})^d + \int_{\underline{u}}^{u} (u-s)^d a^{(1+d)}(s)ds\right).$$

Defining

$$I_n(y) = \frac{1}{d!}\int_0^1 (1-r)^d a^{(1+d)}(\underline{u} + r\,\delta\, g(y))dr,$$

$$J_n(y) = \frac{1}{d!}\int_0^1 r(1-r)^d a^{(2+d)}(\underline{u} + r\,\delta\, g(y))dr,$$

we get then :

$$\theta_t(y) = y + t\delta^d g(y)^d \left(\frac{1}{d!}a^{(d)}(\underline{u}) + \delta g(y) I_n(y)\right).$$

Note that g, I_n, J_n are bounded on $[0,1]$.

For $y \neq 0$, since $b\,d = 1+c$, we have $\dfrac{|g(y)|^d}{y} = O\left(y^c\right)$ at 0. Thus θ_t is differentiable at 0 and $\dfrac{d\theta_t}{dy}(0) = 1$. For $y \neq 0$, we have

$$\frac{d\theta_t}{dy}(y) = 1 + t\delta^d h_n(y),$$

where

$$h_n(y) = g(y)^{d-1} g'(y) \left(\frac{1}{(d-1)!} a^{(d)}(\underline{u}) + (d+1)\delta g(y) I_n(y) + \delta^2 g(y)^2 J_n(y) \right).$$

For $y \neq 0$, since $b\,d = 1+c$, we have

$$\begin{aligned} g(y)^{d-1} g'(y) &= \left(y^b \cos\left(\frac{\pi}{y^c}\right)\right)^{d-1} \left(b\, y^{b-1} \cos\left(\frac{\pi}{y^c}\right) + \pi c y^{b-c-1} \sin\left(\frac{\pi}{y^c}\right)\right), \\ \left|g(y)^{d-1} g'(y)\right| &\leq \left|\cos\left(\frac{\pi}{y^c}\right)\right|^{d-1} \left(b\,|y|^c \left|\cos\left(\frac{\pi}{y^c}\right)\right| + \pi c \left|\sin\left(\frac{\pi}{y^c}\right)\right|\right). \end{aligned}$$

Thus $g(y)^{d-1} g'(y)$ is bounded on $[0,1]$.

As h_n is bounded on $[0,1]$, there exists $T_\delta > 0$ such that for all $y \in [0,1]$ and for all $t \in]0,T]$, $\dfrac{d\theta_t}{dy}(y) > 0$. Notice that $\lim\limits_{\delta \to 0} T_\delta = +\infty$. We can take $\delta > 0$ small enough such that $T_\delta > T$.

Thus for all $t \in]0,T]$, θ_t is an homeomorphism between $[0,1]$ and $[0, 1 + ta(\underline{u} - \delta)]$. Then $u(t,x)$ is a continuous solution of equation (1) on $[0,T] \times \mathbb{R}$. Furthermore, since $u_0 \in BV^s(I)$ and $u_0 \notin BV^{s+\eta}(I)$, where $I = [0,1]$, we deduce that for all $t \in]0,T]$, $u(t,\cdot) \in BV^s(J)$ and $u(t,\cdot) \notin BV^{s+\eta}(J)$, where $J = \theta_t(I) = [0, 1 + ta(\underline{u} - \delta)]$. Finally, as $u(t,\cdot)$ is constant outside J, we have proved that $u(t,\cdot) \in BV^s(\mathbb{R})$ and $u(t,\cdot) \notin BV^{s+\eta}(\mathbb{R})$. $\square$

Remark 5 *As in Remark 4, Theorem 6 is restricted for critical exponent s such that $\dfrac{1}{s} \in \mathbb{N}$. To obtain all exponent $s \in]0,1]$, following [12], we can consider a power-law flux with $p = \dfrac{1}{s}$: $f(u) = |u|^{1+p}$. Our construction is quite similar as in the proof of Theorem 6 with $\underline{u} = 0$ and $\delta > 0$ small enough.*

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