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# DIVISION OF HOLOMORPHIC FUNCTIONS AND GROWTH CONDITIONS

WILLIAM ALEXANDRE AND EMMANUEL MAZZILLI

ABSTRACT. Let  $D$  be a strictly convex domain of  $\mathbb{C}^n$ ,  $f_1$  and  $f_2$  be two holomorphic functions defined on a neighborhood of  $\bar{D}$  and set  $X_l = \{z, f_l(z) = 0\}$ ,  $l = 1, 2$ . Suppose that  $X_l \cap bD$  is transverse for  $l = 1$  and  $l = 2$ , and that  $X_1 \cap X_2$  is a complete intersection. We give necessary conditions when  $n \geq 2$  and sufficient conditions when  $n = 2$  under which a function  $g$  can be written as  $g = g_1 f_1 + g_2 f_2$  with  $g_1$  and  $g_2$  in  $L^q(D)$ ,  $q \in [1, +\infty)$ , or  $g_1$  and  $g_2$  in  $BMO(D)$ . In order to prove the sufficient condition, we explicitly write down the functions  $g_1$  and  $g_2$  using integral representation formulas and new residue currents.

## 1. INTRODUCTION

In this article, we are interested in ideals of holomorphic functions and corona type problems. More precisely, if  $D$  is a domain of  $\mathbb{C}^n$  and  $f_1, \dots, f_k$  are  $k$  holomorphic functions defined in a neighborhood of  $\bar{D}$ , we are looking for condition(s), as close as possible to being necessary and sufficient, under which a function  $g$ , holomorphic on  $D$ , can be written as

$$(1) \quad g = f_1 g_1 + \dots + f_k g_k,$$

with  $g_1, \dots, g_k$  holomorphic on  $D$  and satisfying growth conditions at the boundary of  $D$ . This kind of problem has been widely studied by many authors under different assumptions.

When  $D$  is strictly pseudoconvex and when  $f_1, \dots, f_k$  are holomorphic and bounded functions on  $D$ , which satisfy  $|f|^2 = |f_1|^2 + \dots + |f_k|^2 \geq \delta^2 > 0$ , for a given holomorphic and bounded function  $g$ , finding functions  $g_1, \dots, g_k$  bounded on  $D$  is a question known as the Corona Problem. When  $D$  is the unit ball of  $\mathbb{C}$ , the Corona Problem was solved in 1962 by Carleson in [8]. This question is still open for  $n > 1$ , even for two generators  $f_1$  and  $f_2$ , and even when  $D$  is the unit ball of  $\mathbb{C}^n$ .

For  $p \in [1, +\infty)$ , we denote by  $H^p(D)$  the Hardy space of  $D$ . When  $n > 1$ ,  $k = 2$  and  $|f| \geq \delta > 0$ , Amar proved in [2] that for any  $g \in H^p(D)$ , (1) can be solved with  $g_1$  and  $g_2$  in  $H^p(D)$ . Andersson and Carlsson in [4] generalized this result to any strictly pseudoconvex domain in  $\mathbb{C}^n$  and to any  $k \geq 2$  and also obtained the  $BMO$ -result already announced by Varopoulos in [21]. In [6], they studied the dependence of the  $g_i$ 's on the lower bound  $\delta$  of  $|f|$  and they explicitly obtained a constant  $c_\delta$  such that for all  $i$ ,  $\|g_i\|_{H^p(D)} \leq c_\delta \|g\|_{H^p(D)}$ . Of course  $c_\delta$  goes to infinity when  $\delta$  goes to 0. In [3], when  $|f|$  does not have a positive

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lower bound, Amar and Bruna formulated a sufficient condition in term of the admissible maximum function of  $|f|^2 |\log |f||^{2+\varepsilon}$ ,  $\varepsilon > 0$ , under which the  $g_i$ 's belong to  $H^p(D)$ .

The corona problem was also studied in the case of the Bergman space  $A^p(D)$ , the space of holomorphic functions which belong to  $L^p(D)$ , and in the case of the Zygmund space  $\Lambda_\gamma(D)$  by Krantz and Li in [12], and in the case of Hardy-Sobolev spaces by Fàbrega and Ortega in [13].

In the above papers, the first step of the proof in the case of two generators  $f_1$  and  $f_2$ , is to find two smooth functions on  $D$ ,  $\varphi_1$  and  $\varphi_2$ , such that

$$(2) \quad \varphi_1 f_1 + \varphi_2 f_2 = 1;$$

and then to solve the equation

$$(3) \quad \bar{\partial}\varphi = g \frac{\overline{f_1} \bar{\partial}\varphi_2 - \overline{f_2} \bar{\partial}\varphi_1}{|f_1|^2 + |f_2|^2}.$$

Then setting  $g_1 = g\varphi_1 + \varphi f_2$  and  $g_2 = g\varphi_2 - \varphi f_1$ , (1) holds and, provided  $\varphi$  belongs to the appropriate space,  $g_1$  and  $g_2$  will belong to  $H^p(D)$ ,  $A^p(D)$ ,  $\dots$ . So the problem is reduced to solve the Bezout equation (2) and then to solve the  $\bar{\partial}$ -equation (3) with an appropriate regularity.

In [5], Andersson and Carlsson used an alternative technique. They constructed a division formula  $g = f_1 T_1(g) + \dots + f_k T_k(g)$  where for all  $i$ ,  $T_i$  is a well chosen Berndtsson-Andersson integral operator, and, still under the assumption  $|f| \geq \delta > 0$ , they proved that  $T_i(g)$  belongs to  $H^p(D)$  (resp.  $BMO(D)$ ) when  $g$  belongs to  $H^p(D)$  (resp.  $BMO(D)$ ). The same kind of technics was also used in [7] by Bonneau, Cumenge and Zérialhi who studied the equation (1) in Lipschitz spaces and in the space  $B_M(D) = \{g, \|g\|_{B_M(D)} = \sup_{z \in D} (|g(z)|d(z, bD)^M) < \infty\}$ . In this later work, the generators  $f_1, \dots, f_k$  may have common zeroes but  $\partial f_1 \wedge \dots \wedge \partial f_k$  can not vanish on  $bD \cap \{z, f_1(z) = \dots = f_k(z) = 0\}$ .

The case of generators having common zeroes has also been investigated by Skoda in [20] for weighted  $L^2$ -spaces. Using and adapting the  $L^2$ -techniques developed by Hörmander, for  $D$  pseudoconvex in  $\mathbb{C}^n$ ,  $\psi$  a plurisubharmonic weight on  $D$ ,  $f_1, \dots, f_k$  holomorphic in  $D$ ,  $q = \inf(n, k)$ ,  $\alpha > 1$  and  $g$  holomorphic in  $D$  such that  $\int_D \frac{|g|^2}{|f|^{2\alpha q+2}} e^{-\psi} < \infty$ , Skoda showed that there exist  $g_1, \dots, g_k \in \mathcal{O}(D)$  such that (1) holds and such that for all  $i$ ,  $\int_D \frac{|g_i|^2}{|f|^{2\alpha q}} e^{-\psi} \leq \frac{\alpha}{\alpha-1} \int_D \frac{|g|^2}{|f|^{2\alpha q+2}} e^{-\psi}$ . Moreover the result also holds when  $k$  is infinite and there is no restriction on  $\partial f_1, \dots, \partial f_k$ . However, if one takes  $g = f_1$  for example,  $g$  does not satisfy the assumption of Skoda's theorem in general.

In this article we restrict ourself to a strictly convex domain  $D$  of  $\mathbb{C}^n$  and we consider the case of two generators  $f_1$  and  $f_2$ , holomorphic in a neighborhood of  $\bar{D}$ . We denote by  $X_1$  the set  $X_1 = \{z, f_1(z) = 0\}$ , and by  $X_2$  the set  $X_2 = \{z, f_2(z) = 0\}$ . We assume that the intersections  $X_1 \cap bD$  and  $X_2 \cap bD$  are transverse in the sense of tangent cones and that  $X_1 \cap X_2$  is a complete intersection. Let us recall that an analytic subset  $A$  of pure co-dimension  $m$  in  $\mathbb{C}^n$  is said to be a complete intersection if there are  $m$  holomorphic functions  $h_1, \dots, h_m$  such that  $A = \cap_{i=1}^m \{z, h_i(z) = 0\}$ ; and that the intersection  $X_l \cap D$ ,  $l = 1$  or  $l = 2$ , is said to be transverse if for every  $p \in X_l \cap bD$ , the complex tangent space to  $bD$  at  $p$  and the tangent cone to  $X_l$  at  $p$  span  $T_p \mathbb{C}^n$ .

Our goal here is to find assumptions on  $g$ , holomorphic in  $D$ , as close as possible to being

necessary and sufficient, under which we can write  $g$  as  $g = g_1 f_1 + g_2 f_2$  with  $g_1$  and  $g_2$  holomorphic and belonging to  $BMO(D)$  or  $L^q(D)$ ,  $q \in [1, +\infty)$ .

Let us write  $D$  as  $D = \{z \in \mathbb{C}^n, \rho(z) < 0\}$  where  $\rho$  is a smooth strictly convex function defined on  $\mathbb{C}^n$  such that the gradient of  $\rho$  does not vanish in a neighborhood  $\mathcal{U}$  of  $bD$ . We denote by  $D_r$ ,  $r \in \mathbb{R}$ , the set  $D_r = \{z \in \mathbb{C}^n, \rho(z) < r\}$ , by  $\eta_\zeta$  the outer unit normal to  $bD_{\rho(\zeta)}$  at a point  $\zeta \in \mathcal{U}$  and by  $v_\zeta$  a smooth unitary complex vector field tangent at  $\zeta$  to  $bD_{\rho(\zeta)}$ . As a first result, we show:

**Theorem 1.1.** *Let  $D$  be a strictly convex domain of  $\mathbb{C}^2$ ,  $f_1$  and  $f_2$  be two holomorphic functions defined on a neighborhood of  $\bar{D}$  and set  $X_l = \{z, f_l(z) = 0\}$ ,  $l = 1, 2$ . Suppose that  $X_l \cap bD$  is transverse for  $l = 1$  and  $l = 2$ , and that  $X_1 \cap X_2$  is a complete intersection. Then there exist two integers  $k_1, k_2 \geq 1$  depending only on  $f_1$  and  $f_2$  such that if  $g$  is any holomorphic function on  $D$  which belongs to the ideal generated by  $f_1$  and  $f_2$  and for which there exist two  $C^\infty$  smooth functions  $\tilde{g}_1$  and  $\tilde{g}_2$  such that*

- (i)  $g = \tilde{g}_1 f_1 + \tilde{g}_2 f_2$  on  $D$ ,
- (ii) there exists  $N \in \mathbb{N}$  such that  $|\rho|^N \tilde{g}_1$  and  $|\rho|^N \tilde{g}_2$  vanish to order  $k_2$  on  $bD$ ,
- (iii) there exists  $q \in [1, +\infty]$  such that for  $l = 1, 2$ ,  $\left| \frac{\partial^{\alpha+\beta} \tilde{g}_l}{\partial \bar{\eta}_\zeta^\alpha \partial \bar{v}_\zeta^\beta} \right| |\rho|^{\alpha+\frac{\beta}{2}}$  belongs to  $L^q(D)$  for all non-negative integers  $\alpha$  and  $\beta$  with  $\alpha + \beta \leq k_1$ ,

then there exist two holomorphic functions  $g_1, g_2$  on  $D$  which belong to  $L^q(D)$  if  $q < +\infty$  and to  $BMO(D)$  if  $q = +\infty$ , such that  $g_1 f_1 + g_2 f_2 = g$  on  $D$ .

The number  $k_1$  and  $k_2$  are almost equal to the maximum of the multiplicities of the singularity of  $X_1$  and  $X_2$ . The functions  $g_1$  and  $g_2$  will be obtained via integral operators acting on  $\tilde{g}_1$  and  $\tilde{g}_2$ . These operators are a combination of a Berndtsson-Andersson kernel and of two (2,2)-currents  $T_1$  and  $T_2$  such that  $f_1 T_1 + f_2 T_2 = 1$ . So instead of first solving the Bezout equation (2) in the sense of smooth functions, we solve it in the sense of currents and then, instead of solving a  $\bar{\partial}$ -equation, we “holomorphy” the smooth solutions  $\tilde{g}_1$  and  $\tilde{g}_2$  of the equation  $g = \tilde{g}_1 f_1 + \tilde{g}_2 f_2$  with integral operators using  $T_1$  and  $T_2$ . As we will see in Section 4, these operators can be constructed starting from any currents  $\tilde{T}_1$  and  $\tilde{T}_2$  such that  $f_1 \tilde{T}_1 + f_2 \tilde{T}_2 = 1$ . However, not all such currents will give operators such that  $g_1$  and  $g_2$  belongs to  $L^q(D)$  or  $BMO(D)$ ; as we will see in Section 3, they have to be constructed taking into account the interplay between  $X_1$  and  $X_2$ . Moreover, if  $\tilde{g}_1$  and  $\tilde{g}_2$  are already holomorphic and satisfy the assumptions (i) – (iii) of Theorem 1.1, then  $g_1 = \tilde{g}_1$  and  $g_2 = \tilde{g}_2$ .

Observe that in Theorem 1.1, we do not make any assumption on  $f_1$  or  $f_2$  except that the intersection  $X_1 \cap bD$  and  $X_2 \cap bD$  are transverse in the sense of tangent cones, and that  $X_1 \cap X_2$  is a complete intersection. This later assumption can be removed provided we add a fourth assumption on  $\tilde{g}_1$  and  $\tilde{g}_2$ . If we moreover assume that

- (iv)  $\frac{\partial^{\alpha+\beta} \tilde{g}_1}{\partial \bar{\eta}_\zeta^\alpha \partial \bar{v}_\zeta^\beta} = 0$  on  $X_2 \cap D$  and  $\frac{\partial^{\alpha+\beta} \tilde{g}_2}{\partial \bar{\eta}_\zeta^\alpha \partial \bar{v}_\zeta^\beta} = 0$  on  $X_1 \cap D$  for all non negative integers  $\alpha$  and  $\beta$  with  $0 < \alpha + \beta \leq k_1$ ,

then Theorem 1.1 also holds whenever  $X_1 \cap X_2$  is not complete. However, it then becomes very difficult to find  $\tilde{g}_1$  and  $\tilde{g}_2$  which satisfy this fourth assumption, except if  $X_1 \cap X_2$  is actually complete.

Indeed, the main difficulty in order to be able to apply Theorem 1.1 is to find the two functions  $\tilde{g}_1$  and  $\tilde{g}_2$  satisfying (i)-(iii). The canonical choice when  $|f| \geq \delta > 0$  is to set  $\tilde{g}_1 = \overline{gf_1}|f|^{-2}$  and  $\tilde{g}_2 = \overline{gf_2}|f|^{-2}$ . If  $|f| \geq \delta > 0$  and if  $g$  belongs to  $L^q(D)$ , then  $\tilde{g}_1$  and  $\tilde{g}_2$  will satisfy (i)-(iii) and we can then apply Theorem 1.1. However, if  $|f|$  does not admit a positive lower bound, this will not be necessarily the case. For example, when  $D = \{z \in \mathbb{C}^2, \rho(z) = |z_1 - 1|^2 + |z_2|^2 - 1 < 0\}$ ,  $f_1(z) = z_2$ ,  $f_2(z) = z_2 - z_1^2$  and  $g = f_1$ , we can obviously find  $\tilde{g}_1$  and  $\tilde{g}_2$  which satisfy the assumption of Theorem 1.1 but if we make the canonical choices for  $\tilde{g}_1$  and  $\tilde{g}_2$ , they do not fulfill (iii) for  $q = \infty$  because  $\frac{\partial \tilde{g}_1}{\partial \bar{z}_2} |\rho|^{\frac{1}{2}}$  is not bounded near 0.

Therefore the question of the existence of  $\tilde{g}_1$  and  $\tilde{g}_2$  may itself become a problem that we have to solve. Using first Koranyi balls, we will reduce this global question to a local one and then, using divided differences, we will give numerical conditions under which there indeed exist functions satisfying the hypothesis of Theorem 1.1. We will also prove that these conditions are necessary in order to solve Equation (1) with the  $g_i$ 's belonging to  $L^q(D)$ ,  $q \in [1, +\infty]$ , even in  $\mathbb{C}^n$ . This leads us to an effective way of construction of the solutions of (1) belonging to  $L^q(D)$  or  $BMO(D)$ .

The Koranyi balls in  $\mathbb{C}^2$  are defined as follows. We call the coordinates system centered at  $\zeta$  of basis  $\eta_\zeta, v_\zeta$  the Koranyi coordinates at  $\zeta$ . We denote by  $(z_1^*, z_2^*)$  the coordinates of a point  $z$  in the Koranyi coordinates at  $\zeta$ . The Koranyi ball centered in  $\zeta$  of radius  $r$  is the set  $\mathcal{P}_r(\zeta) := \{\zeta + \lambda \eta_\zeta + \mu v_\zeta, |\lambda| < r, |\mu| < r^{\frac{1}{2}}\}$ . We observe that, by convexity,  $\mathcal{P}_{\alpha\kappa|\rho(\zeta)}(\zeta)$  is included in  $D$  if  $\alpha$  is small enough.

The following theorem enables us to go from a local division formula in  $L^\infty$  to a global division formula in  $BMO$ .

**Theorem 1.2.** *Let  $D$  be a strictly convex domain of  $\mathbb{C}^2$ ,  $f_1$  and  $f_2$  be two holomorphic functions defined on a neighborhood of  $\bar{D}$  and set  $X_l = \{z, f_l(z) = 0\}$ ,  $l = 1, 2$ . Suppose that  $X_1 \cap bD$  and  $X_2 \cap bD$  are transverse, and that  $X_1 \cap X_2$  is a complete intersection. Let  $g$  be a function holomorphic on  $D$  and assume that there exists  $\kappa > 0$  such that for all  $z \in D$ , there exist two functions  $\hat{g}_1$  and  $\hat{g}_2$ , depending on  $z$ ,  $C^\infty$ -smooth on  $\mathcal{P}_{\kappa|\rho(z)}(z)$ , such that*

(a)  $g = \hat{g}_1 f_1 + \hat{g}_2 f_2$  on  $\mathcal{P}_{\kappa|\rho(z)}(z)$ ;

(b) for all non negative integers  $\alpha, \beta, \bar{\alpha}$  and  $\bar{\beta}$ , there exist  $c > 0$ , not depending on  $z$ ,

$$\text{such that } \sup_{\mathcal{P}_{\kappa|\rho(z)}(z)} \left| \frac{\partial^{\alpha+\bar{\alpha}+\beta+\bar{\beta}} \hat{g}_l}{\partial z_1^{*\alpha} \partial z_2^{*\beta} \partial \bar{z}_1^{*\bar{\alpha}} \partial \bar{z}_2^{*\bar{\beta}}} \right| \leq c |\rho(z)|^{-\alpha-\frac{\beta}{2}} \text{ for } l = 1 \text{ and } l = 2.$$

Then there exist two smooth functions  $\tilde{g}_1$  and  $\tilde{g}_2$  which satisfy the assumptions (i)-(iii) of Theorem 1.1 for  $q = +\infty$ .

An analogous theorem holds true in the  $L^q$ -case (see Theorem 6.1). We observe that if, for all  $z \in D$ , there exist two functions  $\hat{g}_1$  and  $\hat{g}_2$ , holomorphic and bounded on  $\mathcal{P}_{2\kappa|\rho(z)}(z)$  by a constant  $c$  which does not depend on  $z$ , and such that  $g = \hat{g}_1 f_1 + \hat{g}_2 f_2$  on  $\mathcal{P}_{2\kappa|\rho(z)}(z)$ , then Cauchy's inequalities implies that  $\hat{g}_1$  and  $\hat{g}_2$  satisfy the assumption of Theorem 1.2 on  $\mathcal{P}_{\kappa|\rho(z)}(z)$  for all  $z$ . Therefore Theorem 1.2 implies that the global solvability of (1) in the  $BMO$  space of  $D$  is nearly equivalent to its uniform local solvability. In order to prove Theorem 1.2, we will cover  $D$  with Koranyi balls and using a suitable partition of unity,

we will glue together the  $\hat{g}_1$  and  $\hat{g}_2$  which we got on each ball. We point out that when we glue together the local  $\hat{g}_1$ 's, except if  $X_1 \cap X_2$  is a complete intersection, in general the “fourth” assumption (iv) of Theorem 1.1 is not satisfied. This is why we chose to present Theorem 1.1 as we did.

When looking for necessary conditions in order to solve Equation (1) with  $g_1$  and  $g_2$  bounded, we first observe that  $g$  is trivially bounded by  $\max(\|g_1\|_{L^\infty}, \|g_2\|_{L^\infty})(|f_1| + |f_2|)$ . Therefore, in order for  $g$  to be written as  $g = g_1 f_1 + g_2 f_2$  with  $g_1$  and  $g_2$  bounded, it is necessary that  $\frac{|g|}{|f_1| + |f_2|}$  be bounded. However this condition alone does not suffice in general. Consider for example the ball  $D := \{z \in \mathbb{C}^2, \rho(z) = |z_1 - 1|^2 + |z_2|^2 - 1 < 0\}$ ,  $f_1(z) = z_2^2$ ,  $f_2(z) = z_2^2 - z_1^q$  and  $g(z) = z_1^{\frac{q}{2}} z_2$  where  $q \geq 3$  is an odd integer. Then  $g(z) = z_2 z_1^{-\frac{q}{2}} f_1(z) - z_2 z_1^{-\frac{q}{2}} f_2(z)$ , so  $g$  belongs to the ideal generated by  $f_1$  and  $f_2$ , and  $\frac{|g|}{|f_1| + |f_2|}$  is bounded on  $D$  by  $\frac{3}{2}$ ; in particular, the classical choice  $\tilde{g}_1 = \frac{g f_1}{|f_1|^2 + |f_2|^2}$  and  $\tilde{g}_2 = \frac{g f_2}{|f_1|^2 + |f_2|^2}$  gives two functions  $\tilde{g}_1$  and  $\tilde{g}_2$  which are smooth and bounded on  $D$ . However, (1) can not be solved with  $g_1$  and  $g_2$  bounded on  $D$ . In order to see this, a good tool is divided differences.

On the one hand, if  $g = g_1 f_1 + g_2 f_2$ , then  $g_1 = g \cdot f_1^{-1}$  on  $X_2 \setminus X_1$ .

On the other hand, for all  $z \in D$ , all unit vector  $v$  tangent to  $bD_{-\rho(z)}$  at  $z$ , all complex numbers  $\lambda_1$  and  $\lambda_2$  with  $\rho(z + \lambda_1 v) < \rho(z)$  and  $\rho(z + \lambda_2 v) < \rho(z)$ , Montel [19] asserts that the modulus of the divided difference  $\frac{g_1(z + \lambda_1 v) - g_1(z + \lambda_2 v)}{\lambda_1 - \lambda_2}$  behaves like  $\left| \frac{\partial g_1}{\partial v} \right|$  at some point  $z + \mu v$  where  $\mu$  is an element of the segment  $[\lambda_1, \lambda_2]$ . Cauchy's inequalities then imply that, up to a uniform multiplicative constant,  $\frac{g_1(z + \lambda_1 v) - g_1(z + \lambda_2 v)}{\lambda_1 - \lambda_2}$  is bounded by  $\|g_1\|_{L^\infty(D)} |\rho(z)|^{-\frac{1}{2}}$ .

So when we compute the divided differences of  $g_1$  at points  $z + \lambda_1 v$  and  $z + \lambda_2 v$  which belong to  $X_2 \setminus X_1$ , whatever  $g_1$  and  $g_2$  may be, we actually compute the divided difference of  $g \cdot f_1^{-1}$ . And if  $g_1$  is bounded, this divided difference times  $|\rho(z)|^{\frac{1}{2}}$  must be bounded by some uniform constant. But in our example, this is not the case because for small  $\varepsilon > 0$ , setting  $z = (\varepsilon, 0)$ ,  $v = (0, 1)$ ,  $\lambda_1 = \varepsilon^{\frac{q}{2}}$  and  $\lambda_2 = -\varepsilon^{\frac{q}{2}}$ , we have that  $\frac{(g f_1^{-1})(z + \lambda_1 v) - (g f_1^{-1})(z + \lambda_2 v)}{\lambda_1 - \lambda_2} |\rho(z)|^{\frac{1}{2}} = \varepsilon^{\frac{1-q}{2}}$  which is unbounded when  $\varepsilon$  goes to zero.

In  $\mathbb{C}^n$ , we will prove that the divided differences of any order of  $g \cdot f_1^{-1}$  and  $g \cdot f_2^{-1}$  must satisfy some boundedness properties when (1) is solvable with  $g_1$  and  $g_2$  in  $L^q(D)$ ,  $q \in [1, +\infty]$  (see Theorems 6.3 and 6.5 for precise statements). Conversely, in  $\mathbb{C}^2$ , if those boundedness properties are satisfied, by polynomial interpolation and on any Koranyi balls, we construct two functions  $\tilde{g}_1$  and  $\tilde{g}_2$  which satisfy the assumptions of Theorem 1.2. It must be mentioned that the error term we will get during the interpolation process will be very difficult to handle. Although the interpolation procedure is a holomorphic one, we will not get two holomorphic functions  $\tilde{g}_1$  and  $\tilde{g}_2$  because we will have to split the error term in a appropriate way in two parts, which will lead to  $C^\infty$ -smooth but not holomorphic functions. Then it will follow from Theorem 1.1 that there exist two functions  $g_1$  and  $g_2$  holomorphic on  $D$ , belonging to  $BMO(D)$  such that  $g = g_1 f_1 + g_2 f_2$ . An analogue result for holomorphic functions in  $L^q(D)$ ,  $q \in [1, +\infty]$ , will be also proved. These two results are precisely stated in Theorem 6.4 and 6.6.

The article is organized as follows. In Section 2, we recall some tools needed for the construction and the estimation of the division formula. In Section 3, we construct the currents which enable us to construct our division formula in Section 4. In Section 5 we establish Theorem 1.1 and finally, in Section 6, we prove the theorems related to local division in the  $L^\infty$  and  $L^q$  case.

## 2. NOTATIONS AND TOOLS

**2.1. Koranyi balls.** The Koranyi balls centered at a point  $z$  in  $D$  have properties linked with distance from  $z$  to the boundary of  $D$  in a direction  $v$ . They were generalized in the case of convex domains of finite type by McNeal in [17] and [18]. A strictly convex domain being in particular a convex domain of finite type, we will adopt the formalism of convex domain of finite type.

For  $z \in \mathbb{C}^n$ ,  $v$  a unit vector in  $\mathbb{C}^n$ , and  $\varepsilon > 0$ , the distance from  $z$  to  $bD_{\rho(z)+\varepsilon}$  in the direction  $v$  is defined by

$$\tau(z, v, \varepsilon) = \sup\{\tau > 0, \rho(z + \lambda v) - \rho(z) < \varepsilon \text{ for all } \lambda \in \mathbb{C}, |\lambda| < \tau\}.$$

Thus  $\tau(z, v, \varepsilon)$  is the maximal radius  $r > 0$  such that the disc  $\Delta_{z,v}(r) = \{z + \lambda v, |\lambda| < r\}$  is included in  $D_{\rho(z)+\varepsilon}$ ; if  $v$  is a tangent vector to  $bD_{\rho(z)}$  at  $z$ , then  $\tau(z, v, \varepsilon)$  is comparable to  $\varepsilon^{\frac{1}{2}}$  and  $\tau(z, \eta_z, \varepsilon)$  is comparable to  $\varepsilon$ .

Before we recall the properties of the Koranyi balls we will need, we adopt the following notation. We write  $A \lesssim B$  if there exists some constant  $c > 0$  such that  $A \leq cB$ . Each time we will mention on which parameters  $c$  depends. We will write  $A \approx B$  if  $A \lesssim B$  and  $B \lesssim A$  both holds. The following propositions are part of well known properties of Koranyi balls and McNeal polydiscs. The interested reader can find a proof of each statements in [17] in the case of convex domains of finite type, keeping in mind that a strictly convex domain is a convex domain of type 2.

**Proposition 2.1.** *There exists a neighborhood  $\mathcal{U}$  of  $bD$  and positive real numbers  $\kappa$  and  $c_1$  such that*

- (i) *for all  $\zeta \in \mathcal{U} \cap D$ ,  $\mathcal{P}_{4\kappa|\rho(\zeta)|}(\zeta)$  is included in  $D$ .*
- (ii) *for all  $\varepsilon > 0$ , all  $\zeta, z \in \mathcal{U}$ ,  $\mathcal{P}_\varepsilon(\zeta) \cap \mathcal{P}_\varepsilon(z) \neq \emptyset$  implies  $\mathcal{P}_\varepsilon(z) \subset \mathcal{P}_{c_1\varepsilon}(\zeta)$ .*
- (iii) *for all  $\varepsilon > 0$  sufficiently small, all  $z \in \mathcal{U}$ , all  $\zeta \in \mathcal{P}_\varepsilon(z)$  we have  $|\rho(z) - \rho(\zeta)| \leq c_1\varepsilon$ .*
- (iv) *for all  $\varepsilon > 0$ , all unit vectors  $v \in \mathbb{C}^n$ , all  $z \in \mathcal{U}$  and all  $\zeta \in \mathcal{P}_\varepsilon(z)$ ,  $\tau(z, v, \varepsilon) \approx \tau(\zeta, v, \varepsilon)$  uniformly with respect to  $\varepsilon, z$  and  $\zeta$ .*

For  $\mathcal{U}$  given by Proposition 2.1 and  $z$  and  $\zeta$  belonging to  $\mathcal{U}$ , we set  $\delta(z, \zeta) = \inf\{\varepsilon > 0, \zeta \in \mathcal{P}_\varepsilon(z)\}$ . Proposition 2.1 implies that  $\delta$  is a pseudo-distance in the following sense:

**Proposition 2.2.** *For  $\mathcal{U}$  and  $c_1$  given by Proposition 2.1 and for all  $z, \zeta$  and  $\xi$  belonging to  $\mathcal{U}$  we have*

$$\frac{1}{c_1}\delta(\zeta, z) \leq \delta(z, \zeta) \leq c_1\delta(\zeta, z)$$

and

$$\delta(z, \zeta) \leq c_1(\delta(z, \xi) + \delta(\xi, \zeta))$$

**2.2. Berndtsson-Andersson reproducing kernel in  $\mathbb{C}^2$ .** Berndtsson-Andersson's kernel will be one of our most important ingredients in the construction of the functions  $g_1$  and  $g_2$  of Theorem 1.1. We now recall its definition for  $D$  a strictly convex domain of  $\mathbb{C}^2$ . We set  $h_1(\zeta, z) = -\frac{1}{2} \frac{\partial \rho}{\partial \zeta_1}(\zeta)$ ,  $h_2(\zeta, z) = -\frac{1}{2} \frac{\partial \rho}{\partial \zeta_2}(\zeta)$ ,  $h = \sum_{i=1,2} h_i d\zeta_i$  and  $\tilde{h} = \frac{1}{\rho} h$ . For a  $(1, 0)$ -form  $\beta(\zeta, z) = \sum_{i=1,2} \beta_i(\zeta, z) d\zeta_i$  we set  $\langle \beta(\zeta, z), \zeta - z \rangle = \sum_{i=1,2} \beta_i(\zeta, z) (\zeta_i - z_i)$ . Then we define the Berndtsson-Andersson reproducing kernel by setting for an arbitrary positive integer  $N$ ,  $n = 1, 2$  and all  $\zeta, z \in D$ :

$$P^{N,n}(\zeta, z) = C_{N,n} \left( \frac{1}{1 + \langle \tilde{h}(\zeta, z), \zeta - z \rangle} \right)^{N+n} (\bar{\partial} \tilde{h})^n,$$

where  $C_{N,n} \in \mathbb{C}$  is a suitable constant. We also set  $P^{N,n}(\zeta, z) = 0$  for all  $z \in D$  and all  $\zeta \notin D$ . Then the following theorem holds true (see [9]):

**Theorem 2.3.** *For all  $g \in \mathcal{O}(D) \cap C^\infty(\bar{D})$  we have*

$$g(z) = \int_D g(\zeta) P^{N,2}(\zeta, z).$$

In order to find an upper bound for this kernel, we will need lower bound for  $1 + \langle \tilde{h}(\zeta, z), \zeta - z \rangle$ . This classical bound in the field is given by the following proposition. We include its proof for the reader convenience.

**Proposition 2.4.** *The following inequality holds uniformly for all  $\zeta$  and  $z$  in  $D$*

$$|\rho(\zeta) + \langle h(\zeta, z), \zeta - z \rangle| \gtrsim \delta(\zeta, z) + |\rho(\zeta)| + |\rho(z)|$$

*Proof:* We write  $z$  as  $z = \zeta + \lambda \eta_\zeta + \mu v_\zeta$  where  $\eta_\zeta$  is the unit outer normal and where  $v_\zeta$  belongs to  $T_\zeta^{\mathbb{C}} bD_{\rho(\zeta)}$ . With this notation,  $\delta(\zeta, z) \approx |\lambda| + |\mu|^2$ ,  $\text{Re } \lambda \approx \text{Re} \langle h(\zeta, z), \zeta - z \rangle$  and  $\text{Im } \lambda \approx \text{Im} \langle h(\zeta, z), \zeta - z \rangle$ .

Since  $\rho$  is convex, there exists  $c$  positive and small such that for all  $z$  and  $\zeta$  in  $D$

$$\begin{aligned} \rho(z) - \rho(\zeta) &\geq 2\text{Re}(\partial \rho(\zeta) \cdot (z - \zeta)) + c|\zeta - z|^2 \\ (4) \qquad \qquad \qquad &= 4\text{Re} \langle h(\zeta, z), \zeta - z \rangle + c|\zeta - z|^2. \end{aligned}$$

If  $\text{Re } \lambda < 0$ , we get from (4)

$$\begin{aligned} |\rho(\zeta) + \langle h(\zeta, z), \zeta - z \rangle| &\geq -\rho(\zeta) - \text{Re} \langle h(\zeta, z), \zeta - z \rangle + |\text{Im} \langle h(\zeta, z), \zeta - z \rangle| \\ &\gtrsim -\rho(z) - \rho(\zeta) + c|\zeta - z|^2 + |\lambda| \\ &\gtrsim \delta(\zeta, z) + |\rho(\zeta)| + |\rho(z)|. \end{aligned}$$

If  $\text{Re } \lambda > 0$ , (4) now yields

$$\begin{aligned} |\rho(\zeta) + \langle h(\zeta, z), \zeta - z \rangle| &\gtrsim -\rho(\zeta) - 2\text{Re} \langle h(\zeta, z), \zeta - z \rangle + \text{Re} \langle h(\zeta, z), \zeta - z \rangle + |\text{Im} \langle h(\zeta, z), \zeta - z \rangle| \\ &\gtrsim -\rho(z) - \rho(\zeta) + c|\zeta - z|^2 + |\lambda| \\ &\gtrsim \delta(\zeta, z) + |\rho(\zeta)| + |\rho(z)|. \quad \square \end{aligned}$$

We will also need an upper bound for  $\tilde{h}$  and thus for  $h$ . In order to get this bound, for a fixed  $z \in D$ , we write  $h$  in the Koranyi coordinates at  $z$ . We denote by  $(\zeta_1^*, \zeta_2^*)$  the Koranyi coordinates of  $\zeta$  at  $z$ . We set  $h_1^* = -\frac{1}{2} \frac{\partial \rho}{\partial \zeta_1^*}(\zeta)$  and  $h_2^* = -\frac{1}{2} \frac{\partial \rho}{\partial \zeta_2^*}(\zeta)$  so that



$h(\zeta, z) = \sum_{i=1,2} h_i^*(\zeta, z) d\zeta_i^*$ . The following Proposition is then a direct consequence of the smoothness of  $\rho$ .

**Proposition 2.5.** *For all  $\zeta \in \mathcal{P}_\varepsilon(z)$  we have uniformly with respect to  $z, \zeta$  and  $\varepsilon$*

- (i)  $|h_1^*(\zeta, z)| \lesssim 1, |h_2^*(\zeta, z)| \lesssim \varepsilon^{\frac{1}{2}},$
- (ii)  $\left| \frac{\partial h_k^*}{\partial \zeta_l^*}(\zeta, z) \right| \lesssim 1$  for  $k, l \in \{1, 2\}.$

### 3. CONSTRUCTION OF THE CURRENTS

If  $f_1$  and  $f_2$  are two holomorphic functions near the origin in  $\mathbb{C}^n$ , Mazzilli constructed in [16] two currents  $T$  and  $S$  such that  $f_1 T = 1, f_2 S = \bar{\partial} T$  and  $f_1 S = 0$  on a sufficiently small neighborhood  $\mathcal{U}$  of 0. He also proved that if  $T$  and  $S$  are any currents satisfying these three hypothesis, then any function  $g$  holomorphic on  $\mathcal{U}$  can be written as  $g = f_1 g_1 + f_2 g_2$  on  $\mathcal{U}$  if and only if  $g \bar{\partial} S = 0$ . Moreover,  $g_1$  and  $g_2$  can be explicitly written down using  $T$  and  $S$ .

Here, when  $f_1$  and  $f_2$  are holomorphic on a domain  $D$ , we first want to obtain a decomposition  $g = g_1 f_1 + g_2 f_2$  on the whole domain  $D$  and then secondly we want to obtain growth estimates on  $g_1$  and  $g_2$ . As a first approach, we could try to globalize the currents  $T$  and  $S$  of [16] in order to have a global decomposition. However, such an approach would fail to give the growth estimates we want.

In [16],  $f_1$  plays a leading role and  $T$  is constructed independently of  $f_2$ , using only  $f_1$ . Then  $S$  is constructed using  $f_1$  and  $f_2$ . If we assume for example that  $f_1$  vanishes at a point  $\zeta_0$  near  $bD$ , because  $T$  is constructed independently of  $f_2$ , it seems difficult to prove that  $g_1$  obtained using  $T$  is bounded except if we require that  $g$  vanishes at  $\zeta_0$  too; but considering  $g = f_2$ , we easily see that in general this condition is not necessary when one wants to write  $g$  as  $g = g_1 f_1 + g_2 f_2$  with  $g_1$  and  $g_2$  bounded for example. So the currents in [16] probably do not give a good decomposition.

Actually, it appears that the role of  $f_2$  must be emphasized in the construction of the currents near a boundary point  $\zeta_0$  such that  $f_1(\zeta_0) = 0$  and  $f_2(\zeta_0) \neq 0$ , or more generally when  $f_2$  is in some sense greater than  $f_1$  and conversely. Following this idea, we construct two currents  $T_1$  and  $T_2$  such that  $f_1 T_1 + f_2 T_2 = 1$  on  $D$ . These currents are defined locally and using a suitable partition of unity we glue together the local currents and get a global current. We now define these local currents.

Let  $\varepsilon_0$  be a small positive real number to be chosen later and let  $\zeta_0$  be a point in  $\bar{D}$ . We distinguish three cases.

First case: If  $\zeta_0$  belongs to  $D_{-\varepsilon_0}$ , i.e. if  $\zeta_0$  is far from the boundary, we do not need to be careful. Using Weierstrass' preparation theorem when  $\zeta_0$  belongs to  $X_1$ , we write  $f_1 = u_{0,1} P_{0,1}$  where  $u_{0,1}$  is a non vanishing holomorphic function in a neighborhood  $\mathcal{U}_0 \subset D_{-\frac{\varepsilon_0}{2}}$  of  $\zeta_0$  and  $P_{0,1}(\zeta) = \zeta_2^{i_{0,1}} + \zeta_2^{i_{0,1}-1} a_{0,1}^{(1)}(\zeta_1) + \dots + a_{0,1}^{(i_{0,1})}(\zeta_1), a_{0,1}^{(k)}$  holomorphic on  $\mathcal{U}_0$  for all  $k$ . If  $\zeta_0$  does not belong to  $X_1$ , we set  $P_{0,1} = 1, i_{0,1} = 0, u_{0,1} = f_1$  and we still have  $f_1 = u_{0,1} P_{0,1}$  with  $u_{0,1}$  which does not vanish on some neighborhood  $\mathcal{U}_0$  of  $\zeta_0$ .

For a smooth  $(2, 2)$ -form  $\varphi$  compactly supported in  $\mathcal{U}_0$  we set

$$\begin{aligned}\langle T_{0,1}, \varphi \rangle &= \frac{1}{c_0} \int_{\mathcal{U}_0} \frac{\overline{P_{0,1}(\zeta)}}{f_1(\zeta)} \frac{\partial^{i_{0,1}} \varphi}{\partial \bar{\zeta}_2^{i_{0,1}}}(\zeta), \\ \langle T_{0,2}, \varphi \rangle &= 0,\end{aligned}$$

where  $c_0$  is a suitable constant (see [16]). Integrating by parts we get  $f_1 T_{0,1} + f_2 T_{0,2} = 1$  on  $\mathcal{U}_0$ .

Second case: If  $\zeta_0$  belongs to  $bD \setminus (X_1 \cap X_2)$ , i.e. if  $\zeta_0$  is “far” from  $X_1 \cap X_2$ , without restriction we assume that  $f_1(\zeta_0) \neq 0$ . Let  $\mathcal{U}_0$  be a neighborhood of  $\zeta_0$  such that  $f_1$  does not vanish in  $\mathcal{U}_0$ . As in the first case when  $f_1(\zeta_0) \neq 0$ , we set  $P_{0,1} = 1$ ,  $i_{0,1} = 0$ ,  $u_{0,1} = f_1$  and for any smooth  $(2, 2)$ -form  $\varphi$  compactly supported in  $D \cap \mathcal{U}_0$  we put

$$\begin{aligned}\langle T_{0,1}, \varphi \rangle &= \frac{1}{c_0} \int_{\mathcal{U}_0} \frac{\overline{P_{0,1}(\zeta)}}{f_1(\zeta)} \frac{\partial^{i_{0,1}} \varphi}{\partial \bar{\zeta}_2^{i_{0,1}}}(\zeta), \\ \langle T_{0,2}, \varphi \rangle &= 0.\end{aligned}$$

where as previously  $c_0$  is a suitable constant. Again, we have  $f_1 T_{0,1} + f_2 T_{0,2} = 1$  on  $\mathcal{U}_0 \cap D$ .

Third case: If  $\zeta_0$  belongs to  $X_1 \cap X_2 \cap bD$ , the situation is more intricate. As in [1], we cover a neighborhood  $\mathcal{U}_0$  of  $\zeta_0$  by a family of polydiscs  $\mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$ ,  $j \in \mathbb{N}$  and  $k \in \{1, \dots, n_j\}$  such that:

- (i) For all  $j \in \mathbb{N}$ , and all  $k \in \{1, \dots, n_j\}$ ,  $z_{j,k}$  belongs to  $bD_{-(1-c\kappa)^j \varepsilon_0}$  where  $c$  is small positive real constant.
- (ii) For all  $j \in \mathbb{N}$ , all  $k, l \in \{1, \dots, n_j\}$ ,  $k \neq l$ , we have  $\delta(z_{j,k}, z_{j,l}) \geq c\kappa(1 - c\kappa)^j \varepsilon_0$ .
- (iii) For all  $j \in \mathbb{N}$ , all  $z \in bD_{-(1-c\kappa)^j \varepsilon_0}$ , there exists  $k \in \{1, \dots, n_j\}$  such that  $\delta(z, z_{j,k}) < c\kappa(1 - c\kappa)^j \varepsilon_0$ ,
- (iv)  $D \cap \mathcal{U}_0$  is included in  $\cup_{j=0}^{+\infty} \cup_{k=1}^{n_j} \mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$ ,
- (v) there exists  $M \in \mathbb{N}$  such that for  $z \in D \setminus D_{-\varepsilon_0}$ ,  $\mathcal{P}_{4\kappa|\rho(z)|}(z)$  intersect at most  $M$  Koranyi balls  $\mathcal{P}_{4\kappa|\rho(z_{j,k})|}(z_{j,k})$ .

Such a family of polydiscs will be called a  $\kappa$ -covering.

We define on each polydisc  $\mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$  two currents  $T_{0,1}^{(j,k)}$  and  $T_{0,2}^{(j,k)}$  such that  $f_1 T_{0,1}^{(j,k)} + f_2 T_{0,2}^{(j,k)} = 1$  as follows. We denote by  $\Delta_\xi(\varepsilon)$  the disc of center  $\xi$  and radius  $\varepsilon$  and by  $(\zeta_{0,1}^*, \zeta_{0,2}^*)$  the coordinates of  $\zeta_0$  in the Koranyi basis at  $z_{j,k}$ . In [1] were proved the next two propositions:

**Proposition 3.1.** *If  $\kappa > 0$  is small enough and if  $\mathcal{P}_{4\kappa|\rho(z_{j,k})|}(z_{j,k}) \cap X_l \neq \emptyset$  then  $|\zeta_{0,1}^*| \geq 4\kappa|\rho(z_{j,k})|$ .*

We assume  $\kappa$  so small that Proposition 3.1 holds for both  $X_1$  and  $X_2$  with the same  $\kappa$ . When  $|\zeta_{0,1}^*| \geq 4\kappa|\rho(z_{j,k})|$  then  $X_l$  can be parametrized as follows (see [1]):

**Proposition 3.2.** *If  $|\zeta_{0,1}^*| \geq 4\kappa|\rho(z_{j,k})|$ , for  $l = 1$  and  $l = 2$ , there exists  $p_l$  functions  $\alpha_{l,1}^{(j,k)}, \dots, \alpha_{l,p_l}^{(j,k)}$  holomorphic on  $\Delta_0(4\kappa|\rho(z_{j,k})|)$ , there exists  $r > 0$ , depending neither on  $j$  nor on  $k$ , and there exists  $u_l^{(j,k)}$  holomorphic on the ball of center  $\zeta_0$  and radius  $r$ , bounded and bounded away from 0, such that:*

- (i)  $\frac{\partial \alpha_{l,i}^{(j,k)}}{\partial \zeta_1^*}$  is bounded on  $\Delta_0(4\kappa|\rho(z_{j,k})|)$  uniformly with respect to  $j$  and  $k$ ,
- (ii) for all  $\zeta \in \mathcal{P}_{4\kappa|\rho(z_{j,k})|}(z_{j,k})$ ,  $f_l(\zeta) = u_l^{(j,k)}(\zeta) \prod_{i=1}^{p_l} (\zeta_2^* - \alpha_{l,i}^{(j,k)}(\zeta_1^*))$ .

Now we define  $T_{0,1}^{(j,k)}$  and  $T_{0,2}^{(j,k)}$  with the following settings.

If  $|\zeta_{0,1}^*| < 4\kappa|\rho(z_{j,k})|$ , then for  $l = 1$  or  $l = 2$ ,  $\mathcal{P}_{4\kappa|\rho(z_{j,k})|}(z_{j,k}) \cap X_l = \emptyset$ , which means that  $z_{j,k}$  is “far” from  $X_1$  and  $X_2$ . In this case we set for  $l = 1$  and  $l = 2$ :

$$\begin{aligned} I_l^{(j,k)} &:= \emptyset, \\ i_l^{(j,k)} &:= 0, \\ P_l^{(j,k)}(\zeta) &:= 1. \end{aligned}$$

If  $|\zeta_{0,1}^*| \geq 4\kappa|\rho(z_{j,k})|$ , then we may have  $\mathcal{P}_{4\kappa|\rho(z_{j,k})|}(z_{j,k}) \cap X_l \neq \emptyset$  for  $l = 1$  or  $l = 2$ . In that case we set for  $l = 1$  and  $l = 2$ :

$$\begin{aligned} I_l^{(j,k)} &:= \{i, \exists z_1^* \in \mathbb{C}, |z_1^*| < 2\kappa|\rho(z_{j,k})| \text{ and } |\alpha_{l,i}^{(j,k)}(z_1^*)| < (\frac{5}{2}\kappa|\rho(z_{j,k})|)^{\frac{1}{2}}\}, \\ i_l^{(j,k)} &:= \#I_l^{(j,k)}, \text{ the cardinal of } I_l^{(j,k)}, \\ P_l^{(j,k)}(\zeta) &:= \prod_{i \in I_l^{(j,k)}} (\zeta_2^* - \alpha_{l,i}^{(j,k)}(\zeta_1^*)). \end{aligned}$$

In both case we set

$$\begin{aligned} \mathcal{U}_1^{(j,k)} &:= \left\{ \zeta \in \mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k}), \left| \frac{f_1(\zeta)|\rho(z_{j,k})|^{\frac{i_1^{(j,k)}}{2}}}{P_1^{(j,k)}(\zeta)} \right| > \frac{1}{3} \left| \frac{f_2(\zeta)|\rho(z_{j,k})|^{\frac{i_2^{(j,k)}}{2}}}{P_2^{(j,k)}(\zeta)} \right| \right\}, \\ \mathcal{U}_2^{(j,k)} &:= \left\{ \zeta \in \mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k}), \frac{2}{3} \left| \frac{f_2(\zeta)|\rho(z_{j,k})|^{\frac{i_2^{(j,k)}}{2}}}{P_2^{(j,k)}(\zeta)} \right| > \left| \frac{f_1(\zeta)|\rho(z_{j,k})|^{\frac{i_1^{(j,k)}}{2}}}{P_1^{(j,k)}(\zeta)} \right| \right\}, \end{aligned}$$

so that  $\mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k}) = \mathcal{U}_1^{(j,k)} \cup \mathcal{U}_2^{(j,k)}$ .

These open sets are designed in order to quantify where  $f_1$  is “bigger” than  $f_2$  and conversely. The idea is the following.

If  $i$  belongs to  $I_l^{(j,k)}$  then  $|\zeta_2^* - \alpha_{l,i}^{(j,k)}(\zeta_1^*)| \lesssim |\rho(z_{j,k})|^{\frac{1}{2}}$  for all  $\zeta \in \mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$ . Thus each zero of  $f_l$  in  $\mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$  brings in some sense a factor  $|\rho(z_{j,k})|^{\frac{1}{2}}$  in  $f_l(\zeta)$ . In the definition of  $\mathcal{U}_l^{(j,k)}$ , we take into account the zeros of  $f_1$  and  $f_2$  which are in the polydisc

$\mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$  with the term  $|\rho(z_{j,k})|^{\frac{i_1^{(j,k)}}{2}}$  and  $|\rho(z_{j,k})|^{\frac{i_2^{(j,k)}}{2}}$ . This means in particular that all the zeros in the polydisc are treated in the same way, we don’t care if they are close from each others, from the boundary of the polydisc or not. The zeros which are outside the polydisc are taken into account by  $\frac{f_l(\zeta)}{P_l^{(j,k)}(\zeta)}$ , which will also measure how far they are from the polydisc.

Therefore,  $\mathcal{U}_1^{(j,k)}$  is the open set where  $f_1$  is bigger than  $f_2$  for an order such that the zeros which are outside of the polydisc are taken into account with the term  $\frac{f_1(\zeta)}{P_1^{(j,k)}(\zeta)}$  and the

zeros which are inside with the term  $|\rho(z_{j,k})|^{\frac{i_l^{(j,k)}}{2}}$ .

For  $l = 1, 2$  and for a smooth  $(2, 2)$ -form  $\varphi$  compactly supported in  $\mathcal{U}_l^{(j,k)}$  we set

$$\langle T_{0,l}^{(j,k)}, \varphi \rangle := \int_{\mathbb{C}^2} \frac{\overline{P_l^{(j,k)}(\zeta)}}{f_l(\zeta)} \frac{\partial^{i_l^{(j,k)}} \varphi}{\partial \overline{\zeta_2}^{i_l^{(j,k)}}}(\zeta).$$

Integrating  $i_l^{(j,k)}$ -times by parts, we get  $f_l T_{0,l}^{(j,k)} = c_l^{(j,k)}$  on  $\mathcal{U}_l^{(j,k)}$  where  $c_l^{(j,k)}$  is an integer bounded by  $i_l^{(j,k)}$ ! (see [16]).

Now we glue together the currents  $T_{0,l}^{(j,k)}$  in order to define the current  $T_{0,l}$ ,  $l = 1, 2$ , such that  $f_1 T_{0,1} + f_2 T_{0,2} = 1$  on  $D \cap \mathcal{U}_0$ . Let  $(\tilde{\chi}_{j,k})_{\substack{j \in \mathbb{N} \\ k \in \{1, \dots, n_j\}}}$  be a partition of unity subordinated to the covering  $(\mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k}))_{\substack{j \in \mathbb{N} \\ k \in \{1, \dots, n_j\}}}$  of  $\mathcal{U}_0$ . Without restriction, we assume that  $\left| \frac{\partial^{\alpha+\beta+\bar{\alpha}+\bar{\beta}} \tilde{\chi}_{j,k}}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta} \partial \bar{\zeta}_1^{*\bar{\alpha}} \partial \bar{\zeta}_2^{*\bar{\beta}}}(\zeta) \right| \lesssim \frac{1}{|\rho(z_{j,k})|^{\alpha+\bar{\alpha}+\frac{\beta+\bar{\beta}}{2}}}$ . Let also  $\chi$  be a smooth function on  $\mathbb{C}^2 \setminus \{0\}$  such that  $\chi(z_1, z_2) = 1$  if  $|z_1| > \frac{2}{3}|z_2|$  and  $\chi(z_1, z_2) = 0$  if  $|z_1| < \frac{1}{3}|z_2|$  and let us define

$$\begin{aligned} \chi_1^{(j,k)}(\zeta) &= \tilde{\chi}_{j,k}(\zeta) \cdot \chi \left( \frac{f_1(\zeta)|\rho(z_{j,k})|^{\frac{i_1^{(j,k)}}{2}}}{P_1^{(j,k)}(\zeta)}, \frac{f_2(\zeta)|\rho(z_{j,k})|^{\frac{i_2^{(j,k)}}{2}}}{P_2^{(j,k)}(\zeta)} \right), \\ \chi_2^{(j,k)}(\zeta) &= \tilde{\chi}_{j,k}(\zeta) \cdot \left( 1 - \chi \left( \frac{f_1(\zeta)|\rho(z_{j,k})|^{\frac{i_1^{(j,k)}}{2}}}{P_1^{(j,k)}(\zeta)}, \frac{f_2(\zeta)|\rho(z_{j,k})|^{\frac{i_2^{(j,k)}}{2}}}{P_2^{(j,k)}(\zeta)} \right) \right). \end{aligned}$$

For  $l = 1$  and  $l = 2$ , the support of  $\chi_l^{(j,k)}$  is included in  $\mathcal{U}_l^{(j,k)}$  so we can put

$$T_{0,l} = \sum_{\substack{j \in \mathbb{N} \\ k \in \{1, \dots, n_j\}}} \frac{1}{c_l^{(j,k)}} \chi_l^{(j,k)} T_{0,l}^{(j,k)}$$

and we have  $f_1 T_{0,1} + f_2 T_{0,2} = 1$  on  $\mathcal{U}_0 \cap D$ .

Now for all  $\zeta_0 \in bD \cup \overline{D}_{-\varepsilon_0}$  we have constructed a neighborhood  $\mathcal{U}_0$  of  $\zeta_0$  and two currents  $T_{0,1}$  and  $T_{0,2}$  such that  $f_1 T_{0,1} + f_2 T_{0,2} = 1$  on  $\mathcal{U}_0 \cap D$ . If  $\varepsilon_0 > 0$  is sufficiently small, we can cover  $\overline{D}$  by finitely many open sets  $\mathcal{U}_1, \dots, \mathcal{U}_n$ . Let  $\chi_1, \dots, \chi_n$  be a partition of unity subordinated to this family of open sets and  $T_{1,1}, \dots, T_{n,1}$  and  $T_{1,2}, \dots, T_{n,2}$  be the corresponding currents defined on  $\mathcal{U}_1, \dots, \mathcal{U}_n$ . We glue together this current and we set

$$T_1 = \sum_{j=1}^n \chi_j T_{j,1} \text{ and } T_2 = \sum_{j=1}^n \chi_j T_{j,2},$$

so that  $f_1 T_1 + f_2 T_2 = 1$  on  $D$ . Moreover  $T_1$  and  $T_2$  are currents supported in  $\overline{D}$  thus they have a finite order  $k_2$  and we can apply  $T_1$  and  $T_2$  to functions of class  $C^{k_2}$  with support in  $\overline{D}$ . This gives  $k_2$  from Theorem 1.1.

#### 4. THE DIVISION FORMULA

In this part, given any two currents  $T_1$  and  $T_2$  of order  $k_2$  such that  $f_1 T_1 + f_2 T_2 = 1$ , assuming that  $g$  is a holomorphic function on  $D$  which belongs to the ideal generated by  $f_1$  and  $f_2$ , and which can be written as  $g = \tilde{g}_1 f_1 + \tilde{g}_2 f_2$ , where  $\tilde{g}_1$  and  $\tilde{g}_2$  are two  $C^\infty$ -smooth functions on  $D$  such that  $|\rho|^N \tilde{g}_1$  and  $|\rho|^N \tilde{g}_2$  vanish to order  $k_2$  on  $bD$  for some  $N \in \mathbb{N}$  sufficiently big, we write  $g$  as  $g = g_1 f_1 + g_2 f_2$  with  $g_1$  and  $g_2$  holomorphic on  $D$ . We point out that the formula we get is valid for any  $T_1$  and  $T_2$  of order  $k_2$  such that  $f_1 T_1 + f_2 T_2 = 1$ .

Under our assumptions, for  $k = 1$  and  $k = 2$  and all fixed  $z \in D$ ,  $\tilde{g}_1 P^{N,k}(\cdot, z)$  and  $\tilde{g}_2 P^{N,k}(\cdot, z)$  can be extended by zero outside  $D$  and are of class  $C^{k_2}$  on  $\mathbb{C}^2$ . So we can apply  $T_1$  and  $T_2$  to  $\tilde{g}_1 P^{N,k}(\cdot, z)$  and  $\tilde{g}_2 P^{N,k}(\cdot, z)$ .

For  $l = 1, 2$ , we denote by  $b_l = b_{l,1} d\zeta_1 + b_{l,2} d\zeta_2$  a  $(1, 0)$ -form such that  $f_l(z) - f_l(\zeta) = \sum_{i=1,2} b_{l,i}(\zeta, z)(z_i - \zeta_i)$ . For the estimates, we will take  $b_{l,i}(\zeta, z) = \int_0^1 \frac{\partial f_l}{\partial \zeta_i}(\zeta + t(z - \zeta)) dt$ , but this is not necessary to get a division formula.

In order to construct the formula, we will need the following lemma which was proved in [15], Lemma 3.1:

**Lemma 4.1.** *Let  $Q = \sum_{i=1}^n Q_i d\zeta_i$  be a  $(1, 0)$  form of  $\mathbb{C}^n$ , let  $H_1, \dots, H_p$  be  $p$   $(1, 0)$ -forms in  $\mathbb{C}^n$  and let  $W_1, \dots, W_{p-1}$  be  $p-1$   $(0, 1)$ -forms in  $\mathbb{C}^n$ . Then the following equality holds*

$$\begin{aligned} \bar{\partial}(\langle Q, z - \zeta \rangle) (\bar{\partial} Q)^{n-p} \wedge H_p \wedge \bigwedge_{k=1}^{p-1} W_k \wedge H_k \\ = \frac{1}{n-p+1} \langle H_p, z - \zeta \rangle (\bar{\partial} Q)^{n-p+1} \wedge \bigwedge_{k=1}^{p-1} W_k \wedge H_k \\ + \frac{1}{n-p+1} \sum_{l=1}^{p-1} \langle H_l, z - \zeta \rangle (\bar{\partial} Q)^{n-p+1} H_p \wedge W_l \wedge \bigwedge_{\substack{k=1 \\ k \neq l}}^{p-1} W_k \wedge H_k. \end{aligned}$$

We now establish the division formula. From Theorem 2.3, we have for all  $z \in D$ :

$$g(z) = \int_D g(\zeta) P^{N,2}(\zeta, z)$$

and since  $g = \tilde{g}_1 f_1 + \tilde{g}_2 f_2$

$$\begin{aligned} g(z) &= f_1(z) \int_D \tilde{g}_1(\zeta) P^{N,2}(\zeta, z) + f_2(z) \int_D \tilde{g}_2(\zeta) P^{N,2}(\zeta, z) \\ (5) \quad &+ \int_D \tilde{g}_1(\zeta) (f_1(\zeta) - f_1(z)) P^{N,2}(\zeta, z) + \int_D \tilde{g}_2(\zeta) (f_2(\zeta) - f_2(z)) P^{N,2}(\zeta, z). \end{aligned}$$

Now from Lemma 4.1, there exists  $\tilde{c}_{N,2}$  such that

$$(f_1(\zeta) - f_1(z)) P^{N,2}(\zeta, z) = \tilde{c}_{N,2} b_1(\zeta, z) \wedge \bar{\partial} P^{N,1}(\zeta, z)$$

and since by assumption  $\tilde{g}_1 P^{N,1}$  vanishes on  $bD$ , Stokes' Theorem yields

$$(6) \quad \int_D \tilde{g}_1(\zeta) (f_1(\zeta) - f_1(z)) P^{N,2}(\zeta, z) = \tilde{c}_{N,2} \int_D \bar{\partial} \tilde{g}_1(\zeta) \wedge b_1(\zeta, z) \wedge P^{N,1}(\zeta, z).$$

We now use the fact that  $f_1T_1 + f_2T_2 = 1$  in order to rewrite this former integral:

$$\begin{aligned}
& \int_D \bar{\partial}\tilde{g}_1(\zeta) \wedge b_1(\zeta, z) \wedge P^{N,1}(\zeta, z) \\
&= \langle f_1T_1 + f_2T_2, \bar{\partial}\tilde{g}_1 \wedge b_1(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle \\
&= \langle f_1T_1, \bar{\partial}\tilde{g}_1 \wedge b_1(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle + f_2(z) \langle T_2, \bar{\partial}\tilde{g}_1 \wedge b_1(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle \\
(7) \quad &+ \langle T_2, (f_2 - f_2(z)) \bar{\partial}\tilde{g}_1 \wedge b_1(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle.
\end{aligned}$$

Again from Lemma 4.1, there exists  $\tilde{c}_{N,1}$  such that

$$\begin{aligned}
& (f_2(\zeta) - f_2(z)) b_1(\zeta, z) \wedge \bar{\partial}\tilde{g}_1 \wedge P^{N,1}(\zeta, z) - (f_1(\zeta) - f_1(z)) b_2(\zeta, z) \wedge \bar{\partial}\tilde{g}_1 \wedge P^{N,1}(\zeta, z) \\
&= \tilde{c}_{N,1} b_1(\zeta, z) \wedge b_2(\zeta, z) \wedge \bar{\partial}\tilde{g}_1 \wedge \bar{\partial}P^{N,0}(\zeta, z).
\end{aligned}$$

So

$$\begin{aligned}
& \langle T_2, (f_2 - f_2(z)) \bar{\partial}\tilde{g}_1 \wedge b_1(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle \\
&= -f_1(z) \langle T_2, \bar{\partial}\tilde{g}_1 \wedge b_2(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle + \langle T_2, f_1 \bar{\partial}\tilde{g}_1 \wedge b_2(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle \\
(8) \quad &+ \tilde{c}_{N,1} \langle T_2, \bar{\partial}\tilde{g}_1 \wedge b_1(\cdot, z) \wedge b_2(\cdot, z) \wedge \bar{\partial}P^{N,0}(\cdot, z) \rangle
\end{aligned}$$

We plug together (6), (7) and (8) and their analogue for  $\int_D g_2(\zeta) (f_2(\zeta) - f_2(z)) P^{N,2}(\zeta, z)$  in (5) and we get

$$\begin{aligned}
g(z) &= f_1(z) \int_D \tilde{g}_1(\zeta) P^{N,2}(\zeta, z) - \tilde{c}_{N,2} f_1(z) \langle T_2, \bar{\partial}\tilde{g}_1 \wedge b_2(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle \\
&+ \tilde{c}_{N,2} f_2(z) \langle T_2, \bar{\partial}\tilde{g}_1 \wedge b_1(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle \\
&+ f_2(z) \int_D \tilde{g}_2(\zeta) P^{N,2}(\zeta, z) - \tilde{c}_{N,2} f_2(z) \langle T_1, \bar{\partial}\tilde{g}_2 \wedge b_1(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle \\
&+ \tilde{c}_{N,2} f_1(z) \langle T_1, \bar{\partial}\tilde{g}_2 \wedge b_2(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle \\
(9) \quad &+ \tilde{c}_{N,2} \langle T_1, f_1 \bar{\partial}\tilde{g}_1 \wedge b_1(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle + \tilde{c}_{N,2} \langle T_2, f_1 \bar{\partial}\tilde{g}_1 \wedge b_2(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle \\
(10) \quad &+ \tilde{c}_{N,2} \langle T_2, f_2 \bar{\partial}\tilde{g}_2 \wedge b_2(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle + \tilde{c}_{N,2} \langle T_1, f_2 \bar{\partial}\tilde{g}_2 \wedge b_1(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle \\
&+ \tilde{c}_{N,2} \tilde{c}_{N,1} \langle \bar{\partial}\tilde{g}_1 \wedge T_2 - \bar{\partial}\tilde{g}_2 \wedge T_1, b_1(\cdot, z) \wedge b_2(\cdot, z) \wedge \bar{\partial}P^{N,0}(\cdot, z) \rangle
\end{aligned}$$

Now since  $\bar{\partial}g = f_1\bar{\partial}\tilde{g}_1 + f_2\bar{\partial}\tilde{g}_2 = 0$ , the line (9) and (10) vanish. Therefore in order to get our division formula, it suffices to prove that  $\bar{\partial}(\bar{\partial}\tilde{g}_1 \wedge T_2 - \bar{\partial}\tilde{g}_2 \wedge T_1) = 0$ .

When  $X_1 \cap X_2$  is not a complete intersection and when assumption (iv) in the introduction is satisfied by  $\tilde{g}_1$  and  $\tilde{g}_2$ , one can prove that  $\bar{\partial}\tilde{g}_1 \wedge \bar{\partial}T_2 = 0$  and  $\bar{\partial}\tilde{g}_2 \wedge \bar{\partial}T_1 = 0$ .

When  $X_1 \cap X_2$  is a complete intersection, we prove that for any  $\zeta_0 \in D$  there exists a neighborhood  $\mathcal{U}_0$  of  $\zeta_0$  such that for all (2, 1)-form  $\varphi$ , smooth and supported in  $\mathcal{U}_0$ , we have  $\langle \bar{\partial}\tilde{g}_1 \wedge T_2 - \bar{\partial}\tilde{g}_2 \wedge T_1, \bar{\partial}\varphi \rangle = 0$ .

Let  $\zeta_0$  be a point in  $D$ . By assumption on  $g$ , there exists a neighborhood  $\mathcal{U}_0$  of  $\zeta_0$  and two holomorphic functions  $\gamma_1$  and  $\gamma_2$  such that  $g = \gamma_1 f_1 + \gamma_2 f_2$  on  $\mathcal{U}_0$ . We now use the following lemma whose proof is postponed to the end of this section:

**Lemma 4.2.** *Let  $f_1$  and  $f_2$  be two holomorphic functions defined in a neighborhood of 0 in  $\mathbb{C}^2$ ,  $X_1 = \{z, f_1(z) = 0\}$  and  $X_2 = \{z, f_2(z) = 0\}$ . We assume that  $X_1 \cap X_2$  is a complete intersection and that 0 belongs to  $X_1 \cap X_2$ . Let  $\varphi_1$  and  $\varphi_2$  be two  $C^\infty$ -smooth*

functions such that  $f_1\varphi_1 = f_2\varphi_2$ .

Then,  $\frac{\varphi_1}{f_2}$  and  $\frac{\varphi_2}{f_1}$  are  $C^\infty$ -smooth in a neighborhood of 0.

Lemma 4.2 implies that the function  $\psi = \frac{\tilde{g}_1 - \gamma_1}{f_2} = \frac{\gamma_2 - \tilde{g}_2}{f_1}$  is smooth on a perhaps smaller neighborhood of  $\zeta_0$  still denoted by  $\mathcal{U}_0$ . Thus

$$\begin{aligned} \langle \bar{\partial}\tilde{g}_1 \wedge T_2 - \bar{\partial}\tilde{g}_2 \wedge T_1, \bar{\partial}\varphi \rangle &= \langle \bar{\partial}(\tilde{g}_1 - \gamma_1) \wedge T_2 + \bar{\partial}(\gamma_2 - \tilde{g}_2) \wedge T_1, \bar{\partial}\varphi \rangle \\ &= \langle \bar{\partial}(f_2\psi) \wedge T_2 + \bar{\partial}(f_1\psi) \wedge T_1, \bar{\partial}\varphi \rangle \\ &= \langle f_2T_2 + f_1T_1, \bar{\partial}\psi \wedge \bar{\partial}\varphi \rangle \\ &= \int_{\mathcal{U}_0} \bar{\partial}\psi \wedge \bar{\partial}\varphi \end{aligned}$$

and since  $\varphi$  is supported in  $\mathcal{U}_0$  we have  $\int_{\mathcal{U}_0} \bar{\partial}\psi \wedge \bar{\partial}\varphi = -\int_{\mathcal{U}_0} d(\varphi\bar{\partial}\psi) = 0$  and so

$$\langle \bar{\partial}\tilde{g}_1 \wedge T_2 - \bar{\partial}\tilde{g}_2 \wedge T_1, \bar{\partial}\varphi \rangle = 0.$$

Now we set

$$\begin{aligned} g_1(z) &= \int_D \tilde{g}_1(\zeta) P^{N,2}(\zeta, z) \\ &\quad + \tilde{c}_{N,2} (\langle T_1, \bar{\partial}\tilde{g}_2 \wedge b_2(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle - \langle T_2, \bar{\partial}\tilde{g}_1 \wedge b_2(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle) \\ g_2(z) &= \int_D \tilde{g}_2(\zeta) P^{N,2}(\zeta, z) \\ &\quad + \tilde{c}_{N,2} (\langle T_2, \bar{\partial}\tilde{g}_1 \wedge b_1(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle - \langle T_1, \bar{\partial}\tilde{g}_2 \wedge b_1(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle) \end{aligned}$$

and we have

$$g = g_1f_1 + g_2f_2$$

with  $g_1$  and  $g_2$  holomorphic on  $D$ . We notice that if  $\tilde{g}_1$  and  $\tilde{g}_2$  are already holomorphic functions then  $g_1 = \tilde{g}_1$  and  $g_2 = \tilde{g}_2$ .

*Proof of Lemma 4.2:* Maybe after a unitary change of coordinates if needed, using Weierstrass' preparation Theorem, we can assume that for  $l = 1, 2$ , the function  $f_l$  is given by  $f_l(z, w) = z^{k_l} + a_1^{(l)}(w)z^{k_l-1} + \dots + a_{k_l}^{(l)}(w)$  where  $a_1^{(l)}, \dots, a_{k_l}^{(l)}$  are holomorphic near 0 and vanish at 0. Moreover, since the intersection  $X_1 \cap X_2$  is transverse,  $P_1$  and  $P_2$  are relatively prime. Thus there exists two polynomials  $\alpha_1$  and  $\alpha_2$  with holomorphic coefficients in  $w$  and a function  $\beta$  of  $w$  not identically zero such that

$$\alpha_1(z, w)f_1(z, w) + \alpha_2(z, w)f_2(z, w) = \beta(w).$$

Multiplying this equality by  $\varphi_1$  we get

$$f_2(\alpha_1\varphi_2 + \alpha_2\varphi_1) = \beta\varphi_1.$$

We now prove that  $\beta$  divides the function  $\psi := \alpha_1\varphi_2 + \alpha_2\varphi_1$ .

If  $\beta(0) \neq 0$ , there is nothing to do. Otherwise, since  $\beta$  is not identically zero, there exists  $k \in \mathbb{N}$  such that  $\beta(w) = w^k\gamma(w)$  where  $\gamma(0) \neq 0$ .

For all  $j \in \mathbb{N}$  we have

$$(11) \quad f_2(z, w) \frac{\partial^j \psi}{\partial \bar{w}^j}(z, w) = \beta(w) \frac{\partial \varphi_1}{\partial \bar{w}^j}(z, w)$$

and for  $w = 0$  and all  $z$  we thus get  $\frac{\partial^j \psi}{\partial \bar{w}^j}(z, 0) = 0$ .

By induction we then deduce from (11) that  $\frac{\partial^{i+j} \psi}{\partial w^i \partial \bar{w}^j}(z, 0) = 0$  for all  $i \in \{0, \dots, k-1\}$  and all  $j \in \mathbb{N}$ . For any integer  $n \geq k$  we therefore can write for all  $z$  and all  $w$

$$\frac{\psi(z, w)}{w^k} = \sum_{\substack{k \leq i+j \leq n \\ i \geq k}} w^{i-k} \bar{w}^j \frac{\partial^{i+j} \psi}{\partial w^i \partial \bar{w}^j}(z, 0) + \sum_{i+j=n+1} w^{i-k} \bar{w}^j \int_0^1 \frac{\partial^{n+1} \psi}{\partial w^i \partial \bar{w}^j}(z, tw) dt.$$

Now, it is easy to check by induction that the function  $w \mapsto \frac{\bar{w}^{i+j}}{w^i}$  is of class  $C^{j-1}$  for all positive integer  $j$  and all non negative integer  $i$ . This implies that  $\frac{\psi(z, w)}{w^k}$  is of class  $C^n$  for all positive integer  $n$  and therefore  $\frac{\varphi_1}{f_2} = \frac{\psi}{\beta}$  is of class  $C^\infty$ .  $\square$

## 5. PROOF OF THE MAIN RESULT

In order to prove Theorem 1.1, for any  $k$  and  $l$  in  $\{1, 2\}$  and any  $q \in [1, +\infty]$ , we have to prove that if  $h$  is a smooth function such that, for all non-negative integers  $\alpha$  and  $\beta$ ,  $\left| \frac{\partial^{\alpha+\beta} h}{\partial \bar{\eta}_\zeta^\alpha \partial v_\zeta^\beta} \right| |\rho|^{\alpha+\frac{\beta}{2}}$  belongs to  $L^q(D)$ , then the function

$$z \mapsto \langle T_l, \bar{\partial} h \wedge b_k(\cdot, z) \wedge P^{N,1}(\cdot, z) \rangle$$

belongs to  $L^q(D)$  if  $q < \infty$  and to  $BMO(D)$  if  $q = +\infty$ .

As usually, since the modulus of the denominator in  $P^{N,1}$  is greater than  $|\rho(z)| + |\rho(\zeta)| + \delta(z, \zeta)$ , the difficulties occurs when we integrate for  $\zeta$  near  $z$  and when  $z$  is near  $bD$ . Moreover, by construction of  $T_1$  and  $T_2$ , the main difficulty is when, in addition,  $z$  is near a point  $\zeta_0$  which belongs to  $bD \cap X_1 \cap X_2$  and we only consider that case.

We assume that  $z$  belongs to the neighborhood  $\mathcal{U}_0$  of a point  $\zeta_0 \in bD \cap X_1 \cap X_2$  and we use the same notations as in Section 3 for the construction of the currents. Moreover, we assume that the *Koranyi* basis at  $\zeta_0$  is the canonical basis of  $\mathbb{C}^2$  and that  $\zeta_0$  is the origin of  $\mathbb{C}^2$ .

We will need an upper bound of  $\frac{P_l^{(j,k)}}{f_l} \frac{\partial^{\alpha+\beta} f_l}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}}$  in order to estimate  $\frac{P_l^{(j,k)}}{f_l} b_m$  and the derivatives of  $\chi_l^{(j,k)}$ . We set  $Q_l^{(j,k)} = \frac{f_l}{P_l^{(j,k)}}$  and we begin with the following lemma:

**Lemma 5.1.** *For all  $j \in \mathbb{N}$ , all  $k \in \{1, \dots, n_j\}$ , all  $\alpha$  and  $\beta$  in  $\mathbb{N}$ ,  $l = 1, 2$ , and all  $\zeta$  in  $\mathcal{P}_{2\kappa|\rho(z_{j,k})|}(z_{j,k})$ , we have uniformly with respect to  $j, k, l$ , and  $\zeta$*

$$\left| \frac{1}{Q_l^{(j,k)}(\zeta)} \frac{\partial^{\alpha+\beta}}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}} \left( Q_l^{(j,k)}(\zeta) \right) \right| \lesssim |\rho(z_{j,k})|^{-\alpha-\frac{\beta}{2}}.$$

*Proof:* We denote by  $(\zeta_{0,1}^*, \zeta_{0,2}^*)$  the coordinates of  $\zeta_0$  in the *Koranyi* coordinates at  $z_{j,k}$ . The definition of  $P_l^{(j,k)}$  forces us to distinguish three cases:

*First case:* If  $|\zeta_{0,1}^*| > 4\kappa|\rho(z_{j,k})|$ , let  $\alpha_{l,i}^{(j,k)}$ ,  $i = 1, \dots, p_l$ , be the family of parametrization given by Proposition 3.2. In this case, we actually seek an upper bound for

$$\frac{1}{\prod_{i \notin I_l^{(j,k)}} \left( \zeta_2^* - \alpha_{l,i}^{(j,k)}(\zeta_1^*) \right)} \frac{\partial^{\alpha+\beta}}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}} \left( \prod_{i \notin I_l^{(j,k)}} \left( \zeta_2^* - \alpha_{l,i}^{(j,k)}(\zeta_1^*) \right) \right),$$



and it suffices to prove for all  $i \notin I_l^{(j,k)}$  and all  $\alpha$  and  $\beta$  that

$$(12) \quad \left| \frac{1}{\zeta_2^* - \alpha_{l,i}^{(j,k)}(\zeta_1^*)} \frac{\partial^{\alpha+\beta}}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}} (\zeta_2^* - \alpha_{l,i}^{(j,k)}(\zeta_1^*)) \right| \lesssim |\rho(z_{j,k})|^{-\alpha - \frac{\beta}{2}}.$$

By definition of  $I_l^{(j,k)}$ , we have  $|\alpha_{l,i}^{(j,k)}(\zeta_1^*)| \geq (\frac{5}{2}\kappa|\rho(z_{j,k})|)^{\frac{1}{2}}$  for all  $\zeta_1^* \in \Delta_0(2\kappa|\rho(z_{j,k})|)$  so  $|\zeta_2^* - \alpha_{l,i}^{(j,k)}(\zeta_1^*)| \gtrsim |\rho(z_{j,k})|^{\frac{1}{2}}$  and (12) holds true for  $\alpha = 0$  and  $\beta = 1$ .

According to Proposition 3.2,  $\frac{\partial \alpha_{l,i}^{(j,k)}}{\partial \zeta_1^*}$  is uniformly bounded on  $\Delta_0(4\kappa|\rho(z_{j,k})|)$ . Cauchy's inequalities then yields  $\left| \frac{\partial^\alpha \alpha_{l,i}^{(j,k)}}{\partial \zeta_1^{*\alpha}}(\zeta_1^*) \right| \lesssim |\rho(z_{j,k})|^{1-\alpha}$ . Since  $|\zeta_2^* - \alpha_{l,i}^{(j,k)}(\zeta_1^*)| \gtrsim |\rho(z_{j,k})|^{\frac{1}{2}}$ , (12) holds true for  $\alpha > 0$  and  $\beta = 0$ . Since the other cases are trivial, we are done in this case.

When  $|\zeta_{0,1}^*| < 4\kappa|\rho(z_{j,k})|$ , we do not have a parametrization of  $X_l$  but according to proposition 3.1,  $\mathcal{P}_{4\kappa|\rho(z_{j,k})|}(z_{j,k}) \cap X_l$  is empty, which means that any  $\zeta \in \mathcal{P}_{2\kappa|\rho(z_{j,k})|}(z_{j,k})$  is far from  $X_l$ . We then have to distinguish two cases, depending on what "far" means. Before, we notice that, since  $\mathcal{P}_{4\kappa|\rho(z_{j,k})|}(z_{j,k}) \cap X_l = \emptyset$ ,  $I_l^{(j,k)}$  is also empty and  $P_l^{(j,k)} = 1$ .

*Second case:* If  $|\zeta_{0,1}^*| < 4\kappa|\rho(z_{j,k})|$  and  $|\zeta_{0,2}^*| < (4\kappa|\rho(z_{j,k})|)^{\frac{1}{2}}$ , then  $\delta(z_{j,k}, \zeta_0) \lesssim |\rho(z_{j,k})|$  and thus for all  $\zeta \in \mathcal{P}_{2\kappa|\rho(z_{j,k})|}(z_{j,k})$ ,  $\delta(\zeta, \zeta_0) \lesssim |\rho(z_{j,k})|$ . In particular, all  $\zeta$  belonging to  $\mathcal{P}_{2\kappa|\rho(z_{j,k})|}(z_{j,k})$  is almost at the same (pseudo-)distance from  $z_{j,k}$  as from  $X_l$ .

For all  $\varepsilon > 0$  and all  $\zeta \in \mathcal{P}_\varepsilon(\zeta_0)$ , it is then easy to see that  $|f_l(\zeta)| \lesssim \varepsilon^{\frac{p_l}{2}}$ . Therefore, Cauchy's inequalities give

$$\left| \frac{\partial^{\alpha+\beta} f_l}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}}(\zeta) \right| \lesssim |\rho(z_{j,k})|^{\frac{p_l}{2} - \alpha - \frac{\beta}{2}}$$

for all  $\zeta \in \mathcal{P}_{2\kappa|\rho(z_{j,k})|}(z_{j,k})$ . Moreover, since  $|\zeta_{0,1}^*| < 4\kappa|\rho(z_{j,k})|$ , on the one hand  $f_l = Q_l^{(j,k)}$ . On the other hand it follows from Proposition 3.1 that  $\mathcal{P}_{4\kappa|\rho(z_{j,k})|}(z_{j,k}) \cap X_l = \emptyset$ . This yields  $|f_l(\zeta)| \gtrsim |\rho(z_{j,k})|^{\frac{p_l}{2}}$  for all  $\zeta \in \mathcal{P}_{2\kappa|\rho(z_{j,k})|}(z_{j,k})$ , thus  $\left| \frac{1}{Q_l^{(j,k)}(\zeta)} \frac{\partial^{\alpha+\beta}}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}} \left( Q_l^{(j,k)}(\zeta) \right) \right| \lesssim |\rho(z_{j,k})|^{-\alpha - \frac{\beta}{2}}$ .

*Third case:* If  $|\zeta_{0,1}^*| < 4\kappa|\rho(z_{j,k})|$  and  $|\zeta_{0,2}^*| \geq (4\kappa|\rho(z_{j,k})|)^{\frac{1}{2}}$ , then all  $\zeta \in \mathcal{P}_{3\kappa|\rho(z_{j,k})|}(z_{j,k})$  is far from  $\zeta_0^*$  and  $Q_l^{(j,k)} = f_l$ . We will see that  $|f_l(\zeta)|$  is comparable to  $|\zeta_{0,2}^*|^{p_l}$  for all  $\zeta \in \mathcal{P}_{3\kappa|\rho(z_{j,k})|}(z_{j,k})$ .

We set  $a(z_{j,k}) = \frac{\partial \rho}{\partial \zeta_1}(z_{j,k})$ ,  $b(z_{j,k}) = \frac{\partial \rho}{\partial \zeta_2}(z_{j,k})$  and

$$P(z_{j,k}) = \frac{1}{\sqrt{|a(z_{j,k})|^2 + |b(z_{j,k})|^2}} \begin{pmatrix} a(z_{j,k}) & b(z_{j,k}) \\ -b(z_{j,k}) & a(z_{j,k}) \end{pmatrix}.$$

Then we have  $\zeta^* = P(z_{j,k})(\zeta - z_{j,k})$  and moreover  $|a(z_{j,k})| \approx 1$  and  $b(z_{j,k})$  tends to 0 when  $z_{j,k}$  goes to  $\zeta_0$ , hence,  $b(z_{j,k})$  is arbitrary small provided  $\mathcal{U}_0$  is sufficiently small.

Therefore, if  $\mathcal{U}_0$  is sufficiently small, for all  $\zeta \in \mathcal{P}_{3\kappa|\rho(z_{j,k})|}(z_{j,k})$ ,

$$\begin{aligned} |\zeta_2| &\geq \frac{1}{\sqrt{|a(z_{j,k})|^2 + |b(z_{j,k})|^2}} (|a(z_{j,k})||\zeta_{0,2}^*| - |b(z_{j,k})||\zeta_{0,1}^*| - |b(z_{j,k})||\zeta_1^*| - |a(z_{j,k})||\zeta_2^*|) \\ &\gtrsim |\zeta_{0,2}^*|. \end{aligned}$$

We also trivially have  $|\zeta_2| \lesssim |\zeta_{0,2}^*|$  and so  $|\zeta_2| \approx |\zeta_{0,2}^*|$ . On the other hand

$$\begin{aligned} |\zeta_1| &\leq \frac{1}{\sqrt{|a(z_{j,k})|^2 + |b(z_{j,k})|^2}} (|a(z_{j,k})|(|\zeta_{0,1}^*| + |\zeta_1^*|) + |b(z_{j,k})|(|\zeta_{0,2}^*| + |\zeta_2^*|)) \\ &\leq 6\kappa|\rho(z_{j,k})| + |b(z_{j,k})|(|\zeta_{0,2}^*| + (2\kappa|\rho(z_{j,k})|)^{\frac{1}{2}}) \\ &\leq c|\zeta_{0,2}^*| \end{aligned}$$

where  $c$  depends neither on  $z_{j,k}$  nor on  $\zeta$  and is arbitrarily small provided  $\mathcal{U}_0$  is small enough.

Now let  $\alpha \in \mathbb{C}$  be such that  $f_l(\zeta_1, \alpha) = 0$ . Since the intersection  $X_l \cap bD$  is transverse, there exists a positive constant  $C$  depending neither on  $\zeta$ , nor on  $\alpha$ , nor on  $j$  and nor on  $k$  such that  $|\alpha| \leq C|\zeta_1|$ .

Therefore if  $\mathcal{U}_0$  is small enough,  $|\alpha| \leq \frac{1}{2}|\zeta_2|$ . This yields, for all  $\zeta \in \mathcal{P}_{3\kappa|\rho(z_{j,k})|}(z_{j,k})$ ,

$$\begin{aligned} |f_l(\zeta)| &\approx \prod_{\alpha/f_l(\zeta_1, \alpha)=0} |\zeta_2 - \alpha| \\ &\approx |\zeta_{0,2}^*|^{p_l}. \end{aligned}$$

Cauchy's inequalities then give for all  $\zeta \in \mathcal{P}_{2\kappa|\rho(z_{j,k})|}(z_{j,k})$

$$\left| \frac{\partial^{\alpha+\beta} f_l}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}}(\zeta) \right| \lesssim |\zeta_{0,2}^*|^{p_l} |\rho(z_{j,k})|^{-\alpha-\frac{\beta}{2}},$$

and since  $Q_l^{(j,k)} = f_l$ , we are done in this case and the lemma is shown.  $\square$

Lemma 5.1 gives us an upper bound for the derivatives of  $\chi_l^{(j,k)}$ :

**Corollary 5.2.** *For all  $j \in \mathbb{N}$ , all  $k \in \{1, \dots, n_j\}$ , all  $\alpha$  and  $\beta$  in  $\mathbb{N}$ ,  $l = 1, 2$  and all  $\zeta \in \mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$ , we have uniformly with respect to  $j, k, l$  and  $\zeta$*

$$\left| \frac{\partial^{\alpha+\beta} \chi_l^{(j,k)}}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}}(\zeta) \right| \lesssim |\rho(z_{j,k})|^{-\alpha-\frac{\beta}{2}}.$$

*Proof:* Since by construction  $\left| \frac{\partial^{\alpha+\beta} \tilde{\chi}_{j,k}}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}}(\zeta) \right| \lesssim |\rho(z_{j,k})|^{-\alpha-\frac{\beta}{2}}$ , we only have to consider

$$\frac{\partial^{\alpha+\beta}}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}} \chi \left( \frac{f_1(\zeta)}{P_1^{(j,k)}(\zeta)} |\rho(z_{j,k})|^{i_1^{(j,k)}}, \frac{f_2(\zeta)}{P_2^{(j,k)}(\zeta)} |\rho(z_{j,k})|^{i_2^{(j,k)}} \right).$$

The derivative  $\frac{\partial^{\gamma+\delta} \chi}{\partial z_1^\gamma \partial z_2^\delta}(z_1, z_2)$  is bounded up to a uniform multiplicative constant by  $\frac{1}{|z_1|^\gamma |z_2|^\delta}$  when  $\frac{1}{3}|z_2| < |z_1| < \frac{2}{3}|z_2|$  and is zero otherwise.

Therefore, we can estimate  $\left| \frac{\partial^{\alpha+\beta} \chi_l^{(j,k)}}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}} \right|$  by a sum of products of  $\left| \frac{1}{Q_l^{(j,k)}} \frac{\partial^{\tilde{\gamma}+\tilde{\delta}}}{\partial \zeta_1^{*\tilde{\gamma}} \partial \zeta_2^{*\tilde{\delta}}} \left( Q_l^{(j,k)} \right) \right|$  where the sum of the  $\tilde{\gamma}$ 's equals  $\alpha$  and the sum of the  $\tilde{\delta}$ 's equals  $\beta$ . Lemma 5.1 then gives the wanted estimates.  $\square$

**Corollary 5.3.** *For any smooth function  $h$ , we can write*

$$\frac{\partial_l^{i_l^{(j,k)}}}{\partial \zeta_2^{*i_l^{(j,k)}}} \left( \chi_l^{(j,k)}(\zeta) \bar{\partial} h(\zeta) \wedge P^{N,1}(\zeta, z) \right) = \psi_1^{(j,k,l)}(\zeta, z) d\zeta_1^* + \psi_2^{(j,k,l)}(\zeta, z) d\zeta_2^*$$

with  $\psi_1^{(j,k,l)}$  and  $\psi_2^{(j,k,l)}$  two  $(0,2)$ -forms supported in  $\mathcal{U}_l^{(j,k)}$  satisfying uniformly with respect to  $j, k, z$  and  $\zeta \in \mathcal{U}_l^{(j,k)}$ :

$$\begin{aligned} \left| \psi_1^{(j,k,l)}(\zeta, z) \right| &\lesssim |\rho(z_{j,k})|^{-\frac{i_l^{(j,k)}}{2} - \frac{5}{2}} \left( \frac{|\rho(z_{j,k})|}{|\rho(z_{j,k})| + |\rho(z)| + \delta(z_{j,k}, z)} \right)^N \tilde{h}(\zeta), \\ \left| \psi_2^{(j,k,l)}(\zeta, z) \right| &\lesssim |\rho(z_{j,k})|^{-\frac{i_l^{(j,k)}}{2} - 2} \left( \frac{|\rho(z_{j,k})|}{|\rho(z_{j,k})| + |\rho(z)| + \delta(z_{j,k}, z)} \right)^N \tilde{h}(\zeta), \end{aligned}$$

and, for  $\nabla_z$  a differential operators of order 1 acting on  $z$ ,

$$\begin{aligned} \left| \nabla_z \psi_1^{(j,k,l)}(\zeta, z) \right| &\lesssim |\rho(z_{j,k})|^{-\frac{i_l^{(j,k)}}{2} - \frac{7}{2}} \left( \frac{|\rho(z_{j,k})|}{|\rho(z_{j,k})| + |\rho(z)| + \delta(z_{j,k}, z)} \right)^N \tilde{h}(\zeta), \\ \left| \nabla_z \psi_2^{(j,k,l)}(\zeta, z) \right| &\lesssim |\rho(z_{j,k})|^{-\frac{i_l^{(j,k)}}{2} - 3} \left( \frac{|\rho(z_{j,k})|}{|\rho(z_{j,k})| + |\rho(z)| + \delta(z_{j,k}, z)} \right)^N \tilde{h}(\zeta), \end{aligned}$$

where  $\tilde{h}(\zeta) = \max_{n \in \{0, \dots, i_l^{(j,k)}\}} \left( \left| \frac{\partial^{n+1} h}{\partial \zeta_2^{*n+1}}(\zeta) |\rho(\zeta)|^{\frac{n+1}{2}} \right|, \left| \frac{\partial^{n+1} h}{\partial \zeta_1^* \partial \zeta_2^{*n}}(\zeta) |\rho(\zeta)|^{\frac{n}{2} + 1} \right| \right)$ .

*Proof:* Propositions 2.4 and 2.5 imply that  $\frac{\partial^n}{\partial \zeta_2^{*n}} P^{N,1}(\zeta, z) = \sum_{p,q=1,2} \tilde{\psi}_{p,q}^{(n,N)}(\zeta, z) d\zeta_p^* \wedge d\bar{\zeta}_q^*$  where

$$|\tilde{\psi}_{p,q}^{n,N}(\zeta, z)| \lesssim \left( \frac{|\rho(\zeta)|}{|\rho(\zeta)| + |\rho(z)| + \delta(\zeta, z)} \right)^N |\rho(\zeta)|^{-\frac{1}{p} - \frac{1}{q} - \frac{n}{2}}.$$

From proposition 2.1, if  $\kappa$  is small enough, we have for all  $\zeta \in \mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$ ,  $\frac{1}{2}|\rho(z_{j,k})| \leq |\rho(\zeta)|$  and thus, provided  $\kappa$  is small enough:

$$\begin{aligned} |\rho(\zeta)| + \delta(\zeta, z) &\geq \frac{1}{2}|\rho(z_{j,k})| + \frac{1}{c_1}\delta(z, z_{j,k}) - \delta(z_{j,k}, \zeta) \\ &\gtrsim |\rho(z_{j,k})| + \delta(z, z_{j,k}) \end{aligned}$$

and so  $|\tilde{\psi}_{p,q}^{n,N}(\zeta, z)| \lesssim \left( \frac{|\rho(z_{j,k})|}{|\rho(z_{j,k})| + |\rho(z)| + \delta(z_{j,k}, z)} \right)^N |\rho(z_{j,k})|^{-\frac{1}{p} - \frac{1}{q} - \frac{n}{2}}$ . This inequality and Corollary 5.2 now yield the two first estimates. The two others can be shown in the same way.

□

In order to estimate  $\frac{P_l^{(j,k)}}{f_l} b_m$ , we need the following lemma:

**Lemma 5.4.** *For all  $j \in \mathbb{N}$ , all  $k \in \{1, \dots, n_j\}$ , all  $\alpha$  and  $\beta$  in  $\mathbb{N}$ ,  $l = 1, 2$  and all  $\zeta \in \mathcal{P}_{2\kappa|\rho(z_{j,k})|}(z_{j,k})$  we have uniformly with respect to  $j, k, l$  and  $\zeta$*

$$\left| \frac{\partial^{\alpha+\beta}}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}} \left( \prod_{i \in I_l^{(j,k)}} (\zeta_i^* - \alpha_{l,i}^{(j,k)}(\zeta_i^*)) \right) \right| \lesssim |\rho(z_{j,k})|^{\frac{i_l^{(j,k)}}{2} - \alpha - \frac{\beta}{2}}.$$

*Proof:* For  $i \in I_l^{(j,k)}$ , there exists  $z_1^* \in \Delta_0(2\kappa|\rho(z_{j,k})|)$  such that  $|\alpha_{l,i}^{(j,k)}(z_1^*)| < \frac{5}{2}\kappa|\rho(z_{j,k})|^{\frac{1}{2}}$ . Since  $\left|\frac{\partial\alpha_{l,i}^{(j,k)}}{\partial\zeta_1^*}\right|$  is uniformly bounded on  $\Delta_0(4\kappa|\rho(z_{j,k})|)$ , for all  $\zeta \in \mathcal{P}_{4\kappa|\rho(z_{j,k})|}(z_{j,k})$ , we have  $\prod_{i \in I_l^{(j,k)}} \left|\zeta_2^* - \alpha_{l,i}^{(j,k)}(\zeta_1^*)\right| \lesssim |\rho(z_{j,k})|^{\frac{i_l^{(j,k)}}{2}}$ . Cauchy's inequalities then give the results.  $\square$

As a direct corollary of Lemma 5.1 and 5.4 we get

**Corollary 5.5.** *For all  $j \in \mathbb{N}$ , all  $k \in \{1, \dots, n_j\}$ , all  $\alpha$  and  $\beta$  in  $\mathbb{N}$ ,  $l = 1, 2$  and all  $\zeta \in \mathcal{P}_{2\kappa|\rho(z_{j,k})|}(z_{j,k})$  we have uniformly with respect to  $j, k, l$  and  $\zeta$*

$$\left| \frac{P_l^{(j,k)}(\zeta)}{f_l(\zeta)} \frac{\partial^{\alpha+\beta} f_l}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}}(\zeta) \right| \lesssim |\rho(z_{j,k})|^{\frac{i_l^{(j,k)}}{2} - \alpha - \frac{\beta}{2}}.$$

In the following corollary, we give estimates for  $l, m \in \{1, 2\}$  of  $\frac{P_l^{(j,k)}}{f_l} b_m$ , which do not depend on  $m$  thanks to the covering  $\mathcal{U}_1^{(j,k)}, \mathcal{U}_2^{(j,k)}$  of  $\mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$ .

**Corollary 5.6.** *For  $l, m \in \{1, 2\}$ , we can write  $\frac{P_l^{(j,k)}}{f_l} b_m = \varphi_1^{(j,k,l,m)} d\zeta_1^* + \varphi_2^{(j,k,l,m)} d\zeta_2^*$  with  $\varphi_1^{(j,k,l,m)}$  and  $\varphi_2^{(j,k,l,m)}$  satisfying for all  $\zeta \in \mathcal{U}_l^{(j,k)}$*

$$\begin{aligned} \left| \varphi_1^{(j,k,l,m)}(\zeta, z) \right| &\lesssim \sum_{0 \leq \alpha + \beta \leq \max(p_1, p_2)} |\rho(z_{j,k})|^{\frac{i_l^{(j,k)}}{2} - 1} \left| \frac{\delta(\zeta, z)}{\rho(z_{j,k})} \right|^{\alpha + \frac{\beta}{2}}, \\ \left| \varphi_2^{(j,k,l,m)}(\zeta, z) \right| &\lesssim \sum_{0 \leq \alpha + \beta \leq \max(p_1, p_2)} |\rho(z_{j,k})|^{\frac{i_l^{(j,k)}}{2} - \frac{1}{2}} \left| \frac{\delta(\zeta, z)}{\rho(z_{j,k})} \right|^{\alpha + \frac{\beta}{2}}, \end{aligned}$$

and for all differential operators  $\nabla_z$  of order 1 acting on  $z$ ,

$$\begin{aligned} \left| \nabla_z \varphi_1^{(j,k,l,m)}(\zeta, z) \right| &\lesssim \sum_{0 \leq \alpha + \beta \leq \max(p_1, p_2)} |\rho(z_{j,k})|^{\frac{i_l^{(j,k)}}{2} - 2} \left| \frac{\delta(\zeta, z)}{\rho(z_{j,k})} \right|^{\alpha + \frac{\beta}{2}}, \\ \left| \nabla_z \varphi_2^{(j,k,l,m)}(\zeta, z) \right| &\lesssim \sum_{0 \leq \alpha + \beta \leq \max(p_1, p_2)} |\rho(z_{j,k})|^{\frac{i_l^{(j,k)}}{2} - \frac{3}{2}} \left| \frac{\delta(\zeta, z)}{\rho(z_{j,k})} \right|^{\alpha + \frac{\beta}{2}}, \end{aligned}$$

uniformly with respect to  $\zeta, z, j$  and  $k$ .

*Proof:* Without restriction we assume  $l = 1$  and for  $m = 1, 2$ , we write  $b_m(\zeta, z) = b_{m,1}^*(\zeta, z) d\zeta_1^* + b_{m,2}^*(\zeta, z) d\zeta_2^*$  where  $b_{m,n}^* = \int_0^1 \frac{\partial f_m}{\partial \zeta_n^*}(\zeta + t(z - \zeta)) dt$ . So

$$\begin{aligned} &b_{m,n}^*(\zeta, z) \\ &= \sum_{0 \leq \alpha + \beta \leq \max(p_1, p_2)} \frac{1}{\alpha + \beta + 1} \frac{\partial^{\alpha+\beta+1} f_m}{\partial \zeta_n^* \partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}}(\zeta) (z_1^* - \zeta_1^*)^\alpha (z_2^* - \zeta_2^*)^\beta + o\left(|z - \zeta|^{\max(p_1, p_2)}\right) \end{aligned}$$

and Corollary 5.5 yields for all  $\zeta \in \mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$ :

$$\left| \frac{\overline{P_1^{(j,k)}(\zeta)}}{f_1(\zeta)} b_{1,1}(\zeta, z) \right| \lesssim \sum_{0 \leq \alpha + \beta \leq \max(p_1, p_2)} |\rho(z_{j,k})|^{\frac{i_1^{(j,k)}}{2} - 1} \left| \frac{\delta(\zeta, z)}{\rho(z_{j,k})} \right|^{\alpha + \frac{\beta}{2}}$$

uniformly with respect to  $z, \zeta, j$  and  $k$ . The proof of the inequality for  $\left| \frac{\overline{P_1^{(j,k)}(\zeta)}}{f_1(\zeta)} b_{1,2}(\zeta, z) \right|$  is exactly the same. The one for  $\left| \frac{\overline{P_1^{(j,k)}(\zeta)}}{f_1(\zeta)} b_{2,1}(\zeta, z) \right|$  uses the definition of  $\mathcal{U}_1^{(j,k)}$ .

On  $\mathcal{U}_1^{(j,k)}$ , we have  $\left| \frac{P_1^{(j,k)}}{f_1} \right| \lesssim \left| \frac{P_2^{(j,k)}}{f_2} \right| |\rho(z_{j,k})|^{\frac{i_1^{(j,k)} - i_2^{(j,k)}}{2}}$  and again Corollary 5.5 yields

$$\begin{aligned} \left| \frac{\overline{P_1^{(j,k)}(\zeta)}}{f_1(\zeta)} b_{2,1}(\zeta, z) \right| &\lesssim \left| \frac{P_2^{(j,k)}(\zeta)}{f_2(\zeta)} b_{2,1}(\zeta, z) \right| |\rho(z_{j,k})|^{\frac{i_1^{(j,k)} - i_2^{(j,k)}}{2}} \\ &\lesssim \sum_{0 \leq \alpha + \beta \leq \max(p_1, p_2)} |\rho(z_{j,k})|^{\frac{i_1^{(j,k)}}{2} - 1} \left| \frac{\delta(\zeta, z)}{\rho(z_{j,k})} \right|^{\alpha + \frac{\beta}{2}} \end{aligned}$$

uniformly with respect to  $z, \zeta, j$  and  $k$ . Again, the inequality for  $\left| \frac{\overline{P_1^{(j,k)}(\zeta)}}{f_1(\zeta)} b_{2,2}(\zeta, z) \right|$  can be obtained in the same way.  $\square$

Corollary 5.3 and 5.6 imply for some  $N'$  arbitrarily large, provided  $N$  is large enough, and for all  $\zeta \in \mathcal{P}_{\kappa|\rho(z_{j,k})|}(z_{j,k})$  that

$$\begin{aligned} \left| \frac{\overline{P_l^{(j,k)}(\zeta)}}{f_l(\zeta)} b_m(\zeta, z) \wedge \frac{\partial^{i_l^{(j,k)}}}{\partial \zeta_2^{*i_l^{(j,k)}}} \left( \chi_l^{(j,k)}(\zeta) \bar{\partial} h(\zeta) \wedge P^{N,1}(\zeta, z) \right) \right| \\ \leq |\rho(z_{j,k})|^{-3} \left( \frac{|\rho(z_{j,k})|}{|\rho(z_{j,k})| + |\rho(z)| + \delta(z_{j,k}, z)} \right)^{N'} \tilde{h}(\zeta) \end{aligned}$$

and for  $\nabla_z$  a differential of order 1

$$\begin{aligned} \left| \nabla_z \left( \frac{\overline{P_l^{(j,k)}(\zeta)}}{f_l(\zeta)} b_m(\zeta, z) \wedge \frac{\partial^{i_l^{(j,k)}}}{\partial \zeta_2^{*i_l^{(j,k)}}} \left( \chi_l^{(j,k)}(\zeta) \bar{\partial} h(\zeta) \wedge P^{N,1}(\zeta, z) \right) \right) \right| \\ \leq |\rho(z)|^{-1} |\rho(z_{j,k})|^{-3} \left( \frac{|\rho(z_{j,k})|}{|\rho(z_{j,k})| + |\rho(z)| + \delta(z_{j,k}, z)} \right)^{N'} \tilde{h}(\zeta) \end{aligned}$$

where  $\tilde{h}(\zeta) = \max_{n \in \{0, \dots, i_l^{(j,k)}\}} \left( \left| \frac{\partial^{n+1} h}{\partial \zeta_2^{*n+1}}(\zeta) |\rho(\zeta)|^{\frac{n+1}{2}} \right|, \left| \frac{\partial^{n+1} h}{\partial \zeta_1^{*n} \partial \zeta_2^{*n}}(\zeta) |\rho(\zeta)|^{\frac{n}{2}+1} \right| \right)$ . We conclude as in the proof of Theorem 1.1 of [1] that Theorem 1.1 holds true.

## 6. LOCAL DIVISION

**6.1. Local holomorphic division.** In this subsection we will prove Theorem 1.2 and his analogue in the  $L^q$  case, the following theorem.

**Theorem 6.1.** *When  $n = 2$ , let  $g$  be a holomorphic function defined on  $D$ . Assume that  $X_1 \cap X_2$  is a complete intersection and that there exist  $\kappa > 0$ , a real number  $q \geq 1$  and a locally finite covering  $(\mathcal{P}_{\kappa|\rho(\zeta_j)}(\zeta_j))_{j \in I}$  of  $D$  such that for all  $j \in I$ , there exist two function  $\hat{g}_1^{(j)}$  and  $\hat{g}_2^{(j)}$ ,  $C^\infty$ -smooth on  $\mathcal{P}_{\kappa|\rho(\zeta_j)}(\zeta_j)$ , which satisfy*

- (a)  $g = \hat{g}_1^{(j)} f_1 + \hat{g}_2^{(j)} f_2$  on  $\mathcal{P}_{\kappa|\rho(\zeta_j)|}(\zeta_j)$ ;
- (b)  $\sum_{j \in I} \int_{\mathcal{P}_{\kappa|\rho(\zeta_j)|}(\zeta_j)} \left| \frac{\partial^{\alpha+\beta} \hat{g}_l^{(j)}(z)}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}} \right| |\rho(\zeta_j)|^{\alpha+\frac{\beta}{2}} dV(z) < \infty$  for  $l = 1$  and  $l = 2$  and all integers  $\alpha$  and  $\beta$ ;
- (c) for  $l = 1$  and  $l = 2$ , for all non negatives integers  $\alpha, \bar{\alpha}, \beta$  and  $\bar{\beta}$ , there exist  $N \in \mathbb{N}$  and  $c > 0$  such that for all  $j$ ,  $|\rho(\zeta_j)|^N \sup_{\mathcal{P}_{\kappa|\rho(\zeta_j)|}(\zeta_j)} \left| \frac{\partial^{\alpha+\bar{\alpha}+\beta+\bar{\beta}} \hat{g}_l^{(j)}}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta} \partial \zeta_1^{*\bar{\alpha}} \partial \zeta_2^{*\bar{\beta}}} \right| \leq c$ .

Then there exist two smooth functions  $\tilde{g}_1$  and  $\tilde{g}_2$  which satisfy (i)-(iii) of Theorem 1.1 with  $q$ .

*Proof:* It suffices to glue together all the  $\hat{g}_1^{(j)}$  and  $\hat{g}_2^{(j)}$  using a suitable partition of unity. Let  $(\chi_j)_{j \in \mathbb{N}}$  be a partition of unity subordinated to  $(\mathcal{P}_{\kappa|\rho(\zeta_j)|}(\zeta_j))_{j \in \mathbb{N}}$  such that for all  $j$  and all  $\zeta \in \mathcal{P}_{\kappa|\rho(\zeta_j)|}(\zeta_j)$ , we have  $\left| \frac{\partial^{\alpha+\bar{\alpha}+\beta+\bar{\beta}} \chi_j}{\partial z_1^{*\alpha} \partial z_2^{*\beta} \partial z_1^{*\bar{\alpha}} \partial z_2^{*\bar{\beta}}}(\zeta) \right| \lesssim \frac{1}{|\rho(\zeta_j)|^{\alpha+\bar{\alpha}+\frac{\beta+\bar{\beta}}{2}}}$ , uniformly with respect to  $\zeta_j$  and  $\zeta$ . We set  $\tilde{g}_1 = \sum_j \chi_j \hat{g}_1^{(j)}$  and  $\tilde{g}_2 = \sum_j \chi_j \hat{g}_2^{(j)}$  and thus we get the two functions defined on  $D$  which satisfy (i), (ii) and (iii) by construction.  $\square$

The proof of Theorem 1.2 is exactly the same so we omit it.

**6.2. Divided differences and division.** In order to apply Theorem 1.2 and 6.1, we will use divided differences and find numerical conditions on  $g$  which ensure the existence of local smooth division formula in  $L^\infty$  and in  $L^q$ . We define the divided differences using the following settings.

We set

$$\Lambda_{z,v}^{(1)} = \{\lambda \in \mathbb{C}, |\lambda| < \tau(z, v, 3\kappa|\rho(z)|) \text{ and } z + \lambda v \in X_2 \setminus X_1\}$$

Thus the points  $z + \lambda v$ ,  $\lambda \in \Lambda_{z,v}^{(1)}$ , are the points of  $X_2 \setminus X_1$  which belong to the disc  $\Delta_{z,v}(\tau(z, v, 3\kappa|\rho(z)|))$ , so they all belong to  $D$  as soon as  $\kappa < \frac{1}{3}$ . We analogously define

$$\Lambda_{z,v}^{(2)} = \{\lambda \in \mathbb{C}, |\lambda| < \tau(z, v, 3\kappa|\rho(z)|) \text{ and } z + \lambda v \in X_1 \setminus X_2\}.$$

For a function  $h$  defined on a subset  $\mathcal{U}$  of  $\mathbb{C}^n$ ,  $z \in \mathbb{C}^n$ ,  $v$  a unit vector of  $\mathbb{C}^n$  and  $\lambda \in \mathbb{C}$  such that  $z + \lambda v$  belongs to  $\mathcal{U}$ , we set  $h_{z,v}[\lambda] = h(z + \lambda v)$ .

If for  $\mu_1, \dots, \mu_k$  pairwise distinct  $h_{z,v}[\mu_1, \dots, \mu_k]$  is defined, for  $\lambda_1, \dots, \lambda_{k+1} \in \mathbb{C}$  pairwise distinct such that  $z + \lambda_i v$  belongs to  $\mathcal{U}$  for all  $i$ , we set

$$h_{z,v}[\lambda_1, \dots, \lambda_{k+1}] := \frac{h_{z,v}[\lambda_1, \dots, \lambda_k] - h_{z,v}[\lambda_2, \dots, \lambda_{k+1}]}{\lambda_1 - \lambda_{k+1}}.$$

Now, for  $z \in X_2 \setminus X_1$  (resp.  $z \in X_1 \setminus X_2$ ) let us define  $g^{(2)}(z) = \frac{g(z)}{f_2(z)}$  (resp.  $g^{(1)}(z) = \frac{g(z)}{f_1(z)}$ ). For  $l = 1$  or  $l = 2$ , the quantity  $g_{z,v}^{(l)}[\lambda_1, \dots, \lambda_k]$  make sense for all  $\lambda_1, \dots, \lambda_k \in \Lambda_{z,v}^{(l)}$  pairwise distinct.

We first prove a lemma we will need in this section.

**Lemma 6.2.** *Let  $\alpha$  and  $\beta$  be two functions defined on a subset  $\mathcal{U}$  of  $\mathbb{C}$ . Then, for all  $z_1, \dots, z_n$  pairwise distinct points of  $\mathcal{U}$  we have*

$$(\alpha \cdot \beta)[z_1, \dots, z_n] = \sum_{k=1}^n \alpha[z_1, \dots, z_k] \cdot \beta[z_k, \dots, z_n].$$

*Proof:* We prove the lemma by induction on  $n$ , the case  $n = 1$  being trivial. We assume the lemma proved for  $n$  points,  $n \geq 1$ . Let  $z_1, \dots, z_{n+1}$  be  $n + 1$  points of  $\mathcal{U}$ . Then

$$\begin{aligned}
& (\alpha \cdot \beta)[z_1, \dots, z_{n+1}] \\
&= \frac{(\alpha \cdot \beta)[z_1, z_3, \dots, z_{n+1}] - (\alpha \cdot \beta)[z_2, \dots, z_{n+1}]}{z_1 - z_2} \\
&= \frac{1}{z_1 - z_2} \left( \sum_{k=3}^{n+1} \alpha[z_1, z_3, \dots, z_k] \beta[z_k, \dots, z_{n+1}] + \alpha[z_1] \beta[z_3, \dots, z_{n+1}] \right) \\
&\quad - \frac{1}{z_1 - z_2} \sum_{k=2}^{n+1} \alpha[z_2, \dots, z_k] \beta[z_k, \dots, z_{n+1}] \\
&= \sum_{k=3}^{n+1} \frac{\alpha[z_1, z_3, \dots, z_k] - \alpha[z_2, \dots, z_k]}{z_1 - z_2} \beta[z_k, \dots, z_{n+1}] + \\
&\quad \frac{\alpha[z_1] - \alpha[z_2]}{z_1 - z_2} \beta[z_2, \dots, z_{n+1}] + \alpha[z_1] \frac{\beta[z_1, z_3, \dots, z_{n+1}] - \beta[z_2, \dots, z_{n+1}]}{z_1 - z_2}. \quad \square
\end{aligned}$$

**6.2.1. The  $L^\infty - BMO$ -case.** In this subsection, we establish the necessary conditions in  $\mathbb{C}^n$  and the sufficient conditions in  $\mathbb{C}^2$  for a function  $g$  to be written as  $g = g_1 f_1 + g_2 f_2$ , with  $g_1$  and  $g_2$  smooth functions satisfying the hypothesis of Theorem 1.1.

For  $l = 1$  and  $l = 2$  let us define the numbers

$$c_\infty^{(l)}(g) = \sup \left( |g_{z,v}^{(l)}[\lambda_1, \dots, \lambda_k]| \tau(z, v, |\rho(z)|)^{k-1} \right)$$

where the supremum is taken over all  $z \in D$ , all  $v \in \mathbb{C}^n$  with  $|v| = 1$ , all  $k \in \mathbb{N}^*$  and  $\lambda_1, \dots, \lambda_k \in \Lambda_{z,v}^{(l)}$  pairwise distinct.

We have the following necessary conditions in  $\mathbb{C}^n$ ,  $n \geq 2$ .

**Theorem 6.3.** *In  $\mathbb{C}^n$ ,  $n \geq 2$ , let  $g_1, g_2$  be two bounded holomorphic functions on  $D$  and set  $g = g_1 f_1 + g_2 f_2$ . Then*

$$\left\| \frac{g}{\max(|f_1|, |f_2|)} \right\|_{L^\infty(D)} \lesssim \max(\|g_1\|_{L^\infty(D)}, \|g_2\|_{L^\infty(D)})$$

and for  $l = 1, 2$ :

$$c_\infty^{(l)}(g) \lesssim \sup_{b\Delta_{z,v}(\tau(z, v, 4\kappa|\rho(z)|))} |g_l|.$$

*Proof:* The first point is trivial and we only prove the second one for  $l = 1$ . Let  $\lambda_1, \dots, \lambda_k$  be  $k$  pairwise distinct elements of  $\Lambda_{z,v}^{(1)}$ . For all  $i$  we have  $g_{z,v}^{(1)}[\lambda_i] = g_1(z + \lambda_i v)$  because  $f_2(z + \lambda_i v) = 0$ . Therefore,  $g_{z,v}^{(1)}[\lambda_1, \dots, \lambda_k] = (g_1)_{z,v}[\lambda_1, \dots, \lambda_k]$ . Since [19]  $|g_{z,v}^{(1)}[\lambda_1, \dots, \lambda_k]| = \frac{1}{2i\pi} \int_{|\lambda|=\tau(z, v, 4\kappa|\rho(z)|)} \frac{g_1(z+\lambda v)}{\prod_{i=1}^k (\lambda - \lambda_i)} d\lambda$ , it follows that

$$|g_{z,v}^{(1)}[\lambda_1, \dots, \lambda_k]| \lesssim \tau(z, v, |\rho(z)|)^{-k+1} \sup_{b\Delta_{z,v}(\tau(z, v, 4\kappa|\rho(z)|))} |g_1|. \quad \square$$

Now we prove that these conditions are sufficient in  $\mathbb{C}^2$  in order to get a *BMO* division.

**Theorem 6.4.** *In  $\mathbb{C}^2$ , let  $g$  be a holomorphic function on  $D$  which belongs to the ideal of  $\mathcal{O}(D)$  generated by  $f_1$  and  $f_2$  and such that*

- (i)  $c(g) = \sup_{z \in D} \frac{|g(z)|}{\max(|f_1(z)|, |f_2(z)|)} < \infty$ ,  
(ii)  $c_\infty^{(1)}(g)$  and  $c_\infty^{(2)}(g)$  are finite.

There exist two holomorphic functions  $g_1$  and  $g_2$  which belong to  $BMO(D)$  and such that  $g_1 f_1 + g_2 f_2 = g$ .

*Proof:* It suffices to construct, for all  $z$  near  $bD$ , two smooth functions  $\hat{g}_1$  and  $\hat{g}_2$  on  $\mathcal{P}_{\kappa|\rho(z)}(z)$  which satisfy (a) and (b) of Theorem 1.2.

Let  $\zeta_0$  be a point in  $bD$ . If  $f_1(\zeta_0) \neq 0$  then  $f_1$  does not vanish on a neighborhood  $\mathcal{U}_0$  of  $\zeta_0$ . Then we can define  $\hat{g}_1 = \frac{g}{f_1}$ ,  $\hat{g}_2 = 0$  which obviously satisfy (a) and (b) for all  $z \in D$  close to  $\zeta_0$ . We proceed analogously if  $f_2(\zeta_0) \neq 0$ .

If  $\zeta_0$  belongs to  $X_1 \cap X_2 \cap bD$ , since the intersection  $X_1 \cap X_2$  is complete, without restriction we can choose a neighborhood  $\mathcal{U}_0$  of  $\zeta_0$  such that  $X_1 \cap X_2 \cap \mathcal{U}_0 = \{\zeta_0\}$ . Then we fix some point  $z$  in  $\mathcal{U}_0$  and we construct  $\hat{g}_1$  and  $\hat{g}_2$  on  $\mathcal{P}_{\kappa|\rho(z)}(z)$  which satisfy (a) and (b) of Theorem 1.2. We denote by  $p_1$  and  $p_2$  the multiplicity of  $\zeta_0$  as singularity of  $f_1$  and  $f_2$  respectively. We also denote by  $(\zeta_{0,1}^*, \zeta_{0,2}^*)$  the coordinates of  $\zeta_0$  in the Koranyi coordinates at  $z$ .

If  $|\zeta_{0,1}^*| < 4\kappa|\rho(z)|$ , then for  $l = 1$  and  $l = 2$  we set  $I_l = \emptyset$ ,  $i_l = 0$ ,  $P_l(\zeta) = 1$  and  $Q_l(\zeta) = f_l(\zeta)$ .

Otherwise, we use the parametrization  $\alpha_{1,i}$ ,  $i \in \{1, \dots, p_1\}$ , of  $X_1$  and  $\alpha_{2,i}$ ,  $i \in \{1, \dots, p_2\}$ , of  $X_2$  given by Proposition 3.2. We denote by  $I_l$  the set

$$I_l = \{i, \exists z_1^* \in \Delta_0(2\kappa|\rho(z)|) \text{ such that } |\alpha_{l,i}(z_1^*)| \leq (\frac{5}{2}\kappa|\rho(z)|)^{\frac{1}{2}}\},$$

$$i_l = \#I_l, P_l(\zeta) = \prod_{i \in I_l} (\zeta_2^* - \alpha_{l,i}(\zeta_1^*)) \text{ and } Q_l(\zeta) = \frac{f_l}{P_l}.$$

Our first goal is to find  $\tilde{h}_1$  and  $\tilde{h}_2$  in  $C^\infty(\mathcal{P}_{\kappa|\rho(z)}(z))$  such that  $g = \tilde{h}_1 P_1 + \tilde{h}_2 P_2$  on  $\mathcal{P}_{\kappa|\rho(z)}(z)$  and which moreover satisfy good estimates. The function  $g$  belong to the ideal of  $\mathcal{O}(\mathcal{P}_{4\kappa|\rho(z)}(z))$  generated by  $f_1$  and  $f_2$  and so there exist  $h_1$  and  $h_2$  holomorphic in  $\mathcal{P}_{4\kappa|\rho(z)}(z)$  such that  $g = P_1 h_1 + P_2 h_2$ . Moreover, we observe that necessarily  $\tilde{h}_2(\zeta) = h_2(\zeta) = \frac{g(\zeta)}{P_2(\zeta)}$  for all  $\zeta$  such that  $P_1(\zeta) = 0$  and  $P_2(\zeta) \neq 0$ , but we also notice that  $h_2$  may not satisfy good estimates like uniform boundedness for example. Thus, we already know  $\tilde{h}_2(\zeta)$  for such  $\zeta$  and by interpolation, we will reconstruct a “good”  $\tilde{h}_2$  in the whole polydisc  $\mathcal{P}_{\kappa|\rho(z)}(z)$ . We point out that we do not directly divide by  $f_1$  and  $f_2$  because if we do so, we are not able to handle the error term we get during the interpolation procedure.

If  $i_1 = 0$  we set  $\hat{h}_2 = 0$ . Otherwise, without restriction we assume that  $I_1 = \{1, \dots, i_1\}$  and for  $k \leq i_1$  and  $\zeta_1^*$  such that  $P_2(z + \zeta_1^* \eta_z + \alpha_{1,i}(\zeta_1^*) v_z) \neq 0$ , we introduce

$$(13) \quad h_{1,\dots,k}^{(2)}(\zeta_1^*) := \left( \frac{g}{P_2} \right)_{z+\zeta_1^* \eta_z, v_z} [\alpha_{1,1}(\zeta_1^*), \dots, \alpha_{1,k}(\zeta_1^*)].$$

and

$$\hat{h}_2(\zeta) = \sum_{k=1}^{i_2} h_{1,\dots,k}^{(2)}(\zeta_1^*) \prod_{i=1}^{k-1} (\zeta_2^* - \alpha_{1,i}(\zeta_1^*)).$$

We define  $\hat{h}_1$  analogously. Since  $X_1 \cap X_2 \cap \mathcal{U}_0 = \{\zeta_0\}$ ,  $\hat{h}_1$  and  $\hat{h}_2$  are defined on  $\mathcal{P}_{4\kappa|\rho(z)}(z)$ . Moreover,  $\hat{h}_2(\zeta_1^*, \cdot)$  is the polynomial which interpolates  $h_2(\zeta_1^*, \cdot)$  at the points  $\alpha_{1,1}(\zeta_1^*), \dots$ ,



$\alpha_{1,i_1}(\zeta_1^*)$ . Therefore, we get from [19]

$$(14) \quad h_2(\zeta) = \hat{h}_2(\zeta) + P_1(\zeta)e_1(\zeta)$$

with

$$(15) \quad e_1(\zeta) = \frac{1}{2i\pi} \int_{|\xi|=(4\kappa|\rho(z)|)^{\frac{1}{2}}} \frac{h_2(\zeta_1^*, \xi)}{P_1(\zeta_1^*, \xi) \cdot (\xi - \zeta_2^*)} d\xi.$$

We have an analogous expression for  $h_1$  and we point out that (14), (15) and theirs analogue for  $g_1$  also holds if  $i_1 = 0$  or  $i_2 = 0$ .

This yields

$$(16) \quad g(\zeta) = P_1(\zeta)\hat{h}_1(\zeta) + P_2(\zeta)\hat{h}_2(\zeta) + P_1(\zeta)P_2(\zeta)e(\zeta)$$

where

$$\begin{aligned} e(\zeta) &= e_1(\zeta) + e_2(\zeta) \\ &= \frac{1}{2i\pi} \int_{|\xi|=(4\kappa|\rho(z)|)^{\frac{1}{2}}} \frac{g(\zeta_1^*, \xi)}{P_1(\zeta_1^*, \xi) \cdot P_2(\zeta_1^*, \xi) \cdot (\xi - \zeta_2^*)} d\xi. \end{aligned}$$

If we were trying to divide by  $f_1$  and  $f_2$  directly, in the error term above, we wouldn't get  $g$  but  $h_1P_1 + h_2P_2$  that we can not handle.

Of course,  $\hat{h}_2$  will be a part of the function  $\tilde{h}_2$  we are looking for. We first look for an upper bound for  $\hat{h}_2$  using our assumption on the divided differences of  $g^{(2)} = \frac{g}{f_2}$ .

*Fact 1:*  $\hat{h}_2$  satisfies for all  $\zeta \in \mathcal{P}_{2\kappa|\rho(z)}(z)$ , uniformly with respect to  $z$  and  $\zeta$

$$(17) \quad |\hat{h}_2(\zeta)| \lesssim c_\infty^{(2)}(g) \sup_{|\xi|=(4\kappa|\rho(z)|)^{\frac{1}{2}}} |Q_2(z + \zeta_1^*\eta_z + \xi v_z)|$$

Indeed: We have by Lemma 6.2

$$\begin{aligned} &h_{1,\dots,k}^{(2)}(\zeta_1^*) \\ &= \left( \frac{g}{P_2} \right)_{z+\zeta_1^*\eta_z, v_z} [\alpha_{1,1}(\zeta_1^*), \dots, \alpha_{1,k}(\zeta_1^*)] \\ &= \left( g^{(2)}Q_2 \right)_{z+\zeta_1^*\eta_z, v_z} [\alpha_{1,1}(\zeta_1^*), \dots, \alpha_{1,k}(\zeta_1^*)] \\ &= \sum_{j=1}^k g_{z+\zeta_1^*\eta_z, v_z}^{(2)} [\alpha_{1,1}(\zeta_1^*), \dots, \alpha_{1,j}(\zeta_1^*)] (Q_2)_{z+\zeta_1^*\eta_z, v_z} [\alpha_{1,j}(\zeta_1^*), \dots, \alpha_{1,k}(\zeta_1^*)]. \end{aligned}$$

From Montel's theorem [19] on divided differences in  $\mathbb{C}$  and from Cauchy's inequalities, since  $\tau(z, v_z, 4\kappa|\rho(z)|) \approx (4\kappa|\rho(z)|)^{\frac{1}{2}}$ , it follows that

$$|(Q_2)_{z+\zeta_1^*\eta_z, v_z} [\alpha_{1,j}(\zeta_1^*), \dots, \alpha_{1,k}(\zeta_1^*)]| \lesssim |\rho(z)|^{\frac{j-k}{2}} \sup_{|\xi|=(4\kappa|\rho(z)|)^{\frac{1}{2}}} |Q_2(z + \zeta_1^*\eta_z + \xi v_z)|.$$

With the assumption  $c_\infty^{(2)}(g) < \infty$ , this gives for all  $\zeta_1^* \in \Delta_0(2\kappa|\rho(z)|)$ :

$$(18) \quad |h_{1,\dots,k}^{(2)}(\zeta_1^*)| \lesssim c_\infty^{(2)}(g) |\rho(z)|^{\frac{1-k}{2}} \sup_{|\xi|=(4\kappa|\rho(z)|)^{\frac{1}{2}}} |Q_2(z + \zeta_1^*\eta_z + \xi v_z)|$$

and so (17) holds true.

Of course we have the analogous estimate for  $\hat{h}_1$ . Now we have to handle the error term in (16). Since there is a factor  $P_1P_2$  in front of  $e$  in (16), we can put  $P_2e$  either with  $\hat{h}_1$  in  $\tilde{h}_1$  or we can put  $P_1e$  with  $\hat{h}_2$  in  $\tilde{h}_2$ . But in order to have a good upper bound for  $\tilde{h}_1$  and  $\tilde{h}_2$ , we have to cut it in two pieces in a suitable way. This will be done analogously to the construction of the currents. Let

$$\begin{aligned}\mathcal{U}_1 &:= \left\{ \zeta \in \mathcal{P}_{\kappa|\rho(z)|}(z), \left| \frac{f_1(\zeta)|\rho(z)|^{\frac{i_1}{2}}}{P_1(\zeta)} \right| > \frac{1}{3} \left| \frac{f_2(\zeta)|\rho(z)|^{\frac{i_2}{2}}}{P_2(\zeta)} \right| \right\}, \\ \mathcal{U}_2 &:= \left\{ \zeta \in \mathcal{P}_{\kappa|\rho(z)|}(z), \frac{2}{3} \left| \frac{f_2(\zeta)|\rho(z)|^{\frac{i_2}{2}}}{P_2(\zeta)} \right| > \left| \frac{f_1(\zeta)|\rho(z)|^{\frac{i_1}{2}}}{P_1(\zeta)} \right| \right\}.\end{aligned}$$

Let also  $\chi$  be a smooth function on  $\mathbb{C}^2 \setminus \{0\}$  such that  $\chi(z_1, z_2) = 1$  if  $|z_1| > \frac{2}{3}|z_2|$  and  $\chi(z_1, z_2) = 0$  if  $|z_1| < \frac{1}{3}|z_2|$ . We set  $\chi_1(\zeta) = \chi\left(\frac{f_1(\zeta)|\rho(z)|^{\frac{i_1}{2}}}{P_1(\zeta)}, \frac{f_2(\zeta)|\rho(z)|^{\frac{i_2}{2}}}{P_2(\zeta)}\right)$ ,  $\chi_2(\zeta) = 1 - \chi_1(\zeta)$  and at last we define

$$\begin{aligned}\tilde{h}_1(\zeta) &= \hat{h}_1(\zeta) + \chi_1(\zeta)P_2(\zeta)e(\zeta), \\ \tilde{h}_2(\zeta) &= \hat{h}_2(\zeta) + \chi_2(\zeta)P_1(\zeta)e(\zeta).\end{aligned}$$

And we now look for an upper bound for  $P_1(\zeta)e(\zeta)$  on  $\mathcal{U}_1$ .

*Fact 2:* For all  $\zeta$  belonging to  $\mathcal{P}_{4\kappa|\rho(z)|}(z)$ , we have uniformly with respect to  $\zeta$  and  $z$

$$(19) \quad |P_1(\zeta)e(\zeta)| \lesssim c(g) \left( |\rho(z)|^{\frac{i_1-i_2}{2}} \sup_{\mathcal{P}_{4\kappa|\rho(z)|}(z)} |Q_1| + \sup_{\mathcal{P}_{4\kappa|\rho(z)|}(z)} |Q_2| \right).$$

Proof: For  $l = 1$  and  $l = 2$ , for all  $i \in I_l$  and for all  $\zeta_1^* \in \Delta_0(4\kappa|\rho(z)|)$  we have, from Proposition 3.2,  $|\alpha_{l,i}(\zeta_1^*)| \leq (3\kappa|\rho(z)|)^{\frac{1}{2}}$  provided  $\kappa$  is small enough. Hence  $|P_l(\zeta)| \lesssim |\rho(z)|^{\frac{i_l}{2}}$  for all  $\zeta \in \mathcal{P}_{4\kappa|\rho(z)|}(z)$ , and with assumption (i), we get for all  $\zeta \in \mathcal{P}_{4\kappa|\rho(z)|}(z)$

$$\begin{aligned}|g(\zeta)| &\leq c(g)(|f_1(\zeta)| + |f_2(\zeta)|) \\ &\lesssim c(g) \left( |\rho(z)|^{\frac{i_1}{2}} |Q_1(\zeta)| + |\rho(z)|^{\frac{i_2}{2}} |Q_2(\zeta)| \right).\end{aligned}$$

This yields for all  $\zeta \in \mathcal{P}_{\kappa|\rho(z)|}(z)$

$$|e(\zeta)| \lesssim c(g) \left( |\rho(z)|^{-\frac{i_2}{2}} \sup_{\mathcal{P}_{4\kappa|\rho(z)|}(z)} |Q_1| + |\rho(z)|^{-\frac{i_1}{2}} \sup_{\mathcal{P}_{4\kappa|\rho(z)|}(z)} |Q_2| \right)$$

from which (19) follows.

Therefore we have the identity  $g = P_1\tilde{h}_1 + P_2\tilde{h}_2$  and upper bounds for  $\tilde{h}_2$  using (17) and (19), the corresponding one for  $\tilde{h}_1$  being also true of course. But our final goal is to write  $g$  as  $g = \hat{g}_1f_1 + \hat{g}_2f_2$ . So we put  $\hat{g}_1 = \frac{\tilde{h}_1}{Q_1}$  and  $\hat{g}_2 = \frac{\tilde{h}_2}{Q_2}$  so that  $g = \hat{g}_1f_1 + \hat{g}_2f_2$ . Moreover, from (17) and (19), and since  $\chi_2$  has support in  $\mathcal{U}_2$ , it follows for  $\zeta \in \mathcal{P}_{\kappa|\rho(z)|}(z)$

$$(20) \quad |\hat{g}_2(\zeta)| \leq (c_\infty^{(2)}(g) + c(g)) \frac{1}{Q_2(\zeta)} \sup_{\mathcal{P}_{4\kappa|\rho(z)|}(z)} |Q_2| + c(g) \frac{1}{Q_1(\zeta)} \sup_{\mathcal{P}_{4\kappa|\rho(z)|}(z)} |Q_1|.$$

Therefore, in order to prove that  $\tilde{g}_2$  is bounded, we will have to prove that  $\frac{Q_i(\xi)}{Q_i(\zeta)}$  is bounded for  $\zeta \in \mathcal{P}_{\kappa|\rho(z)|}(z)$  and  $\xi \in \mathcal{P}_{4\kappa|\rho(z)|}(z)$ . This is the aim of the following Fact 3.

*Fact 3:* For  $l = 1$  and  $l = 2$ ,  $\zeta \in \mathcal{P}_{2\kappa|\rho(z)|}(z)$  and  $\xi \in \mathcal{P}_{4\kappa|\rho(z)|}(z)$ , we have uniformly with respect to  $z$ ,  $\zeta$  and  $\xi$ :

$$(21) \quad \left| \frac{Q_l(\xi)}{Q_l(\zeta)} \right| \lesssim 1.$$

The proof of fact 3 is analogous to the proof of Lemma 5.1. Without any restriction we assume  $l = 2$ .

*First case:* If  $|\zeta_{0,1}^*| > 4\kappa|\rho(z)|$ , then we have the parametrization of  $X_2$  and it suffices to prove for  $i \notin I_2$  that  $\left| \frac{\xi_2^* - \alpha_{2,i}^*(\xi_1^*)}{\zeta_2^* - \alpha_{2,i}^*(\zeta_1^*)} \right| \lesssim 1$ .

If  $|\alpha_{2,i}(\zeta_1^*)| \geq |\rho(z)|^{\frac{1}{2}}$ , since from Proposition 3.2  $\frac{\partial \alpha_{2,i}}{\partial \zeta_1^*}$  is bounded,  $|\alpha_{2,i}(\zeta_1^*)| \geq \frac{1}{2}|\rho(z)|^{\frac{1}{2}}$  and  $|\alpha_{2,i}(\zeta_1^*)| \geq \frac{1}{2}|\alpha_{2,i}(\xi_1^*)|$ , so  $\left| \frac{\xi_2^* - \alpha_{2,i}^*(\xi_1^*)}{\zeta_2^* - \alpha_{2,i}^*(\zeta_1^*)} \right| \lesssim 1$  is satisfied.

If  $|\alpha_{2,i}(\zeta_1^*)| \leq |\rho(z)|^{\frac{1}{2}}$ , then  $|\xi_2^* - \alpha_{2,i}^*(\xi_1^*)| \lesssim |\rho(z)|^{\frac{1}{2}}$  and since by definition of  $I_2$ ,  $|\alpha_{2,i}(\zeta_1^*)| \geq \frac{5}{2}\kappa|\rho(z)|^{\frac{1}{2}}$  for all  $\zeta_1^* \in \Delta_0(2\kappa|\rho(z)|)$ , we have  $|\zeta_2^* - \alpha_{2,i}(\zeta_1^*)| \gtrsim \kappa|\rho(z)|^{\frac{1}{2}}$  for all  $\zeta \in \mathcal{P}_{2\kappa|\rho(z)|}(z)$  and so the inequality  $\left| \frac{\xi_2^* - \alpha_{2,i}^*(\xi_1^*)}{\zeta_2^* - \alpha_{2,i}^*(\zeta_1^*)} \right| \lesssim 1$  holds true.

*Second case:* If  $|\zeta_{0,1}^*| < 4\kappa|\rho(z)|$  and  $|\zeta_{0,2}^*| < (4\kappa|\rho(z)|)^{\frac{1}{2}}$ , then  $\delta(\xi, \zeta_0) \lesssim \delta(\xi, z) + \delta(z, \zeta_0) \lesssim |\rho(z)|$  and as in the proof of Lemma 5.1,  $|Q_2(\xi)| = |f_2(\xi)| \lesssim |\rho(z)|^{\frac{p_2}{2}}$ . From proposition 3.1,  $\mathcal{P}_{4\kappa|\rho(z)|}(z) \cap X_2 = \emptyset$  so  $|f_2(\zeta)| \gtrsim |\rho(\zeta)|^{\frac{p_2}{2}}$  and again we are done in this case.

*Third case:* If  $|\zeta_{0,1}^*| < 4\kappa|\rho(z)|$  and  $|\zeta_{0,2}^*| \geq (4\kappa|\rho(z)|)^{\frac{1}{2}}$ , then as in the third case of the proof of Lemma 5.1,  $f_2(\xi)$  and  $f_2(\zeta)$  are comparable to  $|\zeta_{0,2}^*|^{p_2}$ . Again we are done in this case and Fact 3 is proved.

From (20) and (21), we get that  $\hat{g}_2$  is uniformly bounded. However, assumption (b) of Theorem 1.2 is a little stronger and we need that the derivatives  $\frac{\partial^{\alpha+\beta+\bar{\alpha}+\bar{\beta}} \hat{g}_2}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta} \partial \bar{\zeta}_1^{\bar{\alpha}} \partial \bar{\zeta}_2^{\bar{\beta}}}$  of  $\hat{g}_2$  do

not explode faster than  $|\rho(z)|^{\alpha+\frac{\beta}{2}}$  is  $\mathcal{P}_{\kappa|\rho(z)|}(z)$  for all  $\alpha, \beta, \bar{\alpha}$  and  $\bar{\beta}$ . Actually, inequality (17) and Cauchy's inequalities implies that, for all  $\zeta \in \mathcal{P}_{\kappa|\rho(z)|}(z)$ ,  $\left| \frac{\partial^{\alpha+\beta} \hat{h}_2}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}}(\zeta) \right| \lesssim |\rho(z)|^{-\alpha-\frac{\beta}{2}} c_{\infty}^{(2)}(g) \sup_{|\xi|=(4\kappa|\rho(z)|)^{\frac{1}{2}}} |Q_2(z + \zeta_1^* \eta_z + \xi v_z)|$ . With Lemma 5.1 and (21), we get  $\left| \frac{\partial^{\alpha+\beta}}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}} \left( \frac{\hat{h}_2}{Q_2} \right) \right| \lesssim |\rho(z)|^{-\alpha-\frac{\beta}{2}} c_{\infty}^{(2)}(g)$ .

Applying the same process with (19) to  $eP_1$ , we get

$$\left| \frac{\partial^{\alpha+\beta} eP_1}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}}(\zeta) \right| \lesssim |\rho(z)|^{-\alpha-\frac{\beta}{2}} c(g) \left( |\rho(z)|^{\frac{i_1-i_2}{2}} \sup_{\mathcal{P}_{4\kappa|\rho(z)|}(z)} |Q_1| + \sup_{\mathcal{P}_{4\kappa|\rho(z)|}(z)} |Q_2| \right).$$

Again Lemma 5.1 and (21) yield  $\left| \frac{\partial^{\alpha+\beta+\bar{\alpha}+\bar{\beta}}}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta} \partial \bar{\zeta}_1^{\bar{\alpha}} \partial \bar{\zeta}_2^{\bar{\beta}}}(\zeta) \left( \chi_2 \frac{eP_1}{Q_2} \right) \right| \lesssim |\rho(z)|^{-\alpha-\bar{\zeta}-\frac{\beta+\bar{\beta}}{2}} c(g)$ .

Therefore,  $\hat{g}_2$  satisfies (b) of Theorem 1.2 and of course,  $\hat{g}_1$  also does.  $\square$

**6.3. The  $L^q$ -case.** The assumption, under which a function  $g$  holomorphic on  $D$  can be written as  $g = g_1 f_1 + g_2 f_2$  with  $g_1$  and  $g_2$  being holomorphic on  $D$  and belonging to  $L^q(D)$ , uses a  $\kappa$ -covering  $\left( \mathcal{P}_{\kappa|\rho(z_j)|}(z_j) \right)_{j \in \mathbb{N}}$  in addition to the divided differences.

By transversality of  $X_1$  and  $bD$ , and of  $X_2$  and  $bD$ , for all  $j$  there exists  $w_j$  in the complex tangent plane to  $bD_{\rho(z_j)}$  such that  $\pi_j$ , the orthogonal projection on the hyperplane

orthogonal to  $w_j$  passing through  $z_j$ , is a covering of  $X_1$  and  $X_2$ . We denote by  $w_1^*, \dots, w_n^*$  an orthonormal basis of  $\mathbb{C}^n$  such that  $w_1^* = \eta_{z_j}$  and  $w_n^* = w_j$  and we set  $\mathcal{P}'_\varepsilon(z_j) = \{z' = z_j + z_1^* w_1^* + \dots + z_{n-1}^* w_{n-1}^*, |z_1^*| < \varepsilon \text{ and } |z_k^*| < \varepsilon^{\frac{1}{2}}, k = 2, \dots, n-1\}$ . We put

$$c_{q,\kappa,(z_j)_{j \in \mathbb{N}}}^{(l)}(g) = \sum_{j=0}^{\infty} \int_{z' \in \mathcal{P}'_{2\kappa|\rho(z_j)}(z_j)} |\rho(z_j)|^{q\frac{k-1}{2}+1} \left| g_{z',w_n^*}^{(l)}[\lambda_1, \dots, \lambda_k] \right|^q dV_{n-1}(z')$$

$\sum_{\substack{\lambda_1, \dots, \lambda_k \in \Lambda_{z', w_n^*} \\ \lambda_i \neq \lambda_j \text{ for } i \neq j}}$

where  $dV_{n-1}$  is the Lebesgue measure in  $\mathbb{C}^{n-1}$  and  $g^{(l)} = \frac{g}{f_l}$ ,  $l = 1$  or  $l = 2$ .

Now we prove the following necessary conditions

**Theorem 6.5.** *Let  $g_1$  and  $g_2$  belonging to  $L^q(D)$ ,  $q \in [1, +\infty[$ , be two holomorphic functions on  $D$  and set  $g = g_1 f_1 + g_2 f_2$ . Then*

- (i)  $\frac{g}{\max(|f_1|, |f_2|)}$  belongs to  $L^q(D)$  and  $\left\| \frac{g}{\max(|f_1|, |f_2|)} \right\|_{L^q(D)} \lesssim \max(\|g_1\|_{L^q(D)}, \|g_2\|_{L^q(D)})$ .
- (ii) For  $l = 1$  or  $l = 2$  and any  $\kappa$ -covering  $(\mathcal{P}_{\kappa|\rho(z_j)}(z_j))_j$ , we have  $c_{q,\kappa,(z_j)_{j \in \mathbb{N}}}^{(l)}(g) \lesssim \|g_l\|_{L^q(D)}^q$ ,

*Proof:* The point (i) is trivial and we only prove (ii). As in the proof of Theorem 6.3, for all  $j \in \mathbb{N}$ , all  $z' \in \mathcal{P}'_{\kappa|\rho(z_j)}(z_j)$  and all  $r \in [\frac{7}{2}\kappa|\rho(z_j)|^{\frac{1}{2}}, 4\kappa|\rho(z_j)|^{\frac{1}{2}}]$ , we have

$$g_{z',w_n^*}^{(l)}[\lambda_1, \dots, \lambda_k] = \frac{1}{2i\pi} \int_{|\lambda|=r} \frac{g_l(z' + \lambda w_n^*)}{\prod_{i=1}^k (\lambda - \lambda_i)} d\lambda.$$

After integration for  $r \in [(\frac{7}{2}\kappa|\rho(z_j)|)^{\frac{1}{2}}, (4\kappa|\rho(z_j)|)^{\frac{1}{2}}]$ , Jensen's inequality yields

$$\left| g_{z',w_n^*}^{(l)}[\lambda_1, \dots, \lambda_k] \right|^q \lesssim |\rho(z_j)|^{\frac{1-k}{2}q-1} \int_{|\lambda| \leq (4\kappa|\rho(z_j)|)^{\frac{1}{2}}} |g_l(z' + \lambda w_n^*)|^q dV_1(\lambda).$$

Now we integrate the former inequality for  $z' \in \mathcal{P}'_{\kappa|\rho(z_j)}(z_j)$  and get

$$\int_{z' \in \mathcal{P}'_{\kappa|\rho(z_j)}(z_j)} \left| g_{z',w_n^*}^{(l)}[\lambda_1, \dots, \lambda_k] \right|^q |\rho(z_j)|^{\frac{k-1}{2}q+1} dV_{n-1}(z) \lesssim \int_{z \in \mathcal{P}_{4\kappa|\rho(z_j)}(z_j)} |g_l(z)|^q dV_n(z).$$

Since  $(\mathcal{P}_{\kappa|\rho(z_j)}(z_j))_{j \in \mathbb{N}}$  is a  $\kappa$ -covering, we deduce from this inequality that  $c_{q,\kappa,(z_j)_{j \in \mathbb{N}}}^{(l)}(g) \lesssim \|g_l\|_{L^q(D)}^q$ .  $\square$

**Theorem 6.6.** *Let  $g$  be a holomorphic function on  $D$  belonging to the ideal generated by  $f_1$  and  $f_2$ , such that  $c_{q,\kappa,(z_j)_{j \in \mathbb{N}}}^{(l)}(g)$  is finite and such that  $\frac{g}{\max(|f_1|, |f_2|)}$  belongs to  $L^q(D)$ . Then there exist two holomorphic functions  $g_1$  and  $g_2$  which belong to  $L^q(D)$  and such that  $g = g_1 f_1 + g_2 f_2$ .*

*Proof:* We aim to apply Theorem 6.1. For all  $j$  in  $\mathbb{N}$ , in order to construct on  $\mathcal{P}_{\kappa|\rho(z_j)}(z_j)$  two functions  $\tilde{g}_1^{(j)}$  and  $\tilde{g}_2^{(j)}$  which satisfy the assumption of Theorem 6.1, we proceed as in the proof of Theorem 6.4. The main difficulty occurs, as in the proof of Theorem 6.4, when we are near a point  $\zeta_0$  which belongs to  $X_1 \cap X_2 \cap bD$ . We denote by  $(\zeta_{0,1}^*, \zeta_{0,2}^*)$  the coordinates of  $\zeta_0$  in the Koranyi coordinates at  $z_j$ . If  $|\zeta_{0,1}^*| < 4\kappa|\rho(z_{j_0})|$ , we set  $i_{1,j} = i_{2,j} = 0$ ,  $I_{1,j} = I_{2,j} = \emptyset$ ,  $P_{1,j} = P_{2,j} = 1$ ,  $Q_{1,j} = f_1$  and  $Q_{2,j} = f_2$ . Otherwise, we use the parametrization  $\alpha_{1,i}^{(j)}$ ,  $i \in \{1, \dots, p_1^{(j)}\}$  of  $X_1$  and  $\alpha_{2,i}^{(j)}$ ,  $i \in \{1, \dots, p_2^{(j)}\}$  of

$X_2$  given by Proposition 2.2 and for  $l = 1$  and  $l = 2$ , we still denote by  $I_{l,j}$  the set  $I_{l,j} = \{i, \exists z_1^* \in \Delta_0(2\kappa|\rho(z_j)|)\}$  such that  $|\alpha_{l,i}^{(j)}(z_1^*)| \leq \frac{5}{2}\kappa|\rho(z_j)|^{\frac{1}{2}}$ ,  $i_{l,j} = \#I_{l,j}$ ,  $P_{l,j}(\zeta) = \prod_{i \in I_{l,j}} (\zeta_2^* - \alpha_{l,i}^{(j)}(\zeta_1^*))$  and  $Q_{l,j} = \frac{f_l}{P_{l,j}}$ . We define  $\hat{h}_1^{(j)}$  and  $\hat{h}_2^{(j)}$  as  $\hat{h}_1$  and  $\hat{h}_2$  in the proof of Theorem 6.4. Instead of defining  $e_1^{(j)}$  and  $e_2^{(j)}$  by integrals over the set  $\{|\xi| = (4\kappa|\rho(z_j)|)^{\frac{1}{2}}\}$  as we defined  $e_1$  and  $e_2$  in the proof of Theorem 6.4, here we integrate over  $\{(\frac{7}{2}\kappa|\rho(z_j)|)^{\frac{1}{2}} \leq |\xi| \leq (4\kappa|\rho(z_j)|)^{\frac{1}{2}}\}$  and set

$$e^{(j)}(z) = \frac{1}{2\pi(2 - \sqrt{\frac{7}{2}})\sqrt{\kappa|\rho(z_j)|}} \int_{\{(\frac{7}{2}\kappa|\rho(z_j)|)^{\frac{1}{2}} \leq |\xi| \leq (4\kappa|\rho(z_j)|)^{\frac{1}{2}}\}} \frac{g(z_1^*, \xi)}{P_{1,j}(z_1^*, \xi)P_{2,j}(z_1^*, \xi)(z_2^* - \xi)} dV(\xi).$$

We therefore have for all  $j$  and all  $z \in \mathcal{P}_{\kappa|\rho(z_j)}(z_j)$ :

$$g(z) = \tilde{h}_1^{(j)}(z)P_{1,j}(z) + \tilde{h}_2^{(j)}(z)P_{2,j}(z) + P_{1,j}(z)P_{2,j}(z)e^{(j)}(z).$$

We split  $\mathcal{P}_{\kappa|\rho(z_j)}(z_j)$  in two parts as in Theorem 6.4 and set

$$\begin{aligned} \mathcal{U}_1^{(j)} &:= \left\{ \zeta \in \mathcal{P}_{\kappa|\rho(z_j)}(z_j), \left| \frac{f_1(\zeta)|\rho(z_j)|^{\frac{i_{1,j}}{2}}}{P_{1,j}(\zeta)} \right| > \frac{1}{3} \left| \frac{f_2(\zeta)|\rho(z_j)|^{\frac{i_{2,j}}{2}}}{P_2(\zeta)} \right| \right\}, \\ \mathcal{U}_2^{(j)} &:= \left\{ \zeta \in \mathcal{P}_{\kappa|\rho(z_j)}(z_j), \frac{2}{3} \left| \frac{f_2(\zeta)|\rho(z_j)|^{\frac{i_{2,j}}{2}}}{P_{2,j}(\zeta)} \right| > \left| \frac{f_1(\zeta)|\rho(z_j)|^{\frac{i_{1,j}}{2}}}{P_{1,j}(\zeta)} \right| \right\}. \end{aligned}$$

We still denote by  $\chi$  a smooth function on  $\mathbb{C}^2 \setminus \{0\}$  such that  $\chi(z_1, z_2) = 1$  if  $|z_1| > \frac{2}{3}|z_2|$  and  $\chi(z_1, z_2) = 0$  if  $|z_1| < \frac{1}{3}|z_2|$ ; and we set  $\chi_1^{(j)}(\zeta) = \chi\left(\frac{f_1(\zeta)|\rho(z_j)|^{\frac{i_{1,j}}{2}}}{P_{1,j}(\zeta)}, \frac{f_2(\zeta)|\rho(z_j)|^{\frac{i_{2,j}}{2}}}{P_2(\zeta)}\right)$ ,  $\chi_2^{(j)}(\zeta) = 1 - \chi_1^{(j)}(\zeta)$  and

$$\begin{aligned} \tilde{g}_1^{(j)}(z) &= \frac{1}{Q_1^{(j)}(z)} \left( \hat{h}_1^{(j)}(z) + \chi_1^{(j)}(z)P_{2,j}(z)e^{(j)}(z) \right), \\ \tilde{g}_2^{(j)}(z) &= \frac{1}{Q_2^{(j)}(z)} \left( \hat{h}_2^{(j)}(z) + \chi_2^{(j)}(z)P_{1,j}(z)e^{(j)}(z) \right). \end{aligned}$$

Therefore  $g = \tilde{g}_1^{(j)}f_1 + \tilde{g}_2^{(j)}f_2$  on  $\mathcal{P}_{\kappa|\rho(z_j)}(z_j)$  and in order to apply Theorem 6.1, the assumptions (b) and (c) are left to be shown.

As in the proof of fact 1, it follows from Lemma 6.2 and (21) that

$$\left| \frac{1}{Q_{2,j}(z)} \hat{h}_2^{(j)}(z) \right| \lesssim \sum_{k=1}^{i_{2,j}} |\rho(z_j)|^{\frac{k-1}{2}} \left| g_{z_j+z_1^*\eta_{z_j}, v_{z_j}}^{(2)}[\alpha_{1,1}(z_1^*), \dots, \alpha_{1,k}(z_1^*)] \right|$$

uniformly with respect to  $z \in \mathcal{P}_{2\kappa|\rho(z_j)}(z_j)$  and  $j \in \mathbb{N}$  and therefore

$$(22) \quad \sum_{j \in \mathbb{N}} \int_{\mathcal{P}_{2\kappa|\rho(z_j)}(z_j)} \left| \frac{1}{Q_{2,j}(z)} \hat{h}_2^{(j)}(z) \right|^q dV(z) \lesssim c_{q,\kappa,(z_j)}^{(l)}(g).$$

In particular  $\frac{1}{Q_{2,j}}\hat{h}_2^{(j)}$  is an holomorphic function with  $L^q$ -norm on  $\mathcal{P}_{2\kappa|\rho(z_j)}(z_j)$  lower than  $(c_{q,\kappa,(z_j)}^{(2)}(g))^{1/q}$ . Thus Cauchy's inequalities imply, for all  $\alpha, \beta \in \mathbb{N}$  and all  $z \in \mathcal{P}_{\kappa|\rho(z_j)}(z_j)$ , that

$$(23) \quad \left| \frac{\partial^{\alpha+\beta}}{\partial z_1^{*\alpha} \partial z_2^{*\beta}} \left( \frac{1}{Q_{2,j}} \hat{h}_2^{(j)}(z) \right) \right| \lesssim (c_{q,\kappa,(z_j)}^{(l)}(g))^{1/q} |\rho(z_j)|^{-\frac{3}{q}-\alpha-\frac{\beta}{2}}.$$

Since  $\frac{g}{\max(|f_1|, |f_2|)}$  belongs to  $L^q(D)$ ,  $g$  itself belongs to  $L^q(D)$  and so

$$\int_{\mathcal{P}_{2\kappa|\rho(z_j)}(z_j)} |e^{(j)}(z)|^q dV(z) \lesssim |\rho(z_j)|^{-q\frac{i_{1,j}+i_{2,j}}{2}} \int_{\mathcal{P}_{4\kappa|\rho(z_j)}(z_j)} |g(z)|^q dV(z).$$

In particular, for all  $\alpha$  and  $\beta$  and all  $z \in \mathcal{P}_{\kappa|\rho(z_j)}(z_j)$ , we have

$$(24) \quad \left| \frac{\partial^{\alpha+\beta} e^{(j)}}{\partial z_1^{*\alpha} \partial z_2^{*\beta}}(z) \right| \lesssim |\rho(z_j)|^{-\frac{3}{q}-\frac{i_{1,j}+i_{2,j}}{2}-\alpha-\frac{\beta}{2}}.$$

The inequalities (23) and (24) imply that the hypothesis (c) of Theorem 6.1 is satisfied by  $\tilde{g}_2^{(j)}$  for some large  $N$ , the same is also true for  $\tilde{g}_1^{(j)}$ .

Now, for  $z$  belonging to  $\mathcal{U}_2^{(j)}$ , we get from (21):

$$\left| \frac{P_1^{(j)}(z)e^{(j)}(z)}{Q_2^{(j)}(z)} \right| \lesssim \frac{1}{|\rho(z_j)|} \int_{(\frac{7}{2}\kappa|\rho(z_j)|)^{\frac{1}{2}} \leq |\xi| \leq (4\kappa|\rho(z_j)|)^{\frac{1}{2}}} \frac{|g(\zeta_1^*, \xi)|}{\max(|f_1(\zeta_1^*, \xi)|, |f_2(\zeta_1^*, \xi)|)} dV(\xi)$$

and so

$$\int_{\mathcal{U}_2 \cap \mathcal{P}_{\kappa|\rho(z_j)}(z_j)} \left| \frac{P_1^{(j)}(z)e^{(j)}(z)}{Q_2^{(j)}(z)} \right|^q dV(z) \lesssim \int_{\mathcal{P}_{4\kappa|\rho(z_j)}(z_j)} \left( \frac{|g(\zeta_1^*, \xi)|}{\max(|f_1(\zeta_1^*, \xi)|, |f_2(\zeta_1^*, \xi)|)} \right)^q dV(\xi).$$

Since  $(\mathcal{P}_{\kappa|\rho(z_j)}(z_j))_{j \in \mathbb{N}}$  is a  $\kappa$ -covering, this yields:

$$(25) \quad \sum_{j \in \mathbb{N}} \int_{\mathcal{U}_2 \cap \mathcal{P}_{\kappa|\rho(z_j)}(z_j)} \left| \frac{P_1^{(j)}(z)e^{(j)}(z)}{Q_2^{(j)}(z)} \right|^q dV(z) \lesssim \left\| \frac{g}{\max(|f_1|, |f_2|)} \right\|_{L^q(D)}^q.$$

Moreover, for all  $\alpha, \beta \in \mathbb{N}$ ,  $\left| \frac{\partial^{\alpha+\beta} \chi_2^{(j)}}{\partial \zeta_1^{*\alpha} \partial \zeta_2^{*\beta}}(z) \right| \lesssim |\rho(z_j)|^{-\alpha-\frac{\beta}{2}}$ , (22) and (25) imply that  $(\tilde{g}_2^{(j)})_{j \in \mathbb{N}}$  satisfy the assumption (b) of Theorem 6.1 that we can therefore apply.  $\square$

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