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THE HOMOTOPY THEORY OF BIALGEBRAS OVER PAIRS OF OPERADS

SINAN YALIN

ABSTRACT. We endow the category of bialgebras over a pair of operads in distribution with a cofibrantly generated model category structure. We work in the category of chain complexes over a field of characteristic zero. We split our construction in two steps. In the first step, we equip coalgebras over an operad with a cofibrantly generated model category structure. This theorem generalizes the one of [1]. In the second one we use the adjunction between bialgebras and coalgebras via the free algebra functor. This result allows us to do classical homotopical algebra in various categories such as associative bialgebras, Lie bialgebras or Poisson bialgebras in chain complexes.

Keywords : operads, bialgebras category, homotopical algebra.

AMS : 18G55 ; 18D50.

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INTRODUCTION

The goal of this paper is to define a model category structure for the categories of bialgebras governed by operads in distribution. The work of Drinfeld on quantum groups (see [2] and [3]) has initiated the study of bialgebra structures where the product and the coproduct belong to various types of algebras. Besides the classical Hopf algebras, examples include their non-commutative non-cocommutative variant, Lie bialgebras and Poisson bialgebras. Applications ranges from knot theory, in topology, to integrable systems in mathematical physics. The theory of operads in distribution, introduced by Fox and Markl in [5], provides a convenient generalization of the classical categories of bialgebras defined by products and coproducts in distribution. The general idea is that there is an operad encoding the operations (where we have several inputs and a single output) and another operad encoding the

cooperations (a single input and several outputs). The distributive law then formalizes the interplay between these operads, i.e the compatibilities between operations and cooperations. We refer the reader to [15] for a detailed survey providing many examples of these generalized bialgebras. One may then wonder how to transpose homotopical algebra methods in this setting, as it has been done successfully for algebras over operads. For this aim, this paper constructs a closed model category structure for this kind of bialgebras. We expect interesting homotopical outcomes in the various fields of mathematics where such bialgebraic structures appear.

The existence of a cofibrantly generated model category structure on algebras over a suitable operad is a classical result, see [11]. When working over a field of characteristic zero, such a structure exists for any operad. Let $Ch_{\mathbb{K}}^+$ be the full subcategory of $Ch_{\mathbb{K}}$ of connective chain complexes. We denote by ${}^PCh_{\mathbb{K}}^+$ the category of P -coalgebras in $Ch_{\mathbb{K}}^+$. In a first step we establish the existence of a model category structure for coalgebras over an operad:

Theorem 0.1. *The category of P -coalgebras ${}^PCh_{\mathbb{K}}^+$ inherits a cofibrantly generated model category structure such that a morphism f of ${}^PCh_{\mathbb{K}}^+$ is*

- (i) *a weak equivalence if $U(f)$ is a weak equivalence in $Ch_{\mathbb{K}}^+$;*
- (ii) *a cofibration if $U(f)$ is a cofibration in $Ch_{\mathbb{K}}^+$;*
- (iii) *a fibration if f has the right lifting property with respect to acyclic cofibrations.*

Note that an analogous result has been proven in [21] in the context of unbounded chain complexes. We follow another simpler approach. We do not address the same level of generality, but we obtain a stronger result. To be more precise, in contrast with [21], we obtain a cofibrantly generated structure. These generating cofibrations are crucial to transfer the model structure on bialgebras. Moreover, we do not need the hypothesis considered in [21] about the underlying operad (see [21], condition 4.3). Our method is close to the ideas of [9]. Such a result also appears in [1], but for coalgebras over a quasi-free cooperad. We prove this theorem via the following steps. First, we prove two crucial results. The first is the structure of the cofree coalgebra over an operad. The second one is based on the construction, for any P -coalgebra A , of its enveloping cooperad. It expresses the coproduct of A with a cofree coalgebra in terms of the evaluation of the associated enveloping cooperad functor. Axioms MC2 and MC3 are obvious. Axiom MC1 is proved in an analogue way than in the case of algebras. The main difficulty lies in the proofs of MC4 and MC5. For this aim, we use proofs inspired from that of [9] and adapted to our operadic setting. In order to produce the desired factorization axioms, our trick here is to use a slightly modified version of the usual small object argument. We use smallness with respect to injections systems.

We denote by ${}^QCh_{\mathbb{K}}^+$ the category of (P, Q) -bialgebras in $Ch_{\mathbb{K}}^+$, where P encodes the operations and Q the cooperations. We use an adjunction

$$U : {}^QCh_{\mathbb{K}}^+ \rightleftarrows {}^PCh_{\mathbb{K}}^+ : Q^*.$$

The model category structure on (P, Q) -bialgebras is then given by our main theorem:

Theorem 0.2. *The category of (P, Q) -bialgebras ${}^QCh_{\mathbb{K}}^+$ inherits a cofibrantly generated model category structure such that a morphism f of ${}^QCh_{\mathbb{K}}^+$ is*

- (i) a weak equivalence if $U(f)$ is a weak equivalence in ${}^{\mathcal{Q}}Ch_{\mathbb{K}}^+$ (i.e. a weak equivalence in $Ch_{\mathbb{K}}^+$ by definition of the model structure on ${}^{\mathcal{Q}}Ch_{\mathbb{K}}^+$);
- (ii) a fibration if $U(f)$ is a fibration in ${}^{\mathcal{Q}}Ch_{\mathbb{K}}^+$;
- (iii) a cofibration if f has the left lifting property with respect to acyclic fibrations.

The main difficulty is the proof of MC5. We use mainly the small object argument for smallness with respect to injections systems, combined with a result about cofibrations in algebras over an operad due to Hinich [11]. We cannot use the usual simplifying hypothesis of smallness with respect to all morphisms.

Organization: the overall setting is reviewed in section 1. We suppose some prerequisites concerning operads (see [16]) and give definitions of algebras and coalgebras over an operad. Then we define distributive laws from the monadic viewpoint, following [5]. Examples of monads and comonads include operads and cooperads. We recall some basic facts about the small object argument in cofibrantly generated model categories, in order to fix useful notations for the following.

The heart of this paper consists of sections 2 and 3, devoted to the proofs of theorem 0.1 and 0.2. The proof of theorem 0.1 relies heavily on the notion of enveloping cooperad, which is defined in 2.1. In 2.3, we follow the argument line of [9], checking carefully where modifications are needed to work at our level of generality. Theorem 0.2 is proved in section 3, by using adjunction properties to transfer the model structure obtained in theorem 0.1. The crux here is a small object argument with respect to systems of injections of coalgebras.

1. RECOLLECTIONS

In this section, we first list some notions and facts about operads and algebras over operads. Then we review the interplay between monads and comonads by means of distributive laws and make the link with operads. It leads us to the crucial definition of bialgebras over pairs of operads in distribution. Finally, we recall a classical tool of homotopical algebra, namely the small object argument, aimed to produce factorizations in model categories. The material of this section is taken from [16], [5] and [13].

1.1. Algebras and coalgebras over operads. Definitions of this subsection are given in the category of non-negatively graded chain complexes $Ch_{\mathbb{K}}$, where \mathbb{K} is a field of characteristic zero. It will be also our base category for the next sections. We adopt most conventions of [16] and freely use the notations of this reference. The only exception is the name Σ -module which denotes the notion of \mathbb{S} -module of [16]. This convention is more usual in topology. We also refer the reader to [16] for the definitions of operads and cooperads.

Operads are used to parametrize various kind of algebraic structures: associative, commutative, Poisson or Lie algebras for instance. There exists several equivalent approaches for the definition of an algebra over an operad. We will use the following one which we recall for convenience:

Definition 1.1. (*the monadic approach*) Let (P, γ, ι) be an operad, where γ is the composition product and ι the unit. A P -algebra is a vector space A endowed with

a linear application $\gamma_A : P(A) \rightarrow A$ such that the following diagrams commute

$$\begin{array}{ccc} (P \circ P)(A) & \xrightarrow{P(\gamma_A)} & P(A) \\ \gamma^{(A)} \downarrow & & \downarrow \gamma_A \\ P(A) & \xrightarrow{\gamma_A} & A \end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{\iota(A)} & P(A) \\ & \searrow \gamma_A & \downarrow \\ & = & A \end{array}$$

We will denote ${}^P\text{Vect}_{\mathbb{K}}$ the category of P -algebras in vector spaces and ${}^P\text{Ch}_{\mathbb{K}}$ the category of P -algebras in non-negatively graded chain complexes. Fundamental examples of operads include the operad As encoding associative algebras, the operad Com of commutative algebras, the operad Lie of Lie algebras and the operad $Pois$ of Poisson algebras.

For every vector space V , we can equip $P(V)$ with a P -algebra structure by setting $\gamma_{P(V)} = \gamma(V) : P(P(V)) \rightarrow P(V)$. As a consequence of the definition, we thus get the free P -algebra functor:

Proposition 1.2. (see [16], proposition 5.2.6) *The P -algebra $(P(V), \gamma(V))$ equipped with the map $\iota(V) : I(V) = V \rightarrow P(V)$ is the free P -algebra on V .*

There is also a notion of coalgebra over a cooperad:

Definition 1.3. (the comonadic approach) Let (C, Δ, η) be a cooperad, where Δ is the decomposition product and η the counit (they define on C a structure of comonoid). A C -coalgebra is a vector space X equipped with a linear application $\rho : X \rightarrow C(X)$ such that the following diagrams commute:

$$\begin{array}{ccc} X & \xrightarrow{\rho} & C(X) \\ \rho \downarrow & & \downarrow \Delta\rho \\ C(X) & \xrightarrow{C(\rho)} & C(C(X)) \end{array}$$

$$\begin{array}{ccc} X & \xrightarrow{\rho} & C(X) \\ & \searrow & \downarrow \eta\rho \\ & = & X \end{array}$$

We can go from operads to cooperads and vice-versa by dualization. Indeed, if C is a cooperad, then the Σ -module P defined by $P(n) = C(n)^* = \text{Hom}_{\mathbb{K}}(C(n), \mathbb{K})$ form an operad. Conversely, suppose that \mathbb{K} is of characteristic zero and P is an operad such that each $P(n)$ is finite dimensional. Then the $P(n)^*$ form a cooperad, in the sense of [10] and [16]. The additional hypotheses are needed because you have to use, for finite dimensional vector spaces V and W , the isomorphism $(V \otimes W)^* \cong V^* \otimes W^*$ to define properly the decomposition product. We can therefore give the following definition of coalgebras over an operad:

Definition 1.4. Let P be an operad. A P -coalgebra is a vector space C equipped with linear applications $\rho_n : P(n) \otimes C \rightarrow C^{\otimes n}$ for every $n \geq 0$. These maps are Σ_n -equivariant and associative with respect to the operadic compositions. If \mathbb{K} is a field of characteristic zero and the $P(n)$ are finite dimensional, then it is equivalent to define applications $\bar{\rho}_n : C \rightarrow P(n)^* \otimes_{\Sigma_n} C^{\otimes n}$.

Now we would like to define a cofree P -coalgebra functor. One of the main difficulties with the construction of cofree objects is that the tensor product is in general not distributive with respect to the categorical product. In chain complexes, only finite products coincide with finite direct sums, and we need to deal with infinite products in the construction of cofree objects. Consequently, we have to restrict ourselves to the subcategory $Ch_{\mathbb{K}}^+$ of chain complexes C satisfying $C_0 = 0$. This hypothesis ensures that the morphisms $P(n) \otimes C_d \rightarrow (C^{\otimes n})_d$ are zero for $n > d$. Then we have a cofree P -coalgebra functor $P^* : Ch_{\mathbb{K}}^+ \rightarrow_P Ch_{\mathbb{K}}^+$, which is by definition the right adjoint to the forgetful functor and is given by the following formula:

Theorem 1.5. *Let V be an object of $Ch_{\mathbb{K}}^+$. Then*

$$P^*(V) = \bigoplus_{r=1}^{\infty} P(r)^* \otimes_{\Sigma_r} V^{\otimes r}$$

inherits a P -coalgebra structure and forms the cofree P -coalgebra.

It is important to note that under the same hypotheses, the notion of coalgebra over the operad P is equivalent to the notion of coalgebra over the cooperad P^* as defined in [10], or the notion of conilpotent P -coalgebras of [16].

1.2. Monads, comonads and distributive laws. In certain cases, bialgebras can be parametrized by a pair of operads in the following way: one operad encodes the operations, the other encodes the cooperations, such that the concerned bialgebra forms an algebra over the first operad and a coalgebra over the second operad. The compatibility relations between operations and cooperations are formalized by the notion of mixed distributive law between the two operads. The Schur functor associated to an operad forms a monad, and the Schur functor associated to a cooperad forms a comonad. The mixed distributive law induces a distributive law between this monad and this comonad. We review briefly the notion of distributive law in the monadic setting. We refer the reader to [5] for definitions of monads, comonads, their algebras and coalgebras.

Let \mathcal{C} be a category. Suppose we have in \mathcal{C} a monad (P, γ, ι) and a comonad (Q^*, δ, ϵ) . We would like to make P and Q^* compatible, that is to define Q^* -coalgebras in P -algebras or conversely P -algebras in Q^* -coalgebras. This compatibility is formalized by the notion of mixed distributive law:

Definition 1.6. A mixed distributive law $\lambda : PQ^* \rightarrow Q^*P$ between P and Q^* is a natural transformation satisfying the following conditions:

- (i) $\lambda \circ \gamma Q^* = Q^* \gamma \circ \lambda$
- (ii) $\delta P \circ \lambda = \lambda \circ P \delta$
- (iii) $\lambda \circ \iota Q^* = Q^* \iota$
- (iv) $\epsilon P \circ \lambda = P \epsilon$

where the $\Lambda : P^m(Q^*)^n \rightarrow (Q^*)^n P^m$, for every natural integers m and n , are the natural transformations obtained by iterating λ . For instance, for $m = 2$ and

$n = 3$ we have

$$P^2(Q^*)^3 \xrightarrow{P\lambda(Q^*)^2} PQ^*P(Q^*)^2 \xrightarrow{\lambda^2 Q^*} Q^*PQ^*PQ^* \xrightarrow{Q^*\lambda^2} (Q^*)^2PQ^*P \xrightarrow{(Q^*)^2\lambda P} (Q^*)^3P^2$$

This conditions allow us to lift P as an endofunctor of the category $Q^* - \text{Coalg}$ of S -coalgebras and Q^* as an endofunctor of the category $P - \text{Alg}$ of T -algebras. These notations are chosen to emphasize the fact that later, the monad P will correspond to an operad P and the comonad Q^* to an operad Q (which gives a comonad Q^* by dualization and the finiteness hypothesis).

Then we can define the notion of bialgebra over a pair (monad,comonad) endowed with a mixed distributive law:

Definition 1.7. (a) Given a monad P , a comonad Q^* and a mixed distributive law $\lambda : PQ^* \rightarrow Q^*P$, a (P, Q^*) -bialgebra (B, β, b) is an object B of \mathcal{C} equipped with two morphisms $\beta : P(B) \rightarrow B$ and $b : B \rightarrow Q^*(B)$ defining respectively a P -algebra structure and a Q^* -coalgebra structure. Furthermore, the maps β and b satisfy a compatibility condition expressed through the commutativity of the following diagram:

$$\begin{array}{ccc} P(Q^*(B)) & \xleftarrow{P(b)} & P(B) \\ \lambda(B) \downarrow & & \downarrow \beta \\ Q^*(P(B)) & & B \\ Q^*(\beta) \downarrow & & \downarrow b \\ Q^*(B) & \xleftarrow{b} & B \end{array}$$

(b) A morphism of (P, Q^*) -bialgebras is a morphism of \mathcal{C} which is both a morphism of P -algebras and a morphism of Q^* -coalgebras.

The category of (P, Q^*) -bialgebras is denoted $(P, Q^*) - \text{Bialg}$.

Remark 1.8. The application $Q^*(\beta) \circ \lambda(B)$ endows $Q^*(B)$ with a P -algebra structure, and the application $\lambda(B) \circ P(b)$ endows $P(B)$ with a Q^* -coalgebra structure. Moreover, given these two structures, the compatibility diagram of definition 1.7 shows that β is a morphism of Q^* -coalgebras and b a morphism P -algebras. The (P, Q^*) -bialgebras can therefore be considered as Q^* -coalgebras in the category $P - \text{Alg}$ of T -algebras or as P -algebras in the category $Q^* - \text{Coalg}$ of S -coalgebras.

In the particular case of operads, the mixed distributive laws can be defined by explicite formulae, for which we refer the reader to [5]. Suppose that \mathbb{K} is of characteristic zero and that every $Q(n)$ is finite dimensional. The notion of Q^* -coalgebra is then exactly the definition of a coalgebra over a comonad. We can therefore define a (P, Q) -bialgebra with a P -algebra structure, a Q -coalgebra structure and compatibilities with respect to the distributive law. The operadic distributive law formalizes the interplay between algebraic operations and coalgebraic cooperations of the bialgebra.

Theorem 1.9. (cf. [5], theorem 11.10) *Let B be a (P, Q) -coalgebra. Then the free P -algebra $P(B)$ has a natural structure of Q -coalgebra and the cofree Q -coalgebra $Q^*(B)$ has a natural structure of P -algebra.*

1.3. Model categories and the small object argument. Model categories are the natural setting to do homotopical algebra. This means that they encode well defined notions of cylinder objects and path objects, homotopy classes, non-abelian cohomology theories and non abelian functor derivation (Quillen's derived functors). We will just recall here some facts about cofibrantly generated model categories and the small object argument, for the purpose to fix conventions and the definition of objects used in our constructions. We refer the reader to the classical reference [20], but also to [4] for a well-written and detailed account on basis of model categories and their homotopy theories, as well as [13] and [12] to push the analysis further. Let us recall briefly the small object argument, which is a general and useful way to produce factorizations with lifting properties with respect to a given class of morphisms. We just sum up the construction given in [4] without detailing the process. It is important to note that, although the sequential colimits used here run over the natural integers, the small object argument works for higher ordinals. We refer the reader to [13] for a detailed treatment in full generality.

Definition 1.10. An object A of \mathcal{M} is sequentially small if for every functor $F : \mathbb{N} \rightarrow \mathcal{M}$, the canonical map

Remark 1.11. A \mathbb{K} -module is sequentially small if and only if it admits a finite presentation, i.e it is isomorphic to the cokernel of a morphism of finitely generated free \mathbb{K} -modules. A chain complex M is sequentially small if and only if a finite number of M_n are non trivial and each M_n has a finite presentation.

Let $\mathcal{F} = \{f_i : A_i \rightarrow B_i\}_{i \in I}$ a set of morphisms of \mathcal{M} . We consider a morphism $p : X \rightarrow Y$ of \mathcal{C} for which we want to produce a factorization $X \rightarrow X' \rightarrow Y$, such that $X' \rightarrow Y$ has the right lifting property with respect to the morphisms of \mathcal{F} . We do not consider the trivial case $X' = Y$. Then there is a recursive construction providing the following commutative diagram:

$$\begin{array}{ccccccc} X & \xrightarrow{i_1} & G^1(\mathcal{F}, p) & \xrightarrow{i_2} & \dots & \xrightarrow{i_k} & G^k(\mathcal{F}, p) & \xrightarrow{i_{k+1}} & \dots \\ p \downarrow & & p_1 \downarrow & & & & p_k \downarrow & & \\ Y & \xlongequal{\quad} & Y & \xlongequal{\quad} & \dots & \xlongequal{\quad} & Y & \xlongequal{\quad} & \dots \end{array}$$

In this recursive procedure, each i_k is obtained by a pushout of the form

$$\begin{array}{ccc} \bigoplus_{\alpha} A_{\alpha} & \longrightarrow & G^{k-1}(\mathcal{F}, p) \\ \bigoplus_{\alpha} f_{\alpha} \downarrow & & \downarrow i_k \\ \bigoplus_{\alpha} B_{\alpha} & \longrightarrow & G^k(\mathcal{F}, p) \end{array}$$

where the f_{α} are morphisms of \mathcal{F} . The category \mathcal{M} is supposed to admit small colimits, so we can consider the infinite composite $i_{\infty} : X \rightarrow G^{\infty}(\mathcal{F}, p)$ of the sequence of maps

$$X \xrightarrow{i_1} G^1(\mathcal{F}, p) \xrightarrow{i_2} \dots \xrightarrow{i_k} G^k(\mathcal{F}, p) \xrightarrow{i_{k+1}} \dots \longrightarrow G^{\infty}(\mathcal{F}, p)$$

where $G^{\infty}(\mathcal{F}, p)$ is the sequential colimit of this system. The morphism $i_{\infty} : X \rightarrow G^{\infty}(\mathcal{F}, p)$ is called a relative \mathcal{F} -cell complex. By universal property of the colimit, the morphism p has a factorization $p = p_{\infty} \circ i_{\infty}$ where $p_{\infty} : G^{\infty}(\mathcal{F}, p) \rightarrow Y$.

Proposition 1.12. (cf. [4], proposition 7.17) *In the preceding situation, suppose that for every $i \in I$, the object A_i is sequentially small in \mathcal{M} . Then the morphism p_∞ has the right lifting property with respect to the morphisms of \mathcal{F} .*

In the remaining sections of our paper, in order to deal with applications of the form of i_∞ we will need the following properties:

Lemma 1.13. (see [12]) (1) *Let us consider a pushout of the form*

$$\begin{array}{ccc} K & \xrightarrow{f} & K' \\ \downarrow i & & \downarrow j \\ L & \xrightarrow{g} & L' \end{array}$$

in a category \mathcal{C} admitting small colimits. Suppose that i has the left lifting property with respect to a given family \mathcal{F} of morphisms of \mathcal{C} . Then j has also the left lifting property with respect to \mathcal{F} . Another way to state this result is to say that the left lifting property with respect to a given family of morphisms is invariant under cobase change.

(2) *Let us consider a sequential direct system*

$$G^0 \xrightarrow{i_1} G^1 \xrightarrow{i_2} \dots \xrightarrow{i_k} G^k \xrightarrow{i_{k+1}} \dots \longrightarrow \operatorname{colim}_k G^k = G^\infty .$$

Let us note $i_\infty : G^0 \rightarrow G^\infty$ the transfinite composite of the i_k . If for every $k \geq 0$, the morphism i_k has the left lifting property with respect to a given family \mathcal{F} of morphisms of \mathcal{C} , then so does i_∞ .

It is time now to give a concrete example of model category. Of course, topological spaces provide the initial example from which the theory of model categories arised. However, the example we will use to illustrate these notions is that of chain complexes. This choice is motivated by two reasons. Firstly, this will be the base category for the remaining part of our paper. Secondly, the model category structures of algebras and coalgebras over operads will be transferred from this one via adjunctions.

Theorem 1.14. (cf. [4], theorem 7.2) *The category $Ch_{\mathbb{K}}$ of chain complexes over a field \mathbb{K} forms a cofibrantly generated model category such that a morphism f of $Ch_{\mathbb{K}}$ is*

(i) *a weak equivalence if for every $n \geq 0$, the induced map $H_n(f)$ in homology is an isomorphism.*

(ii) *a fibration if for every $n > 0$, the map f_n is surjective.*

(iii) *a cofibration if for every $n \geq 0$, the map f_n is injective.*

For $n \geq 1$, the chain complex D^n is defined by

$$D_k^n = \begin{cases} 0 & k \neq n, n-1 \\ \mathbb{K}b_{n-1} & k = n-1 \\ \mathbb{K}e_n & k = n \end{cases}$$

with $\deg(b_{n-1}) = n-1$, $\deg(e_n) = n$, and a differential δ satisfying $\delta(e_n) = b_{n-1}$. The chain complex S^n is defined by

$$S_k^n = \begin{cases} 0 & k \neq n \\ \mathbb{K}b_n & k = n \end{cases}$$

with $\deg(b_n) = n$. We have for every $n \geq 1$ an obvious inclusion $j_n : S^{n-1} \rightarrow D^n$ which is the identity on $\mathbb{K}b_{n-1}$. The sets of generating cofibrations and generating acyclic cofibrations are given by the following proposition:

Proposition 1.15. (cf. [4], proposition 7.19) *A morphism f of $Ch_{\mathbb{K}}$ is*

(i) *a fibration if and only if for every $n \geq 1$, it has the right lifting property with respect to the inclusions $i_n : 0 \rightarrow D^n$.*

(ii) *an acyclic fibration if and only if for every $n \geq 1$, it has the right lifting property with respect to the inclusions $j_n : S^{n-1} \rightarrow D^n$.*

2. THE MODEL CATEGORY OF COALGEBRAS OVER AN OPERAD

We work in the full subcategory $Ch_{\mathbb{K}}^+$ of $Ch_{\mathbb{K}}$ whose objects are the chain complexes C such that $C_0 = 0$, i.e the connective chain complexes. The category $Ch_{\mathbb{K}}^+$ is actually a model subcategory of $Ch_{\mathbb{K}}$. We suppose that P is an operad in $Ch_{\mathbb{K}}$ such that the $P(n)$ are finite dimensional, $P(0) = 0$ and $P(1) = \mathbb{K}$. Note that the commonly used operads satisfy this hypothesis, for instance As (for the associative algebras), Com (for the commutative associative algebras), Lie (for the Lie algebras), $Pois$ (for the Poisson algebras). There are two difficulties appearing here. Firstly, our operad is not defined exactly in the same category than our algebras. Secondly, the category $Ch_{\mathbb{K}}^+$ inherits the symmetric monoidal structure of $Ch_{\mathbb{K}}$ but not the unit (which is \mathbb{K} concentrated in degree 0). However, $Ch_{\mathbb{K}}$ acts on $Ch_{\mathbb{K}}^+$ via the usual tensor product of chain complexes. The convenient notion to deal with such situations is the one of symmetric monoidal category over a base category. We refer the reader to [8], chapter 1, for a precise definition and the associated properties. In our case, we work in the reduced symmetric monoidal category $Ch_{\mathbb{K}}^+$ over the base $Ch_{\mathbb{K}}^+$ (see also [8], 1.1.17). As shown in [8], all the usual definitions and properties of operads and their algebras hold in the reduced setting. The situation is analog for cooperads and their coalgebras. The model category structure on coalgebras is given by the following theorem:

Theorem 2.1. *The category of P -coalgebras ${}^PCh_{\mathbb{K}}^+$ inherits a cofibrantly generated model category structure such that a morphism f of ${}^PCh_{\mathbb{K}}^+$ is*

(i) *a weak equivalence if $U(f)$ is a weak equivalence in $Ch_{\mathbb{K}}^+$;*

(ii) *a cofibration if $U(f)$ is a cofibration in $Ch_{\mathbb{K}}^+$;*

(iii) *a fibration if f has the right lifting property with respect to acyclic cofibrations.*

The three class of morphisms defined in this theorem are clearly stable by composition and contain the identity maps. Axioms MC2 and MC3 are clear, and MC4 (ii) is obvious by definition of the fibrations. It remains to prove axioms MC1, MC4 (i) and MC5. We first need the crucial notion of enveloping cooperad.

2.1. Enveloping cooperad. Let A be a P -coalgebra. We want to construct a particular cooperad associated to A and called the enveloping cooperad of A . This is a "dual version" of the enveloping operad of [7]. Then we will prove a result linking this construction with the coproduct of ${}^PCh_{\mathbb{K}}^+$ which gives the crux of our proof of MC5 (i). We need the following classical result :

Proposition 2.2. (see [16]) *When \mathbb{K} is an infinite field, we have a fully faithful embedding of the category of Σ -modules in the category of endofunctors of $Vect_{\mathbb{K}}$.*

We consider the Σ -module $P^*[A]$ defined by

$$P^*[A](n) = \bigoplus_{r=1}^{\infty} P(n+r)^* \otimes_{\Sigma_r} A^{\otimes r}.$$

We need the following lemma:

Lemma 2.3. *Let A be a P -coalgebra. For every chain complex C of $Ch_{\mathbb{K}}^+$ we have $P^*[A](C) \cong P^*(A \oplus C)$.*

Proof. Let us note

$$Sh_{p,q} = \{\sigma \in \Sigma_{p+q} \mid \sigma(1) < \dots < \sigma(p), \sigma(p+1) < \dots < \sigma(p+q)\}$$

the set of (p, q) -shuffles, then

$$\begin{aligned} P(n)^* \otimes_{\Sigma_n} (A \oplus C)^{\otimes n} &= \left(\bigoplus_{Sh_{p,q}, p+q=n} P(n)^* \otimes A^{\otimes p} \otimes C^{\otimes q} \right)_{\Sigma_n} \\ &= \bigoplus_{p+q=n} P(p+q)^* \otimes_{\Sigma_p \times \Sigma_q} (A^{\otimes p} \otimes C^{\otimes q}) \end{aligned}$$

hence

$$\begin{aligned} P^*(A \otimes C) &= \bigoplus_{n \geq 0} P(n)^* \otimes_{\Sigma} (A \oplus C)^{\otimes n} \\ &= \bigoplus_n \bigoplus_{p+q=n} P(p+q)^* \otimes_{\Sigma_p \times \Sigma_q} (A^{\otimes p} \otimes C^{\otimes q}) \\ &= \bigoplus_n \bigoplus_{p+q=n} (P(p+q)^* \otimes_{\Sigma_p} A^{\otimes p}) \otimes_{\Sigma_q} C^{\otimes q} \\ &= \bigoplus_q P^*[A](q) \otimes_{\Sigma_q} C^{\otimes q} \\ &= P^*[A](C). \end{aligned}$$

□

The P -coalgebra structure morphism $\rho_A : A \rightarrow P^*(A)$ of A induces a Σ -modules morphism

$$d_0 : P^*[A] \rightarrow P^*[P^*(A)]$$

where

$$d_0(n) = \bigoplus_{r=1}^{\infty} id \otimes \rho_A^{\otimes r} : \bigoplus_{r=1}^{\infty} P(n+r)^* \otimes_{\Sigma_r} A^{\otimes r} \rightarrow \bigoplus_{r=1}^{\infty} P(n+r)^* \otimes_{\Sigma_r} P^*(A)^{\otimes r}.$$

The coproduct $\Delta : P^* \rightarrow P^* \circ P^*$ associated to the comonad (P^*, Δ, η) induces another morphism of Σ -modules

$$d_1 : P^*[A] \rightarrow P^*[P^*(A)]$$

(where $d_1(0) = \Delta(A)$) defined in the following way: for every chain complex C we have an application

$$\begin{array}{ccc} P^*[A](C) \cong P^*(A \oplus C) & \xrightarrow{\Delta(A \oplus C)} & P^*(P^*(A \oplus C)) \\ & \searrow^{P^*(P^*(pr_A), \pi \circ P^*(pr_C))} & \\ & & P^*(P^*(A) \oplus C) \cong P^*[P^*(A)](C) \end{array}$$

where π is the projection on the component of arity 1, hence the associated unique morphism of Σ -modules $d_1 : P^*[A] \rightarrow P^*[P^*(A)]$.

The counity $\eta : P^* \rightarrow Id$ induces a morphism of Σ -modules

$$s_0 : P^*[P^*(A)] \rightarrow P^*[A]$$

(where $s_0(0) = P^*(\eta(A))$) defined in the following way: for every chain complex C , we have an application

$$P^*[P^*(A)](C) \cong P^*(P^*(A) \oplus C) \begin{array}{c} \xrightarrow{P^*(P^*(i_A), i_C)} \\ \xrightarrow{P^*(\eta(A \oplus C))} \end{array} \begin{array}{c} P^*(P^*(A \oplus C)) \\ P^*(A \oplus C) \cong P^*[A](C) \end{array}$$

where $i_A : A \rightarrow A \oplus C$ and $i_C : C \rightarrow A \oplus C$, hence the unique associated morphism of Σ -modules $s_0 : P^*[P^*(A)] \rightarrow P^*[A]$. We finally obtain a reflexive pair (d_0, d_1) of morphisms of Σ -modules induced by the associated reflexive pair of morphisms of Schur functors. The enveloping cooperad of A is the coreflexive equalizer

$$U_{P^*}(A) = \ker(d_0 - d_1) \longrightarrow P^*[A] \begin{array}{c} \xleftarrow{s_0} \\ \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} P^*[P^*(A)]$$

in Σ -modules endowed with the cooperad structure induced by that of $P^*[A]$.

Now we want to prove that for every P -coalgebra A and every chain complex C , we have an isomorphism $U_{P^*}(A)(C) \cong A \times P^*(C)$ where \times is the product in ${}^PCh_{\mathbb{K}}^+$. For this aim we need the following lemma:

Lemma 2.4. *Let A be a P -coalgebra and C be a chain complex. The following equalizer defines the product $A \times P^*(C)$ in the category of P -coalgebras:*

$$A \times P^*(C) = \ker(d_0 - d_1) \longrightarrow P^*(A \oplus C) \begin{array}{c} \xleftarrow{s_0} \\ \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} P^*(P^*(A) \oplus C)$$

where $d_0|_A = \rho_A$, $d_0|_C = id_C$, $d_1|_A = \Delta(A)$, $d_1|_C = id_C$, $s_0|_A = \eta(A)$, $s_0|_C = id_C$.

Proof. We clearly have $d_0 \circ s_0 = d_1 \circ s_0 = id$ so (d_0, d_1) is a reflexive pair in ${}^PCh_{\mathbb{K}}^+$. The space $\ker(d_0 - d_1)$ is the coreflexive equalizer of (d_0, d_1) in $Ch_{\mathbb{K}}^+$ and is a sub- P -coalgebra of $P^*(A \oplus C)$, so it is the coreflexive equalizer of (d_0, d_1) in ${}^PCh_{\mathbb{K}}^+$. Let X be a P -coalgebra, $u : X \rightarrow A$ a morphism of P -coalgebras and $v : X \rightarrow C$ a linear map. They induce a map $(u, v) : X \rightarrow A \oplus C$, hence a morphism of P -coalgebras $\varphi_{(u,v)} : X \rightarrow P^*(A \oplus C)$ obtained by the universal property of the cofree P -coalgebra. The proof ends by seeing that $\varphi_{(u,v)}$ admits a unique factorization through $\ker(d_0 - d_1)$. \square

The coreflexive equalizer in Σ -modules defining the enveloping cooperad induces a coreflexive equalizer in P -coalgebras

$$U_{P^*}(A)(C) \longrightarrow P^*[A](C) \begin{array}{c} \xleftarrow{s_0} \\ \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} P^*[P^*(A)](C)$$

where $P^*[A](C) \cong P^*(A \oplus C)$, $P^*[P^*(A)](C) \cong P^*(P^*(A) \oplus C)$ and d_0, d_1, s_0 turn out to be the morphisms of the lemma above. By unicity of the limit, we have proved the following result:

Proposition 2.5. *Let A be a P -coalgebra and C be a chain complex, then $U_{P^*}(A)(C) \cong A \times P^*(C)$.*

We also need the following general result about Σ -modules:

Proposition 2.6. *Let M be a Σ -module and C a chain complex. If $H_*(C) = 0$ then $H_*(M(C)) = H_*(M(0))$.*

Proof. Recall that we work over a field \mathbb{K} of characteristic 0. We use the norm map $N : M(n) \otimes_{\Sigma_n} C^{\otimes n} \rightarrow M(n) \otimes C^{\otimes n}$ defined by

$$N(c \otimes v_1 \otimes \dots \otimes v_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma.c \otimes v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}.$$

If we denote $p : M(n) \otimes C^{\otimes n} \rightarrow M(n) \otimes_{\Sigma_n} C^{\otimes n}$ the projection, then

$$\begin{aligned} (p \circ N)(c \otimes v_1 \otimes \dots \otimes v_n) &= \frac{1}{n!} \sum_{\sigma \in \Sigma_n} p(\sigma.c \otimes v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}) \\ &= \frac{1}{n!} |\Sigma_n| c \otimes v_1 \otimes \dots \otimes v_n \\ &= c \otimes v_1 \otimes \dots \otimes v_n \end{aligned}$$

so $p \circ N = id$. Therefore $M(n) \otimes_{\Sigma_n} C^{\otimes n}$ is a retract of $M(n) \otimes C^{\otimes n}$. For $n \geq 1$, the Künneth formula gives us for every $k \geq 0$

$$H_k(M(n) \otimes C^{\otimes n}) = \bigoplus_{p+q=k} H_p(M(n) \otimes C) \otimes H_q(C^{\otimes n-1}).$$

This is equal to 0 for $n > 1$ because the fact that $H_*(C) = 0$ implies recursively that $H_*(C^{\otimes n}) = 0$ by the Künneth formula. This is also equal to 0 for $n = 1$ because the fact that $H_k(C) = 0$ implies that $H_k(M(1) \otimes C) = 0$. For $n = 0$, we have $H_k(M(0))$. We conclude that $H_k(M(C)) = H_k(M(0))$. \square

We finally reach the crucial result of this section:

Corollary 2.7. *Let A be a P -coalgebra and C be a chain complex. If $H_*(C) = 0$ then the canonical projection $A \times P^*(C) \rightarrow A$ is a weak equivalence in ${}^PCh_{\mathbb{K}}^+$.*

Proof. According to proposition 2.5, we have $U_{P^*}(A)(C) \cong A \times P^*(C)$. We can apply proposition 2.6 to the Σ -module $U_{P^*}(A)$ since $H_*(C) = 0$ by hypothesis, so

$$H_*(A \times P^*(C)) = H_*(U_{P^*}(A)(C)) = H_*(U_{P^*}(A)(0)).$$

It remains to prove that $H_*(U_{P^*}(A)(0)) = H_*(A)$. For this aim we show that $U_{P^*}(A)(0) \cong A$. It comes from a categorical result: in any category with a final object and admitting products, the product of any object A with the final object is isomorphic to A . We apply this fact to $U_{P^*}(A)(0) \cong A \times P^*(0)$. Indeed, the chain complex 0 is final in $Ch_{\mathbb{K}}^+$ so $P^*(0)$ is final in ${}^PCh_{\mathbb{K}}^+$. \square

2.2. Proof of MC1. The forgetful functor creates the small colimits. The proof of this fact is exactly the same as the proof of the existence of small limits in the P -algebras case. To prove the existence of small limits in ${}^PCh_{\mathbb{K}}^+$, we use the following categorical result:

Theorem 2.8. *(see [18]) Let \mathcal{C} be a category. If \mathcal{C} admits the coreflexive equalizers of every pair of arrows and all small coproducts, then \mathcal{C} admits all the small limits.*

Now let us prove that ${}^PCh_{\mathbb{K}}^+$ admits the coreflexive equalizers and the small products.

Lemma 2.9. *Let $(d_0, d_1 : A \rightarrow B, s_0 : B \rightarrow A)$ be a reflexive pair in ${}^PCh_{\mathbb{K}}^+$. Then $\ker(d_0 - d_1)$ is the coreflexive equalizer of (d_0, d_1) in ${}^PCh_{\mathbb{K}}^+$.*

Proof. The subspace $\ker(d_0 - d_1) \subset A$ is the coreflexive equalizer of (d_0, d_1) in $Ch_{\mathbb{K}}^+$. Moreover, it is a sub- P -coalgebra of A and the inclusion is obviously a P -coalgebras morphism, hence the result. \square

Lemma 2.10. *Let $\{R_i\}_{i \in I}$ be a set of P -coalgebras. Let us set*

$$d_0 = P^*\left(\bigoplus \rho_{R_i}\right) : P^*\left(\bigoplus R_i\right) \rightarrow P^*\left(\bigoplus P^*(R_i)\right)$$

and

$$d_1 = \pi \circ \Delta\left(\bigoplus R_i\right) : P^*\left(\bigoplus R_i\right) \rightarrow P^*\left(\bigoplus P^*(R_i)\right)$$

where $\pi : P^*(P^*(\bigoplus R_i)) \rightarrow P^*(\bigoplus P^*(R_i))$ is the canonical projection and Δ the comultiplication of the comonad (P^*, Δ, η) . Then $\times R_i = \ker(d_0 - d_1)$ is the product of the R_i in ${}^PCh_{\mathbb{K}}^+$.

Proof. We prove the lemma in the case of two P -coalgebras R and S . The proof is the same in the general case. Let us set

$$s_0 = P^*(\eta(R) \oplus \eta(S)) : P^*(P^*(R) \oplus P^*(S)) \rightarrow P^*(R \oplus S),$$

then $d_0 \circ s_0 = d_1 \circ s_0 = id$. According to lemma 2.9, the space $\ker(d_0 - d_1)$ is the coreflexive equalizer of (d_0, d_1) in ${}^PCh_{\mathbb{K}}^+$. Let X be a P -coalgebra. Two linear maps $u : X \rightarrow R$ and $v : X \rightarrow S$ induce a map $(u, v) : X \rightarrow R \oplus S$. This map admits a unique factorization through $P^*(R \oplus S)$ to give a P -coalgebras morphism $\varphi_{(u,v)} : X \rightarrow P^*(R \oplus S)$ by the universal property of the cofree P^* -coalgebra. This morphism admits a unique factorization through $\ker(d_0 - d_1)$ if and only if u and v are morphisms of P -coalgebras. By unicity of the limit this concludes our proof, since $\ker(d_0 - d_1)$ satisfies the same universal property than $R \times S$. \square

2.3. Generating (acyclic) cofibrations, proofs of MC4 and MC5. Before specifying the families of generating cofibrations and generating acyclic cofibrations, we prove axioms MC4 (i) and MC5 (i). The cofibrantly generated structure will then be used to prove MC5 (ii) by means of a small object argument, slightly different from the preceding one since we will use smallness with respect to injections systems. The plan and some arguments parallel those of [9]. However, they work in cocommutative differential graded coalgebras. Some care is necessary in our more general setting. This is the reason why we give full details in proofs, in order to see where we can readily follow [9] and where our modifications (for instance the notion of enveloping cooperad) step in.

MC5 (i). We first need a preliminary lemma:

Lemma 2.11. *Every chain complex X of $Ch_{\mathbb{K}}^+$ can be embedded in a chain complex V satisfying $H_*(V) = 0$.*

Proof. Let us set $V_0 = 0$ and $V_n = X_n \oplus X_{n-1}$ for every $n \geq 1$. We define the differential of V by $\partial_n^V : X_n \oplus X_{n-1} \rightarrow X_{n-1} \hookrightarrow X_{n-1} \oplus X_{n-2} = V_{n-1}$ which is the projection followed by the inclusion for every $n \geq 2$, and $\partial_1^V = 0$. We have $\partial_{n+1}^V \circ \partial_n^V = 0$ so $(V, \partial^V) \in Ch_{\mathbb{K}}^+$. Moreover, for every $n \in \mathbb{N}$, we have $X_n \subset V_n$

so X is injected into V . Finally, for every $n \in \mathbb{N}$, $H_n(V) = \ker(\partial_n^V)/\text{im}(\partial_{n+1}^V) \cong X_n/X_n = 0$. \square

This lemma helps us to prove the following result:

Proposition 2.12. (i) *Let C be a P -coalgebra and V be a chain complex such that $H_*(V) = 0$. Then the projection $C \times P^*(V) \rightarrow C$ is an acyclic fibration with the right lifting property with respect to all cofibrations.*

(ii) *Every P -coalgebras morphism $f : D \rightarrow C$ admits a factorization*

$$D \xrightarrow{j} X \xrightarrow{q} C$$

where j is a cofibration and q an acyclic fibration with the right lifting property with respect to all cofibrations (in particular we obtain axiom MC5 (i)).

Proof. (i) According to corollary 2.7, the map $C \times P^*(V) \rightarrow C$ is a weak equivalence so it remains to prove that it has the right lifting property with respect to all cofibrations (which implies in particular that it is a fibration). Let us consider the following commutative square in ${}^PCh_{\mathbb{K}}$:

$$\begin{array}{ccc} A & \xrightarrow{a} & C \times P^*(V) \\ \downarrow i & & \downarrow \\ B & \xrightarrow{b} & C \end{array}$$

where i is a cofibration. A lifting in this square is equivalent to a lifting in each of the two squares

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow i & & \parallel \\ B & \longrightarrow & C \end{array}$$

and

$$\begin{array}{ccc} A & \longrightarrow & P^*(V) \\ \downarrow i & & \downarrow \\ B & \longrightarrow & 0 \end{array}$$

In the first square this is obvious, just take the bottom map $B \rightarrow C$ as a lifting. In the second square, via the adjunction $U : {}^PCh_{\mathbb{K}}^+ \rightleftarrows Ch_{\mathbb{K}}^+ : P^*$, the lifting problem is equivalent to a lifting problem in the following square of $Ch_{\mathbb{K}}^+$:

$$\begin{array}{ccc} U(A) & \longrightarrow & V \\ U(i) \downarrow & & \downarrow \\ U(B) & \longrightarrow & 0 \end{array}$$

The map $V \rightarrow 0$ is degreewise surjective so it is a fibration of $Ch_{\mathbb{K}}^+$, which is acyclic because $H_*(V) = 0$. The map i is a cofibration, so $U(i)$ is a cofibration by definition and has therefore the left lifting property with respect to acyclic fibrations.

(ii) According to lemma 2.11, there exists an injection $i : U(D) \hookrightarrow V$ in $Ch_{\mathbb{K}}^+$ where V is such that $H_*(V) = 0$. Let us set $X = C \times P^*(V)$, $q : X \rightarrow C$ the projection and

$$j = (f, \tilde{i}) : D \rightarrow C \times P^*(V)$$

where $\tilde{i} : D \rightarrow P^*(V)$ is the factorization of i by universal property of the cofree P -coalgebra. We have $q \circ j = f$. According to (i), the map q is an acyclic fibration with the right lifting property with respect to all cofibrations. It remains to prove that j is a cofibration. Let us consider the composite

$$D \xrightarrow{j} C \times P^*(V) \xrightarrow{pr_2} P^*(V) \xrightarrow{\pi} V$$

where pr_2 is the projection on the second component and π the projection associated to the cofree P -coalgebra on V . We have $\pi \circ pr_2 \circ j = \pi \circ \tilde{i} = i$ by definition of \tilde{i} . The map i is injective so j is also injective, which implies that $U(j)$ is a cofibration in $Ch_{\mathbb{K}}^+$. By definition it means that j is a cofibration in ${}^PCh_{\mathbb{K}}^+$. \square

MC4 (i). Let $p : X \rightarrow Y$ be an acyclic cofibration, let us consider the commutative square

$$\begin{array}{ccc} C & \xrightarrow{a} & X \\ i \downarrow & & \downarrow p \\ D & \xrightarrow{b} & Y \end{array}$$

where i is a cofibration. According to proposition 2.12, the map p admits a factorization $p = q \circ j$ where $j : X \rightarrow T$ is a cofibration and $q : T \rightarrow Y$ an acyclic fibration with the right lifting property with respect to all cofibrations. Axiom MC2 implies that j is a weak equivalence. Let us consider the commutative square

$$\begin{array}{ccc} X & \xlongequal{\quad} & X \\ j \downarrow & & \downarrow p \\ T & \xrightarrow{q} & Y \end{array} .$$

According to axiom MC4 (ii), there exists a lifting $r : T \rightarrow X$ in this square and p is consequently a retract of q via the following retraction diagram:

$$\begin{array}{ccccc} X & \xrightarrow{j} & T & \xrightarrow{r} & X \\ p \downarrow & & \downarrow q & & \downarrow p \\ Y & \xlongequal{\quad} & Y & \xlongequal{\quad} & Y \end{array} .$$

A reasoning similar to that of the proof of MC4 (ii) for P -algebras concludes the proof: the map f inherits the property of right lifting property with respect to cofibrations.

Generating (acyclic) cofibrations We first need two preliminary lemmas:

Lemma 2.13. *Let C be a P -coalgebra. For every homogeneous element $x \in C$ there exists a sub- P -coalgebra $D \subset C$ of finite dimension such that $x \in D$ and $D_k = 0$ for every $k > \deg(x)$.*

Proof. Suppose that $x \in C_n$. Let us note $\Delta : C \rightarrow C \otimes C$ the coproduct. We have

$$\Delta(x) = \sum_{i+j=n} (\sum x'_i \otimes x''_j)$$

where $x'_i \in C_i$ and $x''_j \in C_j$. Using Sweedler's notation we have

$$\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$$

where $x_{(1)} \in C_i$, $x_{(2)} \in C_j$ and the sum is indexed by the integers $0 \leq i \leq n$, $0 \leq j \leq n$ such that $i + j = n$.

We do a recursive reasoning on the degree n of x . For $n = 1$, the element x belongs to $C_1 = \mathbb{K}$ which is a sub- P -coalgebra of finite dimension. Now suppose the lemma true for every $k < n$. Let $x \in C_n$ and $\Delta(x)$ as above. By hypothesis, there exists sub- P -coalgebras of finite dimension $D_{(1)}$ and $D_{(2)}$ satisfying the following conditions:

- $x_{(1)} \in D_{(1)}$, $x_{(2)} \in D_{(2)}$;
 - $(D_{(1)})_j = 0$ if $j > \deg(x_{(1)})$ and $(D_{(2)})_j = 0$ if $j > \deg(x_{(2)})$.
- Then we set

$$D = \mathbb{K}.x \oplus \left(\sum D_{(1)} + \sum D_{(2)} \right).$$

This is a finite sum of finite dimensional sub- P -coalgebras so D is a finite dimensional sub- P -coalgebra containing x . Furthermore, since $\deg(x) > \deg(x_{(1)})$ and $\deg(x) > \deg(x_{(2)})$, this construction implies that $D_j = 0$ if $j > \deg(x)$. \square

Lemma 2.14. *Let $j : C \rightarrow D$ be an acyclic cofibration and $x \in D$ a homogeneous element. Then there exists a sub- P -coalgebra $B \subseteq D$ such that:*

- (i) $x \in B$;
- (ii) B is finite dimensional;
- (iii) the injection $C \cap B \hookrightarrow B$ is an acyclic cofibration in ${}^PCh_{\mathbb{K}}^+$ (we denote also by C the image of C under j , since j is injective and thus $j(C) \cong C$).

Proof. We want to define recursively sub- P -coalgebras

$$B(1) \subseteq B(2) \subseteq \dots \subseteq D$$

such that $x \in B(1)$, each $B(n)$ is finite dimensional and the induced map

$$\frac{B(n-1)}{C \cap B(n-1)} \rightarrow \frac{B(n)}{C \cap B(n)}$$

is zero in homology. This map is well defined, since we do the quotient by an intersection of two sub- P -coalgebras which is still a sub- P -coalgebra.

The P -coalgebra $B(1)$ is given by lemma 2.13. Now suppose that for some integer $n \geq 1$ the coalgebra $B(n-1)$ has been well constructed. The space $B(n-1)$ is of finite dimension, so we can choose a finite set of homogeneous cycles $z_i \in B(n-1)$, $\bar{z}_i \in \frac{B(n-1)}{C \cap B(n-1)}$, such that the homology classes of the \bar{z}_i span $H_*\left(\frac{B(n-1)}{C \cap B(n-1)}\right)$. For every i , lemma 2.13 provides us a finite dimensional sub- P -coalgebra $A(z_i) \subseteq D$ containing z_i . We can then define

$$B(n) = B(n-1) + \sum_i A(z_i).$$

The sub- P -coalgebra $B(n)$ is of finite dimension because it is the sum of finite dimensional sub- p -coalgebras. Moreover, the induced map in homology

$$H_*\left(\frac{B(n-1)}{C \cap B(n-1)}\right) \rightarrow H_*\left(\frac{B(n)}{C \cap B(n)}\right)$$

is zero because it sends the homology classes of the \bar{z}_i to 0.

Let us define $B = \bigcup B(n)$ and prove that $C \cap B \hookrightarrow B$ is an acyclic cofibration. First it is injective so it is a cofibration. To prove its acyclicity, let us consider the

following short exact sequence:

$$0 \rightarrow B \cap C \hookrightarrow B \rightarrow \frac{B}{C \cap B} \rightarrow 0.$$

It is sufficient to consider the long exact sequence induced by this sequence in homology and to prove that $H_*\left(\frac{B}{C \cap B}\right) = 0$. Let $z \in B$ such that $\partial(\bar{z}) = 0$ in $H_*\left(\frac{B}{C \cap B}\right)$, where ∂ is the differential of $\frac{B}{C \cap B}$. We have $\partial(z) \in B \cap C = \bigcup B(n) \cap C$ and $B(1) \subseteq \dots \subseteq D$ so there exists an integer n such that $z \in B(n-1)$ and $\partial(z) \in B(n-1) \cap C$. It implies that $[\bar{z}] \in H_*\left(\frac{B(n-1)}{C \cap B(n-1)}\right)$, where $[\bar{z}]$ is the homology class of \bar{z} . Thus $[\bar{z}] = 0$ in $H_*\left(\frac{B(n)}{C \cap B(n)}\right)$, since the map $H_*\left(\frac{B(n-1)}{C \cap B(n-1)}\right) \rightarrow H_*\left(\frac{B(n)}{C \cap B(n)}\right)$ is zero in homology. We deduce that $z = \partial(b) + B(n) \cap C$ for a certain $b \in B(n)$, so $\bar{z} = \partial(\bar{b})$ in $\frac{B}{B \cap C}$ (the projection $x \mapsto \bar{x}$ commutes with the differentials). Finally, it means that every cycle of $\frac{B}{B \cap C}$ is a boundary, i.e that $H_*\left(\frac{B}{B \cap C}\right) = 0$.

It remains to prove that B is finite dimensional. According to lemma 2.13, the chain complex $B(1)$ is concentrated in degrees $k \leq \deg(x)$. Let us suppose that $B(n-1)$ is concentrated in degrees $k \leq \deg(x)$, then so does $\frac{B}{B \cap C}$. The \bar{z}_i are then of degree $\deg(\bar{z}_i) \leq \deg(x)$, and so do the z_i . The $A(z_i)$ are obtained by lemma 2.13 and thus concentrated in degrees $k \leq \deg(\bar{z}_i) \leq \deg(x)$. The chain complex $B(n) = B(n-1) + \sum_i A(z_i)$ is consequently also concentrated in degrees $k \leq \deg(x)$. The sequence $B(1) \subseteq B(2) \subseteq \dots$ is stationary, which allows us to conclude that $B = \bigcup B(n)$ is finite dimensional. \square

Now we can give a characterization of generating cofibrations and generating acyclic cofibrations.

Proposition 2.15. *A morphism $p : X \rightarrow Y$ of ${}^PCh_{\mathbb{K}}^+$ is*

- (i) *a fibration if and only if it has the right lifting property with respect to the acyclic cofibrations $A \hookrightarrow B$ where B is of finite dimension;*
- (ii) *an acyclic fibration if and only if it has the right lifting property with respect to the cofibrations $A \hookrightarrow B$ where B is of finite dimension.*

Proof. (i) One of the two implications is obvious. Indeed, if p is a fibration then it has the right lifting property with respect to acyclic cofibrations by definition. Conversely, suppose that p has the right lifting property with respect to the acyclic cofibrations $A \hookrightarrow B$ where B is finite dimensional. We consider the following lifting problem:

$$\begin{array}{ccc} C & \xrightarrow{f} & X \\ j \downarrow & \nearrow & \downarrow p \\ D & \longrightarrow & Y \end{array}$$

where j is an acyclic cofibration. Let us define Ω as the set of pairs (\bar{D}, g) , where \bar{D} fits in the composite of two acyclic cofibrations

$$C \hookrightarrow \bar{D} \hookrightarrow D$$

such that this composite is equal to j . The map $g : \bar{D} \rightarrow X$ is a lifting in

$$\begin{array}{ccc} C & \xrightarrow{f} & X \\ \downarrow & & \downarrow p \\ \bar{D} & \longrightarrow & D \longrightarrow Y \end{array} .$$

Recall that cofibrations are injective P -coalgebras morphisms. We endow Ω with a partial order defined by $(\overline{D_1}, g_1) \leq (\overline{D_2}, g_2)$ if $\overline{D_1} \subseteq \overline{D_2}$ and $g_2|_{\overline{D_1}} = g_1$. The commutative square

$$\begin{array}{ccc} C & \xrightarrow{f} & X \\ \parallel & & \downarrow p \\ C & \xrightarrow{j} D \longrightarrow & Y \end{array}$$

admits f as an obvious lifting, so $(C, f) \in \Omega$ and thus Ω is not empty. Moreover, any totally ordered subset of Ω admits an upper bound, just take the sum of its elements. We can therefore apply Zorn lemma. Let $(E, g) \in \Omega$ be a maximal element. We know that E is injected in D by definition, and we want to prove that D is injected in E in order to obtain $E = D$.

Let $x \in D$ be a homogeneous element. According to lemma 2.14 applied to the acyclic cofibration $E \hookrightarrow D$, there exists a finite dimensional sub- P -coalgebra $B \subseteq D$ such that $x \in B$ and $E \cap B \hookrightarrow B$ is an acyclic cofibration. The lifting problem

$$\begin{array}{ccccc} E \cap B & \longrightarrow & E & \xrightarrow{g} & X \\ \downarrow & & \searrow h & \dashrightarrow & \downarrow p \\ B & \longrightarrow & D & \longrightarrow & Y \end{array}$$

admits a solution h by hypothesis about p . We therefore extend g into a map $\tilde{g} : E + B \rightarrow X$ such that $\tilde{g}|_E = g$, $\tilde{g}|_B = h$. According to the diagram above, we have $h|_{E \cap B} = g|_{E \cap B}$ so \tilde{g} is well defined. The short exact sequences

$$0 \rightarrow E \cap B \rightarrow B \rightarrow \frac{B}{E \cap B} \rightarrow 0$$

and

$$0 \rightarrow E \rightarrow E + B \rightarrow \frac{E + B}{B} \rightarrow 0$$

induce long exact sequences in homology

$$\dots \rightarrow H_{n+1}\left(\frac{B}{E \cap B}\right) \rightarrow H_n(E \cap B) \rightarrow H_n(B) \rightarrow H_n\left(\frac{B}{E \cap B}\right) \rightarrow \dots$$

and

$$\dots \rightarrow H_{n+1}\left(\frac{E + B}{E}\right) \rightarrow H_n(E) \rightarrow H_n(E + B) \rightarrow H_n\left(\frac{E + B}{E}\right) \rightarrow \dots$$

But $E \cap B \hookrightarrow B$ induces an isomorphism in homology so in the first exact sequence $H_*\left(\frac{B}{E \cap B}\right) = 0$. Furthermore, the isomorphism $\frac{B}{E \cap B} \cong \frac{E + B}{E}$ implies that $H_*\left(\frac{E + B}{E}\right) = 0$. Accordingly, the map $E \hookrightarrow E + B$ in the second exact sequence induces an isomorphism in homology, i.e $E \hookrightarrow E + B$ is an acyclic cofibration. It means that $(E + B, \tilde{g}) \in \Omega$, and by definition of \tilde{g} the inequality $(E, g) \leq (E + B, \tilde{g})$ holds in Ω . Given that (E, g) is supposed to be maximal, we conclude that $E = E + B$, hence $x \in E$ and $E = D$. The map g is the desired lifting. The map p is a fibration.

(ii) If p is an acyclic fibration, then p has the right lifting property with respect to cofibrations according to axiom MC4 (i). Conversely, let us suppose that p has the right lifting property with respect to cofibrations $A \hookrightarrow B$ where B is finite

dimensional. The proof is similar to that of (i) with a slight change in the definition of Ω . Indeed, we consider the lifting problem

$$\begin{array}{ccc} C & \longrightarrow & X \\ j \downarrow & & \downarrow p \\ D & \longrightarrow & Y \end{array}$$

where j is a cofibration. We define Ω as the set of pairs (\overline{D}, g) where \overline{D} fits in a composite of cofibrations $C \hookrightarrow \overline{D} \hookrightarrow D$ such that this composite is equal to j . We define the same partial order on Ω than in (i), and Ω is clearly not empty since $(C, f) \in \Omega$. The set Ω is inductive so we can apply Zorn's lemma. Let (E, g) be a maximal element of Ω , as before E is injected in D and we want to prove that D is injected in E . Let $x \in D$ be a homogeneous element, according to lemma 2.14 there exists a finite dimensional sub- P -coalgebra $B \subseteq D$ containing x . The map p has the right lifting property with respect to $E \cap B \hookrightarrow B$ by hypothesis, so the method of (i) works here. We extend g to $\tilde{g} : E + B \rightarrow X$, we have $(E + B, \tilde{g}) \in \Omega$ and $(E, g) \leq (E + B, \tilde{g})$. The maximality of (E, g) implies that $E = E + B$ and $g : E = D \rightarrow X$ is the desired lifting. \square

MC5 (ii). We need here to use a slightly refined version of the small object argument. We will consider smallness only with respect to injections systems. Suppose that \mathcal{C} is a category admitting small colimits. A direct system of injections

$$\dots \hookrightarrow B(n) \hookrightarrow B(n+1) \hookrightarrow \dots$$

indexed by a set has $\bigcup_n B(n)$ as colimit. For any object A of \mathcal{C} , the functor $\text{Hom}(A, -)$ gives a commutative diagram of injections

$$\begin{array}{ccccc} \dots & \longrightarrow & \text{Hom}(A, B(n)) & \longrightarrow & \text{Hom}(A, B(n+1)) & \longrightarrow & \dots \\ & & \searrow & & \swarrow & & \\ & & \text{Hom}(A, \bigcup_n B(n)) & & & & \end{array}$$

By universal property of the colimit, this diagram induces a canonical map

$$\text{colim}_n \text{Hom}(A, B(n)) = \bigcup_n \text{Hom}(A, B(n)) \rightarrow \text{Hom}(A, \bigcup_n B(n)).$$

We say that A is small with respect to direct systems of injections if this map is a bijection. Consider a morphism f of \mathcal{C} and a family of morphisms $\mathcal{F} = \{f_i : A_i \rightarrow B_i\}_{i \in I}$ such that the A_i are small with respect to injections systems. If we can prove that the i_k obtained in the construction of the $G^k(\mathcal{F}, f)$ (see section 1.3 for the notation) are injections, then we can use this refined version of the small object argument. We then obtain a factorization $f = f_\infty \circ i_\infty$ where f_∞ has the right lifting property with respect to the morphisms of \mathcal{F} and i_∞ is an injection (the injectivity passes to the transfinite composite). This is the argument we are going to use to prove axiom MC5 (ii) in ${}^P\text{Ch}_{\mathbb{K}}^+$.

Recall that the generating acyclic cofibrations of ${}^P\text{Ch}_{\mathbb{K}}^+$ are the acyclic injections $j_i : A_i \hookrightarrow B_i$ of P -coalgebras such that the B_i are finite dimensional. In order to apply the refined small object argument explained above, we need the following lemma:

Lemma 2.16. *Let C be a object of ${}^PCh_{\mathbb{K}}^+$. If $U(C)$ is small with respect to injections systems, then so does C in ${}^PCh_{\mathbb{K}}^+$.*

Proof. Let us consider a system of injections

$$\dots \hookrightarrow B(n) \hookrightarrow B(n+1) \hookrightarrow \dots$$

of P -coalgebras, and let $f : C \rightarrow \bigcup_n B(n)$ be a morphism of P -coalgebras. The chain complex $U(C)$ is small with respect to injections systems, so there exists an integer N such that we have a unique factorization in $Ch_{\mathbb{K}}^+$

$$f : C \xrightarrow{\tilde{f}} B(N) \hookrightarrow \bigcup_n B(n).$$

The map f is a morphism of P -coalgebras and so does $B(N) \hookrightarrow \bigcup_n B(n)$, thus \tilde{f} is a morphism of P -coalgebras. We have the desired factorization in ${}^PCh_{\mathbb{K}}^+$. \square

In the family of generating acyclic cofibrations $\mathcal{F} = \{j_i : A_i \hookrightarrow B_i\}_{i \in I}$, the B_i are finite dimensional so the A_i too, thus the $U(A_i)$ are small. In particular, they are small with respect to injections systems. Lemma 2.17 implies that the A_i are small with respect to injection systems. Now, let $f : X \rightarrow Y$ be a morphism of P -coalgebras. Recall that the construction of $G^k(\mathcal{F}, f)$ is given by a pushout

$$\begin{array}{ccc} \bigvee_i A_i & \longrightarrow & G^{k-1}(\mathcal{F}, f) \\ \bigvee_i j_i \downarrow & & \downarrow i_k \\ \bigvee_i B_i & \longrightarrow & G^k(\mathcal{F}, f) \end{array}$$

The forgetful functor creates the small colimits, so we obtain the same pushout in $Ch_{\mathbb{K}}^+$ by forgetting P -coalgebras structures. By definition of cofibrations and weak equivalences in $Ch_{\mathbb{K}}^+$, given that $\bigvee_i j_i$ is an acyclic cofibration, the map $U(\bigvee_i j_i)$ is an acyclic cofibration in $Ch_{\mathbb{K}}^+$. In any model category, acyclic cofibrations are stable by pushouts, so the $U(i_k)$ are acyclic cofibrations. By definition, it means that the i_k are acyclic cofibrations, i.e in our case acyclic injections of P -coalgebras. We use our refined version of the small object argument to obtain a factorization $f = f_\infty \circ i_\infty$. Injectivity and acyclicity are two properties which passes to the transfinite composite i_∞ , so i_∞ is an acyclic cofibration of ${}^PCh_{\mathbb{K}}^+$. Moreover, the map f_∞ has by construction the right lifting property with respect to the generating acyclic cofibrations and forms consequently a fibration. Our proof is now complete.

Remark 2.17. This method provides us another way to prove MC5 (i), by using this time the family of generating cofibrations.

3. THE MODEL CATEGORY OF BIALGEBRAS OVER A PAIR OF OPERADS IN DISTRIBUTION

Let P be an operad in $Ch_{\mathbb{K}}^+$. Let Q be an operad in $Ch_{\mathbb{K}}^+$ such that $Q(0) = 0$, $Q(1) = \mathbb{K}$ and the $Q(n)$ are of finite dimension for every $n \in \mathbb{K}$. We suppose that there exists a mixed distributive law between P and Q (see definition 1.6). In the following, the operad P will encode the operations of our bialgebras and the operad Q will encode the cooperations.

Recall that there exists a cofibrantly generated model category structure on the category ${}^PCh_{\mathbb{K}}$ of P -algebras:

Theorem 3.1. (see [11] or [8]) *The category of P -algebras ${}_PCh_{\mathbb{K}}$ inherits a cofibrantly generated model category structure such that a morphism f of ${}_PCh_{\mathbb{K}}$ is*

- (i) *a weak equivalence if $U(f)$ is a weak equivalence in $Ch_{\mathbb{K}}$, where U is the forgetful functor;*
 - (ii) *a fibration if $U(f)$ is a fibration in $Ch_{\mathbb{K}}$;*
 - (iii) *a cofibration if it has the left lifting property with respect to acyclic fibrations.*
- We can also say that cofibrations are relative cell complexes with respect to the generating cofibrations.*

where the generating cofibrations and generating acyclic cofibrations are, as expected, the images of the generating (acyclic) cofibrations of $Ch_{\mathbb{K}}$ under the free P -algebra functor P . Recall that the $j_n : S^{n-1} \hookrightarrow D^n$ and the $i_n : 0 \hookrightarrow D^n$ are respectively the generating cofibrations and the generating acyclic cofibrations of $Ch_{\mathbb{K}}$.

Actually, this structure exists via a transfer of cofibrantly generated model category structure via the adjunction $P : Ch_{\mathbb{K}} \rightleftarrows {}_PCh_{\mathbb{K}} : U$ (see [11] and [8]). The forgetful functor creates fibrations and weak equivalences. The free P -algebra functor P preserves generating (acyclic) cofibrations by definition of the generating (acyclic) cofibrations of ${}_PCh_{\mathbb{K}}$. Moreover, it preserves colimits as a left adjoint (it is a general property of adjunctions, see [18] for instance). Thus it preserves all (acyclic) cofibrations, which are relative cell complexes with respect to the generating (acyclic) cofibrations. Such a pair of functors is called a Quillen adjunction, and induces an adjunction at the level of the associated homotopy categories. According to theorem 1.9, we can lift this free P -algebra functor to the category of Q -coalgebras, so the adjunction

$$P : Ch_{\mathbb{K}}^+ \rightleftarrows {}_PCh_{\mathbb{K}}^+ : U$$

becomes an adjunction

$$P : {}^QCh_{\mathbb{K}}^+ \rightleftarrows {}_P^QCh_{\mathbb{K}}^+ : U.$$

Similarly, the adjunction

$$U : {}^QCh_{\mathbb{K}}^+ \rightleftarrows Ch_{\mathbb{K}}^+ : Q^*$$

becomes an adjunction

$$U : {}_P^QCh_{\mathbb{K}}^+ \rightleftarrows {}_PCh_{\mathbb{K}}^+ : Q^*.$$

The model category structure on (P, Q) -bialgebras is then given by the following theorem:

Theorem 3.2. *The category ${}^QCh_{\mathbb{K}}^+$ inherits a cofibrantly generated model category structure such that a morphism f of ${}^QCh_{\mathbb{K}}^+$ is*

- (i) *a weak equivalence if $U(f)$ is a weak equivalence in ${}^QCh_{\mathbb{K}}^+$ (i.e a weak equivalence in $Ch_{\mathbb{K}}^+$ by definition of the model structure on ${}^QCh_{\mathbb{K}}^+$);*
- (ii) *a fibration if $U(f)$ is a fibration in ${}^QCh_{\mathbb{K}}^+$;*
- (iii) *a cofibration if f has the left lifting property with respect to acyclic fibrations.*

It is clear that this three classes of morphisms are stable by composition and contain the identity morphisms. Axioms MC2 and MC3 are clear, axiom MC4 (i) is obvious by definition of the cofibrations. It remains to prove axioms MC1, MC4 (ii) and MC5.

MC1. The forgetful functor $U : {}_P^QCh_{\mathbb{K}}^+ \rightarrow {}^QCh_{\mathbb{K}}^+$ creates the small limits. The proof is the same than in the case of P -algebras. The forgetful functor $U : {}_P^QCh_{\mathbb{K}}^+ \rightarrow {}_PCh_{\mathbb{K}}^+$

$Ch_{\mathbb{K}}^+$ creates the small colimits. The proof is the same than in the case of P -coalgebras, see section 2.2.

Generating (acyclic) cofibrations. The treatment is similar to the case of P -algebras. Let us note $\{j : A \hookrightarrow B\}$ the family of generating cofibrations and $\{i : A \hookrightarrow B\}$ the family of generating acyclic cofibrations. Then the $P(j)$ form the generating cofibrations of ${}^Q_P Ch_{\mathbb{K}}^+$ and the $P(i)$ form the generating acyclic cofibrations:

Proposition 3.3. *Let f be a morphism of ${}^Q_P Ch_{\mathbb{K}}^+$. Then*

(i) *f is a fibration if and only if it has the right lifting property with respect to the $P(i)$, where $i : A \hookrightarrow B$ an acyclic injection of Q -coalgebras such that B is finite dimensional;*

(ii) *f is an acyclic fibration if and only if it has the right lifting property with respect to the $P(j)$, where $j : A \hookrightarrow B$ is an injection of Q -coalgebras such that B is finite dimensional.*

Proof. Part (ii) can be proved in the same way than part (i), so we only give the details for part (i). Suppose that $f : A \rightarrow B$ is a fibration and consider a commutative square

$$\begin{array}{ccc} P(X) & \xrightarrow{u} & A \\ \downarrow P(i) & & \downarrow f \\ P(Y) & \xrightarrow{v} & B \end{array}$$

in ${}^Q_P Ch_{\mathbb{K}}^+$, where i is an acyclic injection of Q -coalgebras. We use the adjunction

$$P : {}^Q Ch_{\mathbb{K}}^+ \rightleftarrows {}^Q_P Ch_{\mathbb{K}}^+ : U.$$

Via the forgetful functor we obtain in ${}^Q Ch_{\mathbb{K}}^+$ a commutative square

$$\begin{array}{ccc} (U \circ P)(X) & \xrightarrow{U(u)} & U(A) \\ \downarrow (U \circ P)(i) & & \downarrow U(f) \\ (U \circ P)(Y) & \xrightarrow{U(v)} & U(B) \end{array}$$

The unit $\eta : id_{{}^Q_P Ch_{\mathbb{K}}^+} \rightarrow U \circ P$ associated to the adjunction between P and U provides a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\eta(X)} & (U \circ P)(X) & \xrightarrow{U(u)} & U(A) \\ \downarrow i & \dashrightarrow & \downarrow (U \circ P)(i) & & \downarrow U(f) \\ Y & \xrightarrow{\eta(Y)} & (U \circ P)(Y) & \xrightarrow{U(v)} & U(B) \end{array}$$

A lifting $\hat{v} : Y \rightarrow U(A)$ exists in this diagram, given that $U(f)$ is a fibration and has therefore the right lifting property with respect to i . By applying P we obtain

a new commutative diagram

$$\begin{array}{ccccc}
 P(X) & \xrightarrow{P\eta(X)} & (P \circ U \circ P)(X) & \xrightarrow{(P \circ U)(u)} & (P \circ U)(A) \\
 P(i) \downarrow & & \nearrow P(\hat{v}) & & \downarrow (P \circ U)(f) \\
 P(Y) & \xrightarrow{P(\eta(Y))} & (P \circ U \circ P)(Y) & \xrightarrow{(P \circ U)(v)} & (P \circ U)(B)
 \end{array}$$

in ${}^Q_P Ch_{\mathbb{K}}^+$. The counity $\epsilon : P \circ U \rightarrow id_{{}^Q_P Ch_{\mathbb{K}}^+}$ associated to the adjunction gives rise to the commutative diagram

$$\begin{array}{ccccccc}
 P(X) & \xrightarrow{P\eta(X)} & (P \circ U \circ P)(X) & \xrightarrow{(P \circ U)(u)} & (P \circ U)(A) & \xrightarrow{\epsilon(A)} & A \\
 P(i) \downarrow & & \nearrow P(\hat{v}) & & \downarrow (P \circ U)(f) & & \downarrow f \\
 P(Y) & \xrightarrow{P(\eta(Y))} & (P \circ U \circ P)(Y) & \xrightarrow{(P \circ U)(v)} & (P \circ U)(B) & \xrightarrow{\epsilon(B)} & B
 \end{array}$$

Moreover, the following diagrams commute:

$$\begin{array}{ccc}
 (P \circ U)(P(Y)) & \xrightarrow{(P \circ U)(v)} & (P \circ U)(B) \\
 \epsilon(P(Y)) \downarrow & & \downarrow \epsilon(B) \\
 P(Y) & \xrightarrow{v} & B
 \end{array}$$

by naturality of ϵ , and

$$\begin{array}{ccc}
 P & \xrightarrow{P\eta} & P \circ U \circ P \\
 & \searrow & \downarrow \epsilon P \\
 & & P
 \end{array}$$

which is a property associated to any adjunction, see [18] for more details. We conclude that $\epsilon(B) \circ (P \circ U)(v) \circ P(\eta(Y)) = v \circ (\epsilon P \circ P\eta)(Y) = v \circ id_{P(Y)} = v$. Thus $\epsilon(A) \circ P(\hat{v}) : P(Y) \rightarrow A$ is the desired lifting.

We have to prove the other direction of the equivalence. Let us suppose that f has the right lifting property with respect to the $P(i)$, where i ranges over the generating cofibrations of ${}^Q_P Ch_{\mathbb{K}}^+$, and consider the commutative square

$$\begin{array}{ccc}
 X & \xrightarrow{u} & U(A) \\
 i \downarrow & & \downarrow U(f) \\
 Y & \xrightarrow{v} & U(B)
 \end{array}$$

By applying P we obtain

$$\begin{array}{ccc}
 P(X) & \xrightarrow{P(u)} & (P \circ U)(A) \\
 P(i) \downarrow & & \downarrow (P \circ U)(f) \\
 P(Y) & \xrightarrow{P(v)} & (P \circ U)(B)
 \end{array}$$

hence via the counity ϵ of the adjunction

$$\begin{array}{ccccc} P(X) & \xrightarrow{P(u)} & (P \circ U)(A) & \xrightarrow{\epsilon(A)} & A \\ P(i) \downarrow & & \nearrow h & & \downarrow f \\ P(Y) & \xrightarrow{P(v)} & (P \circ U)(B) & \xrightarrow{\epsilon(B)} & B \end{array}$$

where h exists by hypothesis about f . We apply U :

$$\begin{array}{ccccc} P(X) & \xrightarrow{(U \circ P)(u)} & (U \circ P \circ U)(A) & \xrightarrow{U(\epsilon(A))} & U(A) \\ (U \circ P)(i) \downarrow & & \nearrow U(h) & & \downarrow U(f) \\ (U \circ P)(Y) & \xrightarrow{(U \circ P)(v)} & (U \circ P \circ U)(B) & \xrightarrow{U(\epsilon(B))} & U(B) \end{array}$$

hence via the unity η of the adjunction:

$$\begin{array}{ccccc} X & \xrightarrow{\eta(X)} & (U \circ P)(X) & \xrightarrow{\quad\quad\quad} & U(A) \\ i \downarrow & & \nearrow U(h) & & \downarrow U(f) \\ Y & \xrightarrow{\eta(Y)} & (U \circ P)(Y) & \xrightarrow{(U \circ P)(v)} & (U \circ P \circ U)(B) & \xrightarrow{U(\epsilon(B))} & U(B) \end{array}$$

Moreover the following diagrams commute:

$$\begin{array}{ccc} Y & \xrightarrow{\eta(Y)} & (U \circ P)(Y) \\ v \downarrow & & \downarrow (U \circ P)(v) \\ U(B) & \xrightarrow{\eta U(B)} & (U \circ P \circ U)(B) \end{array}$$

by naturality of η , and

$$\begin{array}{ccc} U & \xrightarrow{\eta U} & U \circ P \circ U \\ & \searrow & \downarrow U\epsilon \\ & & U \end{array}$$

which is a property associated to any adjunction. We deduce that $U(\epsilon(B)) \circ (U \circ P)(v) \circ \eta(Y) = (U\epsilon \circ \eta U)(B) \circ v = id_{U(B)} \circ v = v$. Therefore $U(h) \circ \eta(Y) : Y \rightarrow U(A)$ is the desired lifting: the morphism $U(f)$ forms a fibration in ${}^{\mathcal{Q}}Ch_{\mathbb{K}}^+$, which implies by definition that f forms a fibration in ${}^{\mathcal{Q}}Ch_{\mathbb{K}}^+$. \square

MC4 (ii). If MC5 (ii) is proved, then MC4 (ii) follows from the same proof than MC4 (ii) in the case of P -algebras.

MC5. The main difficulty here is to prove axiom MC5. Let f be a morphism of ${}^{\mathcal{Q}}Ch_{\mathbb{K}}^+$. Let us note $\mathcal{F} = \{P(j_i), j_i : A_i \hookrightarrow B_i\}_{i \in I}$ the family of generating cofibrations. Recall that the A_i are sequentially small with respect to injections systems. We use the following lemma to prove that the $P(A_i)$ are also sequentially small with respect to injections systems:

Lemma 3.4. *Let C be an object of ${}^QCh_{\mathbb{K}}^+$. If C is sequentially small in ${}^QCh_{\mathbb{K}}^+$ with respect to injections systems, then $P(C)$ is sequentially small in ${}^PCh_{\mathbb{K}}^+$ with respect to injections systems.*

Proof. Let us suppose that C is sequentially small with respect to injections systems in ${}^QCh_{\mathbb{K}}^+$, and let $F : \mathbb{N} \rightarrow {}^QCh_{\mathbb{K}}^+$ be a functor sending every arrow to an injection. For every $n \in \mathbb{N}$,

$$\text{Hom}_{{}^PCh_{\mathbb{K}}^+}(P(C), F(n)) \cong \text{Hom}_{{}^QCh_{\mathbb{K}}^+}(C, (U \circ F)(n))$$

hence

$$\begin{aligned} \text{colim}_n \text{Hom}_{{}^PCh_{\mathbb{K}}^+}(P(C), F(n)) &\cong \text{colim}_n \text{Hom}_{{}^QCh_{\mathbb{K}}^+}(C, (U \circ F)(n)) \\ &\cong \text{Hom}_{{}^QCh_{\mathbb{K}}^+}(C, \text{colim}_n (U \circ F)(n)) \end{aligned}$$

because $U \circ F : \mathbb{N} \rightarrow {}^QCh_{\mathbb{K}}^+$ and C is sequentially small with respect to injections systems. We can equip $\text{colim}_n (U \circ F)(n)$ with a structure of P -algebra, such that with this structure it forms the colimit of the $F(n)$ in ${}^PCh_{\mathbb{K}}^+$. Indeed, we have $\text{colim}_n (U \circ F)(n) = \{[a], a \in F(n)\} / \sim$ where $a \sim b$ (i.e. $[a] = [b]$), $a \in F(n), b \in F(m), n \leq m$, if the application $F(n) \rightarrow F(m)$ in the sequential system sends a to b . Let $[a_1], \dots, [a_r] \in \text{colim}_n (U \circ F)(n)$ such that $a_1 \in F(n_1), \dots, a_r \in F(n_r)$. We consider $F(n)$ for a given $n \geq \max(n_1, \dots, n_r)$ and we set, for $\mu \in P(n)$, $\mu([a_1], \dots, [a_r]) = \mu(a'_1, \dots, a'_r)$ where a'_1, \dots, a'_r are representing elements of $[a_1], \dots, [a_r]$ in $F(n)$. We then obtain a P -algebra structure on $\text{colim}_n (U \circ F)(n)$ (one says that the forgetful functor creates the sequential colimits). Moreover, this P -algebra structure is compatible with the Q -coalgebras structures of the $F(n)$, since it is defined via their P -algebra structures. It is therefore compatible with the Q -coalgebra structure of $\text{colim}_n (U \circ F)(n)$, such that we obtain the colimit in ${}^QCh_{\mathbb{K}}^+$. We can finally write

$$\begin{aligned} \text{colim}_n \text{Hom}_{{}^PCh_{\mathbb{K}}^+}(P(C), F(n)) &\cong \text{Hom}_{{}^QCh_{\mathbb{K}}^+}(C, U(\text{colim}_n F(n))) \\ &\cong \text{Hom}_{{}^PCh_{\mathbb{K}}^+}(P(C), \text{colim}_n F(n)). \end{aligned}$$

□

We want to apply the small object argument to obtain a factorization $f = f_\infty \circ i_\infty$ of f . Recall that for every $k > 0$, the space $G^k(\mathcal{F}, f)$ is obtained by a pushout

$$\begin{array}{ccc} \bigvee_i P(A_i) & \longrightarrow & G^{k-1}(\mathcal{F}, f) \\ \bigvee_i P(j_i) \downarrow & & \downarrow i_k \\ \bigvee_i P(B_i) & \longrightarrow & G^k(\mathcal{F}, f) \end{array}$$

The forgetful functor $U : {}^QCh_{\mathbb{K}}^+ \rightarrow {}^PCh_{\mathbb{K}}^+$ creates small colimits, so we obtain the same pushout diagram in ${}^PCh_{\mathbb{K}}^+$ by forgetting the Q -coalgebras structures. The j_i are cofibrations of ${}^QCh_{\mathbb{K}}^+$, so the underlying chain complexes morphisms are cofibrations of ${}^PCh_{\mathbb{K}}^+$. Thus, via the adjunction $P : {}^QCh_{\mathbb{K}}^+ \rightleftarrows {}^PCh_{\mathbb{K}}^+ : U$, the $P(j_i)$ are cofibrations of ${}^PCh_{\mathbb{K}}^+$ and so does $\bigvee_i P(j_i)$. In any model category, cofibrations are stable by pushouts, so the i_k are cofibrations of ${}^PCh_{\mathbb{K}}^+$. By definition of cofibrations in ${}^PCh_{\mathbb{K}}^+$, we can apply lemma 1.13 to i_∞ to deduce that i_∞ forms a cofibration in ${}^PCh_{\mathbb{K}}^+$. We now use the following proposition:

Proposition 3.5. *An (acyclic) cofibration of ${}_PCh_{\mathbb{K}}^+$ forms an (acyclic) cofibration in $Ch_{\mathbb{K}}^+$.*

Proof. See section 4.6.3 in [11] (Note that for a base field of characteristic zero, every operad is Σ -split in the sense defined by Hinich). \square

The maps i_k (and thus i_∞) forms therefore cofibrations in $Ch_{\mathbb{K}}^+$, i.e injections. This is crucial to apply our version of the small object argument, since the $P(A_i)$ are small only with respect to injections systems. Finally, i_∞ forms a cofibration in ${}_PCh_{\mathbb{K}}^+$. The map f_∞ has the right lifting property with respect to the generating cofibrations and forms thus an acyclic fibration. Axiom MC5 (i) is proved.

The method to prove MC5 (ii) is the same up to two minor changes: we consider the family of generating acyclic cofibrations, and use the stability of acyclic cofibrations under pushouts.

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