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Opening infinitely many nodes

Martin Traizet

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Abstract : we develop a theory of holomorphic differentials on a certain class of non-compact Riemann surfaces obtained by opening infinitely many nodes.

1 Introduction

This paper is about holomorphic 1-forms, on a certain class of non compact Riemann surfaces. The compact case is a classical subject. The space of holomorphic 1-forms (also called holomorphic differentials) on a compact Riemann surface Σ has complex dimension equal to the genus g of Σ . Moreover, a holomorphic differential ω can be defined by prescribing its integrals (also called periods) on a suitable set of cycles, namely the cycles A_1, \dots, A_g of a canonical homology basis.

The non-compact case is mostly unexplored. In this case, the space of holomorphic differentials on Σ is infinite dimensional. For any practical purpose, it is required to put a norm on this space. So what we are interested in are Banach spaces of holomorphic differentials on Σ . The question which we would like to answer is :

Can we define a holomorphic differential by prescribing its periods, and in which Banach space does it live ?

We will answer this question on a class of non-compact Riemann surfaces which are made of infinitely many Riemann spheres connected by small necks. More precisely, consider an infinite graph Γ . Let V denote its set of vertices and E its set of edges. For each vertex $v \in V$, consider a Riemann sphere S_v . For any edge $e \in E$ from the vertex v to the vertex v' , we connect S_v and $S_{v'}$ by a small neck, see figure 1. We do this by opening nodes, a standard explicit construction.

We denote by γ_e the cycle around the neck corresponding to the edge e . Essentially we prove that we can define a holomorphic differential ω by prescribing its periods α_e on the cycles γ_e for $e \in E$, provided the prescribed periods satisfy the obvious homological obstruction obtained from Cauchy theorem in each Riemann sphere.

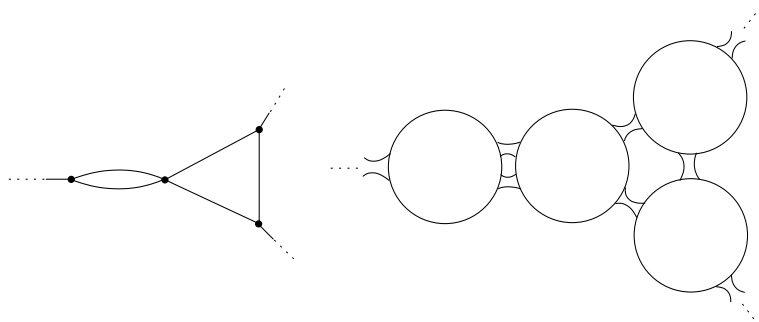


Figure 1: A small portion of an infinite graph and a picture of its associated Riemann surface.

One issue is to put norms on both the space of holomorphic differentials ω and the space of period vectors $\alpha = (\alpha_e)_{e \in E}$. Our choice was to leave the later norm as general as possible, and see what is the right norm on the space of holomorphic differentials so that the operator which maps a holomorphic differential ω to its period vector α is a Banach isomorphism. This result is Theorem 3.

We also prove that the holomorphic differential ω obtained by prescribing its periods depends smoothly (actually, analytically) on all parameters which enter the construction of Σ . Roughly speaking, these parameters can be thought of as the position and size of the necks.

Because our Riemann surface is not compact, one can ask what is the asymptotic behavior of the holomorphic differential ω . This can be done using weighted norms and will be a recurrent question in this paper.

Finally, in the last section of the paper, we consider meromorphic differentials. In the compact case, it is well known that one can define a meromorphic differential by prescribing its poles, principal part at each pole and periods. We prove such a result in our non compact setting (a Mittag Leffler type result).

The theory of holomorphic differentials on non compact Riemann surfaces which we develop in this paper, besides being interesting in its own right, has applications to minimal surfaces theory, via the classical Weierstrass Representation. It can be used to glue infinitely many minimal surfaces together. See [5] for an example.

I would like to thank the referee for valuable comments and suggestions.

2 Opening nodes

Consider a connected graph Γ . It may have multiple edges and edges from a vertex to itself (this is usually called a multi-graph). We denote by V its set of

vertices and E its set of edges. We are mostly interested in the case where V is not finite. We assume that Γ is oriented, which means that each edge has an orientation. The orientation is arbitrary and will only be used to orient certain curves. If $e \in E$ is an oriented edge from vertex v to vertex v' , we say that v is the starting point of e and v' is the endpoint.

For a vertex $v \in V$, the set of edges having v as starting point (resp. endpoint) will be denoted E_v^- (resp. E_v^+). The set of edges adjacent to the vertex v is denoted $E_v = E_v^- \cup E_v^+$. The degree of v is the cardinal of E_v , which we assume is always finite. Note that together with the fact that Γ is connected, this implies that V is countable.

To each vertex $v \in V$ we associate a Riemann sphere $\mathbb{C} \cup \{\infty\}$ denoted S_v . For each oriented edge $e \in E$ from v to v' , we choose a point p_e^- in S_v and a point p_e^+ in $S_{v'}$. We assume that for each vertex v , the points p_e^- for $e \in E_v^-$ and p_e^+ for $e \in E_v^+$ are distinct.

We consider the disjoint union of all S_v for $v \in V$, and for each edge $e \in E$, we identify the points p_e^- and p_e^+ . This creates a node which we call p_e . We call Σ_0 the resulting Riemann surface with nodes.

We open nodes in the standard way as follows, see for example [3]. We use the notation $D(a, R)$ for the disk of center a and radius R in \mathbb{C} . We consider, for each edge $e \in E$ from v to v' , some local complex coordinates $z_e^- : D_e^- \subset S_v \xrightarrow{\sim} D(0, \rho)$ and $z_e^+ : D_e^+ \subset S_{v'} \xrightarrow{\sim} D(0, \rho)$ in a neighborhood of p_e^- and p_e^+ respectively, such that $z_e^-(p_e^-) = 0$ and $z_e^+(p_e^+) = 0$. We assume that for each vertex v , the disks D_e^- for $e \in E_v^-$ and D_e^+ for $e \in E_v^+$ are disjoint in S_v .

Consider again the disjoint union of all Riemann spheres S_v , $v \in V$. For each edge $e \in E$ from v to v' , choose some complex number $t_e \in D(0, \rho^2)$. If $t_e \neq 0$, remove the disk $|z_e^-| \leq \frac{|t_e|}{\rho}$ from D_e^- and $|z_e^+| \leq \frac{|t_e|}{\rho}$ from D_e^+ . We identify each point $z \in D_e^-$ with the point $z' \in D_e^+$ such that $z_e^-(z)z_e^+(z') = t_e$. This creates a neck connecting S_v and $S_{v'}$. If $t_e = 0$, then we identify p_e^- and p_e^+ as before to create a node.

Doing this for all edges defines a (possibly noded) Riemann surface which we call $\Sigma_{\mathbf{t}}$. Here \mathbf{t} denotes the sequence $(t_e)_{e \in E}$. When all t_e are nonzero, $\Sigma_{\mathbf{t}}$ is a genuine Riemann surface. It is compact when Γ is a finite graph.

3 Regular differentials

Regular differentials are the natural generalization of holomorphic 1-forms to Riemann surfaces with nodes.

Definition 1 (Bers [4]) *A differential ω on a Riemann surface with nodes Σ is called regular if it is holomorphic away from the nodes and for each node p (obtained by identifying p^- and p^+), it has simple poles at p^- and p^+ with opposite residues.*

For each edge $e \in E$, the boundary ∂D_e^+ is homologous in $\Sigma_{\mathbf{t}}$ to $-\partial D_e^-$. We define the cycle γ_e as the homology class of ∂D_e^+ . We have

$$\forall v \in V, \quad \sum_{e \in E_v^-} \gamma_e = \sum_{e \in E_v^+} \gamma_e. \quad (1)$$

We want to define a regular differential ω on $\Sigma_{\mathbf{t}}$ by prescribing its periods on the cycles γ_e :

$$\forall e \in E, \quad \int_{\gamma_e} \omega = \alpha_e, \quad \alpha_e \in \mathbb{C}$$

By Cauchy theorem and equation (1), a necessary condition is

$$\forall v \in V, \quad \sum_{e \in E_v^-} \alpha_e = \sum_{e \in E_v^+} \alpha_e. \quad (2)$$

If Γ is a finite graph, equation (2) is the only obstruction to define a regular differential on $\Sigma_{\mathbf{t}}$:

Theorem 1 (Fay [2]) *If Γ is a finite graph, then for \mathbf{t} small enough, the operator $\omega \mapsto (\int_{\gamma_e} \omega)_{e \in E}$ is an isomorphism from the space of regular differentials on $\Sigma_{\mathbf{t}}$ to the space of vectors $(\alpha_e)_{e \in E}$ which satisfy the compatibility condition (2)*

Remark 1 *Fay's theorem is more general, as he does not require the parts S_v to be spheres.*

3.1 Admissible coordinates

In the case of an infinite graph, we need to make some assumptions on the coordinates used to open nodes. We denote by z the standard coordinate in each sphere S_v , and write z_e^\pm to designate either z_e^+ or z_e^- . For each $e \in E$, the ratio $\left| \frac{z_e^\pm}{z - p_e^\pm} \right|$ extends continuously at p_e^\pm so is bounded from above and below in D_e^\pm by positive numbers. We require these numbers to be independent of $e \in E$. The following definition summarizes our requirements on the coordinates.

Definition 2 *We say that the coordinates $(z_e^\pm)_{e \in E}$ are admissible if*

1. all points p_e^- and p_e^+ are different from ∞ ,
2. for each $e \in E$, $z_e^\pm : D_e^\pm \xrightarrow{\sim} D(0, \rho)$ is a diffeomorphism such that $z_e^\pm(p_e^\pm) = 0$ (ρ is independent of e),
3. for each $v \in V$, the disks D_e^- for $e \in E_v^-$ and D_e^+ for $e \in E_v^+$ are disjoint in S_v ,
4. there exists positive constants c_1 and c_2 such that

$$\forall e \in E, \quad \forall z \in D_e^\pm, \quad c_1 |z_e^\pm(z)| \leq |z - p_e^\pm| \leq c_2 |z_e^\pm(z)|.$$

Remark 2 *If the coordinates are admissible, then for each edge $e \in E$, the round disk $D(p_e^\pm, \rho c_1)$ is included in the topological disk D_e^\pm . In particular, point 3 implies that for each vertex $v \in V$, the round disks $D(p_e^+, \rho c_1)$ for $e \in E_v^+$ and $D(p_e^-, \rho c_1)$ for $e \in E_v^-$ are disjoint.*

3.2 The ℓ^∞ case

Next, as $\Sigma_{\mathbf{t}}$ is non compact, the space of regular differentials is infinite dimensional so we need to define a norm on this space. Fix some real number $0 < \epsilon \leq \rho c_1$. For each vertex $v \in V$, let $\Omega_{v,\epsilon}$ be the Riemann sphere S_v minus the disks $D(p_e^-, \epsilon)$ for $e \in E_v^-$ and $D(p_e^+, \epsilon)$ for $e \in E_v^+$. Let $\mathcal{H}_\infty^1(\Sigma_{\mathbf{t}})$ be the space of regular differentials on $\Sigma_{\mathbf{t}}$ such that the norm

$$\|\omega\|_\infty = \sup_{v \in V} \sup_{\Omega_{v,\epsilon}} \left| \frac{\omega}{dz} \right|$$

is finite. Let $\ell^\infty(E)$ be the space of bounded sequences of complex numbers $(\alpha_e)_{e \in E}$ with the sup norm. Our first result, which generalizes Theorem 1 to the infinite case, is

Theorem 2 *Assume that the degree of the vertices of Γ is bounded and that the coordinates are admissible. Then for \mathbf{t} small enough (in ℓ^∞ norm), the operator $\omega \rightarrow (\int_{\gamma_e} \omega)_{e \in E}$ is an isomorphism of Banach spaces from $\mathcal{H}_\infty^1(\Sigma_{\mathbf{t}})$ to the subspace of sequences in $\ell^\infty(E)$ which satisfy (2).*

This result is a corollary of Theorem 3 and proposition 1 below.

Remark 3 *It follows from the theorem that the space $\mathcal{H}_\infty^1(\Sigma_{\mathbf{t}})$ does not depend on the choice of ϵ (taking another ϵ will define an equivalent norm).*

3.3 Examples of admissible coordinates

Example 1 Assume that for each $v \in V$, the points p_e^+ for $e \in E_v^+$ and p_e^- for $e \in E_v^-$ are at distance greater than 2ρ from each other. We can take the coordinates $z_e^\pm(z) = z - p_e^\pm$ on $D_e^\pm = D(p_e^\pm, \rho)$. These coordinates are admissible with $c_1 = c_2 = 1$.

This is probably the most natural choice of coordinates. The proof of Theorem 3 would be significantly simplified if we restricted ourselves to these coordinates, so let me give other examples to motivate the use of more general coordinates.

Example 2 Assume that for each $v \in V$, the points p_e^+ for $e \in E_v^+$ and p_e^- for $e \in E_v^-$ are on the unit circle and are at distance greater than 4ρ from each other. We would like the inversion $\sigma(z) = 1/\bar{z}$ to be well defined on $\Sigma_{\mathbf{t}}$. In this case we take the coordinates

$$z_e^\pm(z) = i \frac{z - p_e^\pm}{z + p_e^\pm}.$$

Then $z_e^\pm \circ \sigma = \overline{z_e^\pm}$, so provided all t_e are real numbers, σ is well defined on $\Sigma_{\mathbf{t}}$. Let D_e^\pm be the inverse image of $D(0, \rho)$ by z_e^\pm . Computations shows that these disks are disjoint, and the coordinates are admissible with $c_1 = \frac{2}{1+\rho}$ and $c_2 = \frac{2}{1-\rho}$.

Example 3 Assume that the graph Γ is bipartite, meaning that the set of vertices has a partition $V = V^+ \cup V^-$, so that all edges go from one vertex in V^- to a vertex in V^+ . Consider on each sphere S_v a meromorphic function f_v with simple zeros at p_e^- for $e \in E_v^-$ and p_e^+ for $e \in E_v^+$. For each $e \in E$ from v to v' , we take $z_e^- = f_v$ and $z_e^+ = f_{v'}$. Some hypothesis must be made on the functions $(f_v)_{v \in V}$ so that these coordinates are admissible. Consider a small complex number t and take $t_e = t^2$ for all $e \in E$. Then we can define a meromorphic function f on $\Sigma_{\mathbf{t}}$ by $f(z) = \frac{f_v(z)}{t}$ if $v \in V^+$ and $f(z) = \frac{t}{f_v(z)}$ if $v \in V^-$. Indeed, for each edge $e \in E$ from v to v' , if $z \in D_e^-$ is identified with $z' \in D_e^+$, we have $f_v(z)f_{v'}(z') = t^2$ so $f(z) = f(z')$ and f is well defined.

The reason this example is interesting is that together with a Riemann surface, we construct a meromorphic function. It is in general not easy to define a meromorphic function on a given Riemann surface (think of Abel's theorem). This is particularly fruitful in the case of minimal surfaces, where the function f will be the Gauss map. See [5], [6] for an illustration of this idea.

3.4 Admissible norms

The ℓ^∞ norm is maybe the most natural one, but for certain applications it is desirable to allow other norms. For example, in section 3.7 we use weighted ℓ^∞ norms to study the decay of regular differentials. In the following definition, we try to consider norms as general as possible.

Let \mathbb{C}^E and \mathbb{C}^V denote the space of sequences of complex numbers $(\alpha_e)_{e \in E}$ and $(u_v)_{v \in V}$ indexed by edges and vertices, respectively. We assume that we are given two norms $\|\cdot\|_E$ and $\|\cdot\|_V$ defined on a subspace \mathcal{B}_E of \mathbb{C}^E and \mathcal{B}_V of \mathbb{C}^V .

Definition 3 We say that the norms $\|\cdot\|_E$ and $\|\cdot\|_V$ are admissible if the following conditions hold :

1. The spaces \mathcal{B}_E and \mathcal{B}_V are Banach spaces.
2. The norms $\|\cdot\|_E$ and $\|\cdot\|_V$ are monotonic, in the sense that if $|\alpha_e| \leq |\alpha'_e|$ for all $e \in E$ then $\|\alpha\|_E \leq \|\alpha'\|_E$, and a similar definition for $\|\cdot\|_V$.
3. There exists a constant $c_3 > 0$ such that for all $\alpha \in \mathcal{B}_E$,

$$\left\| \left(\sum_{e \in E_v} |\alpha_e| \right)_{v \in V} \right\|_V \leq c_3 \|\alpha\|_E.$$

4. There exists a constant $c_4 > 0$ such that for all $u \in \mathcal{B}_V$,

$$\left\| (|u_v| + |u_{v'}|)_{\substack{e \in E \\ e=vv'}} \right\|_E \leq c_4 \|u\|_V.$$

5. The space of sequences $(\lambda_{e,n})_{e \in E, n \in \mathbb{N}}$ for which the norm $\|(\sup_{n \in \mathbb{N}} |\lambda_{e,n}|)_{e \in E}\|_E$ is defined is a Banach space.

3.5 Main result

We define $\mathcal{H}^1(\Sigma_{\mathbf{t}})$ as the space of regular differentials on $\Sigma_{\mathbf{t}}$ such that the norm

$$\|\omega\|_{\Omega} = \left\| \left(\sup_{\Omega_{v,\epsilon}} \left| \frac{\omega}{dz} \right| \right)_{v \in V} \right\|_V$$

is defined. The following theorem, which is the main result of this paper, generalizes Theorem 2 to this more general setup.

Theorem 3 *Assume that the coordinates and the norms are admissible. Then for \mathbf{t} small enough (in ℓ^∞ norm), the operator $\omega \mapsto (\int_{\gamma_e} \omega)_{e \in E}$ is an isomorphism from $\mathcal{H}^1(\Sigma_{\mathbf{t}})$ to the subspace of sequences in \mathcal{B}_E which satisfy (2)*

3.6 Examples of admissible norms

We define weighted ℓ^p spaces for $1 \leq p \leq \infty$ as follows. Consider a function $\sigma : V \rightarrow (0, \infty)$. Given a sequence $u = (u_v)_{v \in V}$, let

$$\|u\|_{p,\sigma} = \left(\sum_{v \in V} \sigma(v)^p |u_v|^p \right)^{1/p} \quad \text{if } p < \infty,$$

$$\|u\|_{\infty,\sigma} = \sup_{v \in V} \sigma(v) |u_v|.$$

We call $\ell^{p,\sigma}(V)$ the space of sequences $u \in \mathbb{C}^V$ for which the above norm is finite. When $\sigma \equiv 1$, these are the usual ℓ^p spaces.

We define a weight function on edges by $\sigma(e) = \frac{1}{2}(\sigma(v) + \sigma(v'))$ if e is an edge from v to v' . The space $\ell^{p,\sigma}(E)$ of sequences $\alpha \in \mathbb{C}^E$ which have finite $\ell^{p,\sigma}$ norm is defined in the obvious way.

Proposition 1 *Assume that there exists a positive number c such that the degree of the vertices of Γ is bounded by c , and that for any adjacent vertices v and v' , $\sigma(v) \leq c\sigma(v')$ and $\sigma(v') \leq c\sigma(v)$. Then the norms $\|\cdot\|_E = \|\cdot\|_{p,\sigma}$ and $\|\cdot\|_V = \|\cdot\|_{p,\sigma}$ are admissible.*

Proof. The first point is standard, the second is clear. Regarding the third

point, in the case $p < \infty$ we write

$$\begin{aligned}
\sum_{v \in V} \sigma(v)^p \left(\sum_{e \in E_v} |\alpha_e| \right)^p &\leq \sum_{v \in V} \sigma(v)^p \deg(v)^{p-1} \sum_{e \in E_v} |\alpha_e|^p \quad \text{by Jensen inequality} \\
&\leq c^{p-1} \sum_{v \in V} \sum_{e \in E_v} \sigma(v)^p |\alpha_e|^p \\
&= c^{p-1} \sum_{e \in E} (\sigma(v)^p + \sigma(v')^p) |\alpha_e|^p \\
&\leq 2^p c^{p-1} \sum_{e \in E} \sigma(e)^p |\alpha_e|^p
\end{aligned}$$

The case $p = \infty$ is straightforward.

Regarding the fourth point, we have for any edge $e \in E$ from v to v' , $\sigma(e) \leq \frac{c+1}{2}\sigma(v)$ and $\sigma(e) \leq \frac{c+1}{2}\sigma(v')$, so

$$\begin{aligned}
\sum_{\substack{e \in E \\ e=vv'}} \sigma(e)^p (|u_v| + |u_{v'}|)^p &\leq \sum_{e \in E} \sigma(e)^p 2^{p-1} (|u_v|^p + |u_{v'}|^p) \\
&\leq \frac{(1+c)^p}{2} \sum_{e \in E} \sigma(v)^p |u_v|^p + \sigma(v')^p |u_{v'}|^p \\
&= \frac{(1+c)^p}{2} \sum_{v \in V} \deg(v) \sigma(v)^p |u_v|^p \\
&\leq \frac{c(1+c)^p}{2} \sum_{v \in V} \sigma(v)^p |u_v|^p
\end{aligned}$$

Finally, point 5 is true by the standard fact that $\ell^\infty(\mathbb{N})$ is a Banach space and an infinite product of Banach spaces is a Banach space for the ℓ^p norm of the norms (cf exercise T page 243 in [1]). \square

3.7 Decay at infinity of normalized differentials

To illustrate the use of weights, let us consider the following example.

Example 4 Let Γ be an infinite graph which contains at least one cycle γ . Give the cycle γ an orientation and orient the rest of Γ arbitrarily. Define the sequence $(\alpha_e)_{e \in E}$ as $\alpha_e = 1$ if e belongs to γ and $\alpha_e = 0$ otherwise. Then $(\alpha_e)_{e \in E}$ satisfies (2), so defines a regular differential $\omega \in \mathcal{H}_\infty^1(\Sigma_{\mathbf{t}})$. We call ω the normalized differential associated to the cycle γ .

When $\mathbf{t} = 0$, ω is supported on the spheres S_v such that the vertex v belongs to the cycle C , so has compact support. It is natural to ask what is the decay of ω at infinity when $\mathbf{t} \neq 0$. For this purpose, we pick an arbitrary vertex $v_0 \in V$ and consider the weight $\sigma(v) = r^{d(v, v_0)}$, where $r > 1$ is a fixed real number, and $d(v, v_0)$ denotes the graphical distance from v_0 to v on Γ . Then $\alpha \in \ell^{\infty, \sigma}(E)$, so by Theorem 3, $\omega \in \mathcal{H}_{\infty, \sigma}^1(\Sigma_{\mathbf{t}})$ for \mathbf{t} small enough, so ω has exponential decay.

More precisely $\sup_{\Omega_{v,\epsilon}} \left| \frac{\omega}{dz} \right|$ is bounded by some constant times $r^{-d(v,v_0)}$. The rate of decay r is arbitrary, but of course the larger r , the smaller \mathbf{t} must be.

4 Proof of Theorem 3

Convention 1 *By a uniform constant, we mean a number which only depends on the numbers ρ , c_1 and c_2 in definition 2 and c_3 , c_4 in definition 3. We use the letter C to denote uniform constants. The same letter C can be used to denote various uniform constants.*

Let L be the operator $\omega \mapsto (\int_{\gamma_e} \omega)_{e \in E}$ from $\mathcal{H}^1(\Sigma_{\mathbf{t}})$ to the subspace of sequences in \mathcal{B}_E which satisfy (2). We prove that

1. L is bounded,
2. L is onto,
3. L is injective.

1) Let e be an edge from v to v' . Let $r = \rho c_1 \geq \epsilon$. By remark 2, we can replace γ_e by the circle $C(p_e^+, r)$, so

$$\left| \int_{\gamma_e} \omega \right| = \left| \int_{C(p_e^+, r)} \omega \right| \leq 2\pi r \sup_{\Omega_{v',\epsilon}} \left| \frac{\omega}{dz} \right| \leq 2\pi r \left(\sup_{\Omega_{v,\epsilon}} \left| \frac{\omega}{dz} \right| + \sup_{\Omega_{v',\epsilon}} \left| \frac{\omega}{dz} \right| \right).$$

By point 4 of the definition of admissible norms, we get $\|L(\omega)\|_E \leq C\|\omega\|_{\Omega}$ for some uniform constant C .

2) Let us prove next that L is onto. Given $(\alpha_e)_{e \in E} \in \mathcal{B}_E$, satisfying (2), we are asked to construct a regular differential ω on $\Sigma_{\mathbf{t}}$ such that $L(\omega) = \alpha$. In each sphere S_v , we define $\omega = \omega_1 + \omega_2$ with

$$\omega_1 = \omega_1^+ + \omega_1^- = \frac{1}{2\pi i} \sum_{e \in E_v^+} \alpha_e \frac{dz}{z - p_e^+} - \frac{1}{2\pi i} \sum_{e \in E_v^-} \alpha_e \frac{dz}{z - p_e^-} \quad (3)$$

$$\omega_2 = \omega_2^+ + \omega_2^- = \sum_{e \in E_v^+} \sum_{n=2}^{\infty} \lambda_{e,n}^+ \frac{\left(\frac{\epsilon}{4}\right)^n dz}{(z - p_e^+)^n} + \sum_{e \in E_v^-} \sum_{n=2}^{\infty} \lambda_{e,n}^- \frac{\left(\frac{\epsilon}{4}\right)^n dz}{(z - p_e^-)^n}. \quad (4)$$

(For $i = 1, 2$, ω_i^+ and ω_i^- denote the first and second sum of ω_i , respectively.) The meromorphic differential ω_1 has the required periods and is entirely determined by the data we are given. Condition (2) ensures that the residue of ω_1 at infinity vanishes, so ω_1 is holomorphic at infinity. The meromorphic differential ω_2 is a corrective term with no periods. The coefficients $\lambda_{e,n}^{\pm}$ are to be determined.

Let us first assume that the series defining ω_2 converge and see what is the condition so that ω is a well defined differential on $\Sigma_{\mathbf{t}}$. Then we solve these equations, and finally prove that the series converge.

For any edge $e \in E$, define

$$\varphi_e^+ = (z_e^+)^{-1} \circ \left(\frac{t_e}{z_e^-} \right), \quad \varphi_e^- = (z_e^-)^{-1} \circ \left(\frac{t_e}{z_e^+} \right) = (\varphi_e^+)^{-1}$$

If $t_e \neq 0$, φ_e^+ is bi-holomorphic from the annulus $\frac{|t_e|}{\rho} \leq |z_e^-| \leq \rho$ to the annulus $\frac{|t_e|}{\rho} \leq |z_e^+| \leq \rho$, and the point z is identified with the point $\varphi_e^+(z)$ when defining $\Sigma_{\mathbf{t}}$. In other words, these are the change of charts for the coordinates z_e^+ and z_e^- on $\Sigma_{\mathbf{t}}$, so ω is well defined on $\Sigma_{\mathbf{t}}$ if $(\varphi_e^+)^*\omega = \omega$ for all edges $e \in E$.

Claim 1 ω is well defined on $\Sigma_{\mathbf{t}}$ if and only if for all edges $e \in E$ such that $t_e \neq 0$

$$\begin{aligned} \forall n \geq 0, \quad \int_{\partial D_e^-} (z - p_e^-)^n ((\varphi_e^+)^*\omega - \omega) &= 0, \\ \forall n \geq 0, \quad \int_{\partial D_e^+} (z - p_e^+)^n ((\varphi_e^-)^*\omega - \omega) &= 0. \end{aligned}$$

Proof. By the theorem on Laurent series, the holomorphic differential $(\varphi_e^+)^*\omega - \omega$ is zero on the annulus $\frac{|t_e|}{\rho} \leq |z_e^-| \leq \rho$ if and only if for all $n \in \mathbb{Z}$,

$$\int_{\partial D_e^-} (z_e^-)^n ((\varphi_e^+)^*\omega - \omega) = 0. \quad (5)$$

Equation (5) for all $n \geq 0$ means that $(\varphi_e^+)^*\omega - \omega$ extends holomorphically to the disk $|z_e^-| \leq \rho$. Using $z - p_e^-$ as a coordinate, this is equivalent to the first condition of the claim. If $n \leq 0$, then by a change of variable

$$\begin{aligned} \int_{\partial D_e^-} (z_e^-)^n ((\varphi_e^+)^*\omega - \omega) &= - \int_{\partial D_e^+} (\varphi_e^-)^* [(z_e^-)^n ((\varphi_e^+)^*\omega - \omega)] \\ &= - \int_{\partial D_e^+} \left(\frac{t_e}{z_e^+} \right)^n (\omega - (\varphi_e^-)^*\omega) \end{aligned}$$

So equation (5) for all $n \leq 0$ means that $\omega - (\varphi_e^-)^*\omega$ extends holomorphically to the disk $|z_e^+| \leq \rho$. Using $z - p_e^+$ as a coordinate, this is equivalent to the second condition of the claim. \square

Claim 2 ω is a well defined regular differential on $\Sigma_{\mathbf{t}}$ if and only if for all $e \in E$ and all $n \geq 2$,

$$\lambda_{e,n}^- = \frac{-1}{2\pi i} \left(\frac{4}{\epsilon} \right)^n \int_{\partial D_e^+} (\varphi_e^- - p_e^-)^{n-1} \omega, \quad (6)$$

$$\lambda_{e,n}^+ = \frac{-1}{2\pi i} \left(\frac{4}{\epsilon} \right)^n \int_{\partial D_e^-} (\varphi_e^+ - p_e^+)^{n-1} \omega. \quad (7)$$

Proof. If $t_e = 0$, then $\varphi_e^+ - p_e^+$ and $\varphi_e^- - p_e^-$ are identically zero. Hence equations (6) and (7) for all $n \geq 2$ mean that ω_2 is holomorphic at p_e^- and p_e^+ , so ω has at most simple poles as required for a regular differential.

If $t_e \neq 0$, we use the previous claim. The definition of $\omega = \omega_1 + \omega_2$ gives

$$\int_{\partial D_e^-} (z - p_e^-)^n \omega = \begin{cases} -\alpha_e & \text{if } n = 0 \\ 2\pi i \lambda_{e,n+1}^- \left(\frac{\epsilon}{4}\right)^{n+1} & \text{if } n \geq 1 \end{cases}$$

By a change of variable,

$$\begin{aligned} \int_{\partial D_e^-} (z - p_e^-)^n (\varphi_e^+)^* \omega &= - \int_{\partial D_e^+} (\varphi_e^-)^* [(z - p_e^-)^n (\varphi_e^+)^* \omega] \\ &= - \int_{\partial D_e^+} (\varphi_e^- - p_e^-)^n \omega \end{aligned}$$

We have similar statements exchanging the roles of the $+$ and $-$ signs. The claim follows. \square

Given a holomorphic differential w on $\Omega_\epsilon = \bigcup_{v \in V} \Omega_{v,\epsilon}$, define for $e \in E$ and $n \geq 2$

$$F_{e,n}^\pm(w) = \frac{-1}{2\pi i} \left(\frac{4}{\epsilon}\right)^n \int_{\partial D_e^\mp} (\varphi_e^\pm - p_e^\pm)^{n-1} w. \quad (8)$$

Let $F^\pm(w) = (F_{e,n}^\pm(w))_{e \in E, n \geq 2}$ and $F(w) = (F^+(w), F^-(w))$. By claim 2, ω is well defined on $\Sigma_{\mathbf{t}}$ if and only if $\lambda = F(w)$. We want to find λ as a fixed point of the map $\lambda \mapsto F(\omega_1 + \omega_2(\lambda))$. To this effect, let \mathcal{B}_L be the space of sequences $\lambda = (\lambda_{e,n}^+, \lambda_{e,n}^-)_{e \in E, n \geq 2}$ such that the following norm is defined :

$$\|\lambda\|_L = \left\| \left(\sup_{n \geq 2} \max\{|\lambda_{e,n}^+|, |\lambda_{e,n}^-|\} \right)_{e \in E} \right\|_E.$$

This is a Banach space by point 5 of definition 3.

Lemma 1 *There exists a uniform constant C such that if $\|\mathbf{t}\|_\infty \leq \frac{c_1 \rho \epsilon}{4c_2^2}$,*

1. $\|\omega_1\|_\Omega \leq \frac{C}{\epsilon} \|\alpha\|_E$,
2. $\|\omega_2\|_\Omega \leq C \|\lambda\|_L$,
3. $\|F(w)\|_L \leq \frac{C}{\epsilon^2} \|\mathbf{t}\|_\infty \|w\|_\Omega$.

Proof of point 1 : we have the straightforward estimate

$$\sup_{\Omega_{v,\epsilon}} \left| \frac{\omega_1}{dz} \right| \leq \frac{1}{2\pi\epsilon} \sum_{e \in E_v} |\alpha_e|.$$

The conclusion follows by point 3 of definition 3.

Proof of point 2 :

$$\sup_{\Omega_{v,\epsilon}} \left| \frac{\omega_2^+}{dz} \right| \leq \sum_{e \in E_v^+} \sum_{n=2}^{\infty} \frac{|\lambda_{e,n}^+|}{\epsilon^n} \left(\frac{\epsilon}{4}\right)^n \leq \left(\sum_{n=2}^{\infty} \left(\frac{1}{4}\right)^n \right) \left(\sum_{e \in E_v^+} \sup_{n \geq 2} |\lambda_{e,n}^+| \right).$$

We estimate ω_2^- in the same way. The conclusion follows by point 3 of definition 3.

Proof of point 3 : Let e be an edge from v to v' . By definition of admissible coordinates, we have

$$|\varphi_e^+(z) - p_e^+| \leq c_2 |z_e^+(\varphi_e^+(z))| = \frac{c_2 |t_e|}{|z_e^-(z)|} \leq \frac{c_2^2 |t_e|}{|z - p_e^-|}.$$

Arguing in the same way on the other side we obtain the following useful estimate

$$\frac{c_1^2 |t_e|}{|z - p_e^-|} \leq |\varphi_e^+(z) - p_e^+| \leq \frac{c_2^2 |t_e|}{|z - p_e^-|} \quad (9)$$

We replace the circle ∂D_e^- in the definition of $F_{e,n}^+$ by the circle $C(p_e^-, r)$ with $r = c_1 \rho$. Then we have for $e \in E$ and $n \geq 2$

$$\begin{aligned} |F_{e,n}^+(w)| &\leq \frac{1}{2\pi} 2\pi r \frac{4}{\epsilon} \left(\frac{4 c_2^2 |t_e|}{\epsilon r} \right)^{n-1} \sup_{\Omega_{v,\epsilon}} \left| \frac{w}{dz} \right| \quad (10) \\ &\leq \frac{4r}{\epsilon} \left(\frac{4c_2^2 |t_e|}{\epsilon r} \right) \sup_{\Omega_{v,\epsilon}} \left| \frac{w}{dz} \right| \\ &\leq \frac{16c_2^2}{\epsilon^2} \|\mathbf{t}\|_\infty \sup_{\Omega_{v,\epsilon}} \left| \frac{w}{dz} \right|. \end{aligned}$$

(The term in the middle parenthesis of the first line is less than one by our hypothesis on \mathbf{t}). We estimate $F_{e,n}^-(w)$ in the same way. The conclusion follows by point 4 of definition 3. \square

It follows from the lemma that if $\|\mathbf{t}\|_\infty$ is small enough (depending on ϵ), the map $\lambda \mapsto F(\omega_1 + \omega_2(\lambda))$ is contracting from \mathcal{B}_L to itself. By the contraction mapping theorem, there exists a unique $\lambda \in \mathcal{B}_L$ such that $F(\omega_1 + \omega_2(\lambda)) = \lambda$.

It remains to consider the convergence of the series defining ω_2 . First of all, we have actually seen in the proof of point 2 of the lemma that this series converges on $\Omega_{v,\epsilon}$, so ω is already defined on each $\Omega_{v,\epsilon}$. It remains to prove that for each edge $e \in E$, ω is defined on the corresponding neck, namely the annular region in $\Sigma_{\mathbf{t}}$ bounded by the circles $C(p_e^-, \epsilon)$ and $C(p_e^+, \epsilon)$.

Since $\|\omega\|_\Omega$ is finite, $\sup_{\Omega_{v,\epsilon}} |\omega|$ is finite for all $v \in V$ (although maybe not uniformly bounded). Since $\lambda = F(\omega)$, we have by equation (10)

$$\left(\frac{\epsilon}{4} \right)^n |\lambda_{e,n}^\pm| \leq r \left(\frac{c_2^2 |t_e|}{r} \right)^{n-1} \sup_{\Omega_{v,\epsilon}} \left| \frac{\omega}{dz} \right|.$$

From this we conclude that if $t_e = 0$, then $\lambda_{e,n}^\pm = 0$ for all $n \geq 2$ so there is no convergence issue. If $t_e \neq 0$ and $|t_e|$ is small enough, then $|z_e^\pm(z)| \geq \frac{1}{2}|t_e|^{1/2}$ implies that $|z - p_e^\pm| > 2\frac{c_2^2 |t_e|}{r}$, which implies by the above estimate that the series

$$\sum_{n=2}^{\infty} \lambda_{e,n}^\pm \frac{\left(\frac{\epsilon}{4} \right)^n}{(z - p_e^\pm)^n}$$

converges. Hence ω is defined on the two domains $|z_e^+| \geq \frac{1}{2}|t_e|^{1/2}$ and $|z_e^-| \geq \frac{1}{2}|t_e|^{1/2}$, which cover the neck. So we have proven that L is onto.

Remark 4 *It follows from the lemma that for $\|\mathbf{t}\|_\infty$ small enough (depending on ϵ), $\|\omega\|_\Omega \leq \frac{C}{\epsilon}\|\alpha\|_E$.*

3) Finally we prove that L is injective for $\|\mathbf{t}\|_\infty$ small enough. Let ω be in the kernel of L . For any $e \in E$, ω is holomorphic in the annulus $c_2 \frac{|t_e|}{\rho} \leq |z - p_e^\pm| \leq c_1 \rho$ (which is included in the annulus $\frac{|t_e|}{\rho} \leq |z_e^\pm| \leq \rho$), so we can write its Laurent series in this annulus as

$$\omega = \sum_{n=-\infty}^{\infty} c_{e,n}^\pm \frac{dz}{(z - p_e^\pm)^n}.$$

The coefficients $c_{e,1}^\pm$ are all zero because $L(\omega) = 0$. The sum for $n \leq 0$ extends holomorphically to the disk $D(p_e^\pm, r)$. The sum for $n \geq 2$ extends holomorphically to the outside of this disk. Therefore, the difference

$$\omega - \sum_{e \in E_v^+} \sum_{n=2}^{\infty} c_{e,n}^+ \frac{dz}{(z - p_e^+)^n} - \sum_{e \in E_v^-} \sum_{n=2}^{\infty} c_{e,n}^- \frac{dz}{(z - p_e^-)^n}$$

extends holomorphically to all of S_v . Since there are no holomorphic 1-form on the sphere, it is zero. Let $\lambda_{e,n}^\pm = \left(\frac{4}{\epsilon}\right)^n c_{e,n}^\pm$, then $\omega = \omega_2(\lambda)$. Since ω is well defined on $\Sigma_{\mathbf{t}}$, we have $\lambda = F(\omega)$. By lemma 1, $\|\lambda\|_L \leq \frac{C}{\epsilon^2} \|\mathbf{t}\|_\infty \|\omega\|_L$, so $\lambda = 0$ if \mathbf{t} is small enough. Hence $\omega = 0$. This concludes the proof of Theorem 3. \square

Remark 5 *We have proven in point 2 that any regular differential $\omega \in \mathcal{H}^1(\Sigma_{\mathbf{t}})$ can be written $\omega = \omega_1(\alpha) + \omega_2(\lambda)$, where ω_1 and ω_2 are defined by (3) and (4). Moreover, $\lambda = F(\omega)$, where $F(\omega)$ is defined by (8).*

5 Smooth dependance on parameters

By Theorem 3, for any $\alpha \in \mathcal{B}_E$ satisfying (2), there exists a unique regular differential ω on $\Sigma_{\mathbf{t}}$ whose periods are prescribed by α . In this section, we prove that ω depends smoothly on the parameters in the construction of $\Sigma_{\mathbf{t}}$, namely $\mathbf{p} = (p_e^+, p_e^-)_{e \in E}$ and $\mathbf{t} = (t_e)_{e \in E}$. Before we can formulate this result, several points need to be addressed.

We assume that the parameter $\mathbf{p} = (p_e^+, p_e^-)_{e \in E}$ is in a neighborhood of some central value $\underline{\mathbf{p}} = (\underline{p}_e^+, \underline{p}_e^-)_{e \in E}$ in ℓ^∞ norm. In this context, we define $\Omega_{v,\epsilon}$ to be the sphere S_v minus the disks $D(\underline{p}_e^-, \epsilon)$ for $e \in E_v^-$ and $D(\underline{p}_e^+, \epsilon)$ for $e \in E_v^+$. We define Ω_ϵ as the disjoint union of all $\Omega_{v,\epsilon}$ for $v \in V$. The norm $\|\cdot\|_\Omega$ is defined as in section 3.4. The point here is that the domain Ω_ϵ is fixed, whereas our former domain Ω_ϵ depends on the parameter \mathbf{p} . Thanks to remark 3, the new norm $\|\cdot\|_\Omega$ on $\mathcal{H}_1(\Sigma_{\mathbf{t}})$ is equivalent to the former one.

The coordinates z_e^\pm must depend in some ways on the parameters, else the requirement $z_e^\pm(p_e^\pm) = 0$ fixes the point p_e^\pm . We assume that for each edge $e \in E$, the coordinates z_e^\pm have the form $z_e^\pm(z) = \zeta_e^\pm(z - p_e^\pm, \xi_e^\pm)$, where ξ_e^\pm is a vector of complex parameters and ζ_e^\pm is a holomorphic function in all its variables. For example 2, we would take $\xi_e^\pm = p_e^\pm$ and $\zeta_e^\pm(z, p_e^\pm) = i \frac{z}{z + 2p_e^\pm}$.

We assume that each parameter ξ_e^\pm is in a finite dimensional complex vector space, possibly depending on the edge, but of uniformly bounded dimension. We write $\xi = (\xi_e^+, \xi_e^-)_{e \in E}$ and we assume that the coordinates z_e^\pm are defined and admissible for all values of ξ in a neighborhood of some central value $\underline{\xi} = (\underline{\xi}_e^+, \underline{\xi}_e^-)_{e \in E}$ in ℓ^∞ norm.

Theorem 4 *The regular differential ω , restricted to Ω_e , depends smoothly on the parameters $(\mathbf{p}, \mathbf{t}, \xi)$ in a neighborhood of $(\underline{\mathbf{p}}, 0, \underline{\xi})$ in ℓ^∞ norm.*

The norm on the domain space is the ℓ^∞ norm. The norm on the target space is the norm $\|\cdot\|_\Omega$.

Remark 6 *By a theorem of Graves-Taylor-Hille-Zorn, a map f from an open set of a complex Banach space E to a complex Banach space F , which is differentiable in the usual (Frechet) sense, is holomorphic, hence smooth. So the word “smoothly” in the theorem can be replaced by the word “analytically”. See the book [7] for the definition of a holomorphic map between complex Banach spaces and Theorem 14.3 for the statement and the proof. Also, Hartog’s theorem holds in the complex Banach space setup : if E_1 and E_2 are Banach spaces and $f : E_1 \times E_2 \rightarrow F$ is separately holomorphic with respect to each of its two variables, then it is holomorphic (Theorem 14.27).*

Proof of Theorem 4 (continued from the proof of Theorem 3) : recall that we found λ as a fixed point of $\lambda \mapsto F(\omega_1) + F(\omega_2(\lambda))$. By the following lemma and the fixed point theorem with parameters, λ depends smoothly on all parameters, so $\omega = \omega_1 + \omega_2(\lambda)$ also depends smoothly on all parameters. \square

Let $\mathcal{H}_1(\Omega_e)$ be the space of holomorphic differentials w on Ω_e such that $\|w\|_\Omega$ is defined.

Lemma 2 *For $(\mathbf{p}, \mathbf{t}, \xi)$ in a neighborhood of $(\underline{\mathbf{p}}, 0, \underline{\xi})$ in ℓ^∞ norm, $\alpha \in \mathcal{B}_E$, $\lambda \in \mathcal{B}_L$ and $w \in \mathcal{H}_1(\Omega_e)$,*

1. *the map $(\mathbf{p}, \alpha) \mapsto \omega_1(\mathbf{p}, \alpha)$ is smooth,*
2. *the map $(\mathbf{p}, \lambda) \mapsto \omega_2(\mathbf{p}, \lambda)$ is smooth,*
3. *the map $(\mathbf{p}, \mathbf{t}, \xi, w) \mapsto F(\mathbf{p}, \mathbf{t}, \xi, w)$ is smooth.*

To prove this lemma, we first consider an abstract result. The setup is the following : we have an infinite family of holomorphic functions $f_n(x_n)$ for $n \in \mathbb{N}$. The variable x_n is in the poly-disk $D(0, R)$ in \mathbb{C}^{d_n} . We assume that the dimension d_n is uniformly bounded. The function f_n takes value in a Banach space F_n . Let $E = \prod_{n \in \mathbb{N}} \mathbb{C}^{d_n}$ and $F = \prod_{n \in \mathbb{N}} F_n$, both with the sup norm. Let $x = (x_n)_{n \in \mathbb{N}}$ and $f(x) = (f_n(x_n))_{n \in \mathbb{N}}$

Lemma 3 *If each $\|f_n\|$ is bounded on the poly-disk $D(0, R)$ by a constant C independent of n , then $f : B(0, R) \rightarrow F$ is smooth, where $B(0, R)$ denotes the ball of radius R in E .*

Proof. Consider some $r < R$. Since the function f_n is holomorphic in the poly-disk $D(0, R)$ and bounded by C , all its partial derivatives of order k in the poly-disk $D(0, r)$ have norm bounded by $Ck!(R-r)^{-k}$ by Cauchy's estimates, which holds true for Banach valued holomorphic functions. The point is that this bound does not depend on n . It is then straightforward to check that f is differentiable in the ball $B(0, r)$, with differential $df(x)(h) = (df_n(x_n)(h_n))_{n \in \mathbb{N}}$. Since r is arbitrary, f is differentiable in the open ball $B(0, R)$. The fact that f is smooth follows by induction (or using the theorem of Graves-Taylor-Hille-Zorn.) \square

Proof of point 1 of lemma 2. We deal with the terms ω_1^- and ω_1^+ separately (see equation (3) for the definition of these differentials).

For each edge $e \in E$ from v to v' , let \mathcal{E}_e be the space of bounded holomorphic functions on $\Omega_{v,\epsilon}$ with the sup norm. This is a Banach algebra for the pointwise product. If $|p_e^- - \underline{p}_e^-| < \frac{\epsilon}{2}$ then for any $z \in \Omega_{v,\epsilon}$, we have $|z - p_e^-| > \frac{\epsilon}{2}$. Let f_e be the holomorphic function $z \mapsto \frac{1}{z - p_e^-}$ on $\Omega_{v,\epsilon}$. The map $p_e^- \mapsto f_e$ is holomorphic and bounded by $\frac{2}{\epsilon}$ from the disk $D(\underline{p}_e^-, \frac{\epsilon}{2})$ to \mathcal{E}_e . Let \mathcal{E} be the product of the spaces \mathcal{E}_e for $e \in E$ with the sup norm. Let $f = (f_e)_{e \in E}$. By lemma 3, the map $\mathbf{p} \mapsto f$ is smooth from the ball $B(\underline{\mathbf{p}}, \frac{\epsilon}{2})$ to \mathcal{E} .

We compose this map with the operator which maps $(\alpha, f) \in \mathcal{B}_E \times \mathcal{E}$ to the differential

$$\sum_{e \in E_v^-} \alpha_e f_e dz \quad \text{in } \Omega_{v,\epsilon}, \text{ for } v \in V.$$

It follows from the definition of the norms that this operator is bilinear bounded from $\mathcal{B}_E \times \mathcal{E}$ to $\mathcal{H}_1(\Omega_\epsilon)$. We deal with ω_1^+ in the same way. This proves the first point of lemma 2.

The proof of point 2 of lemma 2 is very much similar. We first deal with the term ω_2^- . Let $f_{e,n} \in \mathcal{E}_e$ be the function

$$f_{e,n}(z) = \left(\frac{\epsilon}{2}\right)^n \frac{1}{(z - p_e^-)^n}.$$

If $|p_e - \underline{p}_e^-| < \frac{\epsilon}{2}$, then $\|f_{e,n}\|_\infty \leq 1$. Define $f = (f_{e,n})_{e \in E, n \in \mathbb{N}}$. By lemma 3, the map $\mathbf{p} \mapsto f$ is smooth from the ball $B(\underline{\mathbf{p}}, \frac{\epsilon}{2})$ to the space \mathcal{E}' equal to the product of the spaces \mathcal{E}_e for $e \in E$ and $n \in \mathbb{N}$, with the sup norm.

We compose this map with the operator which maps (λ, f) to the differential

$$\sum_{e \in E_v^-} \sum_{n \geq 2} \lambda_{e,n}^- 2^{-n} f_{e,n} dz.$$

It is straightforward to check that this operator is bilinear bounded from $\mathcal{B}_L \times \mathcal{E}'$ to $\mathcal{H}_1(\Omega_\epsilon)$ (the series converges thanks to the term 2^{-n}). We deal with ω_2^+ in the same way. This proves point 2 of lemma 2.

To prove point 3 of the lemma, we replace the integration circle in the definition of $F_{e,n}^+$ by the fixed circle $C(\underline{p}_e^-, r)$ with $r = \frac{\rho c_1}{2}$. For each edge $e \in E$, let A_e^- be the fixed annulus $\frac{r}{2} < |z - \underline{p}_e^-| < \frac{3r}{2}$. Let \mathcal{E}_e'' be the spaces of bounded holomorphic functions on A_e^- . Assume that $\|\mathbf{p} - \underline{\mathbf{p}}\|_\infty < \frac{r}{4}$. If $z \in A_e^-$, then $\frac{r}{4} < |z - \underline{p}_e^-| < 2r$. Assume that $|t_e| < \frac{c_1 \rho^2}{8c_2}$. Then for $z \in A_e^-$, we have $\frac{c_2 |t_e|}{\rho} < |z - \underline{p}_e^-| < c_1 \rho$. Using the definition of admissible coordinates, we get $\frac{|t_e|}{\rho} < |z_e^-(z)| < \rho$, so $\varphi_e^+(z)$ is defined. Let $f_{e,n} \in \mathcal{E}_e''$ be the function

$$f_{e,n}(z) = \left(\frac{4}{\epsilon}\right)^n (\varphi_e^+(z) - p_e^+)^n, \quad z \in A_e^-.$$

By equation (9), we have

$$|\varphi_e^+ - p_e^+| \leq \frac{c_2^2 |t_e|}{|z - \underline{p}_e^-|} \leq \frac{8c_2^2 |t_e|}{\rho c_1}$$

If $|t_e| \leq \frac{\rho c_1 \epsilon}{32c_2^2}$, then $f_{e,n}$ is bounded by 1 on A_e^- . Let \mathcal{E}'' be the product of the spaces \mathcal{E}_e'' for $e \in E$ and $n \in \mathbb{N}$, with the sup norm. Let $f = (f_{e,n})_{e \in E, n \in \mathbb{N}} \in \mathcal{E}''$. By lemma 3, the map $(\mathbf{p}, \mathbf{t}, \xi) \mapsto f$ is smooth from a neighborhood of $(\underline{\mathbf{p}}, 0, \underline{\xi})$ to \mathcal{E}'' .

We compose this map with the operator

$$(f, w) \mapsto \left(\int_{C(\underline{p}_e^-, r)} f_{e,n-1} w \right)_{e \in E, n \geq 2}.$$

This operator is bilinear bounded from $\mathcal{E}'' \times \mathcal{H}_1(\Omega_\epsilon)$ to \mathcal{B}_L . This follows readily from the definition of the norms. This proves that F^+ depends smoothly on parameters. We deal with F^- in the same way. This concludes the proof of lemma 2. \square

Remark 7 We have used the following fact : if $f(z, w)$ is a bounded holomorphic function of $z \in \Omega \subset \mathbb{C}$ and $w \in B(0, R) \subset \mathbb{C}^d$, then the map $w \mapsto f(\cdot, w)$ is holomorphic from the open ball $B(0, R)$ to the space of holomorphic functions on Ω with the sup norm. As in the proof of lemma 3, this follows from the fact that we have uniform estimates for the derivatives of f with respect to w .

Remark 8 The partial differential of ω with respect to \mathbf{t} at $\mathbf{t} = 0$ can be computed as follows : let us fix all variables but \mathbf{t} and use the notation $\frac{\partial}{\partial \mathbf{t}}$ for the differential with respect to \mathbf{t} at $\mathbf{t} = 0$. We have

$$\omega = \omega_1 + \omega_2(F(\mathbf{t}, \omega)).$$

Take the differential with respect to \mathbf{t} :

$$\frac{\partial \omega}{\partial \mathbf{t}} \cdot \mathbf{h} = \omega_2 \left(\frac{\partial F}{\partial \mathbf{t}}(0, \omega) \cdot \mathbf{h} \right) + \omega_2 \left(F(0, \frac{\partial \omega}{\partial \mathbf{t}} \cdot \mathbf{h}) \right).$$

The second term is zero because $F = 0$ when $\mathbf{t} = 0$. To compute the first term we write

$$\frac{\partial \varphi_e^+(z)}{\partial t_e} = \frac{1}{(z_e^+)'(p_e^+)z_e^-(z)}.$$

From this we obtain

$$\frac{\partial \omega}{\partial \mathbf{t}} \cdot \mathbf{h} = \sum_{e \in E_v^+} h_e a_e^+ \frac{dz}{(z - p_e^+)^2} + \sum_{e \in E_v^-} h_e a_e^- \frac{dz}{(z - p_e^-)^2}$$

with

$$a_e^+ = \frac{-1}{2\pi i (z_e^+)'(p_e^+)} \int_{\partial D_e^-} \frac{\omega}{z^-}$$

and a similar definition for a_e^- . In particular, when the coordinates are given as in example 1 by $z_e^\pm = z - p_e^\pm$, we have

$$a_e^+ = \frac{-1}{2\pi i} \int_{\partial D_e^-} \frac{\omega}{z - p_e^-}.$$

6 Estimate of ω on the necks

When $\mathbf{t} = 0$, $\omega_2 = 0$ so $\omega = \omega_1$ is explicitly given by formula 3. Theorem 4 tells us that the restriction of ω to the domain Ω_e depends smoothly on the parameter \mathbf{t} , so gives us good control of ω on Ω_e for small values of \mathbf{t} but says nothing outside of Ω_e , i.e. on the necks.

In a neighborhood of p_e^+ , we expect $\omega \simeq \alpha_e \frac{dz}{z - p_e^+}$. The following proposition gives a precise statement.

Proposition 2 *Assume that the degree of vertices is bounded. There exists a uniform constant C such that for \mathbf{t} small enough and for any edge $e \in E$ one has*

$$\left| \frac{\omega}{dz} - \frac{\alpha_e}{z - p_e^+} \right| \leq C \|\alpha\|_\infty \quad \text{in the annulus } |t_e|^{1/2} \leq |z_e^+| \leq \rho,$$

$$\left| \frac{\omega}{dz} + \frac{\alpha_e}{z - p_e^-} \right| \leq C \|\alpha\|_\infty \quad \text{in the annulus } |t_e|^{1/2} \leq |z_e^-| \leq \rho.$$

Proof : we estimate ω_1 and ω_2 separately. Consider z in the annulus $|t_e|^{1/2} \leq |z_e^+(z)| \leq \rho$. Then for $e' \in E_v^+$, $e' \neq e$, one has $|z - p_{e'}^+| \geq \rho c_1$ and for $e' \in E_v^-$, one has $|z - p_{e'}^-| \geq \rho c_1$. This readily gives the estimate

$$\left| \frac{\omega_1}{dz} - \frac{\alpha_e}{z - p_e^+} \right| \leq C \|\alpha\|_\infty.$$

By remark 4 with $\epsilon = \rho c_1$, we have $\|\omega\|_\infty \leq C \|\alpha\|_\infty$. Using that $\lambda = F(\omega)$ and estimate (10), we have

$$\left(\frac{\epsilon}{4}\right)^n |\lambda_{e,n}^+| \leq Cr \left(\frac{c_2^2 |t_e|}{r}\right)^{n-1} \|\alpha\|_\infty.$$

Using the definition of admissible coordinates, we have $|z - p_e^+| \geq c_1 |t_e|^{1/2}$. For $e' \in E_v^+$, $e' \neq e$, we have $|z - p_{e'}^+| \geq \rho c_1 \geq c_1 |t_e|^{1/2}$. This gives

$$\begin{aligned} \left| \frac{\omega_2^+}{dz} \right| &\leq \sum_{e' \in E_v^+} \sum_{n=2}^{\infty} C r \|\alpha\|_{\infty} \left(\frac{c_2^2 |t_e|}{r} \right)^{n-1} \frac{1}{(c_1 |t_e|^{1/2})^n} \\ &\leq \sum_{e' \in E_v} C \|\alpha\|_{\infty} \frac{2c_2^2}{c_1^2} \end{aligned}$$

provided $|t_e|^{1/2} \leq \frac{rc_1}{2c_2^2}$. We estimate ω_2^- in the same way. This proves the first statement of the proposition. The second statement is similar. \square

Corollary 1 *For any $\epsilon \in (0, c_1 \rho)$ and any edge $e \in E$, if $|\alpha_e| > C\epsilon \|\alpha\|_{\infty}$, then ω has no zero in the annulus bounded in $\Sigma_{\mathbf{t}}$ by the circles $C(p_e^-, \epsilon)$ and $C(p_e^+, \epsilon)$, where C is the constant that appears in proposition 2.*

Proof : if z is in this annulus, then either $|z_e^+(z)| \geq |t_e|^{1/2}$ or $|z_e^-(z)| \geq |t_e|^{1/2}$. Proposition 2 gives $|\frac{\omega}{dz}| > 0$. \square

7 Decay to a bounded differential

Given $\alpha \in \ell^{\infty}(E)$, Theorem 2 gives us a differential $\omega \in \mathcal{H}_{\infty}^1(\Sigma_{\mathbf{t}})$. Consider an edge $e \in E$, and assume that we perturb the parameter t_e while keeping the other parameters $t_{e'}$ for $e' \neq e$ fixed. This perturbs the differential ω , and we would like to understand what is the decay of this perturbation as $v \rightarrow \infty$.

As in example 4, we do this by using weighted spaces $\ell^{\infty, \sigma}$ for the parameter \mathbf{t} . So in this section, we assume that the parameter \mathbf{t} is in a neighborhood of some central value $\underline{\mathbf{t}}$ for the norm \mathcal{B}_E . There are two observations to be made: firstly, the central value $\underline{\mathbf{t}}$ is in ℓ^{∞} but might not be in \mathcal{B}_E , so we write $\dot{\mathbf{t}} = \mathbf{t} - \underline{\mathbf{t}}$ and require that $\dot{\mathbf{t}} \in \mathcal{B}_E$. Second, to open node, we need that $\|\mathbf{t}\|_{\infty}$ remains small, so we require that $\|\dot{\mathbf{t}}\|_{\infty} \leq C \|\dot{\mathbf{t}}\|_E$, or in other words, that $\mathcal{B}_E \hookrightarrow \ell^{\infty}$.

Another motivation is that we will study the smooth dependance of ω on \mathbf{t} for the norm \mathcal{B}_E instead of the norm ℓ^{∞} as in section 5. This is more natural for certain applications we have in mind (more precisely, to construct non-periodic minimal surfaces asymptotic to periodic ones).

The same question can be asked for the parameters \mathbf{p} and ξ . We define a norm on the space of parameters \mathbf{p} and ξ (still denoted $\|\cdot\|_E$ as no confusion can possibly arise) by $\|\mathbf{p}\|_E = \|(\max\{|p_e^+|, |p_e^-|\})_{e \in E}\|_E$ and $\|\xi\|_E = \|(\max\{|\xi_e^+|, |\xi_e^-|\})_{e \in E}\|_E$. We write $\mathbf{p} = \underline{\mathbf{p}} + \dot{\mathbf{p}}$ and $\xi = \underline{\xi} + \dot{\xi}$ with $\underline{\mathbf{p}} \in \mathcal{B}_E$ and $\dot{\xi} \in \mathcal{B}_E$.

Theorem 5 *Assume that $\mathcal{B}_E \hookrightarrow \ell^{\infty}(E)$. Then for $\|\underline{\mathbf{t}}\|_{\infty}$ small enough, $\alpha \in \ell^{\infty}(E)$, and $\|\dot{\mathbf{p}}\|_E$, $\|\dot{\mathbf{t}}\|_E$ and $\|\dot{\xi}\|_E$ small enough, the restriction of $\omega(\underline{\mathbf{p}} + \dot{\mathbf{p}}, \underline{\mathbf{t}} + \dot{\mathbf{t}}, \underline{\xi} + \dot{\xi}, \alpha) - \omega(\underline{\mathbf{p}}, \underline{\mathbf{t}}, \underline{\xi}, \alpha)$ to Ω_{ϵ} belongs to $\mathcal{H}^1(\Omega_{\epsilon})$. Moreover, the map $(\dot{\mathbf{p}}, \dot{\mathbf{t}}, \dot{\xi}, \alpha) \mapsto \omega(\underline{\mathbf{p}} + \dot{\mathbf{p}}, \underline{\mathbf{t}} + \dot{\mathbf{t}}, \underline{\xi} + \dot{\xi}, \alpha) - \omega(\underline{\mathbf{p}}, \underline{\mathbf{t}}, \underline{\xi}, \alpha)$ is smooth.*

The norm on the parameters $\dot{\mathbf{p}}, \dot{\mathbf{t}}, \dot{\xi}$ is the norm $\|\cdot\|_E$. The norm on α is the ℓ^∞ norm. The norm on the target space is the norm $\|\cdot\|_\Omega$. Observe that a priori $\alpha \notin \mathcal{B}_E$ so $\omega(\mathbf{p}, \mathbf{t}, \xi, \alpha) \notin \mathcal{H}^1(\Sigma_{\mathbf{t}})$, which is why we have to subtract $\omega(\underline{\mathbf{p}}, \underline{\mathbf{t}}, \underline{\xi}, \alpha)$.

Returning to the question raised at the beginning of this section, we take \mathcal{B}_E to be the space $\ell^{\infty, \sigma}(E)$ defined in section 3.7. Fix some edge $e \in E$ and assume that $t_{e'} = 0$ for $e' \neq e$. Then $\dot{\mathbf{t}} \in \mathcal{B}_E$, so $\|\omega(\underline{\mathbf{t}} + \dot{\mathbf{t}}, \alpha) - \omega(\underline{\mathbf{t}}, \alpha)\|_{\infty, \sigma}$ is finite. In other words, the influence of the parameter t_e on the restriction of ω to $\Omega_{v, \epsilon}$ decays exponentially as $v \rightarrow \infty$. The same statement holds for the parameters p_e^\pm and ξ_e^\pm .

Proof of Theorem 5. The proof is very similar to the proof of Theorem 4. We need the following analogue of lemma 2.

Lemma 4 *For $\|\underline{\mathbf{t}}\|_\infty$ small enough, $\alpha \in \ell^\infty$, $\lambda \in \ell^\infty$, $w \in \mathcal{H}_\infty^1(\Omega_\epsilon)$, and $\|\dot{\mathbf{p}}\|_E$, $\|\dot{\mathbf{t}}\|_E$ and $\|\dot{\xi}\|_E$ small enough,*

1. $\omega_1(\underline{\mathbf{p}} + \dot{\mathbf{p}}, \alpha) - \omega_1(\underline{\mathbf{p}}, \alpha) \in \mathcal{H}^1(\Omega_\epsilon)$ and depends smoothly on $\dot{\mathbf{p}}$,
2. $\omega_2(\underline{\mathbf{p}} + \dot{\mathbf{p}}, \lambda) - \omega_2(\underline{\mathbf{p}}, \lambda) \in \mathcal{H}^1(\Omega_\epsilon)$ and depends smoothly on $\dot{\mathbf{p}}$,
3. $F(\underline{\mathbf{p}} + \dot{\mathbf{p}}, \underline{\mathbf{t}} + \dot{\mathbf{t}}, \underline{\xi} + \dot{\xi}, w) - F(\underline{\mathbf{p}}, \underline{\mathbf{t}}, \underline{\xi}, w) \in \mathcal{B}_L$ and depends smoothly on $(\dot{\mathbf{p}}, \dot{\mathbf{t}}, \dot{\xi})$.

Proof of point 1. Let $\mathbf{p} = \underline{\mathbf{p}} + \dot{\mathbf{p}}$. We have in S_v

$$\omega_1^-(\mathbf{p}, \alpha) - \omega_1^-(\underline{\mathbf{p}}, \alpha) = \sum_{e \in E_v^-} \alpha_e \frac{p_e^- - \underline{p}_e^-}{(z - p_e^-)(z - \underline{p}_e^-)} dz = \sum_{e \in E_v^-} \alpha_e \dot{p}_e^- g_e dz$$

with $g_e = \frac{1}{(z - p_e^-)(z - \underline{p}_e^-)}$. Let $g = (g_e)_{e \in E}$. As in the proof of point 1 of lemma 2, the map $\mathbf{p} \mapsto g$ is smooth from the ball $\|\mathbf{p} - \underline{\mathbf{p}}\|_\infty < \frac{\epsilon}{2}$ to \mathcal{E} . Since $\mathcal{B}_E \hookrightarrow \ell^\infty$, the map $\dot{\mathbf{p}} \mapsto g$ is smooth in a neighborhood of 0. We compose this map with the operator

$$(\alpha, g, \dot{\mathbf{p}}) \mapsto \sum_{e \in E_v^-} \alpha_e g_e \dot{p}_e^- dz.$$

This operator is trilinear bounded from $\ell^\infty(E) \times \mathcal{E} \times \mathcal{B}_E$ to $\mathcal{H}^1(\Omega_\epsilon)$. We deal with ω_1^+ in the same way.

Proof of point 2. We write in S_v

$$\omega_2^-(\mathbf{p}, \lambda) - \omega_2^-(\underline{\mathbf{p}}, \lambda) = \sum_{e \in E_v^-} \lambda_{e,n}^- 2^{-n} g_{e,n} \dot{p}_e^- dz$$

with

$$g_{e,n} = \frac{f_{e,n}(p_e^-) - f_{e,n}(\underline{p}_e^-)}{p_e^- - \underline{p}_e^-}$$

and $f_{e,n}$ as in the proof of point 2 of lemma 2. Now if a holomorphic function $f(z)$ is bounded by C on the disk $D(0, R)$, then its derivative is bounded by $\frac{2C}{R}$ on the disk $D(0, \frac{R}{2})$ by Cauchy estimate. Hence by the mean value inequality, the function $\frac{f(z)-f(w)}{z-w}$ is bounded by $\frac{2C}{R}$ on $D(0, \frac{R}{2}) \times D(0, \frac{R}{2})$. Hence, from the bound $|f_{e,n}| \leq 1$ for $|p_e^- - \underline{p}_e^-| < \frac{\epsilon}{2}$, we get a uniform bound of $g_{e,n}$ for $|p_e^- - \underline{p}_e^-| < \frac{\epsilon}{4}$. Then we conclude as in the proof of point 1 by composing with a bounded trilinear operator.

Proof of point 3. We first prove that $F(\mathbf{p}, \mathbf{t}, \xi, w) - F(\underline{\mathbf{p}}, \mathbf{t}, \xi, w)$ depends smoothly on $(\dot{\mathbf{p}}, \mathbf{t}, \dot{\xi})$, hence on $(\dot{\mathbf{p}}, \dot{\mathbf{t}}, \dot{\xi})$ since $\mathcal{B}_E \hookrightarrow \ell^\infty$. To do this, we adapt the proof of point 3 of lemma 2 exactly as we did above. Then we prove in the same way that $F(\mathbf{p}, \mathbf{t}, \xi, w) - F(\mathbf{p}, \underline{\mathbf{t}}, \xi, w)$ depends smoothly on $(\dot{\mathbf{t}}, \xi)$, and finally that $F(\mathbf{p}, \mathbf{t}, \xi, w) - F(\mathbf{p}, \underline{\mathbf{t}}, \underline{\xi}, w)$ depends smoothly on $\dot{\xi}$. \square

Proof of Theorem 5. Let

$$G(\mathbf{p}, \mathbf{t}, \xi, \alpha, \lambda) = F(\mathbf{p}, \mathbf{t}, \xi, \omega_1(\mathbf{p}, \alpha) + \omega_2(\mathbf{p}, \lambda)).$$

Recall that G is contracting with respect to λ and that we found λ as a fixed point of $\lambda \mapsto G(\mathbf{p}, \mathbf{t}, \xi, \alpha, \lambda)$. Let $\underline{\lambda}$ be the fixed point of $\lambda \mapsto G(\underline{\mathbf{p}}, \underline{\mathbf{t}}, \underline{\xi}, \alpha, \lambda)$, so that $\omega(\underline{\mathbf{p}}, \underline{\mathbf{t}}, \underline{\xi}, \alpha) = \omega_1(\underline{\mathbf{p}}, \alpha) + \omega_2(\underline{\mathbf{p}}, \underline{\lambda})$. Write $\lambda = \underline{\lambda} + \dot{\lambda}$ and define

$$H(\dot{\mathbf{p}}, \dot{\mathbf{t}}, \dot{\xi}, \alpha, \dot{\lambda}) = G(\mathbf{p}, \mathbf{t}, \xi, \alpha, \lambda) - G(\underline{\mathbf{p}}, \underline{\mathbf{t}}, \underline{\xi}, \alpha, \lambda).$$

Then $\dot{\lambda}$ is a fixed point of $\dot{\lambda} \mapsto H(\dot{\mathbf{p}}, \dot{\mathbf{t}}, \dot{\xi}, \alpha, \dot{\lambda})$. By claim 3 below and the fixed point theorem with parameters, $\dot{\lambda} \in \mathcal{B}_L$ and depends smoothly on $(\dot{\mathbf{p}}, \dot{\mathbf{t}}, \dot{\xi}, \alpha)$. Then we write

$$\omega(\mathbf{p}, \mathbf{t}, \xi, \alpha) - \omega(\underline{\mathbf{p}}, \underline{\mathbf{t}}, \underline{\xi}, \alpha) = (\omega_1(\mathbf{p}, \alpha) - \omega_1(\underline{\mathbf{p}}, \alpha)) + (\omega_2(\mathbf{p}, \lambda) - \omega_2(\underline{\mathbf{p}}, \lambda)) + \omega_2(\underline{\mathbf{p}}, \dot{\lambda}).$$

It follows from lemma 2 and 4 that all three terms are in $\mathcal{H}^1(\Omega_\epsilon)$ and depend smoothly on $(\dot{\mathbf{p}}, \dot{\mathbf{t}}, \dot{\xi}, \alpha)$.

Claim 3 For $\|\dot{\mathbf{t}}\|_\infty$ small enough, $(\dot{\mathbf{p}}, \dot{\mathbf{t}}, \dot{\xi})$ in a neighborhood of 0 and $\dot{\lambda} \in \mathcal{B}_L$, we have $H(\dot{\mathbf{p}}, \dot{\mathbf{t}}, \dot{\xi}, \alpha, \dot{\lambda}) \in \mathcal{B}_L$ and it depends smoothly on $(\dot{\mathbf{p}}, \dot{\mathbf{t}}, \dot{\xi})$. Moreover, it is contracting with respect to $\dot{\lambda}$.

Proof. We write

$$\begin{aligned} H(\dot{\mathbf{p}}, \dot{\mathbf{t}}, \dot{\xi}, \alpha, \dot{\lambda}) &= F(\mathbf{p}, \mathbf{t}, \xi, \omega_1(\mathbf{p}, \alpha)) - F(\underline{\mathbf{p}}, \underline{\mathbf{t}}, \underline{\xi}, \omega_1(\underline{\mathbf{p}}, \alpha)) \\ &\quad + F(\mathbf{p}, \mathbf{t}, \xi, \omega_2(\mathbf{p}, \lambda)) - F(\underline{\mathbf{p}}, \underline{\mathbf{t}}, \underline{\xi}, \omega_2(\underline{\mathbf{p}}, \lambda)). \end{aligned}$$

The first conclusion then follows from lemma 4. To prove H is contracting, we write

$$H(\dot{\mathbf{p}}, \dot{\mathbf{t}}, \dot{\xi}, \alpha, \dot{\lambda}) - H(\dot{\mathbf{p}}, \dot{\mathbf{t}}, \dot{\xi}, \alpha, 0) = F(\mathbf{p}, \mathbf{t}, \xi, \omega_2(\mathbf{p}, \dot{\lambda})).$$

We have already seen in section 4 that this operator is contracting with respect to $\dot{\lambda}$. This proves the claim and Theorem 5. \square

8 Meromorphic differentials

In this section we adapt the results of the previous sections to the case of meromorphic 1 forms.

On a compact Riemann surface, one can define a meromorphic 1-form ω by prescribing its poles and principal parts at each pole, with the only condition that the sum of the residues be zero. Moreover, this defines ω uniquely up to a holomorphic 1-form.

The definition of a meromorphic differential on a Riemann surface with nodes is the same as that of a regular differential, except that it is allowed to have poles away from the nodes.

In each sphere S_v , we select n_v distinct points $q_{v,i}$, $1 \leq i \leq n_v$, distinct from the nodes, to be the poles of ω . One of these points may be ∞ . For each pole $q_{v,i}$, we choose an integer $m_{v,i} \geq 1$, to be the order of the pole. We define the divisor Q as the formal sum $\sum_{v \in V} \sum_{i=1}^{n_v} m_{v,i} q_{v,i}$. We write $(\omega) \geq Q$ to say that ω has at most poles of order $m_{v,i}$ at $q_{v,i}$, and is otherwise regular (so it may also have simple poles at the nodes as a regular differential).

If ω is such a differential, we may write its principal part at $q_{v,i}$ as follows if $q_{v,i} \neq \infty$:

$$P_{v,i}(\omega) = \sum_{n=1}^{m_{v,i}} a_{v,i,n} \frac{dz}{(z - q_{v,i})^n}.$$

By definition this means that $\omega - P_{v,i}(\omega)$ is holomorphic in a neighborhood of $q_{v,i}$. If $q_{v,i} = \infty$, then we use $w = \frac{1}{z}$ as a coordinate in a neighborhood of ∞ to write the principal part :

$$P_{v,i}(\omega) = \sum_{n=1}^{m_{v,i}} a_{v,i,n} \frac{dw}{w^n} = - \sum_{n=1}^{m_{v,i}} a_{v,i,n} z^{n-2} dz.$$

The residue of ω at $q_{v,i}$ is $a_{v,i,1}$. If we let $\alpha_e = \int_{\gamma_e} \omega$, we have by the residue theorem :

$$\forall v \in V, \quad \sum_{e \in E_v^+} \alpha_e - \sum_{e \in E_v^-} \alpha_e + 2\pi i \sum_{i=1}^{n_v} a_{v,i,1} = 0. \quad (11)$$

Essentially, this is the only obstruction to define a meromorphic differential by prescribing its principal parts and periods.

We write $P = (P_{v,i})_{v \in V, 1 \leq i \leq n_v}$. We define \mathcal{B}_P to be the space of principal parts P such that the following norm is defined :

$$\|P\|_P = \|(\sup\{|a_{v,i,n}| : 1 \leq i \leq n_v, 1 \leq n \leq m_{v,i}\})_{v \in V}\|_V.$$

Hypothesis 1 *We assume that*

1. *the coordinates and the norms are admissible,*

2. the number of poles in each sphere, and the orders of the poles, are uniformly bounded,
3. there exists $r > 0$ such that for each vertex $v \in V$, the disks D_e^- for $e \in E_v^-$, D_e^+ for $e \in E_v^+$ and $D(q_{v,i}, r)$ for $1 \leq i \leq n_v$ are disjoint.

In case $q_{v,i} = \infty$, the disk $D(q_{v,i}, r)$ should be understood as the disk $|w| \leq r$, or equivalently, $|z| \geq \frac{1}{r}$. In this setup, the definition of $\Omega_{v,\epsilon}$ must be changed as follows : Given some positive ϵ less than r , we define the domain $\Omega_{v,\epsilon}$ as the Riemann sphere S_v minus the disks $D(p_e^-, \epsilon)$ for $e \in E_v^-$, $D(p_e^+, \epsilon)$ for $e \in E_v^+$ and $D(q_{v,i}, \epsilon)$ for $1 \leq i \leq n_v$. We define Ω_ϵ as the disjoint union of the domains $\Omega_{v,\epsilon}$ for $v \in V$. The norm $\|\cdot\|_\Omega$ is then defined as in section 3.4 with this new definition of $\Omega_{v,\epsilon}$.

Let $\mathcal{M}^1(\Sigma_{\mathbf{t}}, Q)$ be the space of meromorphic differentials ω on $\Sigma_{\mathbf{t}}$ such that $(\omega) \geq Q$ and the norm $\|\omega\|_\Omega$ is defined. The following theorem is the generalization of Theorem 3 to the case of meromorphic differentials.

Theorem 6 *For \mathbf{t} small enough, the operator $\omega \mapsto ((\int_{\gamma_e} \omega)_{e \in E}, P(\omega))$ is an isomorphism from $\mathcal{M}^1(\Sigma_{\mathbf{t}}, Q)$ to the subspace of vectors in $\mathcal{B}_E \times \mathcal{B}_P$ which satisfy the condition (11).*

Proof : the proof of this theorem is exactly the same as the proof of Theorem 3, except that in point 2 of the proof, the definition of ω_1 , equation (3), must be changed to take into account the poles. Given $\alpha \in \mathcal{B}_E$ and $P \in \mathcal{B}_P$, satisfying (11), we are asked to construct a meromorphic differential ω with prescribed periods α and prescribed principal parts P . If the point ∞ in S_v is not a pole, we define ω_1 in S_v as

$$\omega_1 = \frac{1}{2\pi i} \sum_{e \in E_v^+} \alpha_e \frac{dz}{z - p_e^+} - \frac{1}{2\pi i} \sum_{e \in E_v^-} \alpha_e \frac{dz}{z - p_e^-} + \sum_{i=1}^{n_v} \sum_{n=1}^{m_{v,i}} a_{v,i,n} \frac{dz}{(z - q_{v,i})^n}.$$

Equation (11) guarantees that ω_1 is holomorphic at ∞ . If the point ∞ in S_v is a pole, say $q_{v,1} = \infty$, the definition of ω_1 in S_v is the same except that the term corresponding to $i = 1$ in the last sum is replaced by

$$- \sum_{n=2}^{m_{v,i}} a_{v,i,n} z^{n-2} dz.$$

Mind the fact that the sum starts at $n = 2$ and not $n = 1$, so that this term is holomorphic at 0. Equation (11) ensures that the residue of ω_1 at ∞ is $a_{v,1,1}$, so ω_1 has the required principal part at infinity.

With this modification, point 1 of lemma 1 is replaced by

$$\|\omega_1\|_\Omega \leq \frac{C}{\epsilon} \|\alpha\|_E + \frac{C}{\epsilon^k} \|P\|_P$$

where $k = \max\{|m_{v,i}| : v \in V, 1 \leq i \leq n_v\}$. The proof of this fact is straightforward. The rest of the proof is unchanged. \square

Example 5 Choose a vertex v_0 . There exists a unique meromorphic differential on $\Sigma_{\mathbf{t}}$ which has just one double pole at ∞ in S_{v_0} , with principal part dz , and has vanishing periods on all cycles γ_e , $e \in E$ (a normalized differential of the second kind). We can use weighted ℓ^∞ norms to study its decay as in section 3.7, with the same conclusion.

Example 6 Choose two vertices v_1 and v_2 . Let γ be a path from v_1 to v_2 . Orient γ from v_1 to v_2 and orient the rest of Γ arbitrarily. Choose a point q_1 in S_{v_1} and a point q_2 in S_{v_2} . Then there exists a unique meromorphic differential which has simple poles at q_1 and q_2 , with respective residues 1 and -1 , and has period on the cycles γ_e equal to $2\pi i$ if the edge e belongs to γ and zero otherwise (a normalized differential of the third kind).

Remark 9 *All the other results in this paper have a natural generalization to the case of meromorphic differentials. For instance, the generalization of Theorem 4 would be that the restriction of ω to Ω_ϵ depends smoothly on the parameters $(\mathbf{p}, \mathbf{t}, \xi, \mathbf{q})$ in a neighborhood of $(\mathbf{p}, 0, \xi, \mathbf{q})$ in ℓ^∞ norm. It suffices, in the proofs, to replace ω_1 by the above definition. The reason we did not consider meromorphic differentials from the very beginning is that there are more parameters in this case, so although conceptually the same, the proofs would have been longer.*

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