Non-periodic Riemann examples with handles

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FILIPPO MORABITO AND MARTIN TRAIZET

Abstract. We show the existence of 1-parameter families of non-periodic, complete, embedded minimal surfaces in Euclidean space with infinitely many parallel planar ends. In particular we are able to produce finite genus examples and quasi-periodic examples of infinite genus.

1. Introduction

The goal of this paper is to construct families of complete, embedded minimal surfaces in Euclidean space, with infinitely many planar ends. The classical examples of such surfaces have been discovered in 19th century by B. Riemann and are called Riemann minimal examples. They have genus zero and are periodic. W. Meeks, J. Pérez, A. Ros [5] proved that they are the only properly embedded minimal surfaces of genus zero with infinitely many ends.

Periodic examples were constructed by the second author in [7] by adding handles, in a periodic way, to Riemann examples. Our goal in this paper is to follow the same strategy without assuming any periodicity. The first class of examples that we obtain have finite genus:

Theorem 1.1. *For each integer $g \geq 1$, there exists a 1-parameter family of complete, properly embedded minimal surfaces which have genus g and infinitely many planar ends. These surfaces have two limit ends.*

W. Meeks, J. Perez and A. Ros [4] have proven that a properly embedded minimal surface of finite genus and infinitely many ends, must have planar ends and two limit ends. (This later point means, in the finite genus case, that it is homeomorphic to a compact closed surface M punctured in a countable set with precisely two limit points.) Such surfaces have been constructed by L. Hauswirth and F. Pacard [1] using an analytic gluing procedure. Note however that the surfaces we construct are different: their examples degenerate into Costa-Hoffman-Meeks surfaces whereas our examples degenerate into catenoids.

We also obtain examples of infinite genus. In particular we can construct quasi-periodic examples.

Theorem 1.2. *There exists complete, properly embedded minimal surfaces in Euclidean space which are quasi-periodic and non-periodic. These surfaces have infinite genus, infinitely many planar ends and two limit ends.*

Recall that a minimal surface M is periodic if there exists a non-zero translation T such that $T(M) = M$. We say that a minimal surface M is quasi-periodic if there exists a diverging

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sequence of translations $(T_n)_{n\in\mathbb{N}}$ such that the sequence $(T_n(M))_{n\in\mathbb{N}}$ converges smoothly to M on compact subsets of $\mathbb{R}^3$ [3]. (This notion of quasi-periodicity is weaker than the usual notion of quasi-periodicity in cristallography.) An example of quasi-periodic minimal surface in the flat manifold $\mathbb{R}^2 \times \mathbb{S}^1$ was constructed in [2].

To describe the surfaces we construct, we introduce some terminology. Let M be a complete, properly embedded minimal surface with an infinite number of horizontal planar ends.

Definition 1.3. *We say that M is of type $(n_k)_{k\in\mathbb{Z}}$ if there exists an increasing sequence $(h_k)_{k\in\mathbb{Z}}$ such that for each $k \in \mathbb{Z}$,*

- *the intersection of M with the horizontal plane $x_3 = h_k$ is the union of n_k smooth Jordan curves,*
- *the domain $h_k < x_3 < h_{k+1}$ of M has one planar end and is homeomorphic to a planar domain.*

For example, Riemann examples are of type $(1)_{k\in\mathbb{Z}}$. If M is of type $(n_k)_{k\in\mathbb{Z}}$, then its genus is equal to $\sum_{k\in\mathbb{Z}}(n_k - 1)$, possibly infinite. The following theorem is a particular case of our main result, to be stated in the next section.

Theorem 1.4. *Let $(n_k)_{k\in\mathbb{Z}}$ be a sequence of positive integers. Assume that*

- *the sequence $(n_k)_{k\in\mathbb{Z}}$ is bounded,*
- *for all $k \in \mathbb{Z}$, either $n_k = 1$ or $n_{k+1} = 1$.*

Then there exists a family of complete properly embedded minimal surfaces with infinitely many planar ends and of type $(n_k)_{k\in\mathbb{Z}}$. Moreover, these surfaces are periodic (resp. quasi-periodic) if and only if the sequence $(n_k)_{k\in\mathbb{Z}}$ is periodic (resp. quasi-periodic).

We say that a sequence $(u_k)_{k\in\mathbb{Z}}$ is periodic if there exists a positive integer T such that $u_{k+T} = u_k$. We say it is quasi-periodic if there exists a diverging sequence of integers $(T_n)_{n\in\mathbb{N}}$ such that for all $k \in \mathbb{Z}$, $\lim_{n\to\infty} u_{k+T_n} = u_k$. (In the case of a sequence of integer numbers, this means that for all $k \in \mathbb{Z}$, there exists an integer N such that for all $n \geq N$, $u_{k+T_n} = u_k$. In other words, any finite portion of the sequence is repeated infinitely many times.) It is clear that Theorems 1.1 and 1.2 follow from this theorem by choosing appropriate sequences $(n_k)_{k\in\mathbb{Z}}$.

Example 1.5. *Take $n_k = 2$ if k is even and positive, and $n_k = 1$ otherwise. The minimal surfaces we obtain by Theorem 1.4 have a bottom limit end of genus zero and a top limit end of infinite genus. These new examples support a conjecture by W. Meeks on necessary and sufficient topological conditions for an open orientable surface of infinite topology to properly minimally embed in $\mathbb{R}^3$.*

Heuristically, our surfaces are constructed by taking an infinite stack of horizontal planes $(P_k)_{k\in\mathbb{Z}}$, ordered by their height, and gluing n_k catenoidal necks between the planes P_k and P_{k+1} for each $k \in \mathbb{Z}$. It is clear that in this way, we obtain a surface of type $(n_k)_{k\in\mathbb{Z}}$.

The construction follows the lines of the one in the periodic case in [7]. In the periodic case, we worked in the quotient by the period, so we only had to glue a finite number of catenoids and we could use the classical theory of compact Riemann surfaces. In the non-periodic case that we consider in this paper, we have to glue infinitely many catenoids at the same time, and the underlying Riemann surface is not compact. Technically, the construction relies on the theory of *opening infinitely many nodes* developped by the second author in [8].

2. Configurations and forces, main result

Let $(n_k)_{k\in\mathbb{Z}}$ be a sequence of positive integers. A configuration of type $(n_k)_{k\in\mathbb{Z}}$ is a sequence of complex numbers $(p_{k,i})_{1\leq i\leq n_k, k\in\mathbb{Z}}$. The points $p_{k,i}$, $1\leq i\leq n_k$, represent the position of the catenoidal necks that we will create between the planes P_k and P_{k+1}. These points must satisfy a balancing condition which we express in term of forces. Let $c_k=\frac{1}{n_k}$.

Definition 2.1. *The force $F_{k,i}$ exerted on $p_{k,i}$ by the other points of the configuration is defined as*

$$F_{k,i}=2\sum_{\substack{j=1\\ j\neq i}}^{n_k}\frac{c_k^2}{p_{k,i}-p_{k,j}}-\sum_{j=1}^{n_{k+1}}\frac{c_kc_{k+1}}{p_{k,i}-p_{k+1,j}}-\sum_{j=1}^{n_{k-1}}\frac{c_kc_{k-1}}{p_{k,i}-p_{k-1,j}}. \tag{1}$$

A configuration $\{p_{k,i}\}_{k\in\mathbb{Z},i\in\{1,\dots,n_k\}}$ is said to be balanced if all forces $F_{k,i}$ vanish.

For the forces to be defined, we need that for each $k\in\mathbb{Z}$, the points $p_{k,i}$, $1\leq i\leq n_k$ and $p_{k\pm1,i}$, $1\leq i\leq n_{k\pm1}$ are distinct, which we assume from now on. We will see later the existing relationship between the balancing condition and the period problem that we have to solve to construct our family of minimal surfaces.

Example 2.2. Fix some non-zero complex number a. The configuration given by $n_k=1$ and $p_{k,1}=ka$ for all $k\in\mathbb{Z}$ is balanced. This configuration yields the family of Riemann examples.

Moreover, as our construction is based on the implicit function Theorem, we need the differential of the force map to be invertible in some sense. In the case of Example 2.2, the differential of $F_{k,1}$ is given by

$$dF_{k,1}=\frac{1}{a^2}(2dp_{k,1}-dp_{k-1,1}-dp_{k+1,1}).$$

The operator $(u_k)_{k\in\mathbb{Z}}\mapsto(2u_k-u_{k-1}-u_{k+1})_{k\in\mathbb{Z}}$ is neither injective nor surjective from $\ell^\infty(\mathbb{Z})$ to $\ell^\infty(\mathbb{Z})$, where $\ell^\infty(\mathbb{Z})$ is the space of bounded sequences $(u_k)_{k\in\mathbb{Z}}$ with the sup norm. This motivates the following change of variables.

$$\begin{cases} u_{k,i}=p_{k,i}-p_{k,1}, & 1\leq i\leq n_k, \quad k\in\mathbb{Z}\\ \ell_k=p_{k,1}-p_{k-1,1}, & k\in\mathbb{Z}. \end{cases} \tag{2}$$

By definition we have $u_{k,1}=0$. We denote by $\mathbf{U}$ the sequence

$$\dots,\ell_k,u_{k,2},\dots,u_{k,n_k},\ell_{k+1},u_{k+1,2},\dots,u_{k+1,n_{k+1}},\ell_{k+2},\dots \tag{3}$$

The parameter $\mathbf{U}$ determines the configuration up to a translation, which is irrelevant since the forces are invariant by translation of the configuration.

The expression of the forces in terms of the new variables is

$$F_{k,i}=2\sum_{\substack{j=1\\ j\neq i}}^{n_k}\frac{c_k^2}{u_{k,i}-u_{k,j}}-\sum_{j=1}^{n_{k+1}}\frac{c_kc_{k+1}}{u_{k,i}-\ell_{k+1}-u_{k+1,j}}-\sum_{j=1}^{n_{k-1}}\frac{c_kc_{k-1}}{u_{k,i}+\ell_k-u_{k-1,j}}.$$

We define

$$G_k=\sum_{i=1}^{n_k}\sum_{j=1}^{n_{k-1}}\frac{c_kc_{k-1}}{p_{k,i}-p_{k-1,j}}=\sum_{i=1}^{n_k}\sum_{j=1}^{n_{k-1}}\frac{c_kc_{k-1}}{u_{k,i}+\ell_k-u_{k-1,j}}.$$

An elementary computation gives

$$\sum_{i=1}^{n_k} F_{k,i} = G_{k+1} - G_k.$$

Therefore, the configuration is balanced if and only if for all $k \in \mathbb{Z}$, we have $F_{k,i} = 0$ for $2 \leq i \leq n_k$ and $G_k = G_0$. We denote by $\tilde{\mathbf{F}}$ the sequence

$$\ldots, G_k, F_{k,2}, \ldots, F_{k,n_k}, G_{k+1}, F_{k+1,2}, \ldots, F_{k+1,n_{k+1}}, G_{k+2}, \ldots \tag{4}$$

The configuration is balanced if and only if $\tilde{\mathbf{F}} = 0$.

Definition 2.3. *We say that the configuration is non-degenerate if the differential of $\tilde{\mathbf{F}}$ with respect to $\mathbf{U}$, from $\ell^\infty(\mathbb{Z})$ to itself, exists and is an isomorphism.*

Let us return to the configuration of Example 2.2 and see that it is non-degenerate. The configuration is given by $\ell_k = a$. We have $\mathbf{U} = (\ell_k)_{k\in\mathbb{Z}}$ and $\tilde{\mathbf{F}} = (G_k)_{k\in\mathbb{Z}} = (\frac{1}{\ell_k})_{k\in\mathbb{Z}}$. The map $\mathbf{U} \mapsto \tilde{\mathbf{F}}$ is differentiable with differential equal to $\frac{-1}{a^2} id$, so the configuration is non-degenerate.

We are ready to state the main result of this paper.

Theorem 2.4. *Consider a balanced, non-degenerate configuration $\mathbf{U}$ of type $(n_k)_{k\in\mathbb{Z}}$. Further assume that*

1. *the sequence $(n_k)_{k\in\mathbb{Z}}$ is bounded,*
2. *the sequence $\mathbf{U}$ takes a finite number of values,*
3. *for all $k \in \mathbb{Z}$, $\dfrac{1}{n_k}\displaystyle\sum_{i=1}^{n_k} p_{k,i} \neq \frac{1}{n_{k-1}} \sum_{i=1}^{n_{k-1}} p_{k-1,i}$.*

Then there exists a 1-parameter family $(M_t)_{0<t<\varepsilon}$ of complete, properly embedded minimal surfaces with infinitely many planar ends, of type $(n_k)_{k\in\mathbb{Z}}$. Furthermore, each surface M_t is periodic (resp. quasi-periodic) if and only if the configuration is periodic (resp. quasi-periodic).

We say that the configuration is periodic if there exists a positive integer T such that for all $k \in \mathbb{Z}$, $n_{k+T} = n_k$, $\ell_{k+T} = \ell_k$ and $u_{k+T,i} = u_{k,i}$ for all $2 \leq i \leq n_k$. We say the configuration is quasi-periodic if there exists a diverging sequence of integers $(T_n)_{n\in\mathbb{N}}$ such that for all $k \in \mathbb{Z}$, there exists an integer N such that for $n \geq N$, $n_{k+T_n} = n_k$, $\ell_{k+T_n} = \ell_k$ and $u_{k+T_n,i} = u_{k,i}$ for $2 \leq i \leq n_k$. (In other words, any finite part of the configuration is repeated infinitely many times.)

Remark 2.5. Let us discuss the various hypotheses of the theorem.

- We formulated the definition of non-degeneracy by using the ℓ^∞ norm. It is certainly the case that for many configurations, this is not the right norm to use: they are non-degenerate for a suitable choice of the norms on both the domain and the target space of the differential of the force map. We chose the ℓ^∞ norm because this is the most natural one and there are already plenty of configurations which are non-degenerate for this norm.
- It would certainly be interesting to allow for unbounded sequences $(n_k)_{k\in\mathbb{Z}}$, but such configurations cannot be non-degenerate for the ℓ^∞ norm. Maybe a result is possible using a weighted ℓ^∞ norm, with the weight depending in some way on n_k.

- Hypothesis 2 is not required in any fundamental way, but makes the proof of the theorem much easier. It ensures the finiteness Hypothesis 4.1, see Section 4.1. For the examples of configuration that we will consider, hypothesis 2 is a consequence of hypothesis 1, see Remark 3.3.
- We do not know of any example of balanced configuration for which hypothesis 3 fails. It ensures that the Gauss map has multiplicity 2 at the ends and makes the proof slightly simpler at one point (see Section 4.3). Theorem 2.4 definitely holds without this hypothesis.

3. Examples of balanced non-degenerate configurations

In this section, we obtain balanced, non-degenerate configurations by concatenation of finite configurations.

3.1. **Concatenation of finite configurations.** Let h be a positive integer and $(n_0, \cdots, n_h)$ be a finite sequence of positive integers, such that $n_0 = n_h = 1$. A finite configuration of type $(n_0, \cdots, n_h)$ is a collection C of complex numbers $(p_{k,i})_{0\leq k\leq h, 1\leq i\leq n_k}$. We call the points $p_{0,1}$ and $p_{h,1}$ respectively the first and last point of the configuration. We call h the height of the configuration. We call width of the configuration the quantity $\max\{n_0, \cdots, n_h\}$. As in Section 2 we make the change of variables

$$\begin{cases} u_{k,i} = p_{k,i} - p_{k,1}, & 1 \leq i \leq n_k, \quad 0 \leq k \leq h \\ \ell_k = p_{k,1} - p_{k-1,1}, & 1 \leq k \leq h. \end{cases}$$

The forces $F_{k,i}$ for $0 \leq k \leq h$ and $1 \leq i \leq n_k$ are defined as in Section 2, with the convention that $n_{-1} = n_{h+1} = 0$. The quantities G_k for $1 \leq k \leq h$ are defined in the same way.

Definition 3.1. *A finite configuration C of type $(n_0, \cdots, n_h)$ is said to be*

- *balanced if $F_{k,i} = 0$ for $1 \leq k \leq h-1$ and $1 \leq i \leq n_k$. Note that we do not require that the forces $F_{0,1}$ and $F_{h,1}$ vanish. We will denote by F_C the value of $F_{0,1}$ and we will call it the residual force of the configuration.*
- *non-degenerate if the differential of the map which associates to the vector*

$$(\ell_1, u_{1,2}, \ldots, u_{1,n_1}, \ell_2, u_{2,2}, \ldots, u_{2,n_2}, \ell_3, \ldots, \ell_h) \tag{5}$$

the vector

$$(G_1, F_{1,2}, \ldots, F_{1,n_1}, G_2, F_{2,2}, \ldots, F_{2,n_2}, G_3, \ldots, G_h) \tag{6}$$

is an isomorphism.

Proposition 3.2. *If C is a finite, balanced configuration of height h, it holds*

$$F_{h,1} = -F_{0,1} = -F_C,$$

$$(p_{h,1} - p_{0,1})F_C = \sum_{k=1}^{h} \frac{1}{n_k}.$$

In particular, the residual force never vanishes.

Proof. The proposition comes from the following two formulae, which hold for any configuration of height h (not necessarily balanced).

$$\sum_{k=0}^{h}\sum_{i=1}^{n_k} F_{k,i} = 0$$

$$\sum_{k=0}^{h}\sum_{i=1}^{n_k} p_{k,i}F_{k,i} = \sum_{k=0}^{h} n_k(n_k-1)c_k^2 - \sum_{k=0}^{h-1} n_k n_{k+1} c_k c_{k+1} = 1 - \sum_{k=0}^{h}\frac{1}{n_k}.$$

We omit the proof of these formulae. □

Remark 3.3. Let us fix the height h and the type of the configuration, namely the sequence $(n_0, \cdots, n_h)$. We observe that the balancing condition can be written as a finite system of polynomial equations in the unknowns ℓ_k for $1 \leq k \leq h$ and $u_{k,i}$ for $1 \leq k \leq h-1$ and $2 \leq i \leq n_k$. Each polynomial equation defines an algebraic variety in $\mathbb{C}^n$, where n is the number of unknowns. The set of balanced configurations of type $(n_0, \cdots, n_h)$ is the intersection of these varieties. There might be components of non-zero dimension, but a non-degenerate configuration cannot belong to such a component. Basic results of algebraic geometry ensure that there is only a finite number of dimension zero components (which are points), so there is at most a finite number of balanced, non-degenerate configurations of type $(n_0, \cdots, n_h)$.

Definition 3.4. *We will say that two configurations C_1, C_2 of finite height are compatible if their residual forces are equal, that is $F_{C_1} = F_{C_2}$.*

Given a sequence of finite configurations $(C_m)_{m\in\mathbb{Z}}$, we define their concatenation C as follows. We denote by h_m the height of C_m. We write $n_k^{(m)}$, $p_{k,i}^{(m)}$, $\ell_k^{(m)}$, $u_{k,i}^{(m)}$ and $F_{k,i}^{(m)}$ for the quantities associated to the configuration C_m. Let $(\varphi_m)_{m\in\mathbb{Z}}$ be the sequence defined inductively by $\varphi_0 = 0$ and $\varphi_{m+1} = \varphi_m + h_m$ for $m \in \mathbb{Z}$. We define the sequence $(n_k)_{k\in\mathbb{Z}}$ by

$$n_{\varphi_m+k} = n_k^{(m)} \quad \text{for } m \in \mathbb{Z},\ 0 \leq k \leq h_m$$

which makes sense because $n_{h_m}^{(m)} = 1 = n_0^{(m+1)}$. We define the configuration C, of type $(n_k)_{k\in\mathbb{Z}}$, by

$$\begin{cases} \ell_{\varphi_m+k} = \ell_k^{(m)} & \text{for } m \in \mathbb{Z},\ 1 \leq k \leq h_m, \\ u_{\varphi_m+k,i} = u_{k,i}^{(m)} & \text{for } m \in \mathbb{Z},\ 1 \leq k \leq h_m - 1,\ 2 \leq i \leq n_k^{(m)}. \end{cases}$$

This amounts to translate the configurations so that for each $m \in \mathbb{Z}$, the last point of C_m coincides with the first point of C_{m+1}, and identify these two points. The following result is a generalization of Proposition 2.3 in [7].

Proposition 3.5. *Let $(C_m)_{m\in\mathbb{Z}}$ be a sequence of finite configurations. Let C be the configuration obtained by concatenation of these configurations, as explained above. Then:*

- *if all configurations C_m are balanced and compatible, then the configuration C is balanced,*
- *if moreover, all configurations C_m are non-degenerate and have height and width bounded by some number independant of m, then the configuration C is non-degenerate and the sequence $\mathbf{U}$ defined by equation (3) takes a finite number of values.*

Proof. Let us write, for $m \in \mathbb{Z}$,

$$\mathbf{U}^{(m)} = (\ell_1^{(m)}, u_{1,2}^{(m)}, \dots, u_{1,n_1}^{(m)}, \ell_2^{(m)}, u_{2,2}^{(m)}, \dots, u_{2,n_2}^{(m)}, \ell_3^{(m)}, \dots, \ell_h^{(m)}),$$

$$\tilde{\mathbf{F}}^{(m)} = (G_1^{(m)}, F_{1,2}^{(m)}, \dots, F_{1,n_1}^{(m)}, G_2^{(m)}, F_{2,2}^{(m)}, \dots, F_{2,n_2}^{(m)}, G_3^{(m)}, \dots, G_h^{(m)})$$

for the parameters and forces corresponding to the configuration C_m. Then we have, for the configuration C, $\mathbf{U} = (\mathbf{U}^{(m)})_{m\in\mathbb{Z}}$ by definition of the concatenation and $\tilde{\mathbf{F}} = (\tilde{\mathbf{F}}^{(m)})_{m\in\mathbb{Z}}$ by inspection. If all configurations C_m are balanced and compatible, then all G_k are equal and the configuration C is balanced.

Let us assume that the configurations C_m are non-degenerate and have bounded height and width. Then there is only a finite number of possibilities for the types of the configurations. By Remark 3.3, there is only a finite number of configurations C_m for $m \in \mathbb{Z}$ (some of them are repeated infinitely many times in the sequence $(C_m)_{m\in\mathbb{Z}}$). Hence the sequence $\mathbf{U}$ takes only a finite number of values.

To prove that $\tilde{\mathbf{F}}$ is differentiable with respect to $\mathbf{U}$, we observe that for each $m \in \mathbb{Z}$, $\tilde{\mathbf{F}}^{(m)}$ only depends on $\mathbf{U}^{(m)}$. Since there is only a finite number of distinct configurations, the differential of $\tilde{\mathbf{F}}^{(m)}$ with respect to $\mathbf{U}^{(m)}$ has norm bounded by some number independent of m, and the same is true for the second order differential. This implies that $\tilde{\mathbf{F}}$ is differentiable with respect to $\mathbf{U}$ from ℓ^∞ to itself, with differential given by

$$d\tilde{\mathbf{F}}(\mathbf{U})(\mathbf{X}) = \left(d\tilde{\mathbf{F}}^{(m)}(\mathbf{U}^{(m)})(\mathbf{X}^{(m)})\right)_{m\in\mathbb{Z}}.$$

(In other words, the differential has a block diagonal structure.) Again, since there is only a finite number of distinct configurations, the norms of the inverses of $d\tilde{\mathbf{F}}^{(m)}(\mathbf{U}^{(m)})$ are bounded by some number independent of m, so the operator

$$\mathbf{X} \mapsto \left(d\tilde{\mathbf{F}}^{(m)}(\mathbf{U}^{(m)})^{-1}(\mathbf{X}^{(m)})\right)_{m\in\mathbb{Z}}$$

is bounded from ℓ^∞ to itself, so $d\tilde{\mathbf{F}}(\mathbf{U})$ is invertible and the configuration C is non-degenerate. □

3.2. **Examples of finite configurations.**

Example 3.6. A trivial example: fix some non-zero complex number a and some positive integer number h. The configuration of height h and type $(1, 1, \cdots, 1)$ defined by $p_{k,1} = ka$ for $0 \le k \le h$ is balanced, non-degenerate and has residual force equal to $\frac{1}{a}$.

Example 3.7. A nice family of configurations of height 2 and type $(1, n, 1)$, where $n \in \mathbb{N}^*$, which comes from [7]. It is given by

$$p_{0,1} = 0, \quad p_{2,1} = 2\mathrm{i},$$

$$p_{1,j} = \mathrm{i} + \cot\frac{j\pi}{n+1}, \qquad \text{for } 1 \le j \le n.$$

Proposition 3.8. *This configuration is balanced, non-degenerate and has residual force* $\dfrac{n+1}{2n\mathrm{i}}$.

Proof. This configuration is proven to be balanced in [7], Proposition 2.1. Let us prove that it is non-degenerate. It is enough to prove that the differential is injective. Let X be an element in its kernel. Consider a path $\mathbf{U}(t) = (\ell_1(t), u_{1,2}(t), \cdots, u_{1,n}(t), \ell_2(t))$ in the parameter space, such that $\mathbf{U}(0)$ is the given configuration, and $\mathbf{U}'(0) = X$. Then $G_1'(0)$, $G_2'(0)$ and $F_{1,i}'(0)$, $2 \leq i \leq n$ all vanish because X is in the kernel. The points of the configuration at time t are given by $p_{0,1}(t) = 0$, $p_{1,i}(t) = \ell_1(t) + u_{1,i}(t)$ for $1 \leq i \leq n$ and $p_{2,1}(t) = \ell_1(t) + \ell_2(t)$ (with $u_{1,1}(t) = 0$). We extend this configuration into a periodic configuration, denoted $\widetilde{p}_{k,i}(t)$ of period $T(t) = \ell_1(t) + \ell_2(t)$ by writing $\widetilde{p}_{2k,1}(t) = kT(t)$ and $\widetilde{p}_{2k+1,i}(t) = p_{1,i}(t) + kT(t)$ for $k \in \mathbb{Z}$. If we write $\widetilde{F}_{k,i}(t)$ for the forces of this configuration, we have

$$\widetilde{F}_{2k,1}(t) = F_{0,1}(t) + F_{2,1}(t) = G_1(t) - G_2(t),$$
$$\widetilde{F}_{2k+1,i}(t) = F_{2k+1,i}(t).$$

Hence, the derivatives of these forces at time 0 all vanish. Next, we scale this configuration by $1/T(t)$ so that its period is constant by writing $\widehat{p}_{k,i}(t) = \frac{\widetilde{p}_{k,i}(t)}{T(t)}$. If we write $\widehat{F}_{k,i}(t)$ for the forces of this configuration, we have $\widehat{F}_{k,i}'(0) = 0$. Since this periodic configuration is non-degenerate by Proposition 2.1 in [7], in the sense given just after Theorem 1.4 in the same paper, we have $\widehat{p}_{k,i}'(0) = 0$ for all (k, i). From this we get, for $0 \leq k \leq 2$

$$p_{k,i}'(0) = T'(0)\widehat{p}_{k,i}(0) = \lambda p_{k,i}(0) \quad \text{ with } \lambda = \tfrac{T'(0)}{T(0)}.$$

Then we write

$$G_2(t) = \frac{1}{n}\sum_{j=1}^{n} \frac{1}{p_{2,1}(t) - p_{1,j}(t)},$$

$$G_2'(0) = \frac{1}{n}\sum_{j=1}^{n} \frac{-1}{(p_{2,1} - p_{1,j})^2}(\lambda p_{2,1} - \lambda p_{1,j}) = -\lambda G_2(0).$$

Since $G_2'(0) = 0$ and $G_2(0) = F_C \neq 0$, this gives $\lambda = 0$, so $p_{k,i}'(0) = 0$ and $X = 0$. This proves that the configuration is non-degenerate. We compute the residual force using Proposition 3.2. □

Remark 3.9. We can scale these configurations by $\frac{2n}{n+1}$ so that they are compatible. Then by Proposition 3.5, we can concatenate them to obtain balanced configuration whose type is any bounded sequence $(n_k)_{k\in\mathbb{Z}}$ such that $n_k = 1$ for even k. This proves Theorem 1.4.

Example 3.10. *An example of height 3 and type* $(1, 2, 2, 1)$ *given by*

$$p_{0,1} = 0, \quad p_{1,1} = \frac{-\sqrt{2}}{2} + \mathrm{i}, \quad p_{1,2} = \frac{\sqrt{2}}{2} + \mathrm{i},$$
$$p_{2,1} = \frac{-\sqrt{2}}{2} + 2\mathrm{i}, \quad p_{2,2} = \frac{\sqrt{2}}{2} + 2\mathrm{i}, \quad p_{3,1} = 3\mathrm{i}.$$

Proposition 3.11. *This configuration is balanced, non-degenerate and has residual force* $\frac{2}{3\mathrm{i}}$.

The proof is purely computational, we omit the details. (The determinant of the matrix associated to the differential, computed with Mathematica, equals 4/243).

4. Proof of the main theorem

In this section we prove Theorem 2.4.

4.1. **Notations and parameters.** There are six parameters in the construction, denoted by t, a, b, α, β and γ. The parameter t is a positive real number. All other five parameters are sequences of complex numbers of the form $u = (u_{k,i})_{k\in\mathbb{Z},1\leq i\leq n_k}$. We use the notation $u_k = (u_{k,i})_{1\leq i\leq n_k} \in \mathbb{C}^{n_k}$.

The ℓ^∞ norm of the sequence u is defined as usual as $||u||_\infty = \sup\{|u_{k,i}| \,:\, k \in \mathbb{Z},\, 1 \leq i \leq n_k\}$. Each parameter varies in a neighborhood for the ℓ^∞ norm of a central value. The central value is denoted by an upperscript 0, so the central value of the parameters are denoted by a^0, b^0, α^0, β^0 and γ^0. Most of our statements only hold in a small neighborhood of the central values.

The central value of the parameters will be given later. An important point is that the following hypothesis holds. We say that a sequence $u = (u_{k,i})_{k\in\mathbb{Z},1\leq i\leq n_k}$ is finitely valued if the set $\{u_{k,i} \,:\, k \in \mathbb{Z},\, 1 \leq i \leq n_k\}$ is finite.

Hypothesis 4.1 (Finiteness hypothesis)**.** *The central value of each parameter is finitely valued.*

This will be useful to make various statements uniform with respect to $k \in \mathbb{Z}$. By a uniform constant, we mean some number which only depends on the central value of the parameters. We use the notation $D(a, r)$ for the disk of center a and radius r in $\mathbb{C}$.

4.2. **Opening nodes.** Consider infinitely many copies of the Riemann sphere, labelled $(\overline{\mathbb{C}}_k)_{k\in\mathbb{Z}}$. We denote by ∞_k the point ∞ in $\overline{\mathbb{C}}_k$, and $\mathbb{C}_k = \overline{\mathbb{C}}_k \setminus \{\infty_k\}$. For each $k \in \mathbb{Z}$ and each $1 \leq i \leq n_k$, select a point $a_{k,i} \in \mathbb{C}_k$ and a point $b_{k,i} \in \mathbb{C}_{k+1}$. Identify these two points to create a node. This defines a Riemann surface with nodes which we call Σ_0. The parameters involved in this construction are the sequences $a = (a_{k,i})_{k\in\mathbb{Z},1\leq i\leq n_k}$ and $b = (b_{k,i})_{k\in\mathbb{Z},1\leq i\leq n_k}$. The central value of these parameters is given, in term of the given configuration, by

$$a^0_{k,i} = (-1)^k \mathrm{conj}^k(\ell^0_k + u^0_{k,i})$$
$$b^0_{k,i} = (-1)^{k+1} \mathrm{conj}^{k+1}(u^0_{k,i})$$

where $\mathrm{conj}(z) = \overline{z}$ denotes conjugation, so $\mathrm{conj}^k(z)$ is equal to z if k is even and $\overline{z}$ if k is odd. Observe that a^0 and b^0 are finitely valued by hypothesis 2 of Theorem 2.4.

For any $k \in \mathbb{Z}$, the points $a^0_{k,i}$ for $1 \leq i \leq n_k$ and $b^0_{k-1,i}$ for $1 \leq i \leq n_{k-1}$ are distinct in $\mathbb{C}_k$. That follows from the identity

$$l^0_k + u^0_{k,i} - u^0_{k-1,j} = p^0_{k,i} - p^0_{k-1,j}$$

and the fact that second member does not vanish by construction.

Let $\varepsilon_k > 0$ be the smallest value of the distance between these points in $\mathbb{C}_k$. Because of the finiteness hypothesis, $\{\varepsilon_k \,:\, k \in \mathbb{Z}\}$ is finite so we can take $\varepsilon = \min\{\varepsilon_k \,:\, k \in \mathbb{Z}\} > 0$. If $||a - a^0||_\infty < \frac{\varepsilon}{4}$ and $||b - b^0||_\infty < \frac{\varepsilon}{4}$, then for any $k \in \mathbb{Z}$, the points $a_{k,i}$ for $1 \leq i \leq n_k$ and $b_{k-1,i}$ for $1 \leq i \leq n_{k-1}$ are at distance greater than $\frac{\varepsilon}{2}$ from each other, so they remain distinct. We will be using this kind of argument very often. We will not enter in details anymore and simply refer to the finiteness hypothesis.

For each $k \in \mathbb{Z}$ we consider a function g_k defined on $\overline{\mathbb{C}}_k$ by

$$g_k(z) = \sum_{i=1}^{n_{k-1}} \frac{\beta_{k-1,i}}{z - b_{k-1,i}} - \sum_{i=1}^{n_k} \frac{\alpha_{k,i}}{z - a_{k,i}}.$$

The new parameters are the sequences

$$\alpha = (\alpha_{k,i})_{k\in\mathbb{Z},1\leq i\leq n_k} \text{ and } \beta = (\beta_{k,i})_{k\in\mathbb{Z},1\leq i\leq n_k}.$$

We assume that these parameters satisfy the equation

$$\forall k \in \mathbb{Z}, \quad \sum_{i=1}^{n_k} \alpha_{k,i} = \sum_{i=1}^{n_k} \beta_{k,i} = 1. \tag{7}$$

We will see the role of this equation in Section 4.3. The central values of these parameters are given by

$$\alpha^0_{k,i} = \beta^0_{k,i} = \frac{1}{n_k}.$$

Recall that the set $\{n_k \,:\, k \in \mathbb{Z}\}$ is finite, so the sequences α^0 and β^0 are finitely valued as required.

If $||\alpha-\alpha^0||_\infty$ and $||\beta-\beta^0||_\infty$ are small enough, we have $\alpha_{k,i} \neq 0$ and $\beta_{k,i} \neq 0$ for all $k \in \mathbb{Z}$ and $1 \leq i \leq n_k$. Then $g_k(z) \sim \frac{-\alpha_{k,i}}{z-a_{k,i}}$ in a neighborhood of $a_{k,i}$, so $\frac{1}{g_k}$ is a local complex coordinate in a neighborhood of $a_{k,i}$. In the same way, $g_{k+1} \sim \frac{\beta_{k,i}}{z-b_{k,i}}$ in a neighborhood of $b_{k,i}$, so $\frac{1}{g_{k+1}}$ is a local complex coordinate in a neighborhood of $b_{k,i}$. The finiteness hypothesis allows us to find a positive number ρ so that if a, b, α and β are close enough to a^0, b^0, α^0 and β^0 in ℓ^∞ norm, for any $k \in \mathbb{Z}$ and $1 \leq i \leq n_k$, $\frac{1}{g_k}$ is a diffeomorphism from a neighborhood $V_{k,i}$ of $a_{k,i}$ in $\mathbb{C}_k$ to the disk $D(0,\rho)$ and $\frac{1}{g_{k+1}}$ is a diffeomorphism from a neighborhood $W_{k,i}$ of $b_{k,i}$ in $\mathbb{C}_{k+1}$ to the disk $D(0,\rho)$. We define

$$v_{k,i} := \frac{1}{g_k} : V_{k,i} \xrightarrow{\sim} D(0,\rho) \qquad v_{k,i}(a_{k,i}) = 0,$$

$$w_{k,i} := \frac{1}{g_{k+1}} : W_{k,i} \xrightarrow{\sim} D(0,\rho) \qquad w_{k,i}(b_{k,i}) = 0.$$

By taking ρ small enough, and still using the finiteness hypothesis, we can assume that the ratio $\left|\frac{v_{k,i}}{z-a_{k,i}}\right|$ in $V_{k,i}$ and $\left|\frac{w_{k,i}}{z-b_{k,i}}\right|$ in $W_{k,i}$ are bounded from above and below by some uniform positive numbers (by which we mean that they are independent of k, i and the value of the parameters). Hence these coordinates are admissible in the sense of Definition 2 of [8].

We use these coordinates to open the nodes. Consider a real parameter $t \in (0,\rho)$. For each $k \in \mathbb{Z}$ and $1 \leq i \leq n_k$, remove the disks $|v_{k,i}| \leq \frac{t^2}{\rho}$ from $V_{k,i}$ and $|w_{k,i}| \leq \frac{t^2}{\rho}$ from $W_{k,i}$. Identify each point $z \in V_{k,i}$ with the point $z' \in W_{k,i}$ such that

$$v_{k,i}(z)w_{k,i}(z') = t^2.$$

This creates a neck connecting $\overline{\mathbb{C}}_k$ and $\overline{\mathbb{C}}_{k+1}$. We call Σ_t the resulting Riemann surface.

4.3. **The Gauss map.** We define a meromorphic function g on Σ_t, to be the Gauss map, by

$$g(z) = (tg_k(z))^{(-1)^k} = \begin{cases} tg_k(z) \text{ if } z \in \overline{\mathbb{C}}_k, \, k \text{ even} \\ \frac{1}{tg_k(z)} \text{ if } z \in \overline{\mathbb{C}}_k, \, k \text{ odd} \end{cases}$$

This function is well defined because if, say, k is even and $z \in V_{k,i} \subset \mathbb{C}_k$ is identified with $z' \in W_{k,i} \subset \mathbb{C}_{k+1}$, then

$$g(z) = tg_k(z) = \frac{t}{v_{k,i}(z)} = \frac{w_{k,i}(z')}{t} = \frac{1}{tg_{k+1}(z')} = g(z').$$

In a neighborhood of ∞ we have, thanks to the normalization (7)

$$g_k(z) \simeq \frac{1}{z^2}\left(\sum_{i=1}^{n_{k-1}} \beta_{k-1,i} b_{k-1,i} - \sum_{i=1}^{n_k} \alpha_{k,i} a_{k,i}\right)$$

so the Gauss map has at least a double zero or pole at ∞, as required for a planar end. At the central value of the parameters we have

$$\sum_{i=1}^{n_{k-1}} \beta^0_{k-1,i} b^0_{k-1,i} - \sum_{i=1}^{n_k} \alpha^0_{k,i} a^0_{k,i} = (-1)^k \mathrm{conj}^k \left(\frac{1}{n_{k-1}} \sum_{i=1}^{n_{k-1}} p^0_{k-1,i} - \frac{1}{n_k} \sum_{i=1}^{n_k} p^0_{k,i}\right).$$

This is non-zero by hypothesis 3 of Theorem 2.4, so g_k has a zero of multiplicity precisely 2 at ∞_k. By the finiteness hypothesis, this remains true when the parameters are close to their central value in ℓ^∞ norm. Hence the Gauss map has a double zero at ∞_k if k is even and a double pole if k is odd.

4.4. **The height differential.** Fix some small number $\epsilon > 0$. For each $k \in \mathbb{Z}$, let $\Omega_{k,\epsilon}$ be $\mathbb{C}_k$ minus the disks $D(a^0_{k,i}, \epsilon)$ for $1 \le i \le n_k$ and $D(b^0_{k-1,i}, \epsilon)$ for $1 \le i \le n_{k-1}$. Let Ω_ϵ be the disjoint union of the domains $\Omega_{k,\epsilon}$ for $k \in \mathbb{Z}$. We assume that $||a - a^0||_\infty$, $||b - b^0||_\infty$ and t are small enough so that the disks $|v_{k,i}| \le \frac{t^2}{\rho}$ and $|w_{k,i}| \le \frac{t^2}{\rho}$ (which were removed when defining Σ_t) are included respectively in the disks $D(a^0_{k,i}, \epsilon)$ and $D(b^0_{k,i}, \epsilon)$. This allow us to see the fixed domain Ω_ϵ as a domain in Σ_t. Let $\Omega^1(\Sigma_t)$ be the space of holomorphic 1-forms ω on Σ_t such that the norm

$$||\omega||_{L^\infty(\Omega_\epsilon)} = \sup_{k\in\mathbb{Z}} \sup_{z\in\Omega_{k,\epsilon}} \left|\frac{\omega}{dz}\right|$$

is finite. This is well known to be a Banach space.

Next we define natural cycles on Σ_t. For any $k \in \mathbb{Z}$ and $1 \le i \le n_k$, let $A_{k,i}$ be the homology class in Σ_t of the circle $C(b_{k,i}, \epsilon)$ in $\mathbb{C}_{k+1}$. This circle is homologous in Σ_t to the circle $C(a_{k,i}, \epsilon)$ with the opposite orientation. By Theorem 2 in [8], for t small enough, the operator $\omega \mapsto (\int_{A_{k,i}} \omega)_{k\in\mathbb{Z}, 1\le i\le n_k}$ is an isomorphism of Banach spaces from $\Omega^1(\Sigma_t)$ to the set of sequences $\gamma = (\gamma_{k,i})_{k\in\mathbb{Z}, 1\le i\le n_k}$ in ℓ^∞ which satisfy the compatibility relation

$$\forall k \in \mathbb{Z}, \quad \sum_{i=1}^{n_k} \gamma_{k,i} = \sum_{i=1}^{n_{k-1}} \gamma_{k-1,i}. \tag{8}$$

(Equation (8) is what equation (2) of [8] becomes in our case.) So we can define a holomorphic differential ω on Σ_t by prescribing its $A_{k,i}$-periods as

$$\int_{A_{k,i}} \omega = 2\pi \mathrm{i} \gamma_{k,i}, \qquad k \in \mathbb{Z},\ 1 \le i \le n_k.$$

The new parameter is the sequence $\gamma = (\gamma_{k,i})_{k\in\mathbb{Z},1\leq i\leq n_k}$. The central value of this parameter is given by $\gamma^0_{k,i} = \frac{1}{n_k}$. We require that

$$\forall k \in \mathbb{Z}, \quad \sum_{i=1}^{n_k} \gamma_{k,i} = 1$$

so that the compatibility relation (8) is satisfied.

Proposition 4.2. *The differential ω depends smoothly on all parameters involved in this construction, namely a, b, α, β, γ and t, in a neighborhood of their respective central value a^0, b^0, α^0, β^0, γ^0 and 0. Moreover, when $t = 0$, we have $\omega = \omega_k$ in $\mathbb{C}_k$, where*

$$\omega_k = \sum_{i=1}^{n_{k-1}} \frac{\gamma_{k-1,i}}{z - b_{k-1,i}} dz - \sum_{i=1}^{n_k} \frac{\gamma_{k,i}}{z - a_{k,i}} dz.$$

Proof. The smoothness statement is Theorem 4 in [8]. Smoothness is for the norm defined above: in particular this means that the restriction of ω to the (fixed) domain Ω_ϵ depends smoothly on all parameters.

When $t = 0$, ω is a regular differential on Σ_0 (see Definition 1 in [8]) so has simple poles at all points $a_{k,i}$, $b_{k,i}$. The residues are determined by the prescribed periods, and this gives the claimed formula. □

4.5. **The equations we have to solve.** We define the Weierstrass data on Σ_t by the standard formula

$$(\phi_1, \phi_2, \phi_3) = \left(\frac{1}{2}(g^{-1} - g)\omega, \frac{\mathrm{i}}{2}(g^{-1} + g)\omega, \omega\right).$$

The minimal surface is defined by the Weierstrass representation formula :

$$\psi(z) = \mathrm{Re} \int_{z_0}^{z} (\phi_1, \phi_2, \phi_3) : \Sigma_t \to \mathbb{R}^3 \cup \{\infty\}.$$

The points ∞_k for $k \in \mathbb{Z}$ have to be the planar ends of our minimal surface.

For ψ to be a regular immersion, we need that at each zero or pole of the Gauss map g, which does not corresponds to an end, ω has a zero of the same order. At the end ∞_k, the Gauss map has a double zero or pole. To have an embedded planar end, we need that ω is holomorphic (which is already the case) and does not vanish. We deal with these conditions in Section 4.6.

Then we need $\psi(z)$ to be independent of the integration path from z_0 to z, this is the period problem. We fix some small number $\epsilon' > 0$ and we define the cycle $B_{k,i}$ for $k \in \mathbb{Z}$ and $2 \leq i \leq k$ as the composition of the following four paths :

(1) a fixed path from $a^0_{k,i} + \epsilon'$ to $a^0_{k,1} + \epsilon'$,
(2) a path from $a^0_{k,1} + \epsilon'$ to $b^0_{k,1} + \epsilon'$, going through the neck,
(3) a fixed path from $b^0_{k,1} + \epsilon'$ to $b^0_{k,i} + \epsilon'$,
(4) a path from $b^0_{k,i} + \epsilon'$ to $a^0_{k,i} + \epsilon'$, going through the neck.

Note that unlike $A_{k,i}$, the cycle $B_{k,i}$ is not defined when $t = 0$. We need to solve the following period problem :

$$\mathrm{Re} \int_{A_{k,i}} \phi_\nu = 0 \qquad k \in \mathbb{Z},\ 1 \leq i \leq n_k,\ 1 \leq \nu \leq 3,$$

$$\mathrm{Re}\int_{B_{k,i}}\phi_\nu = 0 \qquad k\in\mathbb{Z},\ 2\le i\le n_k,\ 1\le\nu\le 3.$$

Indeed, the first condition ensures that $\mathrm{Re}\int\phi_\nu$ is well defined in each $\mathbb{C}_k$, and in particular that the residue at ∞ vanishes as required for a planar end. The second condition ensures that $\mathrm{Re}\int\phi_\nu$ does not depend on the choice of the path from $\mathbb{C}_k$ to $\mathbb{C}_{k+1}$.

Our strategy consists in proving that these equations, suitably renormalized, extend smoothly at $t=0$, and in solving them using the implicit function Theorem at $t=0$ to determine all parameters as functions of t.

4.6. **Zeros of ω.** Let us first locate the zeros and poles of g. If k is even, then the zeros of g in $\mathbb{C}_k$ are the zeros of g_k, and g has no poles in $\mathbb{C}_k$ (because the poles of g_k were removed when opening nodes). If k is odd, the poles of g in $\mathbb{C}_k$ are the zeros of g_k, and g has no zeros in $\mathbb{C}_k$. What we need is that for each $k\in\mathbb{Z}$, ω has a zero at each zero of g_k in $\mathbb{C}_k$, with the same multiplicity, and has no further zeros.

Proposition 4.3. *For (t,a,b,γ) in a neighborhood of $(0,a^0,b^0,\gamma^0)$, there exist values of the parameters α and β, depending smoothly on (t,a,b,γ), such that for all $k\in\mathbb{Z}$, ω has a zero at each finite zero of g_k in $\mathbb{C}_k$, with the same multiplicity, and has no further zero. Moreover, when $t=0$, we have $\alpha(0,a,b,\gamma)=\beta(0,a,b,\gamma)=\gamma$.*

Proof. Since g_k has a double zero at ∞_k, it has $n_k+n_{k-1}-2$ finite zeros in $\mathbb{C}_k$, counting multiplicities. Let us count the number of zeros of ω in $\mathbb{C}_k$. By the finiteness hypothesis, we may choose $\epsilon>0$ small enough and R large enough (both independent of k and the parameters) so that $U_k=\Omega_{k,\epsilon}\cap D(0,R)$ contains all zeros of g_k. Let us write $\omega=f_k(z)dz$ in $\mathbb{C}_k$. The number of zeros of ω in U_k, counting multiplicities, is given by

$$N_k=\frac{1}{2\pi\mathrm{i}}\int_{\partial U_k}\frac{df_k}{f_k}.$$

At the central value of the parameters, we have $f_k=g_k$, so f_k does not vanish on ∂U_k. Thanks to the finiteness hypothesis, we may find a number $c>0$ independent of k such that for all k, $|f_k|\ge c$ on ∂U_k (still, at the central value of the parameters). By smooth dependence of ω on parameters, we have $|f_k|\ge\frac{c}{2}$ on ∂U_k when the parameters stay close, in ℓ^∞ norm, to their central value. Hence N_k is a smooth, integer valued function of the parameters, so it is constant. This proves that for each $k\in\mathbb{Z}$, ω has $n_k+n_{k-1}-2$ zeros in U_k.

Let us now see that ω has no further zeros. It is proven in [8], Corollary 1, that there exists a uniform constant C (independent of ϵ) such that the following is true for t small enough : for any $k\in\mathbb{Z}$ and any $1\le i\le n_k$, if $|\gamma_{k,i}|\ge C\epsilon||\gamma||_\infty$, then ω has no zero in the annulus in Σ_t bounded by the circles $C(a_{k,i},\epsilon)$ in $\mathbb{C}_k$ and $C(b_{k,i},\epsilon)$ in $\mathbb{C}_{k+1}$ (this annulus is what we call a neck). Recall that the central value of the parameter $\gamma_{k,i}$ is $\gamma^0_{k,i}=\frac{1}{n_k}$. Since the sequence $(n_k)_{k\in\mathbb{Z}}$ is bounded, the ratio $\frac{|\gamma_{k,i}|}{||\gamma||_\infty}$ is bounded from below by a uniform positive number when the parameter γ is close to γ^0. Hence, provided we choose $\epsilon>0$ small enough, ω has no zeros on the necks.

It remains to consider the zeros of ω outside the disk $D(0,R)$ in $\mathbb{C}_k$. We introduce the coordinate $w=1/z$ on this domain and write $\omega=h_k(w)dw$, $w\in D(0,\frac{1}{R})$. At the central value of the parameters, the function h_k has no zero, so $|h_k|$ is bounded from below by a constant

independent of k by the finiteness hypothesis. Hence when the parameters are close to their central value, we have that for any k, ω has no zero outside the disk $D(0,R)$ in $\mathbb{C}_k$.

Next we need to adjust the parameters α and β so that ω vanishes at the zeros of g_k. One problem is that we do not know a priori the multiplicities of the zeros of g_k. The following lemma is useful.

Lemma 4.4. *Let P be a polynomial of degree d and Ω be a domain in $\mathbb{C}$ containing all the zeros of P. Given a holomorphic function f in Ω, let*

$$F_i = \int_{\partial\Omega} \frac{z^i f(z)}{P(z)} dz.$$

If $F_i = 0$ for all $0 \leq i \leq d-1$ then f/P is holomorphic in Ω.

Proof of Lemma 4.4. By the Weierstrass Preparation Theorem, we may write $f = Ph + r$, where h is holomorphic in Ω and r is a polynomial of degree less than d. By Cauchy's Theorem,

$$F_i = \int_{\partial\Omega} \frac{z^i r(z)}{P(z)} dz = -2\pi \mathrm{i}\, \mathrm{Res}_\infty \left(\frac{z^i r(z)}{P(z)} dz \right).$$

To establish last identity, we have used the residue Theorem on the complementary of Ω in $\mathbb{C}$, and the fact that Ω contains all the zeros of P so the only pole is at infinity. Assume that $r \neq 0$ and let $k = \deg(r)$. Take $i = d-k-1$. By a straightforward computation, we get

$$\mathrm{Res}_\infty \frac{z^i r(z) dz}{P(z)} = -\frac{r_k}{p_d}$$

where r_k and p_d are respectively the leading coefficients of the polynomials r and p. Hence $F_i = 0$ implies that $r_k = 0$, contradicting the fact that r has degree k. Hence $r = 0$ so $f/P = h$ is holomorphic in Ω. □

Returning to the proof of Proposition 4.3, let

$$Q_k = \prod_{i=1}^{n_k} (z - a_{k,i}) \prod_{i=1}^{n_{k-1}} (z - b_{k-1,i})$$

and write $g_k = \frac{P_k}{Q_k}$, so P_k is a polynomial of degree $n_k + n_{k-1} - 2$. Define, for $k \in \mathbb{Z}$ and $0 \leq i \leq n_k + n_{k-1} - 3$

$$\mathcal{Z}_{k,i} = \int_{\partial U_k} \frac{z^i \omega}{P_k(z)}.$$

Let $\mathcal{Z} = \mathcal{Z}(t, a, b, \alpha, \beta, \gamma) = (\mathcal{Z}_{k,i})_{k \in \mathbb{Z}, 0 \leq i \leq n_k + n_{k-1} - 3}$. This is a smooth function of all parameters (with the ℓ^∞ norm on the target space). If $\mathcal{Z} = 0$, then by the lemma, for all $k \in \mathbb{Z}$, ω/P_k is holomorphic in U_k, so at each zero of g_k, ω has a zero with at least same order. Since we counted the number of zeros to be the same, the multiplicities are equal as required. The proposition then follows from the following lemma and the implicit function Theorem. (The last statement of the proposition follows from the uniqueness part of the implicit function Theorem.)

Lemma 4.5. *When $t = 0$ and $\alpha = \beta = \gamma$, we have $\mathcal{Z}(0, a, b, \gamma, \gamma, \gamma) = 0$. Moreover, the partial differential of $\mathcal{Z}$ with respect to (α, β) at $(0, a^0, b^0, \alpha^0, \beta^0, \gamma^0)$ is an isomorphism of Banach spaces.*

Proof of Lemma 4.5. If $t = 0$ and $\alpha = \beta = \gamma$, then by Proposition 4.2, $\omega = g_k dz$ in $\mathbb{C}_k$ for all $k \in \mathbb{Z}$, so $\mathcal{Z} = 0$. We write D for the partial differential with respect to (α, β) at the central value of the parameters. Observe that when $t = 0$, Σ_0 does not depend on (α, β), so ω does not depend on (α, β) and $D\omega = 0$. Hence

$$D\left(\frac{z^i\omega}{P_k}\right) = -\frac{z^i\omega}{P_k^2}DP_k = -\frac{z^i dz}{P_k Q_k}DP_k = -\frac{z^i dz}{P_k}Dg_k$$

$$D\mathcal{Z}_{k,i}(\alpha,\beta) = -\int_{\partial U_k}\frac{z^i}{P_k}f_k dz \quad \text{with } f_k = \sum_{j=1}^{n_{k-1}}\frac{\beta_{k-1,j}}{z - b^0_{k-1,j}} - \sum_{j=1}^{n_k}\frac{\alpha_{k,j}}{z - a^0_{k,j}}.$$

From this we see that $D\mathcal{Z}_k(\alpha,\beta)$ only depends on α_k and β_{k-1} (so $D\mathcal{Z}$ has "block diagonal" form). Let us prove that for each k, the operator $(\alpha_k, \beta_{k-1}) \mapsto D\mathcal{Z}_k(\alpha_k, \beta_{k-1})$ is an isomorphism. The domain and target spaces both have finite complex dimension $n_k + n_{k-1} - 2$, because by normalization of the parameters, we have $\sum_{i=1}^{n_k}\alpha_{k,i} = \sum_{i=1}^{n_{k-1}}\beta_{k-1,i} = 0$ (instead of 1 since we are in the tangent space to the parameter space when we compute the differential). Let (α_k, β_{k-1}) be in the kernel. Then by Lemma 4.4, the function $\frac{f_k}{P_k}$ is holomorphic in U_k. Hence the polynomials $f_k Q_k$ and P_k, which have the same degree, have the same zeros, hence are proportional. So there exists $\lambda \in \mathbb{C}$ such that $\alpha_k = \lambda\alpha^0_k$ and $\beta_{k-1} = \lambda\beta^0_{k-1}$. Because of the normalizations this gives $\alpha_k = \beta_{k-1} = 0$. Hence for each k, the operator $(\alpha_k, \beta_{k-1}) \mapsto D\mathcal{Z}_k(\alpha_k, \beta_{k-1})$ is an isomorphism. By the finiteness hypothesis, the inverse of these operators is bounded by a constant independent of k. It readily follows that $D\mathcal{Z}$ is an isomorphism of Banach spaces from ℓ^∞ to ℓ^∞. □

4.7. **The period problem for ω.**

Proposition 4.6. *Assume that α and β are given by Proposition 4.3. For (t, a, b) in a neighborhood of $(0, a^0, b^0)$, there exist values of the parameter γ, depending continuously on (t, a, b), such that the following period problem is solved :*

$$\operatorname{Re}\int_{A_{k,i}}\omega = 0, \qquad k \in \mathbb{Z},\ 1 \le i \le n_k,$$

$$\operatorname{Re}\int_{B_{k,i}}\omega = 0, \qquad k \in \mathbb{Z},\ 2 \le i \le n_k.$$

Moreover, when $t = 0$, we have $\gamma_{k,i}(0, a, b) = \frac{1}{n_k}$.

Proof. The period problem for the cycles $A_{k,i}$ is equivalent to $\gamma_{k,i} \in \mathbb{R}$, which we assume from now on. Regarding the cycles $B_{k,i}$, we need the following

Lemma 4.7. *The function*

$$\left(\operatorname{Re}\int_{B_{k,i}}\omega \ - 2(\gamma_{k,i} - \gamma_{k,1})\log t\right)_{k\in\mathbb{Z}, 2\le i\le n_k}$$

extends smoothly at $t = 0$ to a smooth function of all parameters (with the ℓ^∞ norm on the target space).

The proof of this lemma is technical and given in appendix A. We make the change of variable $t = e^{-1/\tau^2}$, where $\tau > 0$ is in a neighborhood of 0. Define

$$\mathcal{V}_{k,i} = \tau^2 \mathrm{Re} \int_{B_{k,i}} \omega$$

and $\mathcal{V} = (\mathcal{V}_{k,i})_{k\in\mathbb{Z}, 2\leq i\leq n_k}$. By the lemma, the function $\mathcal{V}$ extends smoothly at $\tau = 0$ to a smooth function of the parameters (τ, a, b, γ). Moreover, when $\tau = 0$, we have

$$\mathcal{V}(0, a, b, \gamma) = -2(\gamma_{k,i} - \gamma_{k,1}).$$

For each $k \in \mathbb{Z}$, the partial differential of $\mathcal{V}_k$ with respect to γ only depends on γ_k, and it is straightforward to see that it is an isomorphism. (The domain and target spaces are both real vector spaces of dimension $n_k - 1$: recall the normalization $\sum_{i=1}^{n_k} \gamma_{k,i} = 1$.) The proposition then follows by the implicit function Theorem. □

Remark 4.8. *γ is a smooth function of the parameters (τ, a, b). From now on the parameter τ replaces the parameter t.*

4.8. **The B-period problem for ϕ_1 and ϕ_2.** In this section we solve the period problem

$$\mathrm{Re} \int_{B_{k,i}} \phi_1 = \mathrm{Re} \int_{B_{k,i}} \phi_2 = 0, \qquad k \in \mathbb{Z},\ 2 \leq i \leq n_k.$$

This is equivalent to

$$\int_{B_{k,i}} g^{-1}\omega = \overline{\int_{B_{k,i}} g\omega}, \qquad k \in \mathbb{Z},\ 2 \leq i \leq n_k.$$

We define for $k \in \mathbb{Z}$ and $2 \leq i \leq n_k$

$$\mathcal{H}_{k,i} = t\left(\int_{B_{k,i}} g^{(-1)^k}\omega - \overline{\int_{B_{k,i}} g^{(-1)^{k+1}}\omega}\right).$$

Let $\mathcal{H} = (\mathcal{H}_{k,i})_{k\in\mathbb{Z}, 2\leq i\leq n_k}$. We want to solve the equation $\mathcal{H} = 0$.

Lemma 4.9. *The function $\mathcal{H}$ extends smoothly at $\tau = 0$ to a smooth function of all parameters (for the ℓ^∞ norms). Moreover, when $\tau = 0$ we have*

$$\mathcal{H}_{k,i} = \int_{b_{k,1}}^{b_{k,i}} g_{k+1}^{-1}\omega_{k+1} - \overline{\int_{a_{k,i}}^{a_{k,1}} g_k^{-1}\omega_k}.$$

The proof of this lemma is technical and is given in appendix A.

Assume that the parameters α, β and γ are determined as functions of (τ, a, b) by Propositions 4.3 and 4.6. Then $\mathcal{H}$ extends at $\tau = 0$ to a smooth function of (τ, a, b). Moreover, as $\alpha = \beta = \gamma$ when $\tau = 0$, we have $\omega_k = g_k dz$ so

$$\mathcal{H}_{k,i}(0, a, b) = b_{k,i} - b_{k,1} + \overline{a_{k,i}} - \overline{a_{k,1}}.$$

We normalize the parameter b by requiring that $b_{k,1} = 0$ for all $k \in \mathbb{Z}$. (This may be seen as a normalization of translation in $\mathbb{C}_k$.) Let $\mathcal{H}_k = (\mathcal{H}_{k,i})_{2\leq i\leq n_k}$. For each k, the partial differential of $\mathcal{H}_k$ with respect to b only depends on b_k, and is easily seen to be injective, so is an isomorphism because the domain and range both have complex dimension $n_k - 1$. Hence, by the finiteness hypothesis, the partial differential of $\mathcal{H}$ with respect to b is an isomorphism from ℓ^∞ to ℓ^∞. By the implicit function Theorem, we get

Proposition 4.10. *Assume that the parameters α, β, γ are determined by Propositions 4.3 and 4.6. For (τ, a) in a neighborhood of $(0, a^0)$, there exist values of the parameter b, depending smoothly on (τ, a), such that $\mathcal{H}(\tau, a, b(\tau, a)) = 0$. Moreover, when $\tau = 0$, we have*

$$b_{k,i} = -\overline{a_{k,i}} + \overline{a_{k,1}}.$$

4.9. **The A-period problem for ϕ_1 and ϕ_2.** In this section we solve the period problem

$$\mathrm{Re}\int_{A_{k,i}} \phi_1 = \mathrm{Re}\int_{A_{k,i}} \phi_2 = 0, \qquad k \in \mathbb{Z},\, 1 \leq i \leq n_k.$$

This is equivalent to

$$\int_{A_{k,i}} g^{-1}\omega = \overline{\int_{A_{k,i}} g\omega}, \qquad k \in \mathbb{Z},\, 1 \leq i \leq n_k.$$

Recall that $\mathrm{conj}(z) = \overline{z}$ denotes the complex conjugation. We define

$$\mathcal{F}^-_{k,i} = \frac{-1}{t}\mathrm{conj}^k\left(\int_{A_{k,i}} g^{(-1)^k}\omega\right),$$

$$\mathcal{F}^+_{k,i} = \frac{1}{t}\mathrm{conj}^{k+1}\left(\int_{A_{k,i}} g^{(-1)^{k+1}}\omega\right),$$

$$\mathcal{F}_{k,i} = \mathcal{F}^-_{k,i} + \mathcal{F}^+_{k,i}.$$

We want to solve the equations $\mathcal{F}_{k,i} = 0$ for all $k \in \mathbb{Z}$ and $1 \leq i \leq n_k$. This equation will give us the balancing equation $F_{k,i} = 0$ of Section 2. However, to be able to use the non-degeneracy hypothesis, we have to reformulate this infinite system of equations in a slightly different way, as we did for the balancing condition $F_{k,i} = 0$ in Section 2. Define

$$\mathcal{F}^-_k = \sum_{i=1}^{n_k} \mathcal{F}^-_{k,i},$$

$$\mathcal{F}^+_k = \sum_{i=1}^{n_k} \mathcal{F}^+_{k,i},$$

$$\mathcal{F}_k = \sum_{i=1}^{n_k} \mathcal{F}_{k,i} = \mathcal{F}^-_k + \mathcal{F}^+_k.$$

Lemma 4.11. *It holds, independently of the values of the parameters,*

$$\forall k \in \mathbb{Z}, \qquad \mathcal{F}^-_k + \mathcal{F}^+_{k-1} = 0.$$

Proof. We have, in $\mathbb{C}_k$, $g^{(-1)^k} = tg_k$. This function is holomorphic at ∞_k. Hence, by the residue Theorem,

$$-\sum_{i=1}^{n_k}\int_{A_{k,i}} g^{(-1)^k}\omega + \sum_{i=1}^{n_{k-1}}\int_{A_{k-1,i}} g^{(-1)^k}\omega = 0$$

so

$$t\mathrm{conj}^k(\mathcal{F}^-_k) + t\mathrm{conj}^k(\mathcal{F}^+_{k-1}) = 0.$$

which proves the lemma. □

Let us write $\mathcal{G}_k = -\mathcal{F}_k^-$. Then by the lemma,

$$\mathcal{F}_k = \mathcal{F}_k^- - \mathcal{F}_{k+1}^- = \mathcal{G}_{k+1} - \mathcal{G}_k.$$

Solving $\mathcal{F}_{k,i} = 0$ for all $k \in \mathbb{Z}$ and $1 \leq i \leq n_k$ is then equivalent to solve for all $k \in \mathbb{Z}$, $\mathcal{F}_{k,i} = 0$ for $2 \leq i \leq n_k$ and $\mathcal{G}_k = \mathcal{G}_0$. We write $\mathcal{F} = (\mathcal{F}_{k,i})_{k\in\mathbb{Z}, 2\leq i\leq n_k}$ and $\mathcal{G} = (\mathcal{G}_k)_{k\in\mathbb{Z}}$.

Next we need to introduce the parameters ℓ_k and $u_{k,i}$ of Section 2 to make use of the non-degeneracy hypothesis. We make the change of parameter

$$a_{k,i} = (-1)^k \mathrm{conj}^k(\ell_k + u_{k,i}).$$

We define the parameter $\mathbf{U}$ by equation (3).

Proposition 4.12. *Assume that the parameters α, β, γ and b are determined by Propositions 4.3, 4.6 and 4.10. The functions $\mathcal{F}$ and $\mathcal{G}$ extend at $\tau = 0$ to smooth functions of $(\tau, \mathbf{U})$ (for the ℓ^∞ norms). Moreover, at $\tau = 0$ we have*

$$\mathcal{F}_{k,i}(0, \mathbf{U}) = 4\pi \mathrm{i} F_{k,i}(\mathbf{U}),$$

$$\mathcal{G}_k(0, \mathbf{U}) = 4\pi \mathrm{i} G_k(\mathbf{U})$$

where the functions $F_{k,i}$ and G_k are as in Section 2. Hence, if the configuration is balanced and non-degenerate in the sense of Definition 2.3, for τ in a neighborhood of 0, there exist values of the parameter $\mathbf{U}$, depending smoothly on τ for the ℓ^∞ norm, such that $\mathcal{F}(\mathbf{U}(\tau)) = \mathcal{F}(\mathbf{U}^0) = 0$ and $\mathcal{G}(\mathbf{U}(\tau)) = \mathcal{G}(\mathbf{U}^0) =$ constant, so our period problem is solved.

Proof. We have $g^{(-1)^k} = t g_k$ in $\mathbb{C}_k$, and $A_{k,i}$ is homologous to the circle $C(a_{k,i}, \epsilon)$ in $\mathbb{C}_k$ with the negative orientation, so

$$\mathcal{F}_{k,i}^- = \mathrm{conj}^k \left(\int_{C(a_{k,i},\epsilon)} g_k \omega \right).$$

We see by this formula that $\mathcal{F}_{k,i}^-$ extends at $\tau = 0$. Now ω depends smoothly on all parameters (for the norm $||\omega||_\infty$ defined in Section 4.4), and the function equal to g_k in each $\mathbb{C}_k$ also depends smoothly on all parameters (using the finiteness hypothesis). So $(\mathcal{F}_{k,i}^-)_{k\in\mathbb{Z}, 1\leq i\leq n_k}$ is a smooth function of all parameters by composition with a bounded linear operator.

In the same way, we have $g^{(-1)^{k+1}} = t g_{k+1}$ in $\mathbb{C}_{k+1}$ and $A_{k,i}$ is homologous to the circle $C(b_{k,i}, \epsilon)$ in $\mathbb{C}_{k+1}$, so

$$\mathcal{F}_{k,i}^+ = \mathrm{conj}^{k+1} \left(\int_{C(b_{k,i},\epsilon)} g_{k+1} \omega \right).$$

It follows that $(\mathcal{F}_{k,i}^+)_{k\in\mathbb{Z}, 1\leq i\leq n_k}$ is a smooth function of all parameters. Hence $\mathcal{F}$ and $\mathcal{G}$ are smooth functions of the parameters.

Next assume that $\tau = 0$ and α, β, γ and b are determined by Propositions 4.3, 4.6 and 4.10. Then

$$\alpha_{k,i} = \beta_{k,i} = \gamma_{k,i} = \frac{1}{n_k} = c_k,$$

and $\omega = g_k dz$ in $\mathbb{C}_k$. Also

$$b_{k,i} = (-1)^{k+1} \mathrm{conj}^{k+1}(u_{k,i}).$$

Then we compute $\mathcal{F}^-_{k,i}$ and $\mathcal{F}^+_{k,i}$ as residues :

$$\begin{aligned}
\mathcal{F}^-_{k,i} &= \mathrm{conj}^k\left(2\pi\mathrm{i}\,\mathrm{Res}_{a_{k,i}}(g_k)^2\right)\\
&= 2\pi\mathrm{i}(-1)^k\mathrm{conj}^k\left(2\sum_{j\neq i}\frac{c_k^2}{a_{k,i}-a_{k,j}}-2\sum_{j=1}^{n_{k-1}}\frac{c_kc_{k-1}}{a_{k,i}-b_{k-1,j}}\right)\\
&= 4\pi\mathrm{i}\left(\sum_{j\neq i}\frac{c_k^2}{u_{k,i}-u_{k,j}}-\sum_{j=1}^{n_{k-1}}\frac{c_kc_{k-1}}{\ell_k+u_{k,i}-u_{k-1,j}}\right)\\
\mathcal{F}^+_{k,i} &= \mathrm{conj}^{k+1}\left(2\pi\mathrm{i}\,\mathrm{Res}_{b_{k,i}}(g_{k+1})^2\right)\\
&= 2\pi\mathrm{i}(-1)^{k+1}\mathrm{conj}^{k+1}\left(2\sum_{j\neq i}\frac{c_k^2}{b_{k,i}-b_{k,j}}-2\sum_{j=1}^{n_{k+1}}\frac{c_kc_{k+1}}{b_{k,i}-a_{k+1,j}}\right)\\
&= 4\pi\mathrm{i}\left(\sum_{j\neq i}\frac{c_k^2}{u_{k,i}-u_{k,j}}-\sum_{j=1}^{n_{k+1}}\frac{c_kc_{k+1}}{u_{k,i}-\ell_{k+1}-u_{k+1,j}}\right)
\end{aligned}$$

This gives $\mathcal{F}_{k,i}=4\pi iF_{k,i}$. Regarding $\mathcal{G}_k$, we have

$$\sum_{i=1}^{n_k}\sum_{\substack{j=1\\ j\neq i}}^{n_k}\frac{1}{u_{k,i}-u_{k,j}}=0$$

so $\mathcal{G}_k=4\pi iG_k$. □

4.10. **Embeddedness.** At this point we have constructed a one parameter family of minimal immersion $\psi_t:\Sigma_t\to\mathbb{R}^3$ for $t>0$ small enough. (We switch back to the parameter t.) Let $M_t=\psi_t(\Sigma_t)$.

Proposition 4.13. *M_t is an embedded minimal surface.*

Proof. Fix a complex number O and for $k\in\mathbb{Z}$, let O_k be the point $z=O$ in $\mathbb{C}_k$. By the finiteness hypothesis, we may choose O so that $O_k\in\Omega_{k,\epsilon}$ for all $k\in\mathbb{Z}$. Let $\varphi_t:\mathbb{R}^3\to\mathbb{R}^3$ be the affine transformation $(x_1,x_2,x_3)\mapsto(2tx_1,2tx_2,x_3)$. Define

$$\psi_{k,t}(z)=\varphi_t(\psi_t(z)-\psi_t(O_k)):\Omega_{k,\epsilon}\to\mathbb{R}^3.$$

If k is even, then $g=tg_k$ in $\mathbb{C}_k$ so

$$\psi_{k,t}(z)=\mathrm{Re}\int_O^z\left((g_k^{-1}-t^2g_k)\omega,\mathrm{i}(g_k^{-1}+t^2g_k)\omega,\omega\right).$$

This extends (as a smooth function of τ) at $t=0$, with value

$$\psi_{k,0}(z)=\mathrm{Re}\int_O^z(dz,\mathrm{i}dz,\omega_k)=(\mathrm{Re}(z),-\mathrm{Im}(z),h_k(z)-h_k(O))$$

where the function h_k is defined by

$$h_k(z)=\sum_{i=1}^{n_{k-1}}c_{k-1}\log|z-b^0_{k-1,i}|-\sum_{i=1}^{n_k}c_k\log|z-a^0_{k,i}|.$$

Hence the image $\psi_{k,t}(\Omega_{k,\epsilon})$ converges to the graph of $z \mapsto h_k(\overline{z}) - h_k(\overline{O})$ on $\Omega_{k,\epsilon}$, so it is embedded for t small enough. When k is odd, we have by similar computations that $\psi_{k,t}(\Omega_{k,\epsilon})$ converges to the graph of $z \mapsto h_k(-z) - h_k(-O)$.

Observe that the function h_k is bounded on Ω_k. Provided ϵ is small enough, we can find a number η large enough such that the level line $h_k = \eta$ consists of n_k closed convex curves around the points $a^0_{k,i}$, $1 \leq i \leq n_k$, and the level line $h_k = -\eta$ consists of n_{k-1} closed convex curves around the points $b^0_{k-1,i}$, $1 \leq i \leq n_{k-1}$.

Back to our minimal immersion ψ_t, let $M_{k,t}$ be the intersection of the image $\psi_t(\Omega_{k,\epsilon})$ with the slab $\psi_t(O_k) - \eta < x_3 < \psi_t(O_k) + \eta$. By what we have seen, for t small enough, each $M_{k,t}$ is embedded, and its boundary consists of n_k closed horizontal convex curves on the top and n_{k-1} closed convex curves on the bottom. By Lemma A.2, $\int_{O_k}^{O_{k+1}} \omega \sim -2c_k \log t$, so we see that $M_{k+1,t}$ lies strictly above $M_{k,t}$.

The intersection of M_t with the slab $\psi_t(O_k) + \eta < x_3 < \psi_t(O_{k+1}) - \eta$ is the union of n_k minimal annuli. Each annulus is bounded by convex curves in horizontal planes, so is foliated by horizontal convex curves by a theorem of Shiffman [6], hence embedded.

It remains to see that these n_k annuli are disjoint. Consider one of these annuli. There exists i, $1 \leq i \leq n_k$ such that our annulus is included in the image of the annulus bounded by the circles $C(a^0_{k,i}, \epsilon')$ and $C(b^0_{k,i}, \epsilon')$ (provided we take η large enough). The image of these circles are close to circles of radius $\frac{\epsilon'}{2t}$. By Lemma A.3, $t \int_{a^0_{k,i}+\epsilon'}^{b^0_{k,i}+\epsilon'} g^{\pm 1}\omega$ extends smoothly at $t = 0$ (as a smooth function of τ), with value $\pm\epsilon'$. Hence the boundary circles are inside a vertical cylinder of radius $\frac{2\epsilon'}{t}$ for t small enough. By the convex hull property of minimal surfaces, the annulus is inside this cylinder.

Now from our analysis of $\psi_{k,t}$, the axes of these cylinders are separated by a distance greater than $\frac{c}{t}$ for some uniform positive number c. Hence these cylinders are disjoint provided we take $\epsilon' > 0$ small enough. This proves that the annuli are disjoint, so M_t is embedded. □

Appendix A. Proof of Lemmas 4.7 and 4.9

We start by proving Lemma 4.7. The period of ω on $B_{k,i}$ has four terms corresponding to the four paths in the definition of $B_{k,i}$. The first term is easily dealt with. Indeed, the path from $a^0_{k,i} + \epsilon'$ to $a^0_{k,1} + \epsilon'$ is fixed and may be chosen in the domain $\Omega_{k,\epsilon'}$. The restriction of ω to $\Omega_{\epsilon'}$ depends smoothly on all parameters, and we compose it with the linear operator $\omega \mapsto (\int_{a^0_{k,i}+\epsilon'}^{a^0_{k,1}+\epsilon'} \omega)_{k\in\mathbb{Z}, 2\leq i\leq n_k}$ which is bounded from $L^\infty(\Omega_{\epsilon'})$ to ℓ^∞ hence smooth. We treat the third term, where the integral along a path joining $b^0_{k,1} + \epsilon'$ to $b^0_{k,i} + \epsilon'$ appears, in the same way.

To handle the second and fourth terms, we expand ω in Laurent series and estimate carefully its coefficients. First let us define various constants and in particular explain how we choose ϵ' in the definition of the cycle $B_{k,i}$. Remember that we have fixed a small number $\epsilon > 0$. We choose a constant r independent of k such that for all $k \in \mathbb{Z}$, $|g_k| \leq \frac{r}{2}$ in $\Omega_{k,\epsilon}$. We choose a small number $\epsilon' > 0$ independent of k such that for all $k \in \mathbb{Z}$ and $1 \leq i \leq n_k$, $|g_k| \geq 2r$ in the disk $D(a_{k,i}, 2\epsilon')$ and $|g_{k+1}| \geq 2r$ in the disk $D(b_{k,i}, 2\epsilon')$. If $||a - a^0||_\infty \leq \epsilon'$ and $||b - b^0||_\infty \leq \epsilon'$, we have $|g_k| \geq 2r$ in the disk $D(a^0_{k,i}, \epsilon')$ and $|g_{k+1}| \geq 2r$ in the disk $D(b^0_{k,i}, \epsilon')$. Finally, we choose a

constant r' independent of k such that for all $k \in \mathbb{Z}$, $|g_k| \leq \frac{r'}{2}$ in $\Omega_{k,\epsilon'}$. All this is possible thanks to the finiteness hypothesis.

Lemma A.1. *Let $v = v_{k,i}$. We have, in the annulus $\frac{t^2}{\rho} \leq |v| \leq \rho$,*

$$\omega = -\gamma_{k,i}\frac{dv}{v} + \sum_{n\geq 0} r^{n+1}c^+_{k,i,n}v^n dv + \sum_{n\geq 2}(rt^2)^{n-1}c^-_{k,i,n}\frac{dv}{v^n}.$$

The coefficients $c^\pm_{k,i,n}$ are bounded. More precisely, let $c^+ = (c^+_{k,i,n})_{k\in\mathbb{Z},1\leq i\leq n_k,n\geq 0}$ and $c^- = (c^-_{k,i,n})_{k\in\mathbb{Z},1\leq i\leq n_k,n\geq 2}$. Then c^+ and c^- are in ℓ^∞ and are smooth functions of all parameters.

Proof. Using $v = v_{k,i}$ as a coordinate, we can write the Laurent series of ω in the annulus $\frac{t^2}{\rho} \leq |v| \leq \rho$ as

$$\omega = \sum_{n\in\mathbb{Z}} c_{k,i,n}v^n dv.$$

The coefficient $c_{k,i,n}$ is given by

$$c_{k,i,n} = \frac{1}{2\pi\mathrm{i}}\int_{\partial V_{k,i}} \frac{\omega}{v^{n+1}} = \frac{1}{2\pi\mathrm{i}}\int_{\partial V_{k,i}} \omega g_k^{n+1}.$$

In particular $c_{k,i,-1} = -\gamma_{k,i}$. For $n \geq 0$, we write $c_{k,i,n} = r^{n+1}c^+_{k,i,n}$ with

$$c^+_{k,i,n} = \frac{1}{2\pi\mathrm{i}}\int_{C(a^0_{k,i},\epsilon)} \omega\left(\frac{g_k}{r}\right)^{n+1}.$$

We have used that $\partial V_{k,i}$ is homologous to the circle $C(a^0_{k,i},\epsilon)$. We have $|\frac{g_k}{r}| \leq \frac{1}{2}$ in $\Omega_{k,\epsilon}$ so $(c^+_{k,i,n})_{n\in\mathbb{N}}$ is bounded. The function equal to $\frac{g_k}{r}$ in each $\Omega_{k,\epsilon}$ is in the open unit ball of $L^\infty(\Omega_\epsilon)$ and depends smoothly on parameters. Recall that $L^\infty(\Omega_\epsilon)$ is a Banach algebra for the pointwise product. If $\mathcal{A}$ is a Banach algebra, the map $x \mapsto (x^n)_{n\in\mathbb{N}}$ is smooth from the open unit ball of $\mathcal{A}$ to $\mathcal{A}^{\mathbb{N}}$ with the sup norm (an easy exercice). From this and the fact that ω, restricted to Ω_ϵ, depends smoothly on parameters, and composition with a bounded linear operator, we conclude that c^+ depends smoothly on all parameters.

For $n \leq -2$ we write $m = -n \geq 2$ and $c_{k,i,-m} = (rt^2)^{m-1}c^-_{k,i,m}$ with

$$\begin{aligned} c^-_{k,i,m} &= \frac{1}{2\pi\mathrm{i}}\int_{\partial V_{k,i}} \omega\left(\frac{v_{k,i}}{rt^2}\right)^{m-1} = \frac{-1}{2\pi\mathrm{i}}\int_{\partial W_{k,i}} \omega\left(\frac{1}{rw_{k,i}}\right)^{m-1} \\ &= \frac{-1}{2\pi\mathrm{i}}\int_{C(b^0_{k,i},\epsilon)} \omega\left(\frac{g_{k+1}}{r}\right)^{m-1}. \end{aligned}$$

We have used the fact that $v_{k,i}w_{k,i} = t^2$ and $\partial V_{k,i}$ is homologous to $-\partial W_{k,i}$. From this we conclude in the same way as above that c^- is in ℓ^∞ and depends smoothly on all parameters. □

Lemma 4.7 follows from the following lemma.

Lemma A.2. *The function*

$$\left(\int_{a^0_{k,i}+\epsilon'}^{b^0_{k,i}+\epsilon'} \omega + 2\gamma_{k,i}\log t\right)_{k\in\mathbb{Z},1\leq i\leq n_k}$$

extends smoothly at $t = 0$ to a smooth function of all parameters (for the ℓ^∞ norm).

Proof. Let $\varphi_{k,i} = v_{k,i}(a_{k,i} + \epsilon')$ and $\psi_{k,i} = w_{k,i}(b_{k,i} + \epsilon')$, so $v_{k,i}(b_{k,i} + \epsilon') = \frac{t^2}{\psi_{k,i}}$. Then by Lemma A.1

$$\begin{aligned}
\int_{a^0_{k,i}+\epsilon'}^{b^0_{k,i}+\epsilon'} \omega &= \int_{v=\varphi_{k,i}}^{\frac{t^2}{\psi_{k,i}}} \left(-\gamma_{k,i}\frac{dv}{v} + \sum_{n\geq 0} r^{n+1} c^+_{k,i,n} v^n dv + \sum_{n\geq 2} (rt^2)^{n-1} c^-_{k,i,n} \frac{dv}{v^n} \right) \\
&= -\gamma_{k,i} \log t^2 + \gamma_{k,i} \log(\varphi_{k,i}\psi_{k,i}) \\
&+ \sum_{n\geq 0} \frac{c^+_{k,i,n}}{n+1} \left(\left(\frac{rt^2}{\psi_{k,i}} \right)^{n+1} - (r\varphi_{k,i})^{n+1} \right) \\
&+ \sum_{n\geq 2} \frac{c^-_{k,i,n}}{n-1} \left(\left(\frac{rt^2}{\varphi_{k,i}} \right)^{n-1} - (r\psi_{k,i})^{n-1} \right)
\end{aligned}$$

By our choice of ϵ' we have $|r\varphi_{k,i}| \leq \frac{1}{2}$ and $|r\psi_{k,i}| \leq \frac{1}{2}$. By our choice of r' we have $|\frac{rt^2}{\varphi_{k,i}}| \leq \frac{rr't^2}{2} \leq \frac{1}{2}$ provided $t^2 \leq \frac{1}{rr'}$. In the same way $|\frac{rt^2}{\psi_{k,i}}| \leq \frac{1}{2}$. So the sequences $(r\varphi_{k,i})_{k,i}$, $(r\psi_{k,i})_{k,i}$, $(\frac{rt^2}{\varphi_{k,i}})_{k,i}$ and $(\frac{rt^2}{\psi_{k,i}})_{k,i}$ are all in the ball of radius $\frac{1}{2}$ in ℓ^∞, and they depend smoothly on all parameters (they are given by explicit formula). Let us deal only with the term in the above formula containing $r\varphi_{k,i}$, as the others are similar. We write this term as

$$\sum_{n\geq 0} \frac{c^+_{k,i,n}}{n+1} \left(\frac{1}{\sqrt{2}} \right)^{n+1} (r\sqrt{2}\varphi_{k,i})^{n+1}.$$

The sequence $(r\sqrt{2}\varphi_{k,i})_{k,i}$ is in the open unit ball of ℓ^∞, so by the fact recalled above about Banach algebras, the sequence $((r\sqrt{2}\varphi_{k,i})^{n+1})_{k,i,n}$ is in ℓ^∞ and depends smoothly on parameters. We multiply by the sequence $(c^+_{k,i,n})_{k,i,n}$ which depends smoothly on parameters, and compose with the linear operator which maps the sequence $(x_{k,i,n})_{k\in\mathbb{Z},1\leq i\leq n_k,n\in\mathbb{N}}$ in ℓ^∞ to

$$\left(\sum_{n\geq 0} \frac{1}{n+1} \left(\frac{1}{\sqrt{2}} \right)^{n+1} x_{k,i,n} \right)_{k\in\mathbb{Z},1\leq i\leq n_k}.$$

This operator is bounded, so we conclude that third term in the expansion of the integral of ω depends smoothly on parameters. We deal with the other terms in the same way. □

To prove Lemma 4.9, we need a lemma which is similar to Lemma A.2.

Lemma A.3. *The function*

$$\left(\int_{a^0_{k,i}+\epsilon'}^{b^0_{k,i}+\epsilon'} v_{k,i}\omega \right)_{k\in\mathbb{Z},1\leq i\leq n_k}$$

extends smoothly at $\tau = 0$ to a smooth function of all parameters. Moreover, its value at $\tau = 0$ is

$$\left(\int_{a^0_{k,i}+\epsilon'}^{a_{k,i}} g_k^{-1}\omega_k \right)_{k\in\mathbb{Z},1\leq i\leq n_k}.$$

The function

$$\left(\int_{a^0_{k,i}+\epsilon'}^{b^0_{k,i}+\epsilon'} w_{k,i}\omega\right)_{k\in\mathbb{Z},1\leq i\leq n_k}$$

extends smoothly at $\tau=0$ to a smooth function of all parameters. Moreover, its value at $\tau=0$ is

$$\left(\int_{b_{k,i}}^{b^0_{k,i}+\epsilon'} g_{k+1}^{-1}\omega_{k+1}\right)_{k\in\mathbb{Z},1\leq i\leq n_k}.$$

Proof. Let us prove the first statement. Using Lemma A.1 we have

$$\begin{aligned}
\int_{a^0_{k,i}+\epsilon'}^{b^0_{k,i}+\epsilon'} v_{k,i}\omega &= \int_{v=\varphi_{k,i}}^{\frac{t^2}{\psi_{k,i}}} \left(\sum_{n\geq -1} c^+_{k,i,n} r^{n+1} v^{n+1} dv + \sum_{n\geq 2} (rt^2)^{n-1} c^-_{k,i,n} \frac{dv}{v^{n-1}}\right)\\
&= \frac{1}{r}\sum_{n\geq -1} \frac{c^+_{k,i,n}}{n+2}\left(\left(\frac{rt^2}{\psi_{k,i}}\right)^{n+2} - (r\varphi_{k,i})^{n+2}\right)\\
&+ rt^2 c^-_{k,i,2}(\log t^2 - \log(\varphi_{k,i}\psi_{k,i}))\\
&+ rt^2 \sum_{n\geq 3} \frac{c^-_{k,i,n}}{n-2}\left(\left(\frac{rt^2}{\varphi_{k,i}}\right)^{n-2} - (r\psi_{k,i})^{n-2}\right)
\end{aligned}$$

The terms on the second and fourth line extend smoothly at $t=0$ by the same argument as in Lemma A.2. The term on the third line extends continously at $t=0$, and only as a smooth function of τ because of the term $t^2\log t^2$. The value of the integral when $t=0$ is

$$\frac{-1}{r}\sum_{n\geq -1}\frac{c^+_{k,i,n}}{n+2}(r\varphi_{k,i})^{n+2} = \int_{v=\varphi_{k,i}}^{0}\sum_{n\geq -1} c^+_{k,i,n} r^{n+1} v^{n+1} dv = \int_{a^0_{k,i}+\epsilon'}^{a_{k,i}} g_k^{-1}\omega_k.$$

The proof of the second statement of Lemma A.3 is similar, using $w_{k,i}$ instead of $v_{k,i}$ as a coordinate. □

We are now ready to prove Lemma 4.9. Recall that $g^{(-1)^k}$ is equal to tg_k in $\mathbb{C}_k$, $(tg_{k+1})^{-1}$ in $\mathbb{C}_{k+1}$, and $\frac{w_{k,i}}{t}$ in $W_{k,i}$. So we have

$$t\int_{B_{k,i}} g^{(-1)^k}\omega = t^2\int_{a^0_{k,i}+\epsilon'}^{a^0_{k,1}+\epsilon'} g_k\omega + \int_{a^0_{k,1}+\epsilon'}^{b^0_{k,1}+\epsilon'} w_{k,1}\omega + \int_{b^0_{k,1}+\epsilon'}^{b^0_{k,i}+\epsilon'} g_{k+1}^{-1}\omega + \int_{b^0_{k,i}+\epsilon'}^{a^0_{k,i}+\epsilon'} w_{k,i}\omega.$$

The first and third terms are integrals on fixed paths, so they depend smoothly on parameters by the smooth dependence of ω. The second and fourth terms extend smoothly at $\tau=0$ by the previous lemma. Moreover, the value at $\tau=0$ is

$$\int_{b_{k,1}}^{b^0_{k,1}+\epsilon'} g_{k+1}^{-1}\omega_{k+1} + \int_{b^0_{k,1}+\epsilon'}^{b^0_{k,i}+\epsilon'} g_{k+1}^{-1}\omega_{k+1} + \int_{b^0_{k,i}+\epsilon'}^{b_{k,i}} g_{k+1}^{-1}\omega_{k+1} = \int_{b_{k,1}}^{b_{k,i}} g_{k+1}^{-1}\omega_{k+1}.$$

In the same way, we write

$$t\int_{B_{k,i}} g^{(-1)^{k+1}}\omega = \int_{a^0_{k,i}+\epsilon'}^{a^0_{k,1}+\epsilon'} g_k^{-1}\omega + \int_{a^0_{k,1}+\epsilon'}^{b^0_{k,1}+\epsilon'} v_{k,1}\omega + t^2\int_{b^0_{k,1}+\epsilon'}^{b^0_{k,i}+\epsilon'} g_{k+1}\omega + \int_{b^0_{k,i}+\epsilon'}^{a^0_{k,i}+\epsilon'} v_{k,i}\omega.$$

This function extends smoothly at $\tau = 0$ with value

$$\int_{a^0_{k,1}+\epsilon'}^{a^0_{k,1}+\epsilon'} g_k^{-1}\omega_k + \int_{a^0_{k,1}+\epsilon'}^{a_{k,1}} g_k^{-1}\omega_k + \int_{a_{k,i}}^{a^0_{k,i}+\epsilon'} g_k^{-1}\omega_k = \int_{a_{k,i}}^{a_{k,1}} g_k^{-1}\omega_k.$$

This proves Lemma 4.9. □

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