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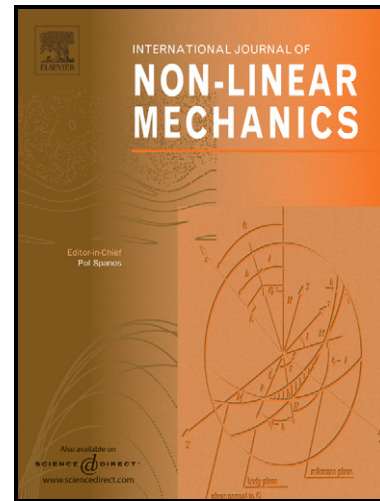
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# ON THE SPATIAL BEHAVIOR OF SOLUTIONS FOR NONLINEAR TYPE II HEAT CONDUCTION

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**Abstract:** In this paper we study the spatial behavior of the solutions for a problem determined by the nonlinear version of the Green and Naghdi type II heat conduction theory. We obtain a spatial decay estimates for the usual boundary-initial-value problem and also an upper bound for the amplitude term of the spatial estimate. Finally, we analyze a non-standard initial value problem defined on a particular family of heat conductors.

**Keywords:** type II thermoelasticity, energy methods.

## 1. INTRODUCTION

Maxwell [30] pointed out that the classical linear theory of heat conduction based on Fourier's law predicts that a thermal perturbation at some point in a material body will be felt instantly at all other points of the body, however distant they are. This is referred to as the *paradox of heat conduction*. It is physically unrealistic since it implies that thermal signals propagate with infinite speed. Thus, given the classical theory's non-causal nature, and causality's fundamental role in modern physics, different theories of heat conduction have been put forth over the course of the 20th century (see, for example, Hetnarski and Ignaczak [13, 14] and the references cited therein). In the book of Ignaczak and Ostoja-Starzewski [20] several studies concerning applicability of nonclassical thermoelastic theories are proposed. Green and Naghdi [9, 10, 11, 12] proposed some of these thermomechanical theories where the heat conduction does not agree with the usual one (see also [36]). They considered three theories labelled as type I, II and III, respectively. These theories were based on an entropy balance law rather than the usual entropy inequality. However, we want to mention the total compatibility of the entropy balance law with the entropy inequality. Their thermodynamics proposed the use of the *thermal displacement*

$$\alpha(\mathbf{x}, t) = \int_{t_0}^t \theta(\mathbf{x}, s) ds + \alpha_0,$$

where  $\theta$  is the empirical temperature.

The type I thermoelasticity coincides with the classical one; in type II, the heat is allowed to propagate by means of thermal waves but without dissipating energy and for this reason it is also known as *thermoelasticity without energy dissipation*; the type III thermoelasticity includes the two above mentioned theories as particular cases. All these theories have recently been the aim of great amount of works (as a matter of illustration see Bergmann *et al.* [2], Chandrasekharaiah [3, 4], Ciarletta *et al.* [5], Iesan [19], Jordan [21], Jordan and Puri [22],

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Jordan and Straughan [23], Lazzari and Nibbi [26], Kalpakides and Maugin [24], Maugin and Kalpakides [29], Quintanilla [33, 34] or Quintanilla and Straughan [35]). In this paper we study the nonlinear version of the type II heat conduction.

The study of edge effects in several thermomechanical situations has deserved much attention in the last three decades. They were motivated by the desire of giving a precise mathematical version of the result known in elastostatics as *Saint-Venant Principle*. This interest was later extended to other thermomechanical fields and, in general, to abstract partial differential equations. Several decay estimates for elliptic [7], parabolic [15, 16], hyperbolic [17, 32] and/or combination of them [27] have been obtained. In these contributions the authors obtain growth/decay estimates for the solutions.

It is worth noting that there are few contributions about the spatial decay in nonlinear dynamical problems. This fact applies to parabolic problems, but it is more marked for hyperbolic problems. Flavin, Knops and Payne [8] provided a method for hyperbolic problems that was later completed by Chirita and Quintanilla [6]. The former contribution applied for nonlinear problems, but the kind of nonlinearity was very restrictive and, then, their technique was used later only for linear problems. General nonlinear problems add extra difficulties not easy to solve. For example, we do not know any contributions of this type for nonlinear elastodynamics. Our aim here is to find a class of nonlinear equations (with a physical meaning) to which the methods of Flavin, Knops and Payne can be applied. The equations that we consider appear in a natural way in the analysis of the nonlinear problem of the Green and Naghdi heat conduction theory. Paying attention to the examples proposed in this paper, the key point of our proposal is the introduction of terms like  $\theta^m$ , for  $m > 2$ , in the energy of the system ( $\theta$  is the temperature). This fact allows us to apply the above cited methods.

As we mentioned above, there are not physical neither mathematical studies to support type II theory. Therefore, at this stage, we think that it is important to analyze different proposals with their respective conclusions. Our work is addressed to this objective from a mathematical point of view. In order to establish the more appropriate scenery for this theory, some initial attempts must be done.

The organization of this paper is as follows. In Section 2 we set the equation we are going to deal with and the assumptions that we impose to it. We also show a couple of examples that satisfy our assumptions. Section 3 is devoted to obtain a spatial decay estimates for the usual boundary-initial-value problem. As a corollary, we get a result of *domain of influence* type. Later, in Section 4, we obtain an upper bound for the amplitude term (of the spatial estimate) by means of the problem's initial data. In Section 5, we find spatial decay estimates for a non-standard initial value problem defined in a particular case following one of our examples. Finally, in the last section, some concluding remarks are stated.

## 2. PRELIMINARIES

Let  $R$  be the semi-infinite cylinder  $(0, \infty) \times D$ , where  $D$  is a two dimensional bounded domain such that the boundary  $\partial D$  is smooth enough to apply the divergence theorem. The finite end face of the cylinder is in the plane  $x_1 = 0$ . Let  $D(z)$  be the cross-section of the points in  $R$  such that  $x_1 = z$ , and let  $R(z)$  be the points of the cylinder such that  $x_1 > z$ . The equations we study here are determined on  $R$ .

In the following, we consider a nonlinear type II heat conduction material. We denote by  $\rho$  the mass density, which depends on the material point, and by  $\eta$  the entropy density. The evolution

equation is given by

$$(2.1) \quad \rho \dot{\eta} = \Phi_{i,i}$$

where

$$(2.2) \quad \rho \eta = -\frac{\partial \Psi}{\partial \theta} \text{ and } \Phi_i = \frac{\partial \Psi}{\partial \alpha_{,i}}.$$

$\Psi(\theta, \alpha_{,i})$  is the *free energy* and  $\Phi_i$  is the entropy flux vector.

If we substitute (2.2) into (2.1) we obtain

$$(2.3) \quad \frac{\partial}{\partial t} \left( -\frac{\partial \Psi}{\partial \theta} \right) = \frac{\partial}{\partial x_i} \left( \frac{\partial \Psi}{\partial \alpha_{,i}} \right).$$

To have a well-posed problem in  $R$  we impose the following initial conditions

$$(2.4) \quad \alpha(x_1, x_2, x_3, 0) = g(x_1, x_2, x_3), \quad \dot{\alpha}(x_1, x_2, x_3, 0) = h(x_1, x_2, x_3)$$

and also the boundary conditions given by

$$(2.5) \quad \alpha(0, x_2, x_3, t) = f(x_2, x_3, t), \quad \forall (x_2, x_3) \in D, \quad \alpha(x_1, x_2, x_3, t) = 0 \quad \forall (x_2, x_3) \in \partial D.$$

In order to avoid difficulties concerning the smoothness of the solutions, we assume that  $f(x_2, x_3, t) = 0$  for  $(x_2, x_3) \in D$ .

In our study, it will be relevant to consider the internal energy function

$$\Sigma = \Psi + \rho \theta \eta.$$

Our aim is to obtain the spatial behavior for the solutions of the problem determined by (2.3) with initial conditions (2.4) and boundary conditions (2.5) whenever we assume that

$$(2.6) \quad \int_R \Sigma dV < \infty$$

at time  $t = 0$ .

The class of functions we are going to work with is restricted to satisfy the following assumptions:

**A1:**  $|\Phi_1| \leq C_1 \Sigma^{1/2} + C_2 \Sigma^p$  with  $C_1, C_2 > 0$  and  $1/2 < p < 1$ .

**A2:**  $\Sigma \geq C_3 \theta^2 + C_4 \theta^q$  with  $C_3, C_4 > 0$  and  $p + 1/q = 1$ .

Let us show several examples that we had in mind and led us to propose the above assumptions.

**Example 2.1.** Let us consider the functions

$$\Psi^{(m)} = -c_{1m} |\theta|^m + c_{2m} |\nabla \alpha|^m + c_{3m} \theta |\nabla \alpha|^{m-1} + \theta (d_{im} \alpha_{,i})^{m-1}, \quad \text{for } m \geq 2,$$

where  $c_{1m}$  and  $c_{2m}$  are positive constants and  $c_{3m}$  and  $d_{im}$  are arbitrary constants. If we define

$$\Psi = \Psi^{(2)} + \Psi^{(m)}, \quad m > 2,$$

then we have:

$$\rho \eta = 2c_{12} \theta - c_{32} |\nabla \alpha| + mc_{1m} |\theta|^{m-2} \theta - c_{3m} |\nabla \alpha|^{m-1} - d_{i2} \alpha_{,i} - (d_{im} \alpha_{,i})^{m-1}$$

and

$$(2.7) \quad \Sigma = c_{12} |\theta|^2 + c_{22} |\nabla \alpha|^2 + (m-1)c_{1m} |\theta|^m + c_{2m} |\nabla \alpha|^m.$$

In this case,

$$\begin{aligned}\Phi_1 = \frac{\partial \Psi}{\partial \alpha_{,1}} = & 2c_{22}\alpha_{,1} + c_{32}\theta \frac{\alpha_{,1}}{|\nabla \alpha|} + mc_{2m}|\nabla \alpha|^{m-1} \frac{\alpha_{,1}}{|\nabla \alpha|} + (m-1)c_{3m}\theta |\nabla \alpha|^{m-2} \frac{\alpha_{,1}}{|\nabla \alpha|} + \\ & + \theta d_{12} + (m-1)\theta d_{1m}(d_{1m}\alpha_{,1})^{m-2}.\end{aligned}$$

Note that from (2.7) we obtain

$$\begin{aligned}|\alpha_{,1}| & \leq |\nabla \alpha| \leq K_1 \Sigma^{1/2}, \\ |\theta| & \leq K_2 \Sigma^{1/2}, \\ |\nabla \alpha|^{m-1} & = \left(|\nabla \alpha|^m\right)^{\frac{m-1}{m}} \leq K_3 \Sigma^{\frac{m-1}{m}}, \\ |\theta| |\nabla \alpha|^{m-2} & = \left(|\theta|^m\right)^{\frac{1}{m}} \left(|\nabla \alpha|^m\right)^{\frac{m-2}{m}} \leq K_4 \Sigma^{\frac{1}{m}} \Sigma^{\frac{m-2}{m}} = K_4 \Sigma^{\frac{m-1}{m}} \text{ and} \\ |\theta| |\alpha_{,1}| & \leq |\theta| |\nabla \alpha|.\end{aligned}$$

Therefore,  $|\Phi_1| \leq C_1 \Sigma^{1/2} + C_2 \Sigma^p$  with  $C_1, C_2 > 0$  and  $1/2 < p < 1$ .

Note also that  $\Sigma \geq C_3 \theta^2 + C_4 \theta^q$  for  $q = m$ .

**Example 2.2.** We define  $\Gamma^{(m)} = \Psi^{(m)} + b_{mp}|\theta|^p |\nabla \alpha|^{m-p}$ . And we take

$$\Psi = \Psi^{(2)} + \Gamma^{(m)}, \quad m > 2.$$

In this case,

$$\rho\eta = 2c_{12}\theta - c_{32}|\nabla \alpha| + mc_{1m}|\theta|^{m-2}\theta - c_{3m}|\nabla \alpha|^{m-1} - pb_{mp}\theta|\theta|^{p-2}|\nabla \alpha|^{m-p} - d_{i2}\alpha_{,i} - (d_{im}\alpha_{,i})^{m-1}.$$

Thus,

$$\Sigma = c_{12}|\theta|^2 + c_{22}|\nabla \alpha|^2 + (m-1)c_{1m}|\theta|^m + c_{2m}|\nabla \alpha|^m + (1-p)b_{mp}|\theta|^p |\nabla \alpha|^{m-p}.$$

### 3. SPATIAL ESTIMATES

In this section we obtain our spatial decay estimate. The domain of influence type result for the solutions will be a corollary.

Our analysis starts by considering the following function:

$$I(z, t) = - \int_0^t \int_D \Phi_1 \theta dA ds.$$

Let us compute the divergence of the vector field  $\Phi_i \theta$ :

$$[\Phi_i \theta]_{,i} = \Phi_{i,i} \theta + \Phi_i \theta_{,i} = \frac{\partial}{\partial t} \left( -\frac{\partial \Psi}{\partial \theta} \right) \theta + \frac{\partial \Psi}{\partial \alpha_{,i}} \dot{\alpha}_{,i}.$$

Or, equivalently,

$$[\Phi_i \theta]_{,i} = -\frac{\partial}{\partial t} \left( \frac{\partial \Psi}{\partial \theta} \theta \right) + \frac{\partial \Psi}{\partial \theta} \dot{\theta} + \frac{\partial \Psi}{\partial \alpha_{,i}} \dot{\alpha}_{,i} = -\frac{\partial}{\partial t} \left( \frac{\partial \Psi}{\partial \theta} \theta \right) + \frac{\partial \Psi}{\partial t}.$$

Using equation (2.2) we obtain

$$[\Phi_i \theta]_{,i} = \frac{\partial}{\partial t} [\rho \theta \eta] + \frac{\partial \Psi}{\partial t} = \dot{\Sigma}.$$

We consider the part of  $R$  between  $x_1 = z^*$  and  $x_1 = z$ . Applying the divergence theorem, we obtain:

$$(3.1) \quad I(z, t) - I(z^*, t) = \int_0^t \int_{z^*}^z \int_D \dot{\Sigma} dA dx_1 ds.$$

And integrating:

$$I(z, t) - I(z^*, t) = - \int_{z^*}^z \int_D \Sigma dV + \int_{z^*}^z \int_D \Sigma(0) dV,$$

where  $dV$  stands for  $dA dx_1$  and  $\Sigma(0)$  for function  $\Sigma$  evaluated at  $t = 0$ . Note that the integral of  $\Sigma(0)$  depends on the initial data of the problem.

It is clear that

$$\begin{aligned} \frac{\partial I}{\partial t} &= - \int_D \Phi_1 \theta dA \text{ and} \\ \frac{\partial I}{\partial z} &= - \int_D \Sigma dA + \int_D \Sigma(0) dA \end{aligned}$$

Applying assumption A1 we obtain:

$$\left| \frac{\partial I}{\partial t} \right| \leq \int_D |\Phi_1| |\theta| dA \leq C_1 \int_D \Sigma^{1/2} |\theta| dA + C_2 \int_D \Sigma^p |\theta| dA.$$

And applying A2,

$$\left| \frac{\partial I}{\partial t} \right| \leq C_1 C_3^{-1/2} \int_D \Sigma^{1/2} \Sigma^{1/2} dA + C_2 C_4^{-1/q} \int_D \Sigma^p \Sigma^{1/q} dA = (C_1 C_3^{-1/2} + C_2 C_4^{-1/q}) \int_D \Sigma dA.$$

If we denote  $C_5 = C_1 C_3^{-1/2} + C_2 C_4^{-1/q}$ , then we have

$$\left| \frac{\partial I}{\partial t} \right| \leq C_5 \left( - \frac{\partial I}{\partial z} + \int_D \Sigma(0) dA \right).$$

Or, equivalently,

$$(3.2) \quad \left| \frac{\partial I}{\partial t} \right| + C_5 \frac{\partial I}{\partial z} \leq C_5 \int_D \Sigma(0) dA.$$

To simplify a little bit the notation, let us denote  $E_1(z) = \int_D \Sigma(0) dA$ . Then, inequality (3.2) implies that

$$(3.3) \quad \frac{\partial I}{\partial t} + C_5 \frac{\partial I}{\partial z} \leq C_5 E_1(z)$$

and

$$(3.4) \quad \frac{\partial I}{\partial t} - C_5 \frac{\partial I}{\partial z} \geq -C_5 E_1(z).$$

These inequalities have been studied in several references before (see, for instance, [8]).

Note that (3.3) and (3.4) can be rewritten, respectively, as

$$(3.5) \quad c \frac{\partial I}{\partial t} + \frac{\partial I}{\partial z} \leq E_1(z)$$

and

$$(3.6) \quad c \frac{\partial I}{\partial t} - \frac{\partial I}{\partial z} \geq -E_1(z)$$

where  $c = 1/C_5$ .

Let us fix  $(z_0, t_0) \in [0, L] \times [0, \infty)$ . If we set  $t = t_0 + c(z - z_0)$ , then the expression on the left hand side of inequality (3.5) can be thought of as

$$(3.7) \quad \frac{\partial}{\partial z} I(z, t_0 + c(z - z_0))$$

Therefore, on integrating (3.5) we obtain

$$(3.8) \quad I(z, t_0 + c(z - z_0)) - I(z_0, t_0) \leq \int_{z_0}^z E_1(\xi) d\xi$$

where  $z \geq z_0$ .

On the other hand, if we set now  $t = t_1 + c(z_1 - z)$ , and integrating (3.6) we obtain

$$(3.9) \quad I(z, t_1 + c(z_1 - z)) - I(z_1, t_1) \geq - \int_z^{z_1} E_1(\xi) d\xi$$

where  $z \leq z_1$ .

Taking  $t_0 = t_1 = 0$ , assuming that the initial data satisfy (2.6), and making  $z \rightarrow \infty$ , from inequalities (3.8) and (3.9) we obtain that, for a finite time  $t$ ,

$$\lim_{z \rightarrow \infty} I(z, t) = 0.$$

Taking into account this result and relation (3.1) we get

$$I(z, t) = \int_{R(z)} \Sigma(t) dV - \int_{R(z)} \Sigma(0) dV.$$

Now inequality (3.3) implies that

$$(3.10) \quad \mathcal{E}(z, t) \leq \mathcal{E}(z^*, 0)$$

where

$$(3.11) \quad \mathcal{E}(z, t) = \int_z^\infty \int_D \Sigma(t) dV$$

and  $z, z^*$  and  $t$  are related by  $t = c(z - z^*)$ . In a similar way, we get,

$$(3.12) \quad \mathcal{E}(z, t) \leq \mathcal{E}(z^{**}, 0)$$

for  $t = c(z^{**} - z)$ . Finally, from inequalities (3.10) and (3.12), we conclude that

$$(3.13) \quad \mathcal{E}(z, t) \leq \mathcal{E}(z^*, t^*)$$

for  $|t - t^*| \leq c(z - z^*)$ .

Therefore, we have the following result.

**Theorem 3.1.** *Let  $\alpha$  be a solution of the initial-boundary-value problem defined by (2.3)–(2.5). Then the energy function  $\mathcal{E}(z, t)$  defined in (3.11) satisfies inequality (3.13) whenever  $|t - t^*| \leq c(z - z^*)$ , provided that the initial data satisfy (2.6).*

If one defines the measure

$$(3.14) \quad \mathcal{E}^*(z, t) = \int_0^t \mathcal{E}(z, s) ds$$

the following inequalities can be obtained (see Horgan and Quintanilla [16]):

$$(3.15) \quad \begin{aligned} \mathcal{E}^*(z, t) &\leq c \int_{z-c^{-1}t}^z \mathcal{E}(p, 0) dp, & c^{-1}t \leq z \\ \mathcal{E}^*(z, t) &\leq c \int_0^z \mathcal{E}(p, 0) dp + \left(1 - \frac{cz}{t}\right) \mathcal{E}^*(0, t), & c^{-1}t \geq z \end{aligned}$$

If  $g$  and  $h$  vanish, we have that  $\mathcal{E}(p, 0) = 0$  for every  $p \geq 0$ . Thus, the last inequalities become

$$(3.16) \quad \begin{aligned} \mathcal{E}^*(z, t) &= 0, & c^{-1}t \leq z \\ \mathcal{E}^*(z, t) &\leq \left(1 - \frac{cz}{t}\right) \mathcal{E}^*(0, t), & c^{-1}t \geq z \end{aligned}$$

Note that the first equality implies that the solution vanishes for  $t \leq cz$ , which is a domain of influence kind of result.

Therefore, we have proved the following property.

**Corollary 3.2.** *If homogeneous initial conditions are supposed, then estimates (3.16) hold. In particular,  $\alpha = 0$  whenever  $c^{-1}t \leq z$ .*

#### 4. THE AMPLITUDE TERM

To have a better description of our estimates, we need an upper bound for the amplitude term by means of the data problem. In this section we obtain a bound for  $\mathcal{E}^*(0, t)$  when the initial data vanish.

In this section we will also assume that

$$\mathbf{A3:} \quad |\rho\eta| \leq D_1 \Sigma^{1/2} + D_2 \Sigma^p \text{ with } D_1, D_2 > 0 \text{ and } 1/2 < p < 1.$$

$$\mathbf{A4:} \quad |\Phi_i| \leq D_1 \Sigma^{1/2} + D_2 \Sigma^p.$$

It is worth noting that the families of functions proposed in examples 2.1 and 2.2 satisfy these conditions.

It is clear that

$$\int_0^t \int_R (\rho\dot{\eta} - \Phi_{i,i}) \theta \, dV \, ds = 0.$$

Thus, making some calculations, we obtain

$$\mathcal{E}(0, t) = - \int_0^t \int_{D(0)} \Phi_1 \theta \, dA \, ds.$$

Therefore, for any arbitrary function  $\Gamma(\mathbf{x}, t)$  which decays in a (sufficiently) fast way as  $x_1$  goes to infinity and such that it agrees with  $\theta(\mathbf{x}, t)$  at  $D(0)$ , we have

$$\mathcal{E}^*(0, t) = - \int_0^t \int_{D(0)} (t-s) \Phi_1 \Gamma \, dA \, ds.$$

Using the divergence theorem we obtain that

$$\mathcal{E}^*(0, t) = \int_0^t \int_R (t-s) \rho\dot{\eta} \Gamma \, dV \, ds + \int_0^t \int_R (t-s) \Phi_i \Gamma_{,i} \, dV \, ds.$$

We denote by  $I_1$  the first integral and by  $I_2$  the second one.  $I_1$  can be rewritten as

$$I_1 = \int_0^t \int_R \rho \eta \Gamma dV ds - \int_0^t \int_R (t-s) \rho \eta \dot{\Gamma} dV ds = I_{11} - I_{12}.$$

We now bound the absolute values of integrals  $I_{11}$ ,  $I_{12}$  and  $I_2$ .

Taking into account assumption A3 we get

$$|I_{11}| \leq \int_0^t \int_R \left( D_1 \Sigma^{1/2} + D_2 \Sigma^p \right) |\Gamma| dV ds.$$

Thus,

$$|I_{11}| \leq D_1 \left( \int_0^t \int_R \Sigma dV ds \right)^{1/2} \left( \int_0^t \int_R |\Gamma|^2 dV ds \right)^{1/2} + D_2 \left( \int_0^t \int_R \Sigma dV ds \right)^p \left( \int_0^t \int_R |\Gamma|^{1/(1-p)} dV ds \right)^{1-p}.$$

Or, equivalently,

$$|I_{11}| \leq D_1 \mathcal{E}^*(0, t)^{1/2} \left( \int_0^t \int_R |\Gamma|^2 dV ds \right)^{1/2} + D_2 \mathcal{E}^*(0, t)^p \left( \int_0^t \int_R |\Gamma|^{1/(1-p)} dV ds \right)^{1-p}.$$

Now we will use the inequality

$$a \cdot b \leq \frac{(\epsilon a)^\alpha}{\alpha} + \frac{(\epsilon^{-1} b)^\beta}{\beta}, \quad \text{when } \frac{1}{\alpha} + \frac{1}{\beta} = 1, \quad \text{for any } \epsilon > 0.$$

In this way, we obtain:

$$|I_{11}| \leq \epsilon_1 \mathcal{E}^*(0, t) + N_1 \int_0^t \int_R |\Gamma|^2 dV ds + \epsilon_2 \mathcal{E}^*(0, t) + N_2 \int_0^t \int_R |\Gamma|^{1/(1-p)} dV ds.$$

In the above expression,  $\epsilon_1$  and  $\epsilon_2$  are two positive constants that we can select as small as we want, and  $N_1$  and  $N_2$  are two calculable positive constants.

In a similar way we have

$$|I_{12}| \leq \int_0^t \int_R (t-s) \left( D_1 \Sigma^{1/2} + D_2 \Sigma^p \right) \dot{\Gamma} dV ds.$$

Using the same reasoning as before, we can obtain

$$|I_{12}| \leq \epsilon_3 \mathcal{E}^*(0, t) + N_3(t) \int_0^t \int_R |\dot{\Gamma}|^2 dV ds + \epsilon_4 \mathcal{E}^*(0, t) + N_4(t) \int_0^t \int_R |\dot{\Gamma}|^{1/(1-p)} dV ds,$$

where  $\epsilon_3$  and  $\epsilon_4$  are positive but as small as we want, and  $N_3(t)$  and  $N_4(t)$  can depend on the time.

Using A4 and the fact that  $|\Gamma_{,i}| \leq (\Gamma_{,i} \Gamma_{,i})^{1/2}$ , we obtain:

$$|I_2| \leq 3 \int_0^t \int_R (t-s) \left( D_1 \Sigma^{1/2} + D_2 \Sigma^p \right) (\Gamma_{,i} \Gamma_{,i})^{1/2} dV ds.$$

Now we bound  $|I_2|$  as before:

$$|I_2| \leq \epsilon_5 \mathcal{E}^*(0, t) + N_5(t) \int_0^t \int_R (\Gamma_{,i} \Gamma_{,i}) dV ds + \epsilon_6 \mathcal{E}^*(0, t) + N_6(t) \int_0^t \int_R (\Gamma_{,i} \Gamma_{,i})^{1/2(1-p)} dV ds,$$

where  $\epsilon_5$ ,  $\epsilon_6$ ,  $N_5(t)$  and  $N_6(t)$  are as above.

In view of the previous estimates and selecting  $\epsilon_i = 1/12$  for  $i = 1, \dots, 6$  we obtain  
(4.1)

$$\mathcal{E}^*(0, t) \leq H(t) \left[ \int_0^t \int_R \left( |\Gamma|^2 + |\dot{\Gamma}|^2 + (\Gamma_{,i}\Gamma_{,i}) + |\Gamma|^{1/(1-p)} + |\dot{\Gamma}|^{1/(1-p)} + (\Gamma_{,i}\Gamma_{,i})^{1/2(1-p)} \right) dV ds \right],$$

where  $H(t)$  is a function calculable in terms of the constitutive constants and the time.

We now need to obtain an upper estimate for the integral on the right hand side of (4.1) in terms of the boundary data. In order to do so, we take

$$\Gamma(x_1, x_2, x_3, t) = e^{-\delta x_1} f(x_2, x_3, t),$$

where  $\delta$  is an arbitrary but positive constant and  $f(x_2, x_3, t)$  is one of our boundary conditions given at (2.5).

Note that

$$\dot{\Gamma}(x_1, x_2, x_3, t) = e^{-\delta x_1} \dot{f}(x_2, x_3, t),$$

and

$$\Gamma_{,i}\Gamma_{,i} = \delta^2 e^{-2\delta x_1} f^2(x_2, x_3, t) + e^{-2\delta x_1} (f_2^2(x_2, x_3, t) + f_3^2(x_2, x_3, t)).$$

Therefore, each one of the integrals on the right hand side of (4.1) can be bounded. In fact, we have:

$$\begin{aligned} \int_R |\Gamma|^k dV &= \int_D \int_0^\infty e^{-k\delta x_1} |f^k(x_2, x_3, t)| dx_1 dA = \frac{1}{k\delta} \int_D |f^k(x_2, x_3, t)| dA, \text{ for } k = 2 \text{ or } k = 1/(1-p). \\ \int_R |\dot{\Gamma}|^k dV &= \int_D \int_0^\infty e^{-k\delta x_1} |\dot{f}^k(x_2, x_3, t)| dx_1 dA = \frac{1}{k\delta} \int_D |\dot{f}^k(x_2, x_3, t)| dA, \text{ for } k = 2 \text{ or } k = 1/(1-p). \\ \int_R (\Gamma_{,i}\Gamma_{,i})^k dV &= \int_D \int_0^\infty e^{-2k\delta x_1} \left[ \delta^2 f^2(x_2, x_3, t) + f_2^2(x_2, x_3, t) + f_3^2(x_2, x_3, t) \right]^k dx_1 dA \\ &= \frac{1}{2k\delta} \int_D \left[ \delta^2 f^2(x_2, x_3, t) + f_2^2(x_2, x_3, t) + f_3^2(x_2, x_3, t) \right]^k dA, \text{ for } k = 1 \text{ or } k = 1/2(1-p). \end{aligned}$$

And, then,

$$\begin{aligned} \mathcal{E}^*(0, t) &\leq H(t) \left[ \int_0^t \int_D \frac{1}{2\delta} \left( (1 + \delta^2) |f|^2 + |\dot{f}|^2 + f_2^2 + f_3^2 \right) + \right. \\ &\quad \left. + \frac{1-p}{\delta} \left( |f|^{1/(1-p)} + |\dot{f}|^{1/(1-p)} + (\delta^2 f^2 + f_2^2 + f_3^2)^{1/2(1-p)} \right) dA ds \right]. \end{aligned} \quad (4.2)$$

## 5. A NON-STANDARD PROBLEM FOR EQUATION (2.3)

In this section we restrict our attention to the problem determined by the functions given at example 2.1:

$$\Psi = \Psi^{(2)} + \Psi^{(m)}, \quad m > 2,$$

where

$$\Psi^{(m)} = -c_{1m} |\theta|^m + c_{2m} |\nabla \alpha|^m + c_{3m} \theta |\nabla \alpha|^{m-1} + \theta (d_{im} \alpha_{,i})^{m-1}, \quad \text{for } m \geq 2.$$

$c_{1m}$  and  $c_{2m}$  are positive constants and  $c_{3m}$  and  $d_{im}$  are arbitrary constants.

We briefly discuss the behavior of the solutions of (2.3) subject to the boundary conditions (2.5) and the non-standard conditions

$$(5.1) \quad \alpha(\mathbf{x}, T) = k\alpha(\mathbf{x}, 0), \quad \theta(\mathbf{x}, T) = k\theta(\mathbf{x}, 0),$$

where  $|k| > 1$ . Such non-standard conditions have been the subject of recent attention. In 2002 Payne and Schaefer [31] proposed the study of a non-standard initial problem associated with elastodynamics. This kind of problems was motivated by regularization of ill-posed Cauchy problems for the backward in time classical heat equation [1].

The analysis begins by considering the function

$$(5.2) \quad F_\gamma(z) = - \int_0^T \int_{D(z)} e^{-2\gamma s} \Phi_1 \theta dA ds$$

where, as established in [17], the positive constant  $\gamma$  is given by

$$\gamma = \frac{1}{T} \ln |k|.$$

It is clear, see (3.1), that

$$F_\gamma(z) - F_\gamma(z^*) = \int_0^T \int_{z^*}^z \int_D e^{-2\gamma s} \dot{\Sigma} dA dx_1 ds,$$

for every  $z \geq z^* \geq 0$ .

Or, equivalently,

$$F_\gamma(z) - F_\gamma(z^*) = \int_0^T \int_{z^*}^z \int_D \frac{d}{ds} [e^{-2\gamma s} \Sigma] dA dx_1 ds + 2\gamma \int_0^T \int_{z^*}^z \int_D e^{-2\gamma s} \Sigma dA dx_1 ds.$$

If we denote by  $R(z, z^*)$  the part of the cylinder from  $z^*$  to  $z$  and we integrate with respect to the time in the first integral above, we get:

$$F_\gamma(z) - F_\gamma(z^*) = \int_{R(z, z^*)} [e^{-2\gamma T} \Sigma(T) - \Sigma(0)] dV + 2\gamma \int_0^T \int_{R(z, z^*)} e^{-2\gamma s} \Sigma dV ds,$$

where function  $\Sigma$  is the one defined by (2.7).

Hence, taking into account the non-standard conditions (5.1) and the value of  $\gamma$  (that gives  $k^2 e^{-2\gamma T} = 1$ ) we obtain:

$$e^{-2\gamma T} \Sigma(T) - \Sigma(0) = (k^{m-2} - 1) [(m-1)c_{1m}|\theta(0)|^m + c_{2m}|\nabla\alpha(0)|^m].$$

As a result,

$$\begin{aligned} F_\gamma(z) - F_\gamma(z^*) &= \int_{R(z, z^*)} (k^{m-2} - 1) [(m-1)c_{1m}|\theta(0)|^m + c_{2m}|\nabla\alpha(0)|^m] dV \\ &\quad + 2\gamma \int_0^T \int_{R(z, z^*)} e^{-2\gamma s} \Sigma dV ds. \end{aligned}$$

And, computing now  $F'_\gamma(z)$  we obtain:

$$F'_\gamma(z) = \int_{D(z)} (k^{m-2} - 1) [(m-1)c_{1m}|\theta(0)|^m + c_{2m}|\nabla\alpha(0)|^m] dA + 2\gamma \int_0^T \int_{D(z)} e^{-2\gamma s} \Sigma dA ds.$$

Being the first integral positive, it is clear that

$$F'_\gamma(z) \geq 2\gamma \int_0^T \int_{D(z)} e^{-2\gamma s} \Sigma dA ds.$$

As we want to estimate the absolute value of  $F_\gamma(z)$  in terms of its derivative, we remind here that functions  $\Phi_1$  and  $\theta$  satisfy assumptions A1 and A2. Therefore

$$\begin{aligned}
 (5.3) \quad |F_\gamma(z)| &\leq \int_0^T \int_{D(z)} e^{-2\gamma s} [C_1 \Sigma^{1/2} + C_2 \Sigma^p] |\theta| dA ds \\
 &\leq \int_0^T \int_{D(z)} e^{-2\gamma s} [C_1 \Sigma^{1/2} C_3^{-1/2} \Sigma^{1/2} + C_2 \Sigma^p C_4^{-1/q} \Sigma^{1/q}] dA ds.
 \end{aligned}$$

Finally,

$$|F_\gamma(z)| \leq \frac{C_1 C_3^{-1/2} + C_2 C_4^{-1/q}}{2\gamma} F'_\gamma(z) = C_\gamma F'_\gamma(z).$$

Hence, we can obtain an alternative of Phragmen-Lindelof type which states (see reference [7]) that the solutions either grow exponentially for  $z$  large enough or solutions decay exponentially in the form

$$\mathcal{E}(z) \leq \mathcal{E}(0) e^{-z/C_\gamma},$$

for all  $z \geq 0$ , where

$$\mathcal{E}(z) = \int_0^T \int_{R(z)} 2\gamma e^{-2\gamma s} \Sigma dV ds + \int_{R(z)} (k^{m-2} - 1) [(m-1)c_{1m}|\theta(0)|^m + c_{2m}|\nabla\alpha(0)|^m] dV.$$

As a final comment, let us say that this result can be extended for  $|k| < 1$ . The idea is to observe that if  $\alpha$  is a solution of our problem, then it is also a solution of the backward in time problem determined by the same boundary conditions, but with the non-standard initial conditions given by

$$k^{-1}\alpha(\mathbf{x}, T) = \alpha(\mathbf{x}, 0), \quad k^{-1}\theta(\mathbf{x}, T) = \theta(\mathbf{x}, 0),$$

for function

$$\Psi = \Psi_*^{(2)} + \Psi_*^{(m)}, \quad m > 2,$$

with

$$\Psi_*^{(m)} = -c_{1m}|\theta|^m + c_{2m}|\nabla\alpha|^m - c_{3m}\theta|\nabla\alpha|^{m-1} - \theta(d_{im}\alpha_i)^{m-1}, \quad \text{for } m \geq 2.$$

In this case,  $k^{-1}$  is now greater than one and the analysis proposed previously can be adapted to this new system.

## 6. CONCLUDING REMARKS

This paper was focussed in the analysis of a mathematical aspect that can appear in the non-linear theory of type II heat conduction. The mathematical structure of this theory is given by a hyperbolic differential equation while, in the classical theory, the structure is given by a parabolic equation. We have seen that, under some constitutive assumptions, the energy does not grow along certain space-time directions. In particular, a result of domain of influence has been obtained. This kind of result cannot be obtained for the classical theory of heat conduction due to the phenomenon of diffusion.

We think that, nowadays, it is impossible to decide which of both theories is appropriate to better describe real situations: it depends on many different aspects. Surely, for certain scenarios one of these theories is more suitable, while for other situations, the other one is preferable. Let us say that the heat conduction phenomena at very low temperatures seem to be well described by the theory of wave propagation at finite speed. For instance, Hwang *et al.* [18], Maïga *et al.* [28], Kim *et al.* [25] or Vadasz *et al.* [37] suggest that a mechanism for the increased heat

transfer characteristics of a nanofluid may be through a hyperbolic equation for the temperature field.

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### Highlights

- We study the spatial behavior of the solutions for a problem determined by the nonlinear version of the Green.
- We get a spatial decay estimate for usual boundary-initial-value problem & high bound for amplitude term of spatial estimate.
- We analyze a non-standard initial value problem defined on a particular family of heat conductors.