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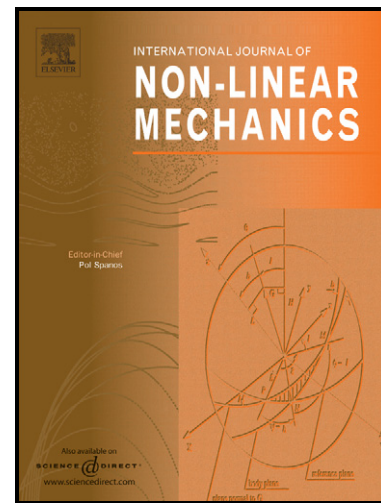
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The unilateral contact buckling problem of continuous beams in the presence of initial geometric imperfections: an analytical approach based on the theory of elastic stability

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Abstract

An analytical method for the treatment of the elastic buckling problem of continuous beams with intermediate unilateral constraints is presented, which is based on the fundamental theory of elastic stability. The study focuses on the unilateral contact buckling problem of beams in the presence of initial geometric imperfections. The mathematical Euler approach, based on the fundamental solution of the boundary value problem of the buckling of continuous beams, is appropriately modified in order to take into account the unilateral contact conditions. Furthermore, in order the obtained analytical solutions to be applicable for practical design cases, the actual strength of the cross section of the beam under combined compression and bending is considered. The implementation of the proposed method is demonstrated through a characteristic example.

Keywords: Buckling; Unilateral contact; Initial geometric imperfections.

1. Introduction

In many cases engineers deal with practical problems where, besides classical boundary conditions (bilateral constraints), unilateral support conditions are also possible. It is obvious that classical structural or mechanical problems with a certain level of complexity become even more complex to be solved when unilateral constraints should also be taken into account. Buckling involving unilateral contact is encountered

in a variety of different practical applications. The unilateral contact buckling problem of delaminated plates in a composite member, the possible contact buckling phenomena in metal forming processes, the buckling of the compressive plates of the steel sheeting in a composite slab, are some of the many structural and mechanical problems where the beginning and the evolution of the buckling phenomena are strongly affected by unilateral contact conditions.

Problems of that type are usually handled using computational techniques based on variational formulations of the governing differential equations ([1], [2], [3], [4], [5], [6], [7], [8]). For the solution of some specific problems in the area of civil engineering, simplified mathematical models have also been developed by Ma et al. [9],[10]. Furthermore, some researchers ([11], [12], [13], [14], [15]) have studied the aforementioned problem from a more theoretical point of view, giving analytical solutions for some classes of problems.

Additionally, the stability and strength of structures in real life applications is influenced by the existence of initial geometric imperfections which develop due to a variety of reasons, as e.g. manufacturing processes, member handling from the factory to the construction site, etc. For this reason, the design against buckling in all the structural design codes considers initial geometric imperfections having certain shapes and amplitudes.

In the present paper an analytical method is developed which can be applied in common practical problems. The aforementioned methodology is based on the linear elastic stability theory, appropriately extended in order to take into account the unilateral constraints. More specifically, the Euler equilibrium method [16], [17] of finding the instability load of continuous beams in the presence of initial geometric imperfections, is applied. The considered continuous beams, besides bilateral constraints, are subjected to unilateral ones.

For the implementation of the proposed method, first arbitrary initial geometric imperfections compatible with the function of the unilateral constraint are introduced in the structure. Then, the governing differential equations describing the bending behaviour of the beam are constructed. The boundary conditions of the problem are formulated appropriately, in order to take into account the unilateral constraints. In the sequel, the arising Boundary Value Problem (BVP) is examined under all the possible contact situations. The solution of the BVP gives the deflection curve of the beam as a function of the applied load. The so-called **instability load** is the one that leads the values of the obtained deflection curve of the beam to infinity. On the other hand, it

should be noticed that the maximum load that a beam can sustain is affected by the actual strength of the beam's section. In this case, the **ultimate load** is the one which marginally satisfies the design strength criteria and depends also on the shape and amplitude of the initial imperfections.

In the following paragraphs the formulation and the implementation of the described method are presented. Without losing generality, in Section 2, the non-homogeneous BVP of a simply supported continuous geometrically imperfect beam with two unequal spans and one intermediate unilateral constraint, subjected to axial compressive load is formulated and investigated. The next part of the paper (Section 3) is devoted to the consideration of the actual strength of the beam under axial compression and bending. In the last part of the paper (Section 4) the implementation of the proposed method is presented through a characteristic example. Due to the fact that the solution of the non-homogeneous BVP is strongly connected with the eigenvalues of the corresponding homogeneous BVP, i.e. the buckling loads of the unilateral contact buckling problem of the geometrically perfect beam, useful details concerning the latter are given in the Appendix.

2. Formulation of the elastic contact buckling of the geometrically imperfect continuous beam

2.1 Formulation

A geometrically imperfect continuous beam with an intermediate constraint is considered, subjected to an axial compressive load (Fig. 1). The beam is divided into two spans, Span I and Span II, having lengths aL and $(1-a)L$ respectively, where L is the total length of the continuous beam. The two spans are equipped with the coordinate systems x_1, w_1 and x_2, w_2 , as it is shown in Fig.1. Here x_1, x_2 measure the position along the axis of the beam and w_1, w_2 denote the transverse deflections of the beam in the two spans.

The initial shape of the imperfect beam is assumed to be described by a Fourier sine series having the following form:

$$w_0(x) = \sum_{k=1}^n b_k \sin\left(\frac{k\pi x}{L}\right) \quad x \in [0, L]. \quad (2.1)$$

In the above relation, L is the total length of the beam and w_0 the initial deflections due to the existence of the imperfection. The arbitrary initial geometric imperfection has to be compatible with the unilateral constraint, therefore, the following inequality should be satisfied:

$$w_0(aL) \leq 0. \quad (2.2)$$

Due to the fact that the method is applied on a continuous beam, the initial geometric imperfection has to be separated into two functions, one for each span of the beam (Span I and Span II):

$$w_{1,0}(x_1) = \sum_{k=1}^n b_k \sin\left(\frac{k\pi x_1}{L}\right) \quad x_1 \in [0, aL] \quad (2.3)$$

$$w_{2,0}(x_2) = -\sum_{k=1}^n b_k \sin\left(\frac{k\pi x_2}{L}\right) (-1)^k \quad x_2 \in [0, (1-a)L]. \quad (2.4)$$

Inequality (2.2) imposes that the Fourier coefficients b_k of the above relations should, in turn, satisfy the following inequalities:

$$\sum_{k=1}^n b_k \sin(k\pi a) \leq 0 \quad \text{or} \quad \sum_{k=1}^n b_k \sin(k\pi(1-a)) \geq 0. \quad (2.5)$$

Then, for the description of the bending behaviour of the beam, the well known Euler's equilibrium method can be applied, which leads to a fourth-order non-homogeneous linear differential equation for the two spans of the beam respectively:

$$\frac{d^4 w(x_1)}{dx_1^4} + \lambda^2 \frac{d^2 w(x_1)}{dx_1^2} = \sum_{k=1}^n b_k \left(\frac{k\pi}{L}\right)^4 \sin\left(\frac{k\pi x_1}{L}\right) \quad x_1 \in [0, aL] \quad (2.6)$$

$$\frac{d^4 w(x_2)}{dx_2^4} + \lambda^2 \frac{d^2 w(x_2)}{dx_2^2} = -\sum_{k=1}^n b_k \left(\frac{k\pi}{L}\right)^4 \sin\left(\frac{k\pi x_2}{L}\right) (-1)^k \quad x_2 \in [0, (1-a)L]. \quad (2.7)$$

In the above fundamental equations the parameter λ is determined through the following equation:

$$\lambda = \sqrt{\frac{P}{EI}}. \quad (2.8)$$

In relation (2.8), E denotes the Young's modulus of the material of the beam, I denotes the moment of inertia of the beam's cross-section for in plane bending and P is the axial compressive load applied on the beam.

The boundary conditions of the problem are formulated taking into account the essential boundary conditions, the natural boundary conditions and the unilateral conditions at the point of the support. Fig. 2 shows the convention for the positive internal forces and the positive transverse deflections.

1. Essential boundary conditions

- Zero vertical displacement at the positions of the classical supports (points A, B):

$$w_1(0) = 0 \quad (2.9)$$

$$w_2(0) = 0 \quad (2.10)$$

- Common vertical displacement at the points of the unilateral support (point C):

$$w_1(aL) = w_2((1-a)L) = u \quad (2.11)$$

2. Natural boundary conditions

- Common rotation at the position of the unilateral support (point C):

$$-w_1'(aL) = w_2'((1-a)L) \quad (2.12)$$

- Zero bending moment at the positions of the classical supports (points A, B):

$$-EIw_1''(0) = 0 \quad (2.13)$$

$$-EIw_2''(0) = 0 \quad (2.14)$$

- Moment equilibrium at the position of the unilateral support (point C):

$$-EIw_1''(aL) + EIw_2''((1-a)L) = 0 \quad (2.15)$$

- Forces equilibrium at the position of the unilateral support (point C).

In order to formulate the boundary conditions that correspond to the unilateral constraint, the support reaction R should be considered with an unknown value

(Fig.2). Obviously the existence of this reaction force R depends on whether the unilateral constraint is active or not.

$$[-EIw_1''(aL) - Pw_1'(aL)] + [-EIw_2''((1-a)L) - Pw_2'((1-a)L)] = R. \quad (2.16)$$

3. Unilateral contact boundary conditions at the position of the unilateral constraint

The unilateral constraint may be represented through the following inequality conditions [18],[19]:

$$w_1(aL) = w_2((1-a)L) = u \leq 0 \quad (2.17)$$

$$R \leq 0 \quad (2.18)$$

$$Ru = 0. \quad (2.19)$$

The solution of equations (2.6), (2.7) with respect to the boundary conditions of the problem gives the transverse deflection w at each point of the beam as a function of the axial compressive load P . Consequently, the unilateral contact elastic buckling problem of the geometrically imperfect continuous beam with one intermediate constraint is formulated through the differential equations (2.6), (2.7), the equations (2.9)-(2.16) which represent the boundary conditions, the additional inequality conditions (2.17), (2.18) and the complementarity condition of equation (2.19).

It has to be noticed that the fundamental equations (2.6) and (2.7) do not take into account the stress in the lateral direction of the beam (Euler-Bernoulli beam assumption). Therefore, they can also be derived by the consideration of the one-dimensional von Karman model [20]. Moreover, these types of models are suitable for pre-buckling analysis under the assumption of infinitesimal deformations. For problems where large deformations have to be considered, different types of beam models should be taken into account [21], [22]. Such models lead to non-linear ordinary differential equations and are suitable for the analysis and simulation of a variety of mechanical problems. However, bifurcation problems of that type are very difficult to be solved analytically and, therefore, computational methods are usually employed.

2.2 Solution of the BVP of the geometrically imperfect beam for all the possible contact situations

For the above described geometrically imperfect structure, the BVP is expected to have one unique solution that gives the deflection curve of the beam. This solution depends on the type and amplitude of the initial imperfections considered. In this case the critical state is denoted by disproportionate large transverse deflections, developing when the axial compressive load P is approaching a certain value, the so-called **instability load** (P_i). Due to the existence of the initial imperfection, the beam has a bending deflection even from the initial stage of loading, in contrast with the perfect structure where the critical state is indicated by a sudden passing from the stable initial straight line configuration to the bended deformed configuration (bifurcation equilibrium state).

The arising solution of the BVP of the imperfect structure is a superposition of a general solution and of a particular solution related with the type of the initial imperfection, i.e.:

$$w_1(x_1) = A_1 \cos \lambda x_1 + B_1 \sin \lambda x_1 + C_1 x_1 + D_1 + \sum_{k=1}^n b_k F_k \sin\left(\frac{k\pi x_1}{L}\right), \quad x_1 \in [0, aL] \quad (2.20)$$

$$w_2(x_2) = A_2 \cos \lambda x_2 + B_2 \sin \lambda x_2 + C_2 x_2 + D_2 - \sum_{k=1}^n b_k F_k \sin\left(\frac{k\pi x_2}{L}\right) (-1)^k, \quad x_2 \in [0, (1-a)L]. \quad (2.21)$$

It is noticed that the solutions w_1, w_2 give the total transverse deflections of the beam, i.e. initial deflections are included in them. In the above solutions the terms F_k are functions of the axial compressive load P , known in the bibliography as magnification factors, which are given by the following relation:

$$F_k = \frac{1}{1 - \frac{P}{P_k}}. \quad (2.22)$$

In the above, P_k are the eigenvalues of the elastic buckling problem of the simply supported beam, i.e.:

$$P_k = \frac{k^2 \pi^2 EI}{L^2}. \quad (2.23)$$

The full expression of the deflection curve of the continuous beam is given after the determination of the coefficients $A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2$. These coefficients are calculated through the boundary conditions of the problem. Applying the boundary conditions at the ends of the beam, the coefficients A_1, A_2, D_1, D_2 are all taking zero values, while the usage of relations (2.11), (2.12), (2.15), (2.16) leads to the determination of the unique values of the coefficients B_1, B_2, C_1, C_2 . However, due to the presence of the inequality conditions, the calculation of the above coefficients requires an appropriate examination. Obviously, the solution of the BVP of the structure depends on the different contact situations which can occur. There exist three possible deformed configurations compatible with the unilateral constraint. The first corresponds to the situation that the unilateral constraint is inactive. In this case the contact reaction force $R=0$ and the transverse deflection $u<0$. When $R<0$ the unilateral constraint is active, therefore the transverse deflection u is equal to zero. Finally, there exists the limit situation, where the beam is in contact with the constraint without producing any reaction force ($R=0, u=0$). It is obvious that the values of the coefficients B_1, B_2, C_1, C_2 depend on the above contact situation. The solution procedure for each one of them is described in the following.

2.2.1 Case of inactive constraint and of neutral contact status ($R=0$ and $u \leq 0$)

In the case where the unilateral constraint is inactive, the normal contact force is equal to zero. Then, applying the necessary conditions $R=0$ and $u \leq 0$ to the BVP, the coefficients B_1, B_2, C_1, C_2 of the general solution are all taking zero values, therefore the final solution is derived through the particular solution only. Notice that this case includes both the case of inactive constraint ($R=0, u < 0$) and the case of neutral contact status ($R=0, u=0$). The arising solution of the BVP is given by the following equations:

$$w_1(x_1) = \sum_{k=1}^n b_k F_k \sin\left(\frac{k\pi x_1}{L}\right) \quad x_1 \in [0, aL] \quad (2.24)$$

$$w_2(x_2) = -\sum_{k=1}^n b_k F_k \sin\left(\frac{k\pi x_2}{L}\right) (-1)^k \quad x_2 \in [0, (1-a)L] . \quad (2.25)$$

The above solution is valid only if the restriction introduced by the following inequality is satisfied:

$$u(aL) \leq 0 \Rightarrow \sum_{k=1}^n b_k F_k \sin(k\pi a) \leq 0 \quad \forall P \geq 0 \quad (2.26)$$

or equivalently,

$$u((1-a)L) \leq 0 \Rightarrow \sum_{k=1}^n b_k F_k \sin(k\pi(1-a))(-1)^k \geq 0 \quad \forall P \geq 0. \quad (2.27)$$

The previous inequalities imply that a specific value of the axial load P that makes the inequality untrue is possible to exist. This load value causes the development of the reaction contact force R and is termed as P_c . This characteristic value of loading is calculated from the limit case where (2.26) (or equivalently (2.27)), holds as equality, i.e.

$$u(aL) = 0 \Rightarrow \sum_{k=1}^n b_k F_k \sin[k\pi a] = 0 \quad (2.28)$$

or equivalently,

$$u((1-a)L) = 0 \Rightarrow \sum_{k=1}^n b_k F_k \sin[k\pi(1-a)](-1)^k = 0. \quad (2.29)$$

The latter equations derive an $(n-1)$ order polynomial algebraic equation with respect to the variable P . Obviously, negative or complex values of load P cannot be admissible solutions. It should also be pointed out that more than one real positive solutions may exist, from which only the smallest is of interest.

2.2.2 Active constraint, $R < 0$ and $u = 0$

In the case where the unilateral constraint is active and the normal contact force is $R < 0$, the arising solution of the BVP is described by the following equations:

$$w_1(x_1) = B_1 \sin \lambda x_1 + C_1 x_1 + \sum_{k=1}^n b_k F_k \sin\left(\frac{k\pi x_1}{L}\right) \quad x_1 \in [0, aL] \quad (2.30)$$

$$w_2(x_2) = B_2 \sin \lambda x_2 + C_2 x_2 - \sum_{k=1}^n b_k F_k \sin\left(\frac{k\pi x_2}{L}\right)(-1)^k \quad x_2 \in [0, (1-a)L] \quad (2.31)$$

$$B_1 = \frac{\left[\sum_{k=1}^n b_k F_k \sin(k\pi a) \right] \sin[\lambda(1-a)L]}{a(1-a)\lambda L \sin[\lambda L] - \sin[\lambda a L] \sin[\lambda(1-a)L]} \quad (2.32)$$

$$B_2 = \frac{B_1 \sin[\lambda a L]}{\sin[\lambda(1-a)L]} \quad (2.33)$$

$$C_1 = \frac{-B_1 \sin[\lambda a L] - \sum_{k=1}^n b_k F_k \sin(k\pi a)}{aL} \quad (2.34)$$

$$C_2 = \frac{C_1 a L}{(1-a)L}. \quad (2.35)$$

Clearly, the above solution is valid only if the restriction introduced by the inequality condition $R < 0$, is satisfied, i.e.:

$$B_1 \cos(\lambda a L) + B_2 \cos(\lambda(1-a)L) < 0 \quad \forall P \geq 0. \quad (2.36)$$

If a value of load P exists so that left side of inequality (2.36) tends to zero, then the beam develops the tendency to be separated from the unilateral constraint. This axial load is termed as P_s . Obviously, the admissible values of P_s should belong into the set of the positive real numbers (\mathbb{R}^+). From all the admissible solutions, only the smallest value is of interest.

2.3 Calculation of the instability loading

In the previous section the formulation and the solution of the BVP of the unilateral contact elastic buckling problem of a geometrically imperfect continuous beam with two equal spans and one intermediate unilateral constraint, was presented. This solution is actually the elastic transverse deflection curve of the beam which is dependent on the axial loading and on the initial imperfection. As it was aforementioned, the critical equilibrium state for an imperfect beam is denoted by disproportionate large values in the deflection curve when the loading tends to the value of the instability load ($P \rightarrow P_i$). The calculation of this load is accomplished via the determination of the poles of the function of the deflection curve. Depending on the contact situation (e.g. active constraint, inactive constraint or neutral condition) the poles of the deflection curve are actually the eigenvalues of the buckling equations of the corresponding bifurcation

problem (i.e. the homogeneous BVP of the geometrically perfect beam) and therefore they have to be determined. The necessary calculations are based on the same considerations as in the already studied case of the imperfect beam. The details concerning the case of the unilateral contact buckling problem of the geometrically perfect beam are given in the Appendix .

As it is obvious from the functions which describe the solution for each case (equations (2.24), (2.25) and (2.30), (2.31)) , it is not clear which value of load is able to cause instability, because this value is strongly dependent on the shape of the initial imperfection. Therefore, the following procedure is proposed in order to determine the instability load.

Step 1

The initial imperfection is applied and the deflection at the position of the unilateral support is examined (i.e. if $u < 0$ or if $u = 0$).

Step 2

If $u < 0$, then relations (2.24)-(2.27) hold. It is then checked whether there exists a valid value P_c yielded by relations (2.28) or (2.29) when they hold as equalities.

a) If these relations do not produce a valid value for P_c , then it is sure that the beam will never come in contact with the unilateral support. In this case the procedure continues with Step 5.

b) If these relations produce a valid value for P_c , then it is examined if there exists an eigenvalue of the corresponding BVP of the perfect structure, producing disproportionate large deflections, located inside the interval $[0, P_c]$. If such an eigenvalue does not exist, the beam is able to sustain more loading till the unstable equilibrium state and the procedure continues with Step 4. On the other hand, if such eigenvalue exists inside the interval $[0, P_c]$, then this is actually the **instability load** of the beam with the given imperfection and the procedure is terminated. In this case the instability load of the imperfect beam coincides with the critical load of the corresponding bifurcation problem.

Step 3

If $u = 0$, then the beam is in contact with the constraint and relations (2.30)-(2.36) hold. Then, it is checked whether there exists a valid value P_s yielded by relation (2.36)

when it holds as equality ($R=0$). In this case the beam develops the tendency to be separated from the unilateral support.

a) If relation (2.36) does not produce a valid value for P_s , then it is sure that the beam will never lose contact with the unilateral support. In this case the procedure is continued with Step 4.

b) If relation (2.36) produces a valid value for P_s , then it is examined if there exists an eigenvalue of the corresponding BVP of the perfect structure that produces disproportionate large deflections, located inside the interval $[0, P_s]$. If not, the procedure continues with Step 5. If yes, then this eigenvalue is the instability load and the procedure is terminated. Similarly, also in this case, the instability load coincides with the critical load of the corresponding bifurcation problem.

Step 4

Having reached in this step, the deflection curve is in contact with the unilateral support and will remain in contact till the maximum value of the loading which leads to unstable equilibrium state has been attained, therefore, $u = 0$ and $R < 0$ and relations (2.30)-(2.36) hold. Now, two different cases are considered depending on the previous step which led to the specific situation. More specifically:

- Case of arriving in Step 4 from Step 2b

In this case the beam has come in contact with the unilateral support for $P=P_c$ and has the ability to sustain more loading. For loads $P > P_c$, the bending behavior of the beam is described by the set of equations (2.30)-(2.36). Now, the deflection shape that was produced by the relations in Step 2 for $P=P_c$, should be used as initial geometric imperfections in equations (2.30)-(2.36). All the eigenvalues which are produced by the corresponding bifurcation problem are possible critical loads. The critical eigenvalue (P_{cr_eig}) is the one that produces deflections tending to infinity. Finally, the instability load results as the sum of P_c and P_{cr_eig} , i.e.:

$$P_i = P_c + P_{cr_eig} \quad (2.37)$$

- Case of arriving in Step 4 from Step 3a

As in the previous case, the beam will undergo disproportionate large deflections for one of the eigenvalues of the corresponding bifurcation problem (P_{cr_eig}). This is eigenvalue is also the instability load.

Step 5

The deflection curve is not in contact with the unilateral support and this situation will not change till the unstable equilibrium state will be attained, therefore $u \leq 0$ and $R = 0$ and relations (2.24)-(2.27) hold. The determination of the instability load is based on the following:

- Case of arriving in step 5 from Step 2a
In this case the bending behavior of the beam is described by equations (2.24) and (2.25) until the unstable equilibrium state. The possible critical loads are eigenvalues, of the bifurcation problem of Section 2.2.1. Depending on the initial imperfection, the critical eigenvalue (P_{cr_eig}), is the one for which the beam develops extremely large deflections. This is also the instability load of the beam for the considered imperfection.
- Case of arriving in Step 5 from Step 3a
In case where the beam separates from the unilateral support (Step 3a), the bending behavior of the beam for a load $P > P_s$ is described by equations (2.24)-(2.25). In order to apply correctly the specific relations, the deflected shape which has been produced from Step 3a for $P = P_s$, has to be considered as the “new” initial imperfection. Then, in order to obtain the instability load of the beam, the load P_s should be added to the value of the critical eigenvalue of the corresponding bifurcation problem (determined as before), thus:

$$P_i = P_s + P_{cr_eig} \quad (2.38)$$

The solution procedure which has been described in this Section will be clearly demonstrated in the example treated in Section 5.

3. Failure axial load of the continuous beam considering the strength of the beam's section

3.1 Generalities

Even though the methodology described in the previous section is able to handle the unilateral contact elastic buckling problem of continuous beams with geometric initial imperfections, from a practical point of view, the calculated instability load is not the

ultimate load that the beam can sustain. As it is obvious from the previous paragraphs, two types of loads causing instability can be calculated so far. The first one, the critical load, is the smallest eigenvalue (compatible with the restriction introduced by the unilateral constraint) calculated from the BVP of the perfect continuous beam (see the Appendix). This load surely does not reflect the realistic buckling load of the beam, due to the fact that it does not take into account the inevitable initial geometric imperfections which are present in real structures. For this reason, a second type of load, the instability load, should be determined for the general case of the imperfect continuous beams. Nevertheless, this load cannot be considered as the maximum load that the beam can sustain when a real design case is under study, due to the fact that it does not take into account the beam's actual strength. Clearly, as the loading increases, the transverse deflections of the beam increase disproportionately, leading eventually to the exhaustion of the ultimate strength of the cross-sections of the beam. Moreover, the exhaustion of the capacity of the beam's cross section should be examined under the coexistence of the compressive axial load with the second order bending moment which, in turn, depends on the magnitude of the developed deflections. Therefore, the ultimate load of a real life problem depends on the specific shape of the initial imperfections, their amplitudes and on the strength of the actual cross-section of the beam. For the determination of this ultimate load, the proposed method should be equipped with design criteria connected with the strength of the cross-sections. Without losing generality, it is assumed here that the beam is made of steel and, therefore, the provisions of Eurocode 3 [23] apply. Therefore, the design criterion is related with the bending moment resistance $M_{N,Rd}$ which is computed considering also the effect of the axial compressive force P . More specifically, at each point of the beam the following relation should be fulfilled

$$M_{Ed} \leq M_{N,Rd} , \quad (3.1)$$

where M_{Ed} is the design second order bending moment that can be determined through the following equations:

$$M_{Ed}(x_1) = -EIw_{1,p}'' = -EI(w_1''(x_1) - w_{1,0}''(x_1)) \quad x_1 \in [0, aL] \quad (3.2)$$

$$M_{Ed}(x_2) = -EIw_{2,p}'' = -EI(w_2''(x_2) - w_{2,0}''(x_2)) \quad x_2 \in [0, (1-a)L] \quad (3.3)$$

where $w_{1,p}''$ and $w_{2,p}''$ are the second derivatives of the elastic transverse deflection which are attributed solely to the axial loading P . The calculation of the bending moment

resistance $M_{N.Rd}$ depends on the type of the cross section and the appropriate relations can be found in Eurocode 3 [23].

3.2 Calculation of the second order bending moment as a function of the unilateral constraint conditions

Using the expressions obtained in Section 2 for the deflections w_1 and w_2 , the following equations are obtained that give the function of the bending moment when the constraint is inactive and active, respectively.

1. Inactive unilateral constraint and neutral contact status, i.e. $R=0$ and $u \leq 0$

$$M_{Ed}(x_1) = -EI \left[\sum_{k=1}^n b_k \left(\frac{k\pi}{L} \right)^2 (F_k - 1) \sin \left(\frac{k\pi x_1}{L} \right) \right] \quad x_1 \in [0, aL] \quad (3.4)$$

$$M_{Ed}(x_2) = -EI \left[\sum_{k=1}^n b_k \left(\frac{k\pi}{L} \right)^2 (F_k - 1) \sin \left(\frac{k\pi x_2}{L} \right) (-1)^k \right] \quad x_2 \in [0, (1-a)L] \quad (3.5)$$

2. Active unilateral constraint, i.e. $R < 0$ and $u = 0$

$$M_{Ed}(x_1) = -EI \left[-B_1 \lambda^2 \sin(\lambda x_1) + \sum_{k=1}^n b_k \left(\frac{k\pi}{L} \right)^2 (1 - F_k) \sin \left(\frac{k\pi x_1}{L} \right) \right] \quad x_1 \in [0, aL] \quad (3.6)$$

$$M_{Ed}(x_2) = -EI \left[-B_2 \lambda^2 \sin(\lambda x_2) + \sum_{k=1}^n b_k \left(\frac{k\pi}{L} \right)^2 (F_k - 1) \sin \left(\frac{k\pi x_2}{L} \right) (-1)^k \right] \quad x_2 \in [0, (1-a)L] \quad (3.7)$$

4. Implementation of the described method through a characteristic example

For the implementation and the better comprehension of the described method, an example of calculating the critical, the instability and the ultimate load of a certain continuous beam with a unilateral support is studied. For this reason the continuous beam of Fig. 3 is considered having a total length of 6m. The beam is divided into two

unequal spans by a unilateral contact support which is positioned in a distance of 4.7 m away from the left end of the beam so that the parameter $a \cong 0.783$. The stiffness rigidity of the beam is $EI=16989 \text{ kNm}^2$. This value corresponds to a HEB220 steel section. The quality of the steel used is S460 N with a yield stress equal to $f_y = 460 \text{ Mpa}$. The bifurcation problem of the perfect continuous beam is first studied and the **critical load** is obtained according to relations given in the Appendix. Then, the similar buckling problem of the imperfect beam using an arbitrary initial geometric imperfection is considered and the **instability load** is calculated. In the sequel, the actual strength of the beam's section is considered and the **ultimate load** that the imperfect beam is able to sustain is determined. It is noticed that bending and stability are considered only with respect to the strong axis of the cross-section. However, the method can easily be generalized in order to treat the case of biaxial bending and stability with respect to both axes of the cross-section.

4.1 Calculation of the critical load of the perfect continuous beam

In this contact buckling problem there exist infinite deformed configurations (eigenmodes) different from the straight line, for which the beam is in equilibrium with the corresponding applied critical loads (eigenvalues). Depending on each possible contact situation, the corresponding buckling equations are applied and the eigenvalues are derived according to relations of Appendix A. The first five eigenmodes of the studied case are presented in Fig. 4. Obviously, the calculated critical loads and the corresponding eigenvalues, apart from the first, critical, mode, are mathematical solutions without any value from the engineering point of view. Nevertheless, when the corresponding imperfect structure is considered, all the eigenvalues are significant and may affect the instability load and the final buckling shape of the unilateral buckling problem, due to the fact that they are poles of the arising deflection curve. The above remark is clearly demonstrated in the following paragraph.

4.2 Calculation of the instability load of the imperfect continuous beam

In the demonstrated example a rather unusual imperfection is taken into account in order to reveal some details of the presented methodology. It is supposed that the initial imperfection of the beam is described by a Fourier sine series of five terms (Fig.5):

$$w_0(x) = 0.00125 \sin\left(\frac{\pi x}{6}\right) + 0.0025 \sin\left(\frac{2\pi x}{6}\right) + 0.0033 \sin\left(\frac{3\pi x}{6}\right) + \\ + 0.0065 \sin\left(\frac{4\pi x}{6}\right) + 0.00125 \sin\left(\frac{5\pi x}{6}\right) \quad x \in [0, 6]. \quad (4.1)$$

The initial imperfection does not violate the intermediate unilateral constraint due to the fact that it satisfies the requirement of inequality (4.2). At the position of the unilateral support there exists a gap equal to 1.68 mm. The above problem is described through equations (2.6) and (2.7). In order to determine the instability load of the beam and the final buckling shape, the following steps are followed according to the calculation procedure of Section 2.3:

- Application of step 1 and step 2 of the procedure described in Section 2.3

Initially, it is checked which of the three sets of equations described in Section 2.2 is satisfied. It is noticed that in the beginning of the loading the constraint is inactive ($u < 0, R = 0$) thus the equations of section 2.2.1 hold and take the form:

$$w_1(x_1) = \sum_{k=1}^5 b_k F_k \sin\left(\frac{k\pi x_1}{6}\right) \quad x_1 \in [0, 3] \quad (4.2)$$

$$w_2(x_2) = -\sum_{k=1}^5 b_k F_k \sin\left(\frac{k\pi x_2}{6}\right) (-1)^k \quad x_2 \in [0, 3]. \quad (4.3)$$

However, the deflection curve of the beam derived by the solution of the BVP and given by the above equations, is not valid for every value of loading P . The validity of this set of equations has a limit which is defined when inequality (2.27) is not fulfilled. This means that if during the increase of the loading there exists a load P which makes (2.27) untrue, then the beam will come in contact with the unilateral constraint and a contact reaction force will appear. The load P for which the beam will come marginally in contact with the unilateral constraint without producing any reaction force (neutral contact status) is calculated through equation (2.28) or (2.29). The calculations for the case treated here give $P_c = 3391.365$ kN. Therefore the previous set of equations is valid only for $P \leq P_c$.

It is noticed that the smallest eigenvalue of the corresponding bifurcation problem is larger than P_c . Therefore, the applied load can be increased and the deflection of the beam will be described by a different set of equations, as follows.

- Application of step 4 of the procedure described in Section 2.3

For $P > P_c$, the set of equations of Section 2.2.2 hold. The deflection curve of the beam is given by the following equations:

$$w_1(x_1) = B_1 \sin \lambda x_1 + C_1 x_1 + \sum_{k=1}^5 b'_k F_k \sin\left(\frac{k\pi x_1}{6}\right) \quad x_1[0,3] \quad (4.4)$$

$$w_2(x_2) = B_2 \sin \lambda x_2 + C_2 x_2 - \sum_{k=1}^5 b'_k F_k \sin\left(\frac{k\pi x_2}{6}\right) (-1)^k \quad x_2[0,3] \quad (4.5)$$

where:

$$b'_k = b_k F_k = b_k \frac{1}{1 - \frac{P_c}{P_k}} \quad (4.6)$$

In the above equations the initial imperfection is actually the deflection curve produced by the equations considered in the previous step for $P = P_c$.

The coefficients B_1, B_2, C_1, C_2 are calculated through equations (2.32)-(2.35) in which b_k should be substituted with b'_k from (4.6). It is noticed that in the above equations, the value of P should be initialized, because they are not "aware" of the developments in the beam till the moment at which the beam came into contact with the unilateral support. Therefore, when in the above equations $P = 0$, the total axial load in the beam is equal to P_c . As the loading increases (starting from zero load), the restriction (2.36) for the present set of equations is satisfied for any value of P . When the value of the load P approaches to the value:

$$P_{cr_eig} = 13108.87 \text{ kN} \quad (4.7)$$

which is one of the eigenvalues of the corresponding BVP, the deflections tend to infinity. Obviously, the instability load of the structure for the given imperfection is equal to:

$$P_i = P_c + P_{cr_eigen} = 16500.235 \quad (4.8)$$

It is noticed that the load P_{cr_eigen} corresponds to the second eigenvalue of Fig. 4 due to the fact that the buckling equation which corresponds to the contact case of the active

constraint is activated (see Appendix A). Fig. 6 presents the deflections of the beam for characteristic values of the axial load P .

It is clearly concluded from the previous example that the methodology proposed in Section 2.3 for the calculation of the instability load in cases where unilateral constraints and arbitrary initial geometric imperfections are present, leads directly to the exact solution, after the elaboration of simple mathematical operations. Similar results could have been derived by using some advanced finite element software, able to take into account geometric nonlinearity phenomena together with the contact conditions. The solution in this case should have been obtained after the consideration of a load incrementation scheme together with a Newton-Raphson iterative procedure. However, it is well known that the simultaneous consideration of the different types of nonlinearities involved in the examined problem, require a careful selection of the load-incrementation scheme (which, of course, is case dependent), otherwise the convergence of the iterative procedure would be questionable. On the contrary, the proposed methodology avoids the utilization of such techniques and it is able to lead to the exact solution after the application of certain steps, which are based on specific formulas. All the necessary mathematical operations can be carried out easily, without the application of any kind of advanced software, offering in that way an extra advantage to engineers who deal with real life applications.

4.3 Calculation of the ultimate load

For the certain initial geometric imperfection, the instability load which was calculated in the previous Section is not actually the ultimate load that the continuous beam is able to sustain, due to the fact that the development of large deformations causes stresses that eventually will exceed the capacity of the cross-section. Therefore, in order to calculate the ultimate load of the beam, its actual strength should be taken into account. The axial force-bending moment interaction diagram for the HEB 220 cross-section is given in Figure 7. The values of this diagram were computed following the provisions of Eurocode 3 [23].

Here, the bending moments along the beam are calculated for the example of the imperfect beam treated in the previous Section. The same steps are followed as before:

- For $P \leq P_c$ the function of the bending moment is given by applying equations (3.4) and (3.5). More specifically,

$$M_{Ed}(x_1) = -EIw_{1,p}(x_1) = -EI \left[\sum_{k=1}^5 b_k \left(\frac{k\pi}{6} \right)^2 (1 - F_k) \sin \left(\frac{k\pi x_1}{6} \right) \right] \quad x_1 \in [0, 3] \quad (4.9)$$

$$M_{Ed}(x_2) = -EIw_{2,p}(x_2) = -EI \left[\sum_{k=1}^5 b_k \left(\frac{k\pi}{6} \right)^2 (F_k - 1) \sin \left(\frac{k\pi x_2}{6} \right) (-1)^k \right] \quad x_2 \in [0, 3]. \quad (4.10)$$

- For $P > P_c$ the function of the bending moment is given by applying equations (3.6) and (3.7):

$$M_{Ed}(x_1) = -EI \left[-B_1 \lambda^2 \sin(\lambda x_1) + \sum_{k=1}^5 a_k \left(\frac{k\pi}{6} \right)^2 (1 - F_k) \sin \left(\frac{k\pi x_1}{6} \right) \right] \quad x_1 \in [0, 3] \quad (4.11)$$

$$M_{Ed}(x_2) = -EI \left[-B_2 \lambda^2 \sin(\lambda x_2) + \sum_{k=1}^5 a_k \left(\frac{k\pi}{6} \right)^2 (F_k - 1) \sin \left(\frac{k\pi x_2}{6} \right) (-1)^k \right] \quad x_2 \in [0, 3]. \quad (4.12)$$

Then, for each pair of values (P, M_{Ed}) it can be checked if the strength criterion of equation (3.1) is satisfied. If a value of the axial load P exists such as the criterion (3.1) holds as equality and, simultaneously, is smaller than the instability load P_i , then this load represents the ultimate load of the beam and is termed as P_{ult} . In case for which such load does not exist the instability load is also the ultimate load of the beam.

Fig. 8, presents the values of the bending moments for various characteristic values of the load P up to failure, which occurs for $P_{ult} = 3440.965 \text{ kN}$. For this value of the axial force, the maximum value of the bending moment at the beam is equal to $M = 52.60 \text{ kNm}$. This pair of axial force and bending moment lies on the boundary of the interaction diagram, signaling the failure of the beam. Therefore, the ultimate load capacity of the specific continuous beam, is far away from the theoretical instability load found earlier ($P_i = 16500.235 \text{ kN}$). This means that in the certain studied case the maximum loading is defined from the failure of the material rather than from instability.

Figure 9, presents the results of a parametric study in which the imperfection amplitude increases, while keeping the shape of the imperfection curve the same. All the imperfection shapes applied here do not violate the unilateral contact support. It is noticed that the failure load decreases as the amplitude of the imperfection curve increases.

5. Conclusions

A method for calculating the buckling loads and the buckling shapes of continuous beams with unilateral intermediate constraints was presented. The paper is based on the theory of elastic stability using the Euler equilibrium method. This method is appropriately modified in order to take into account both the existence of the unilateral support and the presence of arbitrary initial geometric imperfections which do not violate the unilateral constraints. As a result, the instability loads are calculated, depending on the type and amplitude of the initial imperfections. Due to the fact that the increase of the deflections lead to a rather significant development of the second order bending moments, the ultimate load that the beam is able to sustain is calculated after the consideration of the actual strength of the beam's cross section in compression and bending. The method can be easily implemented without the usage of advanced finite element software and can be extended in order to handle continuous beams with more than one unilateral constraints.

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APPENDIX: The case of the geometrically perfect beam – A homogeneous constrained BVP

A.1 Formulation

As it was shown in the paper, the solution of the geometrically imperfect structure (case of a non-homogeneous constrained BVP) is strongly connected with the eigenvalues of the corresponding homogeneous BVP, i.e. with the case of the geometrically perfect beam. This conclusion results from the fact that the poles of the deflection curve of the geometrically imperfect beam constitute eigenvalues of the bifurcation problem of the geometrically perfect beam. Additionally, the utilization of the proposed calculation procedure of Section 2.3, requires the calculation of these eigenvalues.

Following the same considerations as described in the formulation of the unilateral contact buckling problem of the geometrically imperfect beam, the next two homogeneous equations can be constructed for the case of the geometrically perfect beam:

$$\frac{d^4 w_1(x_1)}{dx_1^4} + \lambda^2 \frac{d^2 w_1(x_1)}{dx_1^2} = 0 \quad x_1 \in [0, aL] \quad (\text{A.1})$$

$$\frac{d^4 w_2(x_2)}{dx_2^4} + \lambda^2 \frac{d^2 w_2(x_2)}{dx_2^2} = 0 \quad x_2 \in [0, (1-a)L] \quad (\text{A.2})$$

If a non-trivial solution for the above equations exists, the beam can be in equilibrium in a bended configuration different from the initial straight line one (bifurcation equilibrium state). Therefore, the solution of the above equations gives the transverse deflections w_1 , w_2 of the beam at any point, as a function of the compressive load P . The boundary conditions of the problem are the same as in the case of the imperfect beam. Due to the fact that the intermediate constraint at point C (Fig 1) is of unilateral

type, the buckling problem of the continuous beam with the unilateral constraint is transformed to a constrained BVP, defined by equations (A.1), (A.2), the additional equations (2.9)-(2.16) which describe the equality boundary conditions of the problem, the inequality constraints (2.17) and (2.18) and the complementarity condition (2.19).

A.2 Examination of the BVP for all the possible contact situations

The BVP discussed in the previous paragraph consists of homogeneous fourth order differential equations, which have, respectively, general solutions of the following form:

$$w_1(x_1) = A_1 \cos \lambda x_1 + B_1 \sin \lambda x_1 + C_1 x_1 + D_1 \quad (\text{A.3})$$

$$w_2(x_2) = A_2 \cos \lambda x_2 + B_2 \sin \lambda x_2 + C_2 x_2 + D_2 \quad (\text{A.4})$$

Due to the existence of the unilateral constraint, an examination of the solution for all the possible contact situations is required, which is presented in the sequel.

A.2.1 Inactive constraint, $R=0$ and $u < 0$

In the case where the unilateral constraint is inactive ($R=0$ and $u < 0$) the BVP produces infinite number of solutions. More specifically, the demand of having non-trivial solution leads to:

$$A_1 = A_2 = C_1 = C_2 = D_1 = D_2 = 0 \quad \text{and} \quad B_1 = B_2 \frac{\sin(\lambda(1-a)L)}{\sin(\lambda aL)} \neq 0 \quad (\text{A.5})$$

and to the following buckling equation:

$$\cot(\lambda aL) + \cot(\lambda(1-a)L) = 0 \quad (\text{A.6})$$

The existence of solution for this specific situation indicates a bended deformed equilibrium configuration different from the initial straight line one (bifurcation). The buckling equation (A.6) gives the eigenvalues of the BVP which are the **buckling loads** of the problem under consideration. From the obtained eigenvalues, only these which produce eigenmodes that satisfy the following restrictions are accepted:

$$B_1 \sin(\lambda aL) < 0 \quad \text{and} \quad B_2 \sin(\lambda(1-a)L) < 0 \quad (\text{A.7})$$

A.2.2 Active constraint, $R < 0$ and $u = 0$

When the constraint is active (e.g. $R < 0$ and $u = 0$) the BVP leads to:

$$A_1 = A_2 = D_1 = D_2 = 0, B_1 = B_2 \frac{\sin(\lambda(1-a)L)}{\sin(\lambda aL)} \neq 0, C_1 = \frac{-B_1 \sin(\lambda aL)}{aL}, C_2 = \frac{aC_1}{1-a} \quad (\text{A.8})$$

and to the following buckling equation:

$$\cot(\lambda aL) + \cot(\lambda(1-a)L) = \frac{1}{\lambda(1-a)aL}. \quad (\text{A.9})$$

The extracted eigenvalues are accepted only if the corresponding eigenmodes fulfill the following restriction:

$$R < 0 \Rightarrow EI\lambda^3 [B_1 \cos(\lambda aL) + B_2 \cos(\lambda(1-a)L)] < 0. \quad (\text{A.10})$$

A.2.3 Neutral contact status, $R = 0$ and $u = 0$

This is a special case, where the reaction force R and the transverse displacement u at the position of the unilateral support are simultaneously equal to zero. The solution leads to:

$$A_1 = A_2 = C_1 = C_2 = D_1 = D_2 = 0 \text{ and } B_1 = -B_2 \frac{\cos(\lambda(1-a)L)}{\cos(\lambda aL)} \neq 0. \quad (\text{A.11})$$

In this particular contact situation the infinite number of eigenvalues is calculated directly through the following formula:

$$P_{(k+n)} = \frac{(k+n)^2 \pi^2 EI}{L^2}, \quad k, n \in \mathbb{Z}. \quad (\text{A.12})$$

The eigenvalues obtained from the utilization of equation (A.12) are accepted only if the corresponding eigenmodes satisfy the complementarity condition (2.19).

Figure Captions

Fig.1 The buckling problem of the continuous beam with the unilateral constraint.

Fig.2.The considered conventions for the positive internal forces and displacements.

Fig. 3. The beam of the considered example.

Fig. 4. The first five eigenvalues and the corresponding eigenmodes of the perfect continuous beam.

Fig. 5. The imperfect continuous beam.

Fig. 6. Progressive deflection of the beam due to increasing loading.

Fig. 7. Interaction diagram for the HEB 220 cross-section according to the provisions of Eurocode 3.

Fig. 8. Variation of the second order bending moment for various loading until failure.

Fig. 9. Axial failure load vs. maximum imperfection amplitude.

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Research highlights

- > Analytical treatment of the unilateral buckling problem.
- > Arbitrary geometric imperfections can be considered.
- > General formulation, considering the actual strength of the beam.

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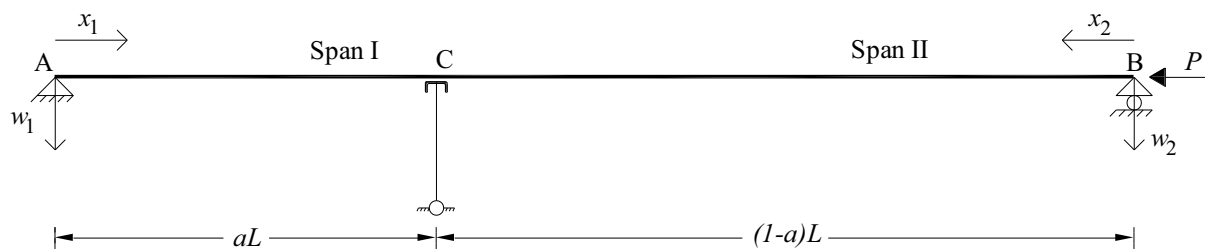


Fig.1 The buckling problem of the continuous beam with the unilateral constraint.

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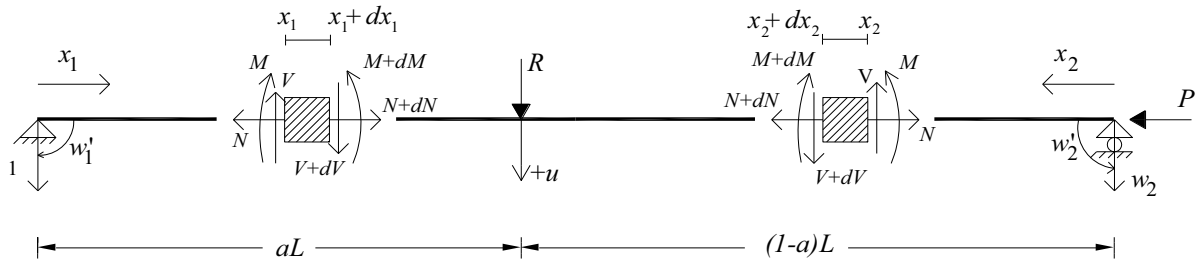


Fig.2. The considered conventions for the positive internal forces and displacements.

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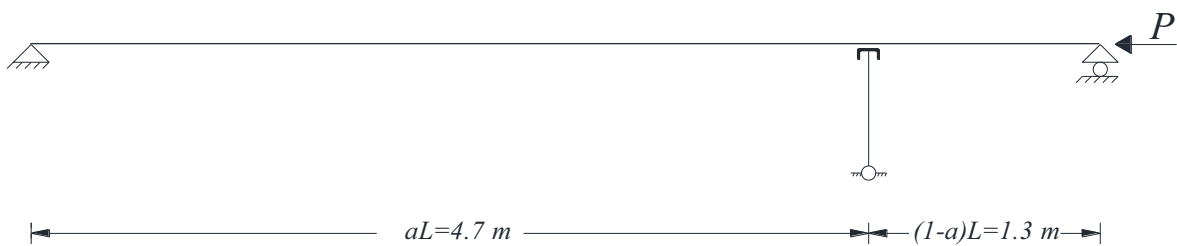


Fig. 3. The beam of the considered example.

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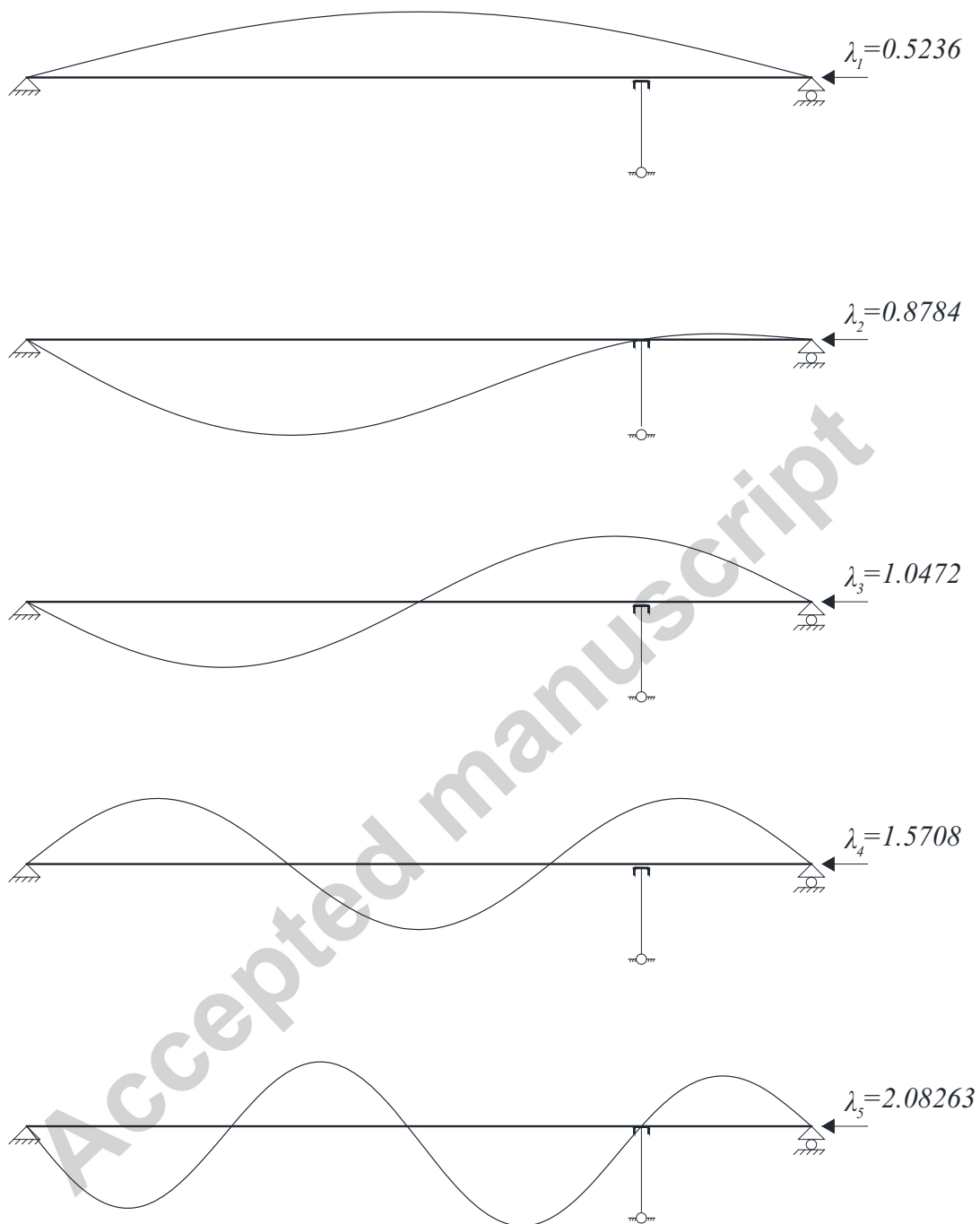


Fig. 4. The first five eigenvalues and the corresponding eigenmodes of the perfect continuous beam.

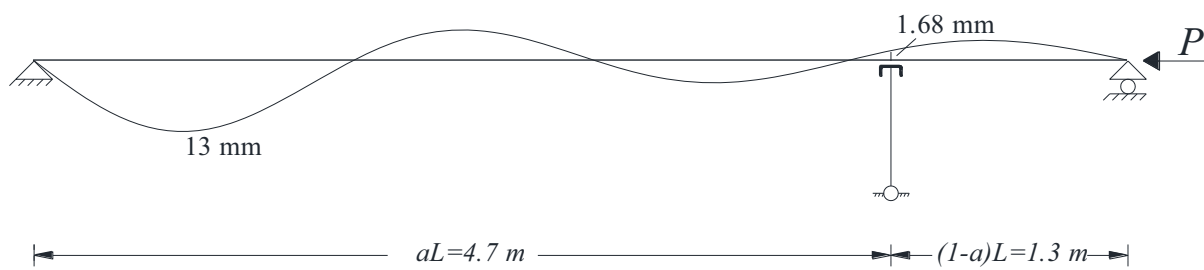


Fig. 5. The imperfect continuous beam.

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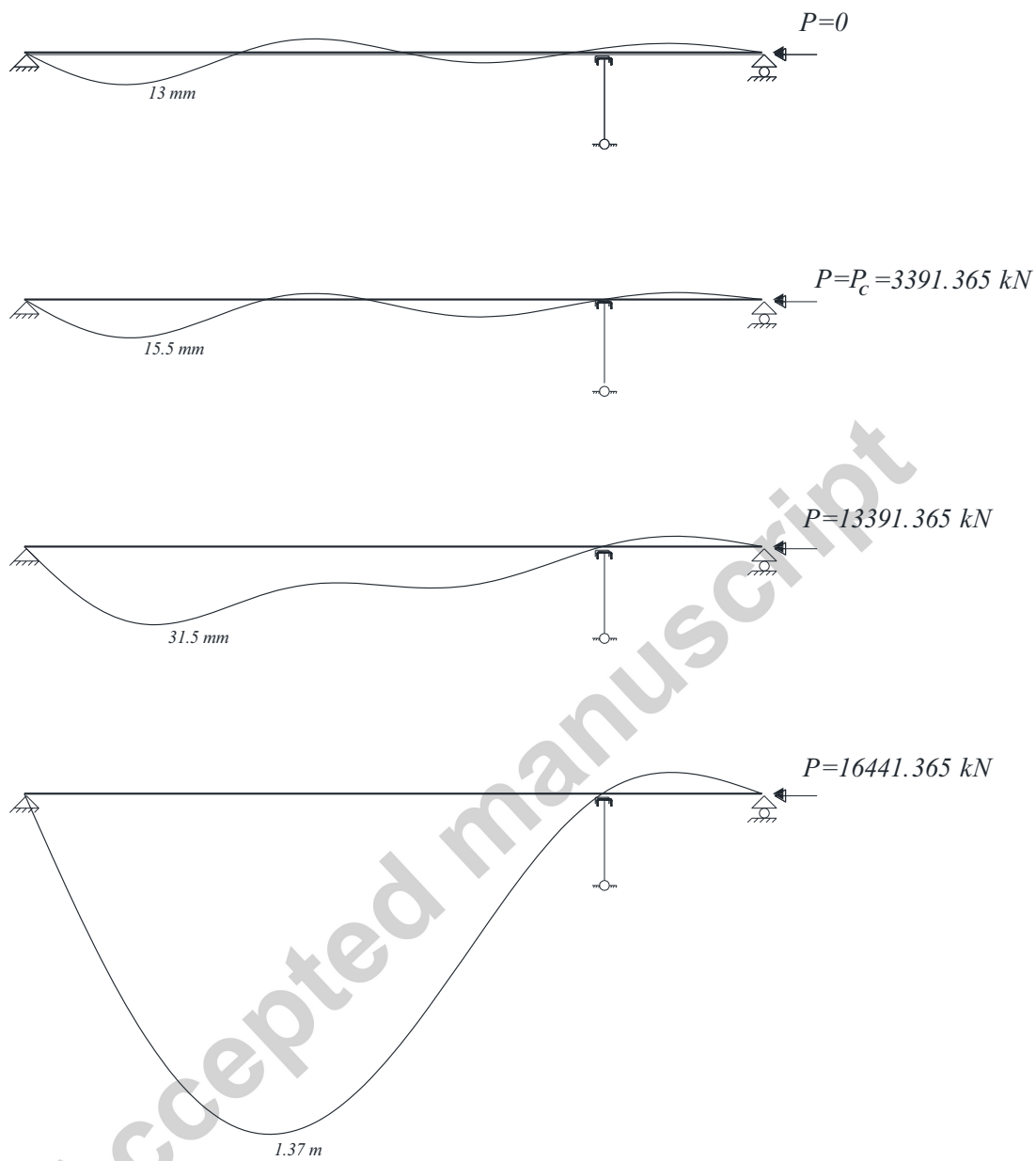


Fig. 6. Progressive deflection of the beam due to increasing loading.

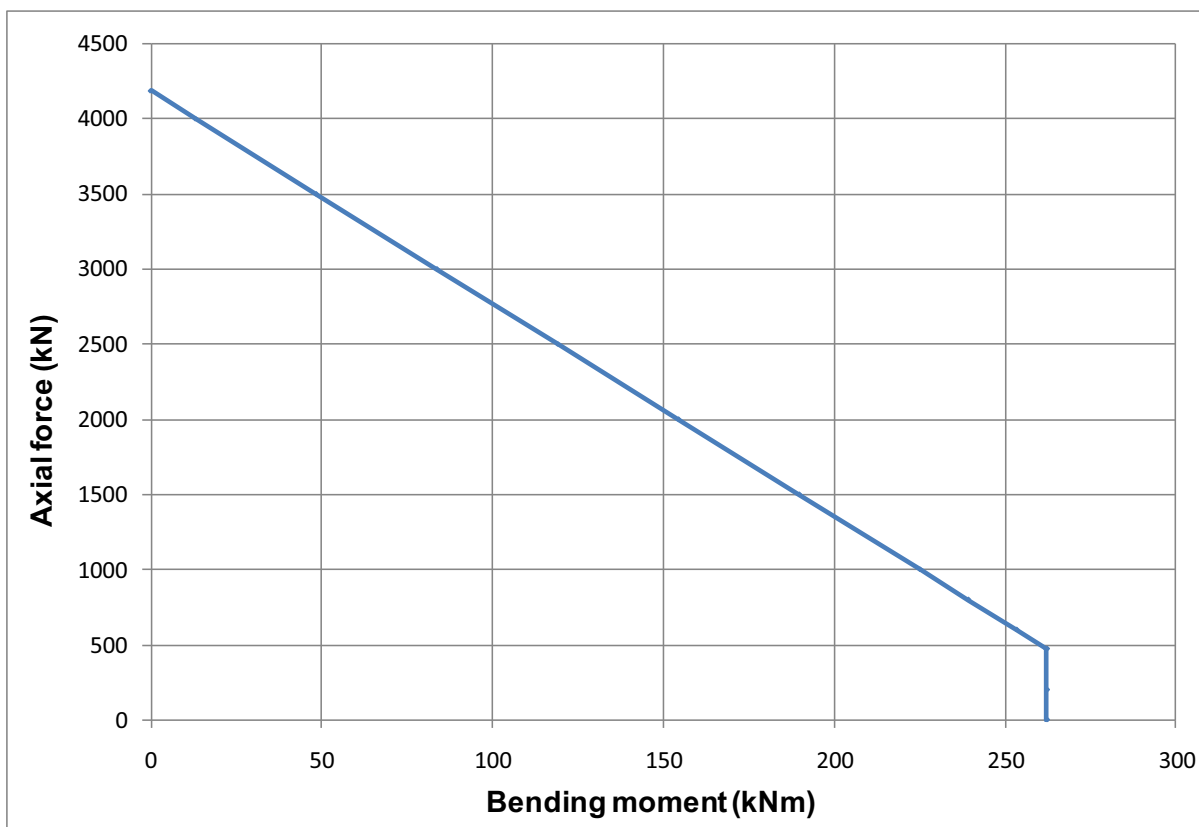


Fig. 7. Interaction diagram for the HEB 220 cross-section according to the provisions of Eurocode 3.

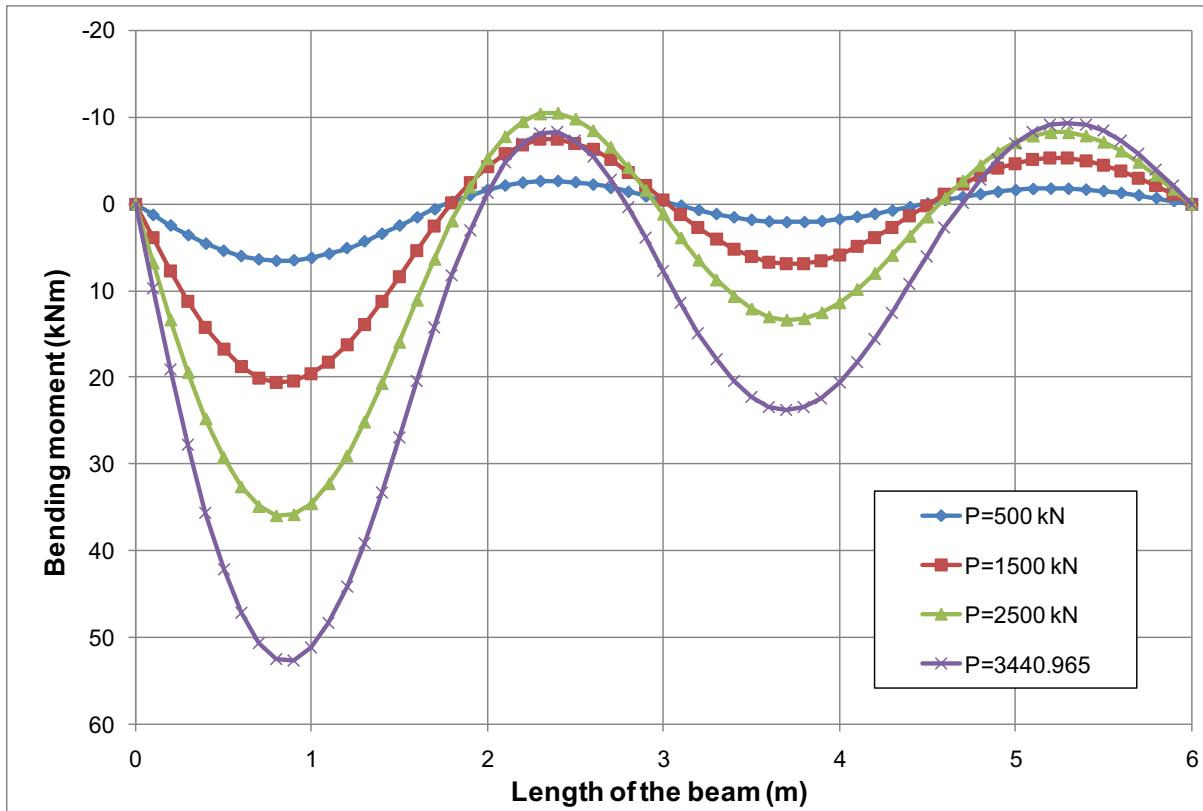


Fig. 8. Variation of the second order bending moment for various loading until failure.

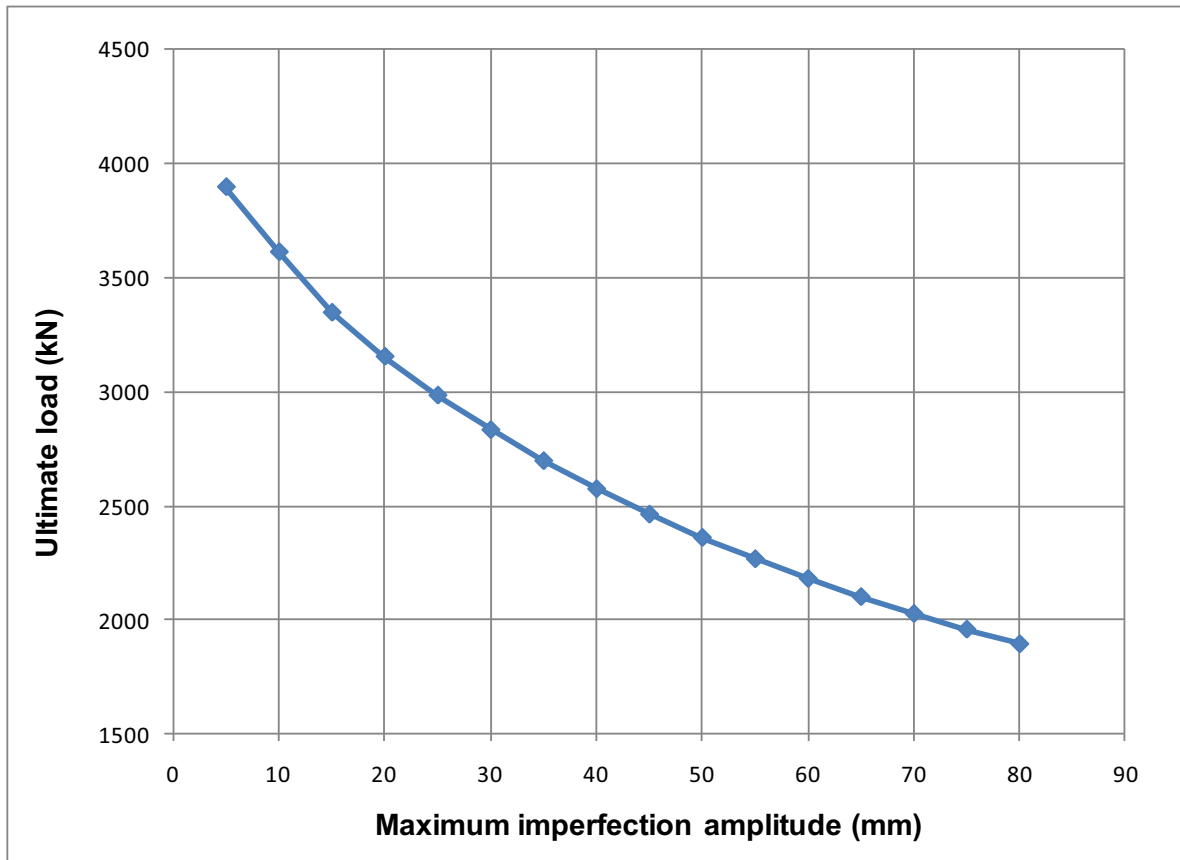


Fig. 9. Axial failure load vs. maximum imperfection amplitude.