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# Total variation distance between two double Wiener-Itô integrals

Rola Zintout\*

*Université de Lorraine*

**Abstract:** Using an approach recently developed by Nourdin and Poly [5], we improve the rate in an inequality for the total variation distance between two double Wiener-Itô integrals originally due to Davydov and Martynova [2]. An application to the rate of convergence of a functional of a correlated two-dimensional fractional Brownian motion towards the Rosenblatt random variable is then given, following a previous study by Maejima and Tudor [3].

**Keywords:** Convergence in total variation; Malliavin calculus; double Wiener-Itô integral; Rosenblatt process.

**2000 Mathematics Subject Classification:** 60F05, 60G15, 60H05, 60H07.

## 1 Introduction

Suppose that  $X = \{X(h), h \in \mathfrak{H}\}$  is an isonormal Gaussian process on a real separable infinite-dimensional Hilbert space  $\mathfrak{H}$ . For any integer  $p \geq 1$ , let  $\mathfrak{H}^{\otimes p}$  be the  $p$ th tensor product of  $\mathfrak{H}$ . Also, denote by  $\mathfrak{H}^{\odot p}$  the  $p$ th symmetric tensor product.

The following statement is due to Davydov and Martynova [2].

**Theorem 1.1.** *Let  $(f_n)$  be a sequence of  $\mathfrak{H}^{\odot 2}$  that converges to  $f_\infty$  in  $\mathfrak{H}^{\otimes 2}$ . Let  $I_2(f_n)$ ,  $n \in \mathbb{N} \cup \{\infty\}$ , denote the double Wiener-Itô integral of  $f_n$  with respect to  $X$ . Then*

$$\sup_{C \in \mathcal{B}(\mathbb{R})} |P(I_2(f_n) \in C) - P(I_2(f_\infty) \in C)| \leq c \sqrt{\|f_n - f_\infty\|_{\mathfrak{H}^{\otimes 2}}}, \quad (1.1)$$

where  $\mathcal{B}(\mathbb{R})$  stands for the set of Borelian sets of  $\mathbb{R}$ .

To each  $f_\infty \in \mathfrak{H}^{\odot 2}$ , one may associate the following Hilbert-Schmidt operator:

$$A_{f_\infty} : \mathfrak{H} \rightarrow \mathfrak{H}, \quad g \mapsto \langle f_\infty, g \rangle_{\mathfrak{H}}. \quad (1.2)$$

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\*Université de Lorraine, Institut Élie Cartan de Lorraine, UMR 7502, Vandoeuvres-lès-Nancy, F-54506, France and CNRS, Institut Élie Cartan de Lorraine, UMR 7502, Vandoeuvres-lès-Nancy, F-54506, France; rola.zintout@hotmail.com

Let  $\lambda_{\infty,k}$ ,  $k \geq 1$ , indicate the eigenvalues of  $A_{f_\infty}$ . In many situations of interest (see below for an explicit example), it happens that the following property, that we label for further use, is satisfied for  $f_\infty$ :

$$\text{the cardinality of } \{k : \lambda_{\infty,k} \neq 0\} \text{ is infinite.} \quad (1.3)$$

The aim of this paper is to take advantage of (1.3) in order to improve (1.1) by a factor 2. More precisely, relying to an approach recently developed by Nourdin and Poly in [5], we shall prove the following result, compare with (1.1):

**Theorem 1.2.** *Let  $f_\infty$  be an element of  $\mathfrak{H}^{\odot 2}$  satisfying (1.3). Let  $(f_n)$  be a sequence of  $\mathfrak{H}^{\odot 2}$  that converges to  $f_\infty$  in  $\mathfrak{H}^{\otimes 2}$ . Then, there exists  $c > 0$  (depending only on  $f_\infty$ ) such that, for all  $n$ ,*

$$\sup_{C \in \mathcal{B}(\mathbb{R})} |P(I_2(f_n) \in C) - P(I_2(f_\infty) \in C)| \leq c \|f_n - f_\infty\|_{\mathfrak{H}^{\otimes 2}}. \quad (1.4)$$

To illustrate the use of Theorem 1.2 in a concrete situation, we consider the following example taken from Maejima and Tudor [3]. Let  $B^{H_1}$ ,  $B^{H_2}$  be two fractional Brownian motions with Hurst parameters  $H_1, H_2$ , respectively. We assume that both  $H_1$  and  $H_2$  are strictly bigger than  $\frac{1}{2}$ . We further assume that the two fractional Brownian motions  $B^{H_1}$  and  $B^{H_2}$  can be expressed as Wiener integrals with respect to the *same* two-sided Brownian motion  $W$ , meaning in particular that  $B^{H_1}$  and  $B^{H_2}$  are *not* independent. Precisely, we set

$$B_t^{H_1} = c(H_1) \int_{\mathbb{R}} dW_y \int_0^t (u-y)_+^{H_1 - \frac{3}{2}} du, \quad t \geq 0, \quad (1.5)$$

$$B_t^{H_2} = c(H_2) \int_{\mathbb{R}} dW_y \int_0^t (u-y)_+^{H_2 - \frac{3}{2}} du, \quad t \geq 0, \quad (1.6)$$

where the constants  $c(H_1)$  and  $c(H_2)$  are chosen so that  $E[(B_1^{H_1})^2] = E[(B_1^{H_2})^2] = 1$ . Define

$$Z_n = n^{1-H_1-H_2} \sum_{k=0}^{n-1} \left[ \frac{(B_{\frac{k+1}{n}}^{H_1} - B_{\frac{k}{n}}^{H_1})(B_{\frac{k+1}{n}}^{H_2} - B_{\frac{k}{n}}^{H_2})}{E \left[ (B_{\frac{k+1}{n}}^{H_1} - B_{\frac{k}{n}}^{H_1})(B_{\frac{k+1}{n}}^{H_2} - B_{\frac{k}{n}}^{H_2}) \right]} - 1 \right]. \quad (1.7)$$

When  $H_1 = H_2 = H$ , observe that (1.7) is related to the quadratic variation of  $B^H$ . In [3], the following extension of a classical result by Taqqu [8] is shown:

**Proposition 1.3.** *Assume that  $H_1 > \frac{1}{2}$ ,  $H_2 > \frac{1}{2}$  and  $H_1 + H_2 > \frac{3}{2}$ . Then,  $Z_n$  converges as  $n \rightarrow \infty$  in  $L^2(\Omega)$  to  $Z_\infty$ , given by*

$$Z_\infty = b(H_1, H_2) \int_{\mathbb{R}^2} dW_x dW_y \int_0^1 (s-x)_+^{H_1-3/2} (s-y)_+^{H_2-3/2} ds, \quad (1.8)$$

where  $b(H_1, H_2)$  is a normalizing explicit constant whose precise value does not matter in the sequel.

In the present paper, by relying to (1.4) we are able to associate an explicit rate to the convergence  $Z_n \xrightarrow{L^2} Z_\infty$  of Proposition 1.3, namely,

$$\sup_{C \in \mathcal{B}(\mathbb{R})} |P(Z_n \in C) - P(Z_\infty \in C)| = O(n^{\frac{3}{2} - H_1 - H_2}). \quad (1.9)$$

When  $H_1 = H_2 = H$ , the rate  $\frac{3}{2} - 2H$  we have obtained in (1.9) is better (by a power 2) than the one computed by Breton and Nourdin in [1], precisely because our inequality (1.4) improves the inequality (1.1) of Davydov and Martynova by a power 2. Actually, we conjecture that our rate in (1.9) is the best possible one can achieve.

The rest of the paper is organized as follows. Section 2 contains some preliminary material on Malliavin calculus. In Section 3 we prove Theorem 1.2. Finally, Section 4 contains our proof of (1.9).

## 2 Preliminaries

Let  $\mathfrak{H}$  be a real separable infinite-dimensional Hilbert space. For any integer  $p \geq 1$ , let  $\mathfrak{H}^{\otimes p}$  be the  $p$ th tensor product of  $\mathfrak{H}$ . Also, we denote by  $\mathfrak{H}^{\odot p}$  the  $p$ th symmetric tensor product.

Suppose that  $X = \{X(h), h \in \mathfrak{H}\}$  is an isonormal Gaussian process on  $\mathfrak{H}$ , defined on some probability space  $(\Omega, \mathcal{F}, P)$ . Assume from now on that  $\mathcal{F}$  is generated by  $X$ . For every integer  $p \geq 1$ , let  $\mathcal{H}_p$  be the  $p$ th Wiener chaos of  $X$ , that is, the closed linear subspace of  $L^2(\Omega)$  generated by the random variables  $\{H_p(X(h)), h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$ , where  $H_p$  is the  $p$ th Hermite polynomial defined by

$$H_p(x) = \frac{(-1)^p}{p!} e^{x^2/2} \frac{d^p}{dx^p} (e^{-x^2/2}).$$

We denote by  $\mathcal{H}_0$  the space of constant random variables. For any  $p \geq 1$ , the mapping  $I_p(h^{\otimes p}) = p! H_p(X(h))$  provides a linear isometry between  $\mathfrak{H}^{\odot p}$  (equipped with the modified norm  $\sqrt{p!} \|\cdot\|_{\mathfrak{H}^{\otimes p}}$ ) and  $\mathcal{H}_p$  (equipped with the  $L^2(\Omega)$  norm). We call  $I_p(f)$  the  $p$ th multiple Wiener-Itô integral of kernel  $f$ . For  $p = 0$ , by convention  $\mathcal{H}_0 = \mathbb{R}$ , and  $I_0$  is the identity map. In particular, when  $f, g \in \mathfrak{H}^{\odot p}$ , observe that

$$E \left[ (I_p(f) - I_p(g))^2 \right] = p! \|f - g\|_{\mathfrak{H}^{\otimes p}}^2. \quad (2.1)$$

It is well-known (Wiener chaos expansion) that  $L^2(\Omega)$  can be decomposed into the infinite orthogonal sum of the spaces  $\mathcal{H}_p$ . That is, any square integrable random variable  $F \in L^2(\Omega)$  admits the following chaotic expansion:

$$F = \sum_{p=0}^{\infty} I_p(f_p), \quad (2.2)$$

where  $f_0 = E[F]$ , and the  $f_p \in \mathfrak{H}^{\odot p}$ ,  $p \geq 1$ , are uniquely determined by  $F$ . For every  $p \geq 0$ , we denote by  $J_p$  the orthogonal projection operator on the  $p$ th Wiener chaos. In particular, if  $F \in L^2(\Omega)$  is as in (2.2), then  $J_p F = I_p(f_p)$  for every  $p \geq 0$ .

Let us now introduce some basic elements of the Malliavin calculus with respect to the isonormal Gaussian process  $X$ . We refer the reader to Nourdin and Peccati [4] or Nualart [6] for a more detailed presentation of these notions. Let  $\mathcal{S}$  be the set of all smooth and cylindrical random variables of the form

$$F = g(X(\phi_1), \dots, X(\phi_n)), \quad (2.3)$$

where  $n \geq 1$ ,  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  is an infinitely differentiable function with compact support, and  $\phi_i \in \mathfrak{H}$ . The Malliavin derivative of  $F$  with respect to  $X$  is the element of  $L^2(\Omega, \mathfrak{H})$  defined as

$$DF = \sum_{i=1}^n \frac{\partial g}{\partial x_i}(X(\phi_1), \dots, X(\phi_n)) \phi_i.$$

By iteration, one can define the  $k$ th derivative  $D^k F$  for every  $k \geq 2$ , which is an element of  $L^2(\Omega, \mathfrak{H}^{\odot k})$ .

For  $k \geq 1$  and  $p \geq 1$ ,  $\mathbb{D}^{k,p}$  denotes the closure of  $\mathcal{S}$  with respect to the norm  $\|\cdot\|_{\mathbb{D}^{k,p}}$ , defined by the relation

$$\|F\|_{\mathbb{D}^{k,p}}^p = E[|F|^p] + \sum_{i=1}^k E\left(\|D^i F\|_{\mathfrak{H}^{\otimes i}}^p\right).$$

The Malliavin derivative  $D$  verifies the following chain rule. If  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuously differentiable with bounded partial derivatives and if  $F = (F_1, \dots, F_n)$  is a vector of elements of  $\mathbb{D}^{1,2}$ , then  $\varphi(F) \in \mathbb{D}^{1,2}$  and

$$D\varphi(F) = \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i}(F) DF_i. \quad (2.4)$$

Observe that (2.4) still holds when  $\varphi$  is Lipschitz and the law of  $F$  has a density with respect to the Lebesgue measure on  $\mathbb{R}^n$  (see, e.g., Proposition 1.2.3 in [6]).

We denote by  $\delta$  the adjoint of the operator  $D$ , also called the divergence operator. A random element  $u \in L^2(\Omega, \mathfrak{H})$  belongs to the domain of  $\delta$ , noted  $\text{Dom}\delta$ , if and only if it verifies

$$|E(\langle DF, u \rangle_{\mathfrak{H}})| \leq c_u \sqrt{E(F^2)}$$

for any  $F \in \mathbb{D}^{1,2}$ , where  $c_u$  is a constant depending only on  $u$ . If  $u \in \text{Dom}\delta$ , then the random variable  $\delta(u)$  is defined by the duality relationship:

$$E(F\delta(u)) = E(\langle DF, u \rangle_{\mathfrak{H}}), \quad (2.5)$$

which holds for every  $F \in \mathbb{D}^{1,2}$ . We will also make use of the following relationships, valid for  $F \in \mathbb{D}^{1,2}$  and  $u \in \text{Dom}\delta$  such that  $Fu \in L^2(\Omega, \mathfrak{H})$ :

$$F\delta(u) = \delta(Fu) + \langle DF, u \rangle_{\mathfrak{H}} \quad (2.6)$$

$$E(\delta(u)^2) = E\|Du\|_{\mathfrak{H}^{\otimes 2}}^2 + E\|u\|_{\mathfrak{H}}^2. \quad (2.7)$$

The operator  $L$  is defined on the Wiener chaos expansion as

$$L = \sum_{q=0}^{\infty} -qJ_q,$$

and is called the infinitesimal generator of the Ornstein-Uhlenbeck semigroup. The domain of this operator in  $L^2(\Omega)$  is the set

$$\text{Dom}L = \left\{ F \in L^2(\Omega) : \sum_{q=1}^{\infty} q^2 \|J_q F\|_{L^2(\Omega)}^2 < \infty \right\} = \mathbb{D}^{2,2}.$$

There is an important relationship between the operators  $D$ ,  $\delta$  and  $L$ . A random variable  $F$  belongs to the domain of  $L$  if and only if  $F \in \text{Dom}(\delta D)$  (i.e.  $F \in \mathbb{D}^{1,2}$  and  $DF \in \text{Dom}\delta$ ), and in this case

$$\delta DF = -LF. \quad (2.8)$$

If  $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$  (with  $\mu$  non-atomic), then the derivative of a random variable  $F$  as in (2.2) can be identified with the element of  $L^2(A \times \Omega)$  given by

$$D_a F = \sum_{q=1}^{\infty} q I_{q-1}(f_q(\cdot, a)), \quad a \in A. \quad (2.9)$$

At this stage, we observe that an easy calculation leads to the following identity for  $F = I_p(f)$  and  $G = I_p(g)$  (with  $f, g \in \mathfrak{H}^{\otimes p}$ ), that we label for further use:

$$E\left(\|DF - DG\|_{\mathfrak{H}}^2\right) = pp! \|f - g\|_{\mathfrak{H}^{\otimes p}}^2. \quad (2.10)$$

We also have the following identity, valid for  $F \in \mathbb{D}^{2,2}$  and  $G \in \mathbb{D}^{1,2}$ :

$$E[LF \times G] = -E[\langle DF, DG \rangle_{\mathfrak{H}}], \quad (2.11)$$

from which we deduce, for  $F_1, F_2, F_3 \in \mathbb{D}^{2,4}$ ,

$$\begin{aligned} E[L(F_1 F_2) F_3] &= -E[\langle D(F_1 F_2), DF_3 \rangle_{\mathfrak{H}}] \\ &= -E[F_1 \langle DF_2, DF_3 \rangle_{\mathfrak{H}}] - E[F_2 \langle DF_1, DF_3 \rangle_{\mathfrak{H}}] \end{aligned}$$

so that

$$E[F_2 \langle DF_1, DF_3 \rangle_{\mathfrak{H}}] = -E[L(F_1 F_2) F_3] - E[F_1 \langle DF_2, DF_3 \rangle_{\mathfrak{H}}]. \quad (2.12)$$

Finally, the following lemma will play a crucial role in our forthcoming calculations.

**Lemma 2.1.** *Let  $F_\infty = I_2(f_\infty)$ , with  $f_\infty \in \mathfrak{H}^{\odot 2}$  satisfying (1.3). Then, for all  $r \geq 1$ , we have*

$$E[|F_\infty|^{2r}] < \infty, \quad E[\|DF_\infty\|_{\mathfrak{H}}^{2r}] < \infty, \quad (2.13)$$

and

$$E\left[\frac{1}{\|DF_\infty\|_{\mathfrak{H}}^{2r}}\right] < \infty. \quad (2.14)$$

*Proof.* The proof of (2.13) is classical and follows directly from the hypercontractivity property of multiple Wiener-Itô integrals. So, let us only focus on (2.14). Let  $e_k$ ,  $k \geq 1$ , be the eigenvectors associated to the eigenvalues  $\lambda_{f_\infty, k}$  of  $A_{f_\infty}$ , see (1.2). Observe that they form an orthonormal system in  $\mathfrak{H}$  and that  $f_\infty$  may be expanded as

$$f_\infty = \sum_{k=1}^{\infty} \lambda_{f_\infty, k} e_k \otimes e_k, \quad (2.15)$$

implying in turn that

$$F_\infty = I_2(f_\infty) = \sum_{k=1}^{\infty} \lambda_{f_\infty, k} (X(e_k)^2 - 1).$$

We have

$$\begin{aligned} E\left[\frac{1}{\|DF_\infty\|_{\mathfrak{H}}^{2r}}\right] &= \int_0^\infty P\left(\frac{1}{\|DF_\infty\|_{\mathfrak{H}}^{2r}} \geq x\right) dx \\ &= \int_0^1 P\left(\frac{1}{\|DF_\infty\|_{\mathfrak{H}}^{2r}} \geq x\right) dx + \int_1^\infty P\left(\frac{1}{\|DF_\infty\|_{\mathfrak{H}}^{2r}} \geq x\right) dx \\ &\leq 1 + \int_1^\infty P\left(\|DF_\infty\|_{\mathfrak{H}}^{2r} \leq x^{-1/r}\right) dx \\ &\leq 1 + r \int_0^1 P\left(\|DF_\infty\|_{\mathfrak{H}}^{2r} \leq u\right) \frac{du}{u^{r+1}}. \end{aligned}$$

As a result, we are left to show that

$$P\left(\|DF_\infty\|_{\mathfrak{H}}^{2r} \leq u\right) = O(u^{r+\frac{1}{2}}) \quad \text{as } u \downarrow 0. \quad (2.16)$$

An immediate calculation leads to

$$\|DF_\infty\|_{\mathfrak{H}}^2 = 4 \sum_{k=1}^{\infty} \lambda_{f_\infty, k}^2 X(e_k)^2, \quad (2.17)$$

where the  $X(e_k)$  are independent  $N(0, 1)$  random variables. Therefore, for any  $u > 0$ ,

$$\begin{aligned} P(\|DF_\infty\|_{\mathfrak{H}}^{2r} \leq u) &\leq P(4\lambda_{f_\infty,1}^2 X(e_1)^2 \leq u, \dots, 4\lambda_{f_\infty,2r+1}^2 X(e_{2r+1})^2 \leq u) \\ &= P\left(|X(e_1)| \leq \frac{\sqrt{u}}{2|\lambda_{f_\infty,1}|}\right) \dots P\left(|X(e_{2r+1})| \leq \frac{\sqrt{u}}{2|\lambda_{f_\infty,2r+1}|}\right) \\ &\leq \frac{u^{r+\frac{1}{2}}}{(2\pi)^{r+\frac{1}{2}}|\lambda_{f_\infty,1}| \dots |\lambda_{f_\infty,2r+1}|} \end{aligned}$$

and (2.16) is checked, thus concluding the proof.  $\square$

### 3 Proof of Theorem 1.2

Let  $f_\infty \in \mathfrak{H}^{\odot 2}$  satisfying (1.3). Let  $(f_n)$  be a sequence of  $\mathfrak{H}^{\odot 2}$  that converges to  $f_\infty$  in  $\mathfrak{H}^{\odot 2}$ . Write  $F_n = I_2(f_n)$  and  $F_\infty = I_2(f_\infty)$ . Our aim in this Section 3 is to show that there exists  $c > 0$  (depending only on  $f_\infty$ ) such that, for all Borelian set  $C$  and all  $n$ ,

$$|P(F_n \in C) - P(F_\infty \in C)| \leq c\|f_n - f_\infty\|_{\mathfrak{H}^{\odot 2}}. \quad (3.18)$$

First of all, relying to the Lebesgue's monotone convergence theorem, we notice that it is not a loss of generality to assume that  $C$  is bounded in (3.18).

Now, we divide the proof of Theorem 1.2 into several steps and we stress that, in what follows, the constant  $c$  shall denote a generic constant only depending on  $f_\infty$  (not on  $n$ !) and whose value may change from one line to another.

*Step 1.* Thanks to (2.17), we have  $\|DF_\infty\|_{\mathfrak{H}}^2 \geq 4\lambda_{f_\infty,k}^2 X(e_k)^2$  for some  $k$  with  $\lambda_{f_\infty,k} \neq 0$  (assumption (1.3)). Since  $X(e_k) \neq 0$  a.s., one has that  $\|DF_\infty\|_{\mathfrak{H}} > 0$  a.s. As a result, one can write

$$|P(F_n \in C) - P(F_\infty \in C)| = \left| E \left[ (1_{F_n \in C} - 1_{F_\infty \in C}) \frac{\|DF_\infty\|_{\mathfrak{H}}}{\|DF_\infty\|_{\mathfrak{H}}} \right] \right|. \quad (3.19)$$

The chain rule for Lipschitz function (for  $n$  large enough, note that  $F_n$  has a density with respect to the Lebesgue measure by Shigekawa theorem [7]) leads to

$$D\left(\int_{-\infty}^{F_n} 1_C(x) dx\right) = 1_C(F_n)DF_n.$$

We then have

$$|P(F_n \in C) - P(F_\infty \in C)| \leq |A_n| + |B_n|, \quad (3.20)$$



with

$$A_n = E \left[ \frac{\left\langle D \left( \int_{F_\infty}^{F_n} 1_C(x) dx \right), DF_\infty \right\rangle_{\mathfrak{H}}}{\|DF_\infty\|_{\mathfrak{H}}^2} \right] \quad (3.21)$$

$$B_n = E \left[ \frac{1_C(F_n) \langle D(F_\infty - F_n), DF_\infty \rangle_{\mathfrak{H}}}{\|DF_\infty\|_{\mathfrak{H}}^2} \right]. \quad (3.22)$$

Step 2 (a bound for  $B_n$ ). Using Cauchy-Schwarz inequality twice, one obtains

$$|B_n| \leq E \left[ \frac{\|D(F_\infty - F_n)\|_{\mathfrak{H}}}{\|DF_\infty\|_{\mathfrak{H}}} \right] \leq \sqrt{E \|D(F_\infty - F_n)\|_{\mathfrak{H}}^2} \sqrt{E \left[ \frac{1}{\|DF_\infty\|_{\mathfrak{H}}^2} \right]}.$$

By (2.10), one has  $E \|D(F_\infty - F_n)\|_{\mathfrak{H}}^2 \leq 4 \|f_\infty - f_n\|_{\mathfrak{H}^{\otimes 2}}^2$ , whereas  $E \left[ \frac{1}{\|DF_\infty\|_{\mathfrak{H}}^2} \right]$  is finite by Lemma 2.1. Thus,

$$|B_n| \leq c \|f_\infty - f_n\|_{\mathfrak{H}^{\otimes 2}}$$

with  $c$  only depending on  $f_\infty$ .

Step 3 (a bound for  $A_n$ ). Using (2.12) with  $F_1 = \|DF_\infty\|_{\mathfrak{H}}^{-2}$ ,  $F_2 = F_\infty$  and  $F_3 = \int_{F_\infty}^{F_n} 1_C(x) dx$ , one obtains that  $|A_n| \leq |Q_n| + |R_n|$ , with

$$Q_n = E \left[ L \left( \frac{F_\infty}{\|DF_\infty\|_{\mathfrak{H}}^2} \right) \int_{F_\infty}^{F_n} 1_C(x) dx \right]$$

$$R_n = E \left[ F_\infty \left\langle D \left( \frac{1}{\|DF_\infty\|_{\mathfrak{H}}^2} \right), D \left( \int_{F_\infty}^{F_n} 1_C(x) dx \right) \right\rangle \right].$$

Step 3a (a bound for  $Q_n$ ). We first observe that  $\left| \int_{F_\infty}^{F_n} 1_C(x) dx \right| \leq |F_n - F_\infty|$ , so that

$$|Q_n| \leq E \left[ \left| L \frac{F_\infty}{\|DF_\infty\|_{\mathfrak{H}}^2} \right| \times |F_n - F_\infty| \right] \leq \sqrt{E \left[ \left( L \frac{F_\infty}{\|DF_\infty\|_{\mathfrak{H}}^2} \right)^2 \right]} \sqrt{E (F_n - F_\infty)^2}$$

$$= \sqrt{E \left[ \left( L \frac{F_\infty}{\|DF_\infty\|_{\mathfrak{H}}^2} \right)^2 \right]} \sqrt{2} \|f_n - f_\infty\|_{\mathfrak{H}^{\otimes 2}},$$

where, in the last line, we have used (2.1).

Let us now check that  $E \left[ \left( L \frac{F_\infty}{\|DF_\infty\|_{\mathfrak{H}}^2} \right)^2 \right]$  is finite. Since  $E \left[ (LG)^2 \right] = E \|DG\|_{\mathfrak{H}}^2 + E \|D^2G\|_{\mathfrak{H}^{\otimes 2}}^2$ , one has, using also that  $D^2F_\infty = 2f_\infty$  and noting  $f_\infty \otimes_1 f_\infty$  the first contraction of  $f_\infty$ ,

$$\begin{aligned}
& E \left[ \left( L \frac{F_\infty}{\|DF_\infty\|_{\mathfrak{H}}^2} \right)^2 \right] \\
&= E \left[ \left\| D \left( \frac{F_\infty}{\|DF_\infty\|_{\mathfrak{H}}^2} \right) \right\|_{\mathfrak{H}}^2 \right] + E \left[ \left\| D^2 \left( \frac{F_\infty}{\|DF_\infty\|_{\mathfrak{H}}^2} \right) \right\|_{\mathfrak{H}^{\otimes 2}}^2 \right] \\
&= E \left[ \left\| -4F_\infty \frac{\langle f_\infty, DF_\infty \rangle_{\mathfrak{H}}}{\|DF_\infty\|_{\mathfrak{H}}^4} + \frac{DF_\infty}{\|DF_\infty\|_{\mathfrak{H}}^2} \right\|_{\mathfrak{H}}^2 \right] \\
&+ E \left[ \left\| D \left[ -4F_\infty \frac{\langle f_\infty, DF_\infty \rangle_{\mathfrak{H}}}{\|DF_\infty\|_{\mathfrak{H}}^4} + \frac{DF_\infty}{\|DF_\infty\|_{\mathfrak{H}}^2} \right] \right\|_{\mathfrak{H}^{\otimes 2}}^2 \right] \\
&= E \left[ \left\| -4F_\infty \frac{\langle f_\infty, DF_\infty \rangle_{\mathfrak{H}}}{\|DF_\infty\|_{\mathfrak{H}}^4} + \frac{DF_\infty}{\|DF_\infty\|_{\mathfrak{H}}^2} \right\|_{\mathfrak{H}}^2 \right] \\
&+ E \left[ \left\| -8DF_\infty \otimes \frac{\langle f_\infty, DF_\infty \rangle_{\mathfrak{H}}}{\|DF_\infty\|_{\mathfrak{H}}^4} - 8F_\infty \frac{f_\infty \otimes_1 f_\infty}{\|DF_\infty\|_{\mathfrak{H}}^4} + 32F_\infty \frac{\langle f_\infty, DF_\infty \rangle_{\mathfrak{H}}^{\otimes 2}}{\|DF_\infty\|_{\mathfrak{H}}^6} + 2 \frac{f_\infty}{\|DF_\infty\|_{\mathfrak{H}}^2} \right\|_{\mathfrak{H}^{\otimes 2}}^2 \right].
\end{aligned}$$

Using Lemma 2.1, it is now clear that  $E \left[ \left( L \frac{F_\infty}{\|DF_\infty\|_{\mathfrak{H}}^2} \right)^2 \right]$  is finite. As a result, one deduces that  $|Q_n| \leq c \|f_n - f_\infty\|_{\mathfrak{H}^{\otimes 2}}$ , with  $c$  only depending on  $f_\infty$ .

Step 3b (a bound for  $R_n$ ). Using (2.12), one has

$$\begin{aligned}
R_n &= E \left[ \int_{F_\infty}^{F_n} 1_C(x) dx \times F_\infty \left( -L \left( \frac{1}{\|DF_\infty\|^2} \right) \right) \right] \\
&\quad - E \left[ \int_{F_\infty}^{F_n} 1_C(x) dx \left\langle DF_\infty, D \left( \frac{1}{\|DF_\infty\|^2} \right) \right\rangle_{\mathfrak{H}} \right]
\end{aligned}$$

implying in turn, using Cauchy-Schwarz inequality and the isometry property (2.1),

$$\begin{aligned}
|R_n| &\leq 2 \|f_n - f_\infty\|_{\mathfrak{H}^{\otimes 2}} \tag{3.23} \\
&\quad \times \left[ \sqrt{E \left[ \left( F_\infty L \frac{1}{\|DF_\infty\|_{\mathfrak{H}}^2} \right)^2 \right]} + \sqrt{E \left[ \left\langle DF_\infty, D \left( \frac{1}{\|DF_\infty\|_{\mathfrak{H}}^2} \right) \right\rangle_{\mathfrak{H}}^2 \right]} \right].
\end{aligned}$$

Hence, in order to prove that  $|R_n| \leq c \|f_n - f_\infty\|_{\mathfrak{H}^{\otimes 2}}$  we are left to check that both  $\eta = E \left[ \left\langle DF_\infty, D \left( \frac{1}{\|DF_\infty\|_{\mathfrak{H}}^2} \right) \right\rangle_{\mathfrak{H}}^2 \right]$  and  $\zeta = E \left[ \left( F_\infty L \frac{1}{\|DF_\infty\|_{\mathfrak{H}}^2} \right)^2 \right]$  are finite.

One has

$$\begin{aligned}\eta &= 64 E \left( \left\langle DF_\infty, \frac{\langle f_\infty, DF_\infty \rangle_{\mathfrak{H}}}{\|DF_\infty\|_{\mathfrak{H}}^4} \right\rangle_{\mathfrak{H}}^2 \right) = 64 E \left( \frac{\langle f_\infty, DF_\infty \otimes DF_\infty \rangle_{\mathfrak{H} \otimes 2}^2}{\|DF_\infty\|_{\mathfrak{H}}^8} \right) \\ &\leq 64 \|f_\infty\|_{\mathfrak{H} \otimes 2}^2 E \left( \frac{1}{\|DF_\infty\|_{\mathfrak{H}}^4} \right),\end{aligned}$$

which is finite by Lemma 2.1. Similarly, using (2.6) and then (2.7), one can write

$$\begin{aligned}\zeta &= E \left[ \left( \delta \left( F_\infty D \frac{1}{\|DF_\infty\|_{\mathfrak{H}}^2} \right) + \left\langle DF_\infty, D \frac{1}{\|DF_\infty\|_{\mathfrak{H}}^2} \right\rangle_{\mathfrak{H}} \right)^2 \right] \\ &\leq 2E \left[ \delta \left( F_\infty D \frac{1}{\|DF_\infty\|_{\mathfrak{H}}^2} \right)^2 \right] + 2\eta \leq 2(\gamma + \beta + \eta),\end{aligned}$$

with

$$\gamma = E \left[ \left\| D \left( F_\infty D \frac{1}{\|DF_\infty\|_{\mathfrak{H}}^2} \right) \right\|_{\mathfrak{H} \otimes 2}^2 \right] \quad \text{and} \quad \beta = E \left[ \left\| F_\infty D \frac{1}{\|DF_\infty\|_{\mathfrak{H}}^2} \right\|_{\mathfrak{H}}^2 \right].$$

Using a similar reasoning as above and relying once again to Lemma 2.1, it is straightforward to check that  $\gamma$  and  $\beta$  are finite. As a result,  $\zeta$  is finite. From (3.23), we deduce that  $|R_n| \leq c \|f_n - f_\infty\|_{\mathfrak{H} \otimes 2}$  with  $c$  only depending on  $f_\infty$ .

*Step 4* (conclusion). Since  $|A_n| \leq |Q_n| + |R_n|$  and taking into account (3.20) as well as the results of Steps 2, 3a and 3b, we obtain finally that (3.18) holds true, thus concluding the proof of Theorem 1.2.  $\square$

## 4 Proof of (1.9)

To prove (1.9), we shall apply our Theorem 1.2. The isonormal Gaussian process  $X = \{X(h) : h \in \mathfrak{H}\}$  we consider here is a two-sided Brownian motion  $W = \{W(h) : h \in L^2(\mathbb{R})\}$ . We divide the proof of (1.9) into several steps.

*Step 1.* Recall from (1.7) and (1.8) the definitions of  $Z_n$  and  $Z_\infty$  respectively. In Maejima and Tudor [3], the authors represent  $Z_n$  and  $Z_\infty$  as

$$Z_n = b(H_1, H_2) \times I_2(f_n) \quad \text{and} \quad Z_\infty = b(H_1, H_2) \times I_2(f_\infty),$$

with  $b(H_1, H_2)$  a suitable constant and

$$\begin{aligned}f_n(x, y) &= n \sum_{i=0}^{n-1} \int_{i/n}^{(i+1)/n} \int_{i/n}^{(i+1)/n} (s-x)_+^{H_1-3/2} (s-y)_+^{H_2-3/2} ds \\ f_\infty(x, y) &= \int_0^1 (s-x)_+^{H_1-3/2} (s-y)_+^{H_2-3/2} ds.\end{aligned}$$

We have moreover, see indeed [3, page 180],

$$\|f_n - f_\infty\|_{L^2(\mathbb{R}^2)} = O(n^{\frac{3}{2}-H_1-H_2}) \quad \text{as } n \rightarrow \infty. \quad (4.24)$$

*Step 2.* Let us check that  $f_\infty$  satisfies (1.3). To do so, recall from (2.15) that  $f_\infty$  may be expanded, with  $e_k$  the eigenvectors associated to  $\lambda_{f_\infty, k}$ , as

$$f_\infty(x, y) = \sum_{k \geq 1} \lambda_{\infty, k} e_k(x) e_k(y). \quad (4.25)$$

Let us first show that  $e_k$  is bounded on  $[0, 1]$  when  $\lambda_{\infty, k} \neq 0$ . Indeed, using Cauchy-Schwarz inequality as well as the identity

$$\int_{\mathbb{R}} (t-x)_+^\alpha (s-x)_+^\alpha dx = c_\alpha |t-s|^{2\alpha+1}$$

valid for any  $\alpha > -\frac{1}{2}$  (with  $c_\alpha > 0$  a constant depending only on  $\alpha$ ), one can write

$$\begin{aligned} e_k(y)^2 &= \frac{1}{\lambda_k^2} \left( \int_{\mathbb{R}} e_k(x) dx \int_0^1 ds (s-x)_+^{H_1-3/2} (s-y)_+^{H_2-3/2} \right)^2 \\ &\leq \frac{1}{\lambda_k^2} \int_{\mathbb{R}} dx \int_{[0,1]^2} dt ds (s-x)_+^{H_1-3/2} (s-y)_+^{H_2-3/2} (t-x)_+^{H_1-3/2} (t-y)_+^{H_2-3/2} \\ &= \frac{c_{H_1}}{\lambda_k^2} \int_{[0,1]^2} dt ds (s-y)_+^{H_2-3/2} (t-y)_+^{H_2-3/2} |t-s|^{2H_1-2}, \end{aligned}$$

with  $c_{H_1}$  a constant depending only on  $H_1$ . Thus, for any  $0 \leq y \leq 1$ ,

$$\begin{aligned} e_k(y)^2 &\leq \frac{c_{H_1}}{\lambda_k^2} \int_{[y,1]^2} dt ds (s-y)^{H_2-3/2} (t-y)^{H_2-3/2} |t-s|^{2H_1-2} \\ &= \frac{c_{H_1}}{\lambda_k^2} \int_{[0,1-y]^2} dt ds s^{H_2-3/2} t^{H_2-3/2} |t-s|^{2H_1-2} \\ &\leq \frac{c_{H_1}}{\lambda_k^2} \int_{[0,1]^2} dt ds s^{H_2-3/2} t^{H_2-3/2} |t-s|^{2H_1-2} \\ &= \frac{2c_{H_1}}{\lambda_k^2} \int_0^1 dt t^{2H_1+2H_2-4} \int_0^1 du u^{H_2-3/2} (1-u)^{2H_1-2} < \infty. \end{aligned}$$

Let us now show that  $f_\infty$  is *not* bounded on  $[0, 1]^2$ . If  $x, y \in [0, \frac{1}{2}]$ , then

$$\begin{aligned} &\int_0^1 (s-x)_+^{H_1-3/2} (s-y)_+^{H_2-3/2} ds = \int_{x \vee y}^1 (s-x)^{H_1-3/2} (s-y)^{H_2-3/2} ds \\ &\geq \int_{x \vee y}^1 \frac{ds}{\sqrt{(s-x)(s-y)}} = \int_{x \vee y}^1 \left[ \left( s - \frac{x+y}{2} \right)^2 - \left( \frac{x-y}{2} \right)^2 \right]^{-\frac{1}{2}} ds \\ &= \int_0^{(1-x)(1-y)} \frac{du}{\sqrt{u(u + (\frac{x-y}{2})^2)}} \geq \int_0^{\frac{1}{4}} \frac{du}{\sqrt{u(u + (\frac{x-y}{2})^2)}}. \end{aligned}$$

Using Fatou's lemma, we may conclude that

$$\begin{aligned} \liminf_{y \rightarrow x} \int_0^1 (s-x)_+^{H_1-3/2} (s-y)_+^{H_2-3/2} ds &\geq \liminf_{y \rightarrow x} \int_0^{\frac{1}{4}} \frac{du}{\sqrt{u(u + (\frac{x-y}{2})^2)}} \\ &\geq \int_0^{\frac{1}{4}} \frac{du}{u} = +\infty. \end{aligned}$$

The fact that  $f_\infty$  is not bounded together with the fact that  $e_k$  is bounded when  $\lambda_{\infty,k} \neq 0$  imply, thanks to (4.25), that  $f_\infty$  satisfies (1.3).

Step 3 (conclusion). Since  $f_\infty$  satisfies (1.3), the proof of (1.9) now follows from Theorem 1.2 and (4.24).  $\square$

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