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SINGULAR MEASURE AS PRINCIPAL EIGENFUNCTION OF SOME NONLOCAL OPERATORS

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ABSTRACT. In this paper, we are interested in the spectral properties of the generalised principal eigenvalue of some nonlocal operator. That is, we look for the existence of some particular solution (λ, ϕ) of a nonlocal operator.

$$\int_\Omega K(x,y)\phi(y)\,dy + a(x)\phi(x) = -\lambda\phi(x),$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded connected set, K a nonnegative kernel and a is continuous. We prove that for the generalised principal eigenvalue $\lambda_p := \sup\{\lambda \in \mathbb{R} \,|\, \exists\, \phi \in C(\Omega), \phi > 0$ so that $\mathcal{L}_\Omega[\phi] + a(x)\phi + \lambda\phi \le 0\}$ there exists always a solution (μ, λ_p) of the problem in the space of signed measure. Moreover μ a positive measure. When μ is absolutely continuous with respect to the Lebesgue measure, $\mu = \phi_p(x)$ is called the principal eigenfunction associated to λ_p. In some simple cases, we exhibit some explicit singular measures that are solutions of the spectral problem.

1. Introduction and Main results

In this note we are interested in the spectral properties of some nonlocal operators. That is, we look for solution (ϕ, λ) of

$$(1.1) \qquad \int_\Omega K(x,y)\phi(y)dy + (a(x) + \lambda)\phi(x) = 0.$$

where $\Omega, K,$ and a satisfy the following assumptions :

$\Omega \subset \mathbb{R}^n$ is a bounded open set $\hfill (H1)$

$K \in C(\mathbb{R}^n \times \mathbb{R}^n),\ K \ge 0,\ \exists c_0 > 0, \epsilon_0 > 0 \quad \text{such that } \inf_{x\in\Omega}\left(\inf_{y\in B(x,\epsilon_0)} K(x,y)\right) > c_0, \hfill (H2)$

$a \in C(\bar\Omega) \cap L^\infty(\Omega) \hfill (H3)$

A typical kernel which satisfies such assumptions is given by

$$(1.2) \qquad k(x,y) = J\left(\frac{x-y}{g(y)}\right)\frac{1}{g^n(y)},$$

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where J is a continuous positive probability density and the function g is bounded and positive. Such kernel was introduced by Cortazar *et al.* [6] in order to model a non homogeneous dispersal process.

In the past few years much attention has been drawn to the study of nonlocal reaction-diffusion equations where such type of nonlocal operators is used to model some long range effects. In this context, the nonlocal operator takes often the form

$$(1.3) \qquad \mathcal{D}_\Omega[u] := \int_\Omega K(x,y)u(y)\,dy - u(x)\int_\Omega K(y,x)u(y)\,dy,$$

where $\Omega \subset \mathbb{R}^n$, $k \geq 0$ satisfies $\int_{\mathbb{R}^n} K(y,x)dy < \infty$ for all $x \in \mathbb{R}^n$; see among other references [2, 6, 8, 9, 12, 13, 16, 18, 19]. In ecology these type of diffusion process has been widely used to describe the dispersal of a population through its environment in the following sense. As stated in [11, 13] if $u(y,t)$ is thought of as a density at a location y at a time t and $k(x,y)$ as the probability distribution of jumping from a location y to a location x, then the rate at which the individuals from all other places are arriving to the location x is $\int_\Omega k(x,y)u(y,t)\,dy$. On the other hand, the rate at which the individuals are leaving the location x is $-u(x,t)\int_\Omega k(y,x)u(y)\,dy$. This formulation of the dispersal of individuals finds its justification in many ecological problems of seed dispersion; see for example [4, 5, 10, 17, 18].

The spectral properties and in particular the existence of principal eigenvalue have recently been investigated and some criteria for the existence of principal eigenvalue have been derived [8, 9, 15, 19]. Namely, the principal eigenvalue λ_p for the operator $\mathcal{L}_\Omega[u] + a(x)u$ can be defined by the formula

$$\lambda_p(\mathcal{L}_\Omega + a(x)) := \sup\{\lambda \in \mathbb{R} \,|\, \exists\, \phi \in C(\Omega), \phi > 0 \ \text{ so that } \ \mathcal{L}_\Omega[\phi] + a(x)\phi + \lambda\phi \leq 0\}$$

and as noted in [8], the condition $\frac{1}{\sup_{\bar\Omega}(a(x))-a(x)} \notin L^1_{loc}(\bar\Omega)$ is sufficient to guarantees the existence of a continuous principal eigenfunction. Another useful criteria is

Theorem 1.1 ([9]). *There exists a positive continuous eigenfunction associated to λ_p **if and only if** $\lambda_p(\mathcal{L}_\Omega + a) < -\sup_\Omega a$.*

On some examples, it has also been observed in [8, 15] that λ_p is not always an eigenvalue in the space $L^1(\bar\Omega)$.

In this note, we study the properties of the principal eigenvalue λ_p when no positive continuous eigenfunction exists i.e. when $\lambda_p = -\sup_\Omega a$ and $\frac{1}{\sup_{\bar\Omega}(a(x))-a(x)} \in L^1(\bar\Omega)$. In this situation, we construct a positive measure, solution of the equation

$$(1.4) \qquad \int_\Omega K(x,y)d\mu(y) + (a(x) + \lambda_p(\mathcal{L}_\Omega + a))\mu = 0.$$

This positive measure is a natural extension of the notion of principal eigenfunction associated to the generalised eigenvalue λ_p.

More precisely,

Theorem 1.2. *Assume that K, a satisfies the assumptions $(H1 - H3)$ and so that $\frac{1}{\sup_{\bar\Omega}(a(x))-a(x)} \in L^1_{loc}(\bar\Omega)$. Assume further that $\lambda_p(\mathcal{L}_\Omega + a) = -\sup_\Omega a$ then there exists a positive measure $\mu \in \mathcal{M}^+(\Omega)$, solution to the equation (1.4). Moreover, if μ is singular with respect to the Lebesgue measure and $\#\Omega_0 > 1$ where $\Omega_0 := \{y \in \bar\Omega \,|\, a(y) = \sup_\Omega a\}$ then there exists infinitely many positive measures, solution of (1.4).*

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These measures are of great importance, since there are at the core of the analysis of propagating phenomena in heterogeneous nonlocal reaction diffusion like

$$(1.5) \qquad \frac{\partial u}{\partial t} = \mathcal{D}[u](x,t) + u(x,t)(a(x) - u(x,t)).$$

The existence of a propagating speed seems to be conditioned to the existence of continuous principal eigenfunction [9, 19]. They also appear naturally in the study of demo-genetic models such as

$$(1.6) \qquad \frac{\partial u}{\partial t} = u\left(a(x) - \int_\Omega a(y)u(y,t)\,dy\right) + \int_\Omega m(x,y)[u(y,t) - u(x,t)]\,dy,$$

where it is known that the solution $u(t,x)$ can blow up and converges to a solution of (1.4), see for example [1, 7, 14].

2. **Construction of the solution**

Let $x_0 \in \bar\Omega$ be a point where the function a achieves its maximum. Then (1.4) rewrites

$$(2.1) \qquad \int_\Omega K(x,y)\phi(y)\,dy + (a(x) - a(x_0))\phi = 0.$$

Let us now introduce the operator:

$$\mathcal{K}[u] := \int_\Omega K(x,y)\frac{u(y)}{a(x_0) - a(y)}\,dy.$$

Since $\frac{1}{a(x_0)-a(y)} \in L^1$ and $K \in C(\mathbb{R}^n \times \mathbb{R}^n)$ is non negative, the positive operator $\mathcal{K}$ is well defined in $C(\bar\Omega)$. Moreover the operator $\mathcal{K}$ is compact since $|\Omega_0| = 0$ and K is uniformly continuous in $\bar\Omega$. Therefore its spectrum consists only on isolated eigenvalue [3].

Assume $1 \in \sigma(\mathcal{K})$ then there exists a positive $\psi \in L^1(\Omega)$ solution of (2.1). Indeed, let λ_1 be the $\max\{|\lambda|\,|\,\lambda \in \sigma(\mathcal{K})\}$ then by the Krein-Rutman Theory there exists $\phi_1 \in C(\bar\Omega)$, $\phi_1 > 0$ so that

$$\mathcal{K}[\phi_1] = \lambda_1\phi_1.$$

Therefore there exists $\psi := \frac{\phi}{(a(x_0)-a(x))} \in L^1$ so that for all $x \in \bar\Omega \setminus \Omega_0$

$$\int_\Omega K(x,y)\psi(y)\,dy + (a(x) - a(x_0))\psi = (\lambda_1 - 1)\phi.$$

If $\lambda_1 > 1$, there exists $\epsilon > 0$ and $\psi \geq 0$ so that for all $x \in \bar\Omega \setminus \Omega_0$

$$\int_\Omega K(x,y)\psi(y)\,dy + (a(x) - a(x_0) - \epsilon)\psi = (\lambda_1 - 1 - \epsilon)\phi > 0.$$

Thus all $x \in \bar\Omega \setminus \Omega_0$

$$\frac{1}{\epsilon + a(x_0) - a(x)} \int_\Omega K(x,y)\psi(y)\,dy > \psi(x)$$

and we get the contradiction

$$+\infty > \frac{\|K\|_\infty}{\epsilon} \int_\Omega \psi(y)\,dy \geq \limsup_{x\to x'\in\partial\Omega_0} \psi(x) = +\infty.$$

So $\lambda_1 \leq 1$ and by construction $\lambda_1 = 1$. Hence, in this situation there exists a positive measure ϕ_1, solution to (2.1).

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Now assume that $1 \notin \sigma(\mathcal{K})$. We will construct a positive measure μ, solution of (2.1). We construct a solution of the form $\alpha \delta_{x_0} + f$ for some $\alpha \in \mathbb{R}^+$ and $f \in L^1(\Omega)$. By introducing this Ansatz in (2.1) and after a straightforward computation we see that f should satisfy

$$(2.2) \qquad \alpha K(x, x_0) + \int_\Omega K(x, y) f(y)\, dy + (a(x) - a(x_0)) f = 0.$$

By denoting $g = (a(x_0) - a(x)) f$, we can rewrite (2.2) as follows

$$(2.3) \qquad \mathcal{K}[g] - g = -\alpha K(x, x_0).$$

By assumption $1 \notin \sigma(\mathcal{K})$, so by the Fredholm alternative the operator $\mathcal{K} - id$ is invertible and for any $h \in C(\bar{\Omega})$ there exists a unique $g \in C(\bar{\Omega})$ so that

$$\mathcal{K}[g] - g = h.$$

Moreover, $g \geq 0$ if $h \leq 0$. Indeed, since

$$\mathcal{K}[\phi_1] - \phi_1 = (\lambda_1 - 1)\phi_1 < 0,$$

we have

$$\mathcal{K}[g] - g = \int_\Omega \widetilde{K}(x, y)\phi_1(y)\frac{g(y)}{\phi_1(y)}\, dy - \frac{g(x)}{\phi_1(x)}\phi_1(x)$$
$$= \int_\Omega \widetilde{K}(x, y)\phi_1(y)\left[\frac{g(y)}{\phi_1(y)} - \frac{g(x)}{\phi_1(x)}\right]\, dy + (\lambda_1 - 1)\phi_1(x)\frac{g(x)}{\phi_1(x)}$$

By denoting $w := \frac{g}{\phi_1}$ we have for w

$$\int_\Omega \widetilde{K}(x, y)\phi_1(y)\left[w(y) - w(x)\right]\, dy + (\lambda_1 - 1)\phi_1(x)w(x) = h \leq 0.$$

Since $\lambda_1 < 1$ and $\phi_1 > 0$ we easily conclude that w cannot achieve a non positive minimum without being constant. Thus $w > 0$ which implies that $g > 0$. As a consequence, for all $\alpha \in \mathbb{R}^{+,*}$, there exists a unique positive function $g_\alpha \in C(\bar{\Omega})$ solution of (2.3). Let us denote g_1 the function obtained for $\alpha = 1$. By construction, we have for any $\alpha \in \mathbb{R}$, $g_\alpha = \alpha g_1$. Therefore, if $\alpha \delta_{x_0} + \frac{g_\alpha}{a(x_0) - a(x)}$ is a solution of (2.1) then

$$\alpha \delta_{x_0} + \frac{g_\alpha}{a(x_0) - a(x)} = \alpha\left(\delta_{x_0} + \frac{g_1}{a(x_0) - a(x)}\right)$$

and the constructed solution is unique up to normalisation. Furthermore any element μ of the linear set engendered by $\delta_{x_0} + \frac{g_1}{a(x_0) - a(x)}$, i.e.

$$\mu \in Lin(\delta_{x_0} + \frac{g_1}{a(x_0) - a(x)}) := \left\{\alpha\left(\delta_{x_0} + \frac{g_1}{a(x_0) - a(x)}\right) \mid \alpha \in \mathbb{R}\right\}$$

is a signed-measure, solution of (2.1).

Observe that the above construction holds for any singular measure μ with its support contained in Ω_0. Indeed, by replacing the Dirac measure at x_0 by μ in the above construction, we see that $\mu + g$ is a positive measure solution of (2.1) where g is the positive solution of

$$\mathcal{K}[g] - g = -\int_\Omega K(x, y) d\mu(y).$$

In particular when $\#\Omega_0 > 1$ the set of singular measures with support in Ω_0 is not reduced to δ_{x_0} and we can construct a solution for any points x_0 where the maximum of a its achieves. Thus any element ν in the linear space engendered by $\delta_{x_0} + g_{x_0,1}$ and $\delta_{x_1} + g_{x_1,1}$, i.e.

$$\nu \in \left\{ \alpha \left(\delta_{x_0} + \frac{g_{x_0,1}}{a(x_0) - a(x)} \right) + \beta \left(\delta_{x_1} + \frac{g_{x_1,1}}{a(x_1) - a(x)} \right) \mid \alpha, \beta \in \mathbb{R} \right\}$$

is a signed-measure, solution of (2.1).

$\square$

3. **Two simple examples**

First Example: Let $\Omega = B_1(0)$ be the unit ball of $\mathbb{R}^3$ and let us consider the following eigenvalue problem

$$(3.1) \qquad \rho \int_\Omega u + (a(x) + \lambda)u = 0 \quad \text{in} \quad \Omega$$

with $a(x) = 1 - \|x\|_2^2$ and $\rho > 0$. We can easily check that for $\rho > \frac{1}{4\pi}$ there exists a solution (ϕ_p, λ_p) to (3.1) with $\lambda_p < -sup_{x \in \Omega}(a(x))$ and $\phi_p \in C(\bar{\Omega})$. Indeed, the function $\frac{\rho}{-\lambda_p - a(x)}$ with λ_p so that $\int_\Omega \frac{\rho}{-\lambda_p - a(x)} dx = 1$ is the desired solution. When $\rho = \frac{1}{4\pi}$ then (3.1) has still a positive solution (λ_p, ϕ_p) with $\lambda_p = -\max a$ and $\phi_p = \frac{\rho}{-\lambda_p - a(x)}$, the positive eigenfunction being now in $L^1(\Omega)$ and singular at x_0. For $\rho < \frac{1}{4\pi}$ we can check that there is no L^1 eigenfunction solution of (3.1). However, with $\lambda_p = -\max_\Omega a$, using the above construction for $\alpha = \frac{1}{\rho} - \int_\Omega \frac{dx}{a(x_0) - a(x)}$ we can see that the positive measure

$$\alpha \delta_0 + \frac{1}{a(x_0) - a(x)}$$

is a solution of (3.1) . Since a achieves a unique maximum, this is the only positive measure, solution of (3.1).

Second Example: Let $B_1(0)$ be the unit ball of $\mathbb{R}^2$ and let $\Omega = B_1(0) \times [0,1]$ be the unit cylinder of $\mathbb{R}^3$ and let us consider the following principal eigenvalue problem

$$(3.2) \qquad \rho \int_\Omega u + (a(x) + \lambda)u = 0 \quad \text{in} \quad \Omega$$

with $a(x) = 1 - \sqrt{x_1^2 + x_2^2}$ and $\rho > 0$. The maximum of $a(x)$ is achieved at any point x_0 of the symmetrical axis of the cylinder (i.e. $\Omega_0 := \{x_0 := (0,0,x_3) \mid$ with $x_3 \in (0,1)\}$) and we can check that

$$\int_\Omega \frac{dx}{a(x_0) - a(x)} = \int_0^1 \left(\int_{B_1(0)} \frac{dx_1 dx_2}{\sqrt{x_1^2 + x_2^2}} \right) dx_3 = 2\pi.$$

As in the previous example, for $\rho > \frac{1}{2\pi}$ there exists a solution (ϕ_p, λ_p) to (3.2) with $\lambda_p < -sup_{x \in \Omega}(a(x))$ and $\phi_p \in C(\bar{\Omega})$. When $\rho = \frac{1}{2\pi}$ then (3.2) has still a positive solution (λ_p, ϕ_p) with $\lambda_p = -\max a$ and $\phi_p = \frac{\rho}{-\lambda_p - a(x)}$, the positive eigenfunction being now in $L^1(\Omega)$ and singular at any $x_0 \in \Omega_0$. For $\rho < \frac{1}{2\pi}$ then there is no L^1 eigenfunction solution of (3.2). But with $\lambda_p = -sup_\Omega a$ and for any $x_0 \in \Omega_0$ we can then construct a positive measure

$$\alpha \delta_{x_0} + \frac{1}{a(x_0) - a(x)},$$

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solution of (3.2). On the interval $(0,1)$, let us consider a singular continuous measure μ like the devil staircase or the Ferenc Riez measure on $(0,1)$. Those two positive measures are singular measure with their support in Ω_0. So from our construction $\mu + g_\mu$ is also a positive measure solution to (3.2). In these two cases, the singular measure does not have an atomic part.

From these two examples, we can see how the set of measure that are solution to the spectral equation (1.4) can be rich. It is therefore necessary to develop criteria to discriminate the pertinent solution.

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