

Controllability of quantum systems with a polarizability term

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Outline of the talk

- 1 Quantum systems
- 2 Controllability results
- 3 Conclusion

Bilinear quantum systems

- The system is described by the Schrödinger equation:

$$i \frac{\partial \psi}{\partial t} = (-\Delta + V(x))\psi(x, t) \quad , x \in \Omega$$

- Ω : real manifold (\mathbf{R}^d , Euclidean sphere,...);
- ψ complex valued wave function (with constant L^2 norm)

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Different models:

- Usually, dipolar approximation $\mu(x, u) = u(t)W(x)$ is enough.
- For better accuracy, we have to consider second order terms $\mu(x, u) = u(t)W_1(x) + u^2(t)W_2(x)$.

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Finite dimensional space: Coron, Rouchon, Beauchard,...

Infinite dimensional space: Morancey (Lyapunov methods with W_1 and W_2 bounded, Ω compact subset of \mathbf{R}^d)

Semi-linear quantum systems: abstract form

- The state is described by the *wave function*, a point in some Hilbert space.
- Dynamics given by the Schrödinger equation

$$\frac{d}{dt}\psi = A\psi + u(t)B\psi + u^2(t)C\psi$$

A , B and C are skew-adjoint operators (not necessarily bounded).

Some standard assumptions

- A has purely discrete spectrum $(-i\lambda_k)_k$, $\lambda_k \rightarrow +\infty$;
- $(\phi_k)_k$, Hilbert basis of H made of eigenvectors of A , $A\phi_k = -i\lambda_k\phi_k$;
- for every k , ϕ_k belongs to the domains of B and C ;
- for every u in \mathbf{R}^+ , $A + uB + u^2C$ is skew-adjoint.

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Non-degenerate transition

A couple (j, k) in \mathbf{N}^2 is a non degenerate transition of (A, B) if $\langle \phi_j, B\phi_k \rangle \neq 0$ and $|\lambda_j - \lambda_k| = |\lambda_m - \lambda_n|$ implies $\{j, k\} = \{m, n\}$ or $\langle \phi_m, B\phi_n \rangle = 0$.

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Periodic excitations of bilinear systems

Let (j, k) be a non-degenerate transition of (A, B) . If u is $2\pi/|\lambda_j - \lambda_k|$ periodic with non zero first Fourier coefficient and zero Fourier coefficient for every frequency associated with a transition (m, n) , $|\lambda_m - \lambda_n| \in \mathbf{N} \setminus \{1\}|\lambda_j - \lambda_n|$. Then, there exists T such that $|\langle \phi_j, \Upsilon_{nT}^{u/n} \phi_k \rangle| \rightarrow 1$ as n tends to infinity.

Weakly-coupled systems

Definition

A pair (A, B) of linear operators is weakly-coupled if there exists $\alpha < 1$ such that $\|B\psi\| \leq \| |A|^\alpha \psi \|$ for every ψ in $D(|A|^\alpha)$ and there exists $c_{A,B}$ such that $|\Im \langle A\psi, B\psi \rangle| \leq c_{A,B} |\langle A\psi, B\psi \rangle|$ for every ψ in $D(A)$.

Good Galerkin approximations

Let (A, B) be weakly-coupled. Then for every ψ in H , $\varepsilon, K > 0$, there exists N in \mathbf{N} such that $\|u\|_{L^1} < K \implies \|X_{(N)}^u(t)\psi - \Upsilon_T^u \psi\| < \varepsilon$

All the systems presented in the physics literature are weakly-coupled.

Result

$$\frac{d\psi}{dt} = A\psi + u(t)B\psi + u^2(t)C\psi$$

Good Galerkin approximation

If (A, B) and (A, C) are weakly-coupled, then for every ψ in H , $\varepsilon, K > 0$, there exists N in \mathbf{N} such that

$$\|u\|_{L^1} + \|u^2\|_{L^1} < K \implies \|X_{(N)}^u(t)\psi - \Upsilon_T^u\psi\| < \varepsilon$$

Result with general C (possibly unbounded)

Let (A, B) and (A, C) be weakly-coupled. Let (j, k) be a non-degenerate transition of (A, B) . For almost every α in \mathbf{R} , for every $\varepsilon > 0$, there exists $u_\varepsilon : [0, T_\varepsilon] \rightarrow \{0, \alpha\}$ such that

$$\|\Upsilon_{(A,B,C)}^{u_\varepsilon}(T_\varepsilon)\phi_j - \phi_k\| < \varepsilon.$$

Idea of the proof

$$\frac{d\psi}{dt} = A\psi + u(t)B\psi + u^2(t)C\psi$$

A bilinear result

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Proof of the semi-linear controllability result: For almost every α in \mathbf{R} , (j, k) is a non-degenerate transition of $(A, B + \alpha C)$. Apply the preceding (bilinear) result to $(A, B + \alpha C)$ and notice that

$$u(t)(B + \alpha C) = \begin{cases} \alpha B + \alpha^2 C & \text{if } u(t) = \alpha \\ 0 & \text{if } u(t) = 0 \end{cases}$$

Particular case: C bounded

$$\begin{aligned} \Upsilon^n, \text{ propagator of } \frac{d\psi}{dt} &= A\psi + u(t)/nB\psi + u^2(t)/n^2C\psi \\ Y_n, \text{ propagator of } \frac{d\psi}{dt} &= A\psi + u(t)/nB\psi \end{aligned}$$

Since Y_n and Υ^n are unitary propagators and u is $2\pi/|\lambda_2 - \lambda_1|$ periodic, one gets

$$\left| \Upsilon_{nt,0}^{u/n} \phi_1 - Y_n(nt, 0) \phi_1 \right| \leq \underbrace{\frac{\|C\|}{n}}_{\rightarrow 0} \underbrace{\left(\frac{t|\lambda_2 - \lambda_1|}{2\pi} + 1 \right) \int_0^{\frac{2\pi}{|\lambda_1 - \lambda_2|}} |u(s)|^2 ds}_{\text{bounded}}$$

Numerically: it converges faster with the semi-linear example than with the bilinear system!

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Take home message

- Semi-linear conservative quantum systems can be treated with tools of geometric control theory.
- Results are obvious when the quadratic term C is bounded.
- Proof for general quadratic terms relies on precise time estimates for the finite dimensional Galerkin approximations.

Future works and open question

- Generalization to systems that are not weakly-coupled.
- Generalization to semi-linear systems (not only quadratic).
- What about controllability time?