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# SEMICLASSICAL MEASURE FOR THE SOLUTION OF THE HELMHOLTZ EQUATION WITH AN UNBOUNDED SOURCE

JULIEN ROYER

ABSTRACT. We study the high frequency limit for the dissipative Helmholtz equation when the source term concentrates on a submanifold of  $\mathbb{R}^n$ . We prove that the solution has a unique semi-classical measure, which is precisely described in terms of the classical properties of the problem. This result is already known when the micro-support of the source is bounded, we now consider the general case.

## 1. STATEMENT OF THE RESULT

We consider on  $\mathbb{R}^n$  the Helmholtz equation

$$(H_h - E_h)u_h = S_h, \quad \text{where } H_h = -h^2\Delta + V_1(x) - ihV_2(x). \quad (1.1)$$

Here  $V_1$  and  $V_2$  are smooth and real-valued potentials which go to 0 at infinity. Thus for any  $h \in ]0, 1]$  the operator  $H_h$  is a non-symmetric (unless  $V_2 = 0$ ) Schrödinger operator with domain  $H^2(\mathbb{R}^n)$ . The energy  $E_h$  will be chosen in such a way that for  $h > 0$  small enough,  $\delta > \frac{1}{2}$  and  $S_h \in L^{2,\delta}(\mathbb{R}^n)$  the equation (1.1) has a unique outgoing solution  $u_h \in L^{2,-\delta}(\mathbb{R}^n)$ . Here we denote by  $L^{2,\delta}(\mathbb{R}^n)$  the weighted space  $L^2(\langle x \rangle^{2\delta} dx)$ , where  $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$ .

The source term  $S_h$  we consider is a profile which concentrates on a submanifold  $\Gamma$  of dimension  $d \in \llbracket 0, n-1 \rrbracket$  in  $\mathbb{R}^n$ , endowed with the Lebesgue measure  $\sigma_\Gamma$ . Given an amplitude  $A \in C^\infty(\Gamma)$  and  $S$  in the Schwartz space  $\mathcal{S}(\mathbb{R}^n)$  we set, for  $h \in ]0, 1]$  and  $x \in \mathbb{R}^n$ :

$$S_h(x) = h^{\frac{1-n-d}{2}} \int_\Gamma A(z) S\left(\frac{x-z}{h}\right) d\sigma_\Gamma(z) \quad (1.2)$$

(this definition will make sense with the assumptions on  $\Gamma$  and  $A$  given below). Our purpose is to study the semiclassical measures for the family of corresponding solutions ( $u_h$ ) when the submanifold  $\Gamma$  is allowed to be unbounded.

This work comes after a number of contributions which deal with more and more general situations. The first paper about the subject is [BCKP02], where  $\Gamma = \{0\}$  (see also [Cas05]). The result was generalized in [CPR02] to the case where  $\Gamma$  is an affine subspace of  $\mathbb{R}^n$ , under the assumption that the refraction index is constant ( $V_1 = 0$ ). This restriction was overcome in [WZ06]. In [Fou06] the source term concentrates on two points and in [Fou06] the refraction index is discontinuous along a hyperplane of  $\mathbb{R}^n$ . All these papers study the semiclassical measure of the solution using its Wigner transform.

The approach in [Bon09] is different. The semiclassical measures are defined with pseudo-differential calculus (see (1.14)) and the resolvent is replaced by the integral over positive times of the propagator (as in [Cas05]). We used this point of view in [Roy10] to deal with the case of a non-constant absorption index ( $V_2 \neq 0$ ,  $V_2 \geq 0$ ) and a general bounded submanifold  $\Gamma$ . We also considered in [Roy11] an absorption index  $V_2$  which can take non-positive values. The purpose

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of this paper is now to allow a general unbounded submanifold  $\Gamma$ .

Let us now state more precisely the assumptions. The potentials  $V_1$  and  $V_2$  are respectively of long and of short range: there exist constants  $\rho > 0$  and  $c_\alpha$  for  $\alpha \in \mathbb{N}^n$  such that

$$\forall \alpha \in \mathbb{N}^n, \forall x \in \mathbb{R}^n, \quad |\partial^\alpha V_1(x)| \leq c_\alpha \langle x \rangle^{-\rho-|\alpha|} \quad \text{and} \quad |\partial^\alpha V_2(x)| \leq c_\alpha \langle x \rangle^{-1-\rho-|\alpha|}. \quad (1.3)$$

Then we introduce the hamiltonian flow  $\phi^t$  corresponding to the classical symbol  $p : (x, \xi) \mapsto |\xi|^2 + V_1(x)$  on the phase space  $\mathbb{R}^{2n}$ . For all  $w \in \mathbb{R}^{2n}$ ,  $t \mapsto \phi^t(w) = (X(t, w), \Xi(t, w)) \in \mathbb{R}^{2n}$  is the solution of the system

$$\begin{cases} \partial_t X(t, w) = 2\Xi(t, w), \\ \partial_t \Xi(t, w) = -\nabla V_1(X(t, w)), \\ \phi^0(w) = w. \end{cases} \quad (1.4)$$

For  $I \subset \mathbb{R}$  we set

$$\Omega_b(I) = \left\{ w \in p^{-1}(I) : \sup_{t \in \mathbb{R}} |X(t, w)| < \infty \right\}$$

We also denote by

$$H_p q = \{p, q\} = \nabla_\xi p \cdot \nabla_x q - \nabla_x p \cdot \nabla_\xi q$$

the Poisson bracket of  $p$  with a symbol  $q \in C^\infty(\mathbb{R}^{2n})$ .

We now consider an energy  $E_0 > 0$  such that

$$\forall w \in \Omega_b(\{E_0\}), \exists T > 0, \quad \int_0^T V_2(X(t, w)) dt > 0. \quad (1.5)$$

Let  $\delta > \frac{1}{2}$ . We know (see [Roy11]) that under Assumption (1.5) there exist an open neighborhood  $J$  of  $E_0$ ,  $h_0 > 0$  and  $c \geq 0$  such that for

$$z \in \mathbb{C}_{J,+} = \{z \in \mathbb{C} : \operatorname{Re} z \in J, \operatorname{Im} z > 0\}$$

and  $h \in ]0, h_0]$  the operator  $(H_h - z)$  has a bounded inverse on  $L^2(\mathbb{R}^n)$  and

$$\left\| \langle x \rangle^{-\delta} (H_h - z)^{-1} \langle x \rangle^{-\delta} \right\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq \frac{c}{h}.$$

Here  $\mathcal{L}(L^2(\mathbb{R}^n))$  denotes the space of bounded operators on  $L^2(\mathbb{R}^n)$ . Moreover for any  $\lambda \in J$  the limit

$$(H_h - (\lambda + i0))^{-1} = \lim_{\beta \rightarrow 0^+} (H_h - (\lambda + i\beta))^{-1}$$

exists in  $\mathcal{L}(L^{2,\delta}(\mathbb{R}^n), L^{2,-\delta}(\mathbb{R}^n))$ .

Now let us be more explicit about the source term  $S_h$  we consider. We recall that  $\Gamma$  is a submanifold of dimension  $d \in \llbracket 0, n-1 \rrbracket$  in  $\mathbb{R}^n$ , endowed with the Riemannian structure given by the restriction of the usual structure on  $\mathbb{R}^n$ , and the corresponding Lebesgue measure  $\sigma_\Gamma$ . We assume that there exist  $R_1 > 0$  and  $\sigma_1 \in ]0, 1[$  such that

$$N\Gamma \cap \mathcal{Z}_-(R_1, 0, -\sigma_1) = \emptyset, \quad (1.6)$$

where  $N\Gamma = \{(z, \xi) \in \Gamma \times \mathbb{R}^n : \xi \perp T_z \Gamma\}$  is the normal bundle of  $\Gamma$  and  $\mathcal{Z}_-(R_1, 0, \sigma_1)$  is an incoming region: for  $R \geq 0$ ,  $\nu \geq 0$  and  $\sigma \in [-1, 1]$  we set

$$\mathcal{Z}_\pm(R, \nu, \sigma) = \{(x, \xi) \in \mathbb{R}^{2n} : |x| \geq R, |\xi| \geq \nu \text{ and } \langle x, \xi \rangle \gtrless \sigma |x| |\xi|\}.$$

Note that Assumption (1.6) is satisfied for any bounded submanifold of  $\mathbb{R}^n$ . When  $d = 0$ , it actually implies that  $\Gamma$  is bounded, but this is not the case in higher dimension: this assumption holds for instance for any affine subspace of dimension  $d \in \llbracket 1, n-1 \rrbracket$  in  $\mathbb{R}^n$ . Now that  $\sigma_1$  is fixed, we can choose a smaller neighborhood  $J$  of  $E_0$  and assume that

$$J \subset [E_1, E_2], \quad \text{were } E_1 > 0 \quad \text{and} \quad \left(\frac{1+\sigma_1}{2}\right)^2 E_2 < E_1. \quad (1.7)$$

We assume that

$$\forall z \in \Gamma, \quad V_1(z) > E_0, \quad (1.8)$$

and we define

$$N_E\Gamma = N\Gamma \cap p^{-1}(\{E_0\}).$$

$N_E\Gamma$  is a submanifold of dimension  $(n-1)$  in  $\mathbb{R}^{2n}$ , endowed with the Riemannian structure defined as follows: for  $(z, \xi) \in N_E\Gamma$  and  $(Z, \Xi), (\tilde{Z}, \tilde{\Xi}) \in T_{(z, \xi)}N_E\Gamma \subset \mathbb{R}^{2n}$  we set

$$g_{(z, \xi)}((Z, \Xi), (\tilde{Z}, \tilde{\Xi})) = \left\langle Z, \tilde{Z} \right\rangle_{\mathbb{R}^n} + \left\langle \Xi_{\perp}, \tilde{\Xi}_{\perp} \right\rangle_{\mathbb{R}^n},$$

where  $\Xi_{\perp}, \tilde{\Xi}_{\perp}$  are the orthogonal projections of  $\Xi, \tilde{\Xi} \in \mathbb{R}^n$  on  $(T_z\Gamma \oplus \mathbb{R}\xi)^{\perp}$  (see the discussion in [Roy10]). We denote by  $\sigma_{N_E\Gamma}$  the canonical measure on  $N_E\Gamma$  given by  $g$ , and assume that

$$\sigma_{N_E\Gamma}(\{(z, \xi) \in N_E\Gamma : \exists t > 0, \phi^t(z, \xi) \in N_E\Gamma\}) = 0. \quad (1.9)$$

We now introduce the amplitude  $A \in C^{\infty}(\Gamma)$ . We assume that there exist  $\delta > \frac{1}{2}$  and  $c \geq 0$  such that

$$\int_{\Gamma} \langle z \rangle^{\delta} (|A(z)| + \|d_z A\| + |A(z)| \|\Pi_z\|) d\sigma_{\Gamma}(z) < +\infty. \quad (1.10a)$$

Moreover for all  $r \in ]0, 1]$  and  $x \in \mathbb{R}^n$  we have

$$\int_{B(x, r) \cap \Gamma} \langle z \rangle^{\delta} (|A(z)| + \|d_z A\| + |A(z)| \|\Pi_z\|) d\sigma_{\Gamma}(z) \leq cr^d. \quad (1.10b)$$

Here  $B(x, r)$  is the ball of radius  $r$  and centered at  $x$ ,  $d_z A : T_z\Gamma \rightarrow \mathbb{R}$  is the differential of  $A$  at point  $z$  and  $\Pi$  is the second fundamental form of the submanifold  $\Gamma$ . For any  $z \in \Gamma$ ,  $\Pi_z$  is a bilinear form from  $T_z\Gamma$  to  $N_z\Gamma$  (see Appendix A), and

$$\|\Pi_z\| = \sup_{\|X\|_{T_z\Gamma} = \|Y\|_{T_z\Gamma} = 1} \|\Pi_z(X, Y)\|_{N_z\Gamma}.$$

Note that all these estimates hold when  $A \in C_0^{\infty}(\Gamma)$ . Here  $A$  is allowed to have a non-compact support, but it still has to stay away from the boundary of  $\Gamma$ :

$$\forall \theta \in C_0^{\infty}(\mathbb{R}^n), \quad z \mapsto A(z)\theta(z) \in C_0^{\infty}(\Gamma). \quad (1.11)$$

Then it remains to consider  $S \in \mathcal{S}(\mathbb{R}^n)$  and define the source term  $S_h$  by (1.2).

Let  $(E_h)_{h \in ]0, h_0]}$  be a family of energy in  $\mathbb{C}_{J,+} \cup J$  such that

$$E_h = E_0 + h\tilde{E} + o_{h \rightarrow 0}(h) \quad (1.12)$$

for some  $\tilde{E} \in \overline{\mathbb{C}_+}$ . Since for all  $h \in ]0, h_0]$  the source term  $S_h$  given by (1.2) belongs to  $L^{2, \delta}(\mathbb{R}^n)$  (see Proposition 2.2) we can define

$$u_h = (H_h - (E_h + i0))^{-1} S_h \in L^{2, -\delta}(\mathbb{R}^n). \quad (1.13)$$

Here  $(H_h - (E_h + i0))^{-1}$  stands for  $(H_h - E_h)^{-1}$  when  $E_h \in \mathbb{C}_{J,+}$ .

Our purpose is to study the semiclassical measures for this family  $(u_h)_{h \in ]0, h_0]}$ . In other words, non-negative measures  $\mu$  on the phase space  $\mathbb{R}^{2n} \simeq T^*\mathbb{R}^n$  such that

$$\forall q \in C_0^{\infty}(\mathbb{R}^{2n}), \quad \langle \text{Op}_{h_m}^w(q) u_{h_m}, u_{h_m} \rangle \xrightarrow{m \rightarrow +\infty} \int_{\mathbb{R}^{2n}} q d\mu, \quad (1.14)$$

for some sequence  $(h_m)_{m \in \mathbb{N}} \in ]0, h_0]^{\mathbb{N}}$  such that  $h_m \rightarrow 0$  (see [Gér91]). Here  $\text{Op}_h^w(q)$  denotes the Weyl quantization of the symbol  $q$ :

$$\text{Op}_h^w(q)u(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y, \xi \rangle} q\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi.$$

We will also use the standard quantization:

$$\text{Op}_h(q)u(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y, \xi \rangle} q(x, \xi) u(y) dy d\xi.$$

We denote by  $C_b^{\infty}(\mathbb{R}^{2n})$  the set of smooth symbols whose derivatives are bounded. For  $\delta \in \mathbb{R}$ , we also denote by  $\mathcal{S}(\langle x \rangle^{\delta})$  the set of symbols  $a \in C^{\infty}(\mathbb{R}^{2n})$  such that

$$\forall \alpha, \beta \in \mathbb{N}^n, \exists c_{\alpha, \beta} \geq 0, \forall (x, \xi) \in \mathbb{R}^{2n}, \quad \left| \partial_x^{\alpha} \partial_{\xi}^{\beta} a(x, \xi) \right| \leq c_{\alpha, \beta} \langle x \rangle^{\delta},$$

and by  $\mathcal{S}_\delta(\mathbb{R}^{2n})$  the set of symbols  $a \in C^\infty(\mathbb{R}^{2n})$  such that

$$\forall \alpha, \beta \in \mathbb{N}^n, \exists c_{\alpha, \beta} \geq 0, \forall (x, \xi) \in \mathbb{R}^{2n}, \quad \left| \partial_x^\alpha \partial_\xi^\beta a(x, \xi) \right| \leq c_{\alpha, \beta} \langle x \rangle^{\delta - |\alpha|}.$$

We can similarly define the sets of symbols  $\mathcal{S}(\langle \xi \rangle^\delta)$  for  $\delta \in \mathbb{R}$ . We refer to [Rob87, Zwo12] for more details about semiclassical analysis.

For  $(z, \xi) \in N_E \Gamma$  we set

$$\kappa(z, \xi) = \pi(2\pi)^{d-n} |A(z)|^2 |\xi|^{-1} |\hat{S}(\xi)|^2,$$

where  $\hat{S}$  is the Fourier transform of  $S$ . The theorem we want to prove is the following:

**Theorem 1.1.** *Let  $S_h$  and  $E_h$  be given by (1.2) and (1.12), and  $u_h$  defined by (1.13). Assume that the assumptions (1.3), (1.5), (1.6), (1.7), (1.8), (1.9), (1.10) and (1.11) hold.*

(i) *Then there exists a non-negative Radon measure  $\mu$  on  $\mathbb{R}^{2n}$  such that for all  $q \in C_0^\infty(\mathbb{R}^{2n})$  we have*

$$\langle \text{Op}_h^w(q) u_h, u_h \rangle \xrightarrow{h \rightarrow 0} \int_{\mathbb{R}^{2n}} q d\mu. \quad (1.15)$$

(ii) *This measure is characterized by the following three properties:*

- a.  $\mu$  is supported in  $p^{-1}(\{E_0\})$ .
- b. For all  $\sigma \in ]\sigma_1, 1[$  there exists  $R \geq 0$  such that  $\mu$  is zero in the incoming region  $\mathcal{Z}_-(R, 0, -\sigma)$ .
- c.  $\mu$  satisfies the following Liouville equation:

$$(H_p + 2 \text{Im} \tilde{E} + 2V_2)\mu = \kappa \sigma_{N_E \Gamma}.$$

This means that for all  $q \in C_0^\infty(\mathbb{R}^{2n})$  we have

$$\int_{\mathbb{R}^{2n}} (-H_p + 2 \text{Im} \tilde{E} + 2V_2) q d\mu = \int_{N_E \Gamma} q(z, \xi) \kappa(z, \xi) d\sigma_{N_E \Gamma}(z, \xi).$$

(iii) *These three properties imply that for any  $q \in C_0^\infty(\mathbb{R}^{2n})$  we have*

$$\int_{\mathbb{R}^{2n}} q d\mu = \int_{\mathbb{R}_+} \int_{N_E \Gamma} \kappa(z, \xi) q(\phi^t(z, \xi)) e^{-2t \text{Im} \tilde{E} - 2 \int_0^t V_2(X(s, z, \xi)) ds} d\sigma_{N_E \Gamma}(z, \xi) dt. \quad (1.16)$$

This result is known when  $A$  is compactly supported on  $\Gamma$  (see [Roy10, Roy11]). The idea for the proof is to write the resolvent as the integral of the propagator over positive times, and to approximate  $u_h$  by a partial solution which only takes into account finite times:

$$u_h^T = \frac{i}{h} \int_0^\infty \chi_T(t) e^{-\frac{it}{h}(H_h - E_h)} S_h dt.$$

Here  $\chi_T(t) = \chi(t - T)$ , where  $\chi \in C^\infty(\mathbb{R}, [0, 1])$  is equal to 1 in a neighborhood of  $] - \infty, 0]$  and supported in  $] - \infty, \tau_0]$  for some well-chosen  $\tau_0 \in ]0, 1]$ . Note that for all  $h \in ]0, 1]$  the semi-group  $t \mapsto e^{-\frac{it}{h} H_h}$  is well-defined for all  $t \geq 0$ . However this is not a contractions semi-group since  $V_2$  is not assumed to be non-negative (see for instance Corollary 3.6 in [EN00])

The idea will be the same to deal with the case of an amplitude  $A$  whose support is not bounded. Let  $\Theta \in C_0^\infty(\mathbb{R}^n, [0, 1])$  be equal to 1 on  $B(0, 1)$ . For any  $R > 0$  we set  $A_R : z \in \Gamma \mapsto A(z)\Theta(z/R)$  and

$$S_h^R(x) = h^{\frac{1-n-d}{2}} \int_{z \in \Gamma} A_R(z) S\left(\frac{x-z}{h}\right) d\sigma_\Gamma(z). \quad (1.17)$$

Given any  $R > 0$ , the proof of [Roy10, Roy11] applies for the source term  $S_h^R$ . Since the choice of  $\chi$  mentioned above depends on the support of  $A_R$ , we denote it by  $\chi_{0,R}$ . Moreover  $\chi_{0,R}$  can be chosen non-increasing. Then for any  $T \geq 0$  we set  $\chi_{T,R} : t \mapsto \chi_{0,R}(t - T)$  and for any  $h \in ]0, h_0]$ :

$$u_h^{T,R} = \frac{i}{h} \int_0^\infty \chi_{T,R}(t) e^{-\frac{it}{h}(H_h - E_h)} S_h^R dt.$$

The key point is to prove that in some suitable sense  $u_h^{T,R}$  is a good approximation of  $u_h$  for large  $T$  and  $R$ , and  $h > 0$  small enough.

Let  $R > 0$ . For  $h \in ]0, h_0]$  we set

$$\tilde{u}_h^R = (H_h - (E_h + i0))^{-1} S_h^R \in L^{2, -\delta}(\mathbb{R}^n).$$

Since  $A_R$  is compactly supported on  $\Gamma$ , we know that Theorem 1.1 holds for  $\tilde{u}_h^R$ . In particular there exists a non-negative Radon measure  $\tilde{\mu}_R$  on  $\mathbb{R}^{2n}$  such that

$$\forall q \in C_0^\infty(\mathbb{R}^{2n}), \quad \langle \text{Op}_h^w(q) \tilde{u}_h^R, \tilde{u}_h^R \rangle \xrightarrow{h \rightarrow 0} \int_{\mathbb{R}^{2n}} q d\tilde{\mu}_R.$$

Moreover, according to (1.16),  $\tilde{\mu}_R$  is supported on the classical trajectories coming from  $N_E^R \Gamma = \{(z, \xi) \in N_E \Gamma : z \in \text{supp } A_R\}$ . Let  $K$  be a compact subset of  $\mathbb{R}^{2n}$ . Assumption (1.6) ensures that for  $R_K > 0$  large enough and  $R \geq R_K$ , the trajectories coming from  $N_E^R \Gamma \setminus N_E^{R_K} \Gamma$  do not meet  $K$  (see Proposition 3.1), and hence  $\tilde{\mu}_R = \tilde{\mu}_{R_K}$  on  $K$ . This is the idea we are going to use to prove existence of the semiclassical measure  $\mu$ . And as expected,  $\mu$  will coincide with  $\tilde{\mu}_{R_K}$  on  $K$ .

The plan of this paper is the following. In section 2 we give some estimates for the source term  $S_h$ , and in Section 3 we show that  $u_h^{T,R}$  is a good approximation of  $u_h$  in order to prove Theorem 1.1. In Appendix A we recall some basic facts about differential geometry, and in particular the second fundamental form which appears when integrating by parts on  $\Gamma$ .

## 2. ESTIMATES OF THE SOURCE TERM

In this section we prove that  $S_h$  and  $S_h^R$  for  $R > 0$  are (uniformly) of size  $O(\sqrt{h})$  in  $L^{2, \delta}(\mathbb{R}^n)$ , where  $\delta > \frac{1}{2}$  is given by (1.10). Then we use Assumption (1.6) to prove that if  $\omega_- \in \mathcal{S}_0(\mathbb{R}^{2n})$  is supported in  $\mathcal{Z}_-(R, 0, -\sigma)$  for some  $R > R_1$  and  $\sigma \in ]\sigma_1, 1[$ , then  $\text{Op}_h^w(\omega_-) S_h = O(h^{\frac{3}{2}})$  in  $L^{2, \delta}(\mathbb{R}^n)$ . Since we even have an estimate of size  $O(h^\infty)$  when  $\omega_-$  is compactly supported, this proves in particular that  $S_h$  is microlocally supported outside an incoming region.

**Lemma 2.1.** *Consider  $B \in C^\infty(\Gamma)$ , a family  $(f_h^z)_{z \in \Gamma, h \in ]0, 1]}$  of functions in  $L^{2, 2+\delta+d/2}(\mathbb{R}^n)$ , and assume that for some  $C_1 \geq 0$  we have*

$$\forall h \in ]0, 1], \quad \int_{\Gamma} \langle z \rangle^\delta |B(z)| \left(1 + \|f_h^z\|_{L^{2, 2+\delta+d/2}(\mathbb{R}^n)}^2\right) d\sigma_\Gamma(z) < C_1 \quad (2.1)$$

and

$$\forall x \in \mathbb{R}^n, \forall r \in ]0, 1], \quad \int_{B(x, r) \cap \Gamma} \langle z \rangle^\delta |B(z)| d\sigma_\Gamma(z) \leq C_1 r^d. \quad (2.2)$$

For  $h \in ]0, 1]$  we consider

$$\tilde{S}_h : x \mapsto h^{\frac{1-n-d}{2}} \int_{\Gamma} B(z) f_h^z \left( \frac{x-z}{h} \right) d\sigma_\Gamma(z).$$

Then  $\tilde{S}_h(x)$  is well-defined for all  $h \in ]0, 1]$  and  $x \in \mathbb{R}^n$ , and there exists a constant  $C \geq 0$  which only depends on  $C_1$  and such that

$$\forall h \in ]0, 1], \quad \left\| \tilde{S}_h \right\|_{L^{2, \delta}(\mathbb{R}^n)} \leq C \sqrt{h}.$$

The idea of the proof is the same as for a compactly supported amplitude but we now have to be careful with the decay at infinity for functions on  $\Gamma$ .

*Proof.* We first remark that Assumption (2.2) holds for all  $r > 0$  since for  $x \in \mathbb{R}^n$  and  $r \geq 1$  Assumption (2.1) gives

$$\int_{B(x, r) \cap \Gamma} \langle z \rangle^\delta |B(z)| d\sigma_\Gamma(z) \leq C_1 \leq r^d C_1. \quad (2.3)$$

Let  $h \in ]0, 1]$  and  $x \in \mathbb{R}^n$ . According to Cauchy-Schwarz inequality and (2.3) we can write

$$\begin{aligned} \left( \int_{\Gamma} |B(z)| \left| f_h^z \left( \frac{x-z}{h} \right) \right| d\sigma_{\Gamma}(z) \right)^2 &= \left( \sum_{m \in \mathbb{N}} \int_{mh \leq |x-z| < (m+1)h} |B(z)| \left| f_h^z \left( \frac{x-z}{h} \right) \right| d\sigma_{\Gamma}(z) \right)^2 \\ &\leq c \sum_{m \in \mathbb{N}} \langle m \rangle^2 \left( \int_{mh \leq |x-z| < (m+1)h} |B(z)| \left| f_h^z \left( \frac{x-z}{h} \right) \right| d\sigma_{\Gamma}(z) \right)^2 \\ &\leq c C_1 h^d \sum_{m \in \mathbb{N}} \langle m \rangle^{2+d} \int_{mh \leq |x-z| < (m+1)h} \langle z \rangle^{-\delta} |B(z)| \left| f_h^z \left( \frac{x-z}{h} \right) \right|^2 d\sigma_{\Gamma}(z), \end{aligned}$$

where  $c \geq 0$  stands for different universal constants. Now using (2.1) we obtain

$$\begin{aligned} \|S_h\|_{L^{2,\delta}(\mathbb{R}^n)}^2 &\leq c C_1 h^{1-n} \int_{\mathbb{R}^n} \langle x \rangle^{2\delta} \sum_{m \in \mathbb{N}} \langle m \rangle^{2+d} \int_{mh \leq |x-z| < (m+1)h} \langle z \rangle^{-\delta} |B(z)| \left| f_h^z \left( \frac{x-z}{h} \right) \right|^2 d\sigma_{\Gamma}(z) dx \\ &\leq c C_1 h \sum_{m \in \mathbb{N}} \langle m \rangle^{2+d} \int_{\Gamma} \int_{m \leq |y| < m+1} \langle z+hy \rangle^{2\delta} \langle z \rangle^{-\delta} |B(z)| |f_h^z(y)|^2 dy d\sigma_{\Gamma}(z) \\ &\leq c C_1 h \sum_{m \in \mathbb{N}} \langle m \rangle^{-2} \int_{\Gamma} \langle z \rangle^{\delta} |B(z)| \int_{m \leq |y| < m+1} \langle y \rangle^{4+d+2\delta} |f_h^z(y)|^2 dy d\sigma_{\Gamma}(z) \\ &\leq c C_1^2 h. \end{aligned}$$

□

Applied with  $f_h^z = S$  and  $B = A$  or  $A_R$  for  $R > 0$  this proposition gives:

**Proposition 2.2.** *There exists a constant  $C \geq 0$  such that*

$$\forall h \in ]0, 1], \forall R > 0, \quad \|S_h\|_{L^{2,\delta}(\mathbb{R}^n)} + \|S_h^R\|_{L^{2,\delta}(\mathbb{R}^n)} \leq C\sqrt{h}.$$

For  $z \in \Gamma$  and  $\xi \in \mathbb{R}^n$  we denote by  $\xi_z^T$  the orthogonal projection of  $\xi$  on  $T_z\Gamma$ , and  $\xi_z^N = \xi - \xi_z^T$ .

**Proposition 2.3.** *Let  $q \in C_b^\infty(\mathbb{R}^{2n})$  and assume that for some  $\varepsilon > 0$  we have*

$$\forall (x, \xi) \in \text{supp } q, \forall z \in \text{supp } A, \quad (|x-z| \geq \varepsilon \quad \text{or} \quad |\xi_z^T| \geq \varepsilon).$$

Then we have

$$\|\text{Op}_h(q)S_h\|_{L^{2,\delta}(\mathbb{R}^n)} = O_{h \rightarrow 0} \left( h^{\frac{3}{2}} \right).$$

*Proof.* • We have

$$\text{Op}_h(q)S_h(x) = \frac{h^{\frac{1-n-d}{2}}}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\Gamma} e^{\frac{i}{h}\langle x-y, \xi \rangle} q(x, \xi) A(z) S\left(\frac{y-z}{h}\right) d\sigma_{\Gamma}(z) dy d\xi.$$

We recall that this is defined in the sense of an oscillatory integral. After a finite number of partial integrations with the operator  $\frac{1+i h \xi \cdot \nabla_y}{1+|\xi|^2}$  we can assume that  $q \in \mathcal{S}(\langle \xi \rangle^{-(n+1)})$ . Then we can use Fubini's Theorem and make the change of variables  $y = z + hv$  for any fixed  $z$ . Let  $\chi_1 \in C_0^\infty(\mathbb{R}^n, [0, 1])$  be supported in  $B(0, \varepsilon)$  and equal to 1 on a neighborhood of 0. We set  $\chi_2 = 1 - \chi_1$ . For  $x \in \mathbb{R}^n$  and  $h \in ]0, 1]$  we can write

$$\text{Op}_h(q)S_h(x) = I_1(x, h) + I_2(x, h),$$

where for  $j \in \{1, 2\}$ :

$$I_j(x, h) = \frac{h^{\frac{1-n-d}{2}}}{(2\pi)^n} \int_{\Gamma} A(z) \chi_j(x-z) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-z, \xi \rangle} e^{-i\langle v, \xi \rangle} q(x, \xi) S(v) dv d\xi d\sigma_{\Gamma}(z).$$

• Let  $N \in \mathbb{N}$  and

$$B(x, z, h) = \frac{\chi_2(x-z)}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-z, \xi \rangle} S(v) L^N \left( e^{-i\langle v, \xi \rangle} q(x, \xi) \right) dv d\xi,$$

where for  $f \in C^\infty(\mathbb{R}^n \times \Gamma \times \mathbb{R}^n \times ]0, 1])$  we have set

$$Lv : (x, z, \xi, h) \mapsto -ih \text{div}_{\xi} \left( \frac{(x-z)f(x, z, \xi, h)}{|x-z|^2} \right).$$

Then there exists a constant  $c$  such that

$$\forall h \in ]0, 1], \forall x \in \mathbb{R}^n, \forall z \in \Gamma, \quad |B(x, z, h)| \leq ch^N \langle x - z \rangle^{-N},$$

and hence

$$\begin{aligned} |I_2(x, h)|^2 &= h^{1-n-d} \left| \int_{\Gamma} A(z) B(x, z, h) d\sigma_{\Gamma}(z) \right|^2 \\ &\leq ch^{2N+1-n-d} \left( \int_{\Gamma} \langle z \rangle^{\delta} |A(z)| d\sigma_{\Gamma}(z) \right) \left( \int_{\Gamma} \langle z \rangle^{-\delta} |A(z)| \langle x - z \rangle^{-2N} d\sigma(z) \right), \end{aligned}$$

where  $c$  depends neither on  $x \in \mathbb{R}^n$  nor on  $h \in ]0, 1]$ . We obtain

$$\begin{aligned} \|I_2(h)\|_{L^{2,\delta}(\mathbb{R}^n)}^2 &\leq ch^{2N+1-n-d} \int_{\Gamma} \int_{\mathbb{R}^n} \langle z \rangle^{-\delta} |A(z)| \langle x \rangle^{2\delta} \langle x - z \rangle^{-2N} dx d\sigma(z) \\ &\leq ch^{2N+1-n-d} \int_{\Gamma} \int_{\mathbb{R}^n} \langle z \rangle^{\delta} |A(z)| \langle x - z \rangle^{2\delta} \langle x - z \rangle^{-2N} dx d\sigma(z) \end{aligned}$$

and finally, if  $N$  was chosen large enough:

$$\|I_2(h)\|_{L^{2,\delta}(\mathbb{R}^n)}^2 = O_{h \rightarrow 0}(h^3).$$

• We now turn to  $I_1$ . Let  $x, \xi \in \mathbb{R}^n$ . The function  $z \mapsto \xi_z^T \in T_z\Gamma$  defines a vector field on  $\Gamma$ , which we denote by  $\xi^T$ , and the norm of  $\xi_z^T$  in  $T_z\Gamma$  is the same as in  $\mathbb{R}^n$ . By assumption, if  $A(z)q(x, \xi)\chi_1(x - z) \neq 0$  then  $|\xi_z^T| \geq \varepsilon$ . And we remark that when  $\xi_z^T \neq 0$  we have

$$e^{\frac{i}{h}\langle x-z, \xi \rangle} = ih |\xi_z^T|^{-2} \xi_z^T \cdot e^{\frac{i}{h}\langle x-z, \xi \rangle},$$

where  $\xi_z^T \cdot f(z)$  is the derivative of  $f \in C^\infty(\Gamma)$  at point  $z$  and in the direction of  $\xi_z^T$ . For any  $z \in \text{supp } A \cap \overline{B(x, \varepsilon)}$  (which is compact according to Assumption (1.11)) there exists an open neighborhood  $\mathcal{V}_z$  of  $z$  in  $\Gamma$  which is orientable, and we can find  $z_1, \dots, z_K \in \text{supp } A \cap \overline{B(x, \varepsilon)}$  such that  $\text{supp } A \cap \overline{B(x, \varepsilon)} \subset \bigcup_{k=1}^K \mathcal{V}_{z_k}$ . We consider  $\zeta_1, \dots, \zeta_K \in C_0^\infty(\Gamma, [0, 1])$  such that  $\sum_{k=1}^K \zeta_k = 1$  in a neighborhood of  $\text{supp } A \cap \overline{B(x, \varepsilon)}$  in  $\Gamma$  and  $\zeta_k$  is supported in  $\mathcal{V}_k$  for all  $k \in \llbracket 1, K \rrbracket$ . Let  $k \in \llbracket 1, K \rrbracket$ . According to Green's Theorem A.1 we have

$$\int_{\Gamma} \text{div} \left( e^{\frac{i}{h}\langle x-z, \xi \rangle} \chi_1(x - z) (A\zeta_k)(z) |\xi_z^T|^{-2} \xi_z^T \right) d\sigma_{\Gamma}(z) = 0$$

and hence

$$\begin{aligned} &\int_{\Gamma} e^{\frac{i}{h}\langle x-z, \xi \rangle} (A\zeta_k)(z) \chi_1(x - z) d\sigma_{\Gamma}(z) \\ &= -ih \int_{\Gamma} \text{div} \xi^T(z) |\xi_z^T|^{-2} e^{\frac{i}{h}\langle x-z, \xi \rangle} (A\zeta_k)(z) \chi_1(x - z) d\sigma_{\Gamma}(z) \\ &\quad - ih \int_{\Gamma} e^{\frac{i}{h}\langle x-z, \xi \rangle} \xi_z^T \cdot ((A\zeta_k)(z) \chi_1(x - z)) |\xi_z^T|^{-2} d\sigma_{\Gamma}(z) \\ &\quad + ih \int_{\Gamma} e^{\frac{i}{h}\langle x-z, \xi \rangle} (A\zeta_k)(z) \chi_1(x - z) |\xi_z^T|^{-4} \xi_z^T \cdot |\xi_z^T|^2 d\sigma_{\Gamma}(z). \end{aligned}$$

Taking the sum over  $k \in \llbracket 1, K \rrbracket$  gives

$$I_1(h) = -ih(I_{1,1}(h) + I_{1,2}(h) + I_{1,3}(h))$$

where, for instance,

$$\begin{aligned} I_{1,1}(x, h) &= \frac{h^{\frac{1-n-d}{2}}}{(2\pi)^n} \int_{\Gamma} A(z) \chi_1(x - z) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \text{div} \xi^T(z) |\xi_z^T|^{-2} e^{\frac{i}{h}\langle x-z, \xi \rangle} e^{-i\langle v, \xi \rangle} q(x, \xi) S(v) dv d\xi d\sigma_{\Gamma}(z) \\ &= \frac{h^{\frac{1-n-d}{2}}}{(2\pi)^n} \int_{\Gamma} A(z) \langle \|\Pi_z\| \rangle \chi_1(x - z) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-z, \xi \rangle} e^{-i\langle v, \xi \rangle} q_{1,z}(x, \xi) S(v) dv d\xi d\sigma_{\Gamma}(z) \\ &= h^{\frac{1-n-d}{2}} \int_{\Gamma} A(z) \langle \|\Pi_z\| \rangle \chi_1(x - z) (\text{Op}_h(q_{1,z})S) \left( \frac{x - z}{h} \right) d\sigma_{\Gamma}(z). \end{aligned}$$

Here we have set

$$q_{1,z}(x, \xi) = \langle \|\Pi_z\| \rangle^{-1} \text{div} \xi^T(z) |\xi_z^T|^{-2} q(x, \xi).$$



• We recall that the Levi-Civita connection  $\nabla^\Gamma$  on the submanifold  $\Gamma$  at point  $z$  is given by the orthogonal projection on  $T_z\Gamma$  of the usual differential on  $\mathbb{R}^n$  (see Appendix A). Let  $z \in \Gamma$ . Let  $Y$  be a vector field on a neighborhood of  $z$  in  $\Gamma$  and let  $\bar{Y}$  be an extension of  $Y$  on a neighborhood of  $z$  in  $\mathbb{R}^n$ . Using (A.1) we see that on a neighborhood of  $z$  in  $\Gamma$  we have

$$\begin{aligned} \langle \nabla_Y \xi^T, Y \rangle_{T\Gamma} &= Y \cdot \langle \xi^T, Y \rangle_{T\Gamma} - \langle \xi^T, \nabla_Y Y \rangle_{T\Gamma} = \bar{Y} \cdot \langle \xi, \bar{Y} \rangle_{\mathbb{R}^n} - \langle \xi^T, \nabla_Y Y \rangle_{T\Gamma} \\ &= \langle \xi, \nabla_Y Y + \Pi(Y, Y) \rangle_{\mathbb{R}^n} - \langle \xi^T, \nabla_Y Y \rangle_{T\Gamma} \\ &= \langle \xi^N, \Pi(Y, Y) \rangle_{\mathbb{R}^n}. \end{aligned}$$

Now if  $Y_1, \dots, Y_d$  are vector fields such that  $(Y_1(z), \dots, Y_d(z))$  is an orthonormal basis of  $T_z\Gamma$  we obtain

$$\operatorname{div} \xi^T(z) = \operatorname{Tr} (Y \mapsto \nabla_Y \xi^T(z)) = \sum_{j=1}^d \langle \xi_z^N, \Pi_z(Y_j(z), Y_j(z)) \rangle_{\mathbb{R}^n},$$

and hence

$$|\operatorname{div} \xi^T(z)| \leq d |\xi_z^N| \|\Pi_z\|.$$

We can apply Lemma 2.1 with  $B = A \langle \|\Pi_z\| \rangle$  and  $f_h^z(x) = \chi_1(hx) (\operatorname{Op}_h^w(q_{1,z})S)(x)$ . Indeed, since  $q$  is assumed to be in  $\mathcal{S}(\langle \xi \rangle^{-1})$ ,  $q_{1,z}$  is in  $C_b^\infty(\mathbb{R}^{2n})$  uniformly in  $z \in \Gamma$ , and hence for all  $k \in \mathbb{N}$  the operator  $\operatorname{Op}_h^w(q_{1,z})$  belongs to  $\mathcal{L}(L^{2,k}(\mathbb{R}^n))$  and the norm is uniform in  $z$  (see [Wan88]). This proves that

$$\|I_{1,1}(h)\|_{L^{2,\delta}(\mathbb{R}^n)} = O_{h \rightarrow 0}(\sqrt{h}).$$

• We have

$$\xi_z^T \cdot (A(z)\chi_1(x-z)) = \chi_1(x-z) d_z A(\xi_z^T) + A(z) \xi_z^T \cdot \chi_1(x-z).$$

We set

$$\begin{cases} B_{2,1}(z) = \|d_z A\| \\ q_{z,2,1}(x, \xi) = \chi_1(hx) d_z A(\xi_z^T) \|d_z A\|^{-1} q(x, \xi) \end{cases} \quad \text{and} \quad \begin{cases} B_{2,2}(z) = A(z) \\ q_{z,2,2}(x, \xi) = q(x, \xi) d(\chi_1)_{hx}(\xi_z^T) \end{cases}$$

$(d_z A(\xi_z^T) \|d_z A\|^{-1})$  can be replaced by  $|\xi|$  when  $\|d_z A\| = 0$ . As above, applying Lemma 2.1 with  $B_{2,j}$  and  $f_{h,2,j}^z = \operatorname{Op}_h^w(q_{z,2,j})S$  for  $j \in \{1, 2\}$  gives then

$$\|I_{1,2}(h)\|_{L^{2,\delta}(\mathbb{R}^n)} = O_{h \rightarrow 0}(\sqrt{h}).$$

• We now deal with  $I_{1,3}(h)$ . For any vector field  $Y$  on  $\Gamma$  we have

$$\begin{aligned} Y \cdot \langle \xi^T, \xi^T \rangle_{T\Gamma} &= Y \cdot \langle \xi, \xi^T \rangle_{\mathbb{R}^n} \\ &= \langle \xi, \nabla_Y \xi^T + \Pi(Y, \xi^T) \rangle = \langle \xi^T, \nabla_Y \xi^T \rangle_{T\Gamma} + \langle \xi^N, \Pi(Y, \xi^T) \rangle_{\mathbb{R}^n} \\ &= \frac{1}{2} Y \cdot \langle \xi^T, \xi^T \rangle_{T\Gamma} + \langle \xi^N, \Pi(Y, \xi^T) \rangle_{\mathbb{R}^n}, \end{aligned}$$

and in particular:

$$\xi^T \cdot \langle \xi^T, \xi^T \rangle_{T\Gamma} = 2 \langle \xi^N, \Pi(\xi^T, \xi^T) \rangle_{\mathbb{R}^n}.$$

Thus we can estimate  $I_{1,3}(h)$  as  $I_{1,1}(h)$ , and this concludes the proof.  $\square$

We now check that according to (1.6) the assumptions of Proposition 2.3 are satisfied for a symbol  $q$  supported in an incoming region:

**Proposition 2.4.** *Let  $\sigma_2 \in ]\sigma_1, 1]$ ,  $R_2 > R_1$  and  $\nu_0 > 0$ . Then there exists  $\varepsilon > 0$  such that for all  $(x, \xi) \in \mathcal{Z}_-(R_2, \nu_0, -\sigma_2)$  and  $z \in \Gamma$  we have*

$$|x - z| \geq \varepsilon \quad \text{or} \quad |\xi_z^T| \geq \varepsilon.$$

*Proof.* Let  $(x, \xi) \in \mathcal{Z}_-(R_2, \nu_0, -\sigma_2)$ ,  $z \in \Gamma$ , and assume that

$$|x - z| \leq \min \left( R_2 - R_1, \frac{R_1(\sigma_2 - \sigma_1)}{4} \right).$$

In particular  $|z| \geq R_1$  and hence, according to (1.6), we have

$$\begin{aligned} |z| |\xi_z^T| &\geq -\langle z, \xi_z^T \rangle = \langle z, \xi_z^N - \xi \rangle = \langle z, \xi_z^N \rangle - \langle x, \xi \rangle + \langle x - z, \xi \rangle \\ &\geq -\sigma_1 |z| |\xi_z^N| + \sigma_2 |x| |\xi| - |x - z| |\xi| \geq \left( -\sigma_1 + \sigma_2 \frac{|x|}{|z|} - \frac{|x - z|}{|z|} \right) |z| |\xi| \\ &\geq \left( \sigma_2 - \sigma_1 - (1 + \sigma_2) \frac{|x - z|}{R_1} \right) |z| \nu_0 \\ &\geq |z| \frac{\nu_0 (\sigma_2 - \sigma_1)}{2}. \end{aligned}$$

□

Now we can estimate the solution  $u_h$  in an incoming region. We recall the following result:

**Proposition 2.5.** *Let  $\tilde{R} > 0$ ,  $0 \leq \tilde{\nu} < \nu$  and  $-1 < \tilde{\sigma} < \sigma < 1$ . Then there exists  $R > \tilde{R}$  such that for  $\omega \in \mathcal{S}_0(\mathbb{R}^{2n})$  supported outside  $\mathcal{Z}_-(\tilde{R}, \tilde{\nu}, -\tilde{\sigma})$  and  $\omega_- \in \mathcal{S}_0(\mathbb{R}^{2n})$  supported in  $\mathcal{Z}_-(R, \nu, -\sigma)$ , we have*

$$\sup_{z \in \mathbb{C}_{J,+}} \left\| \langle x \rangle^{-\delta} \text{Op}_h(\omega_-)(H_h - z)^{-1} \text{Op}_h(\omega) \langle x \rangle^{-\delta} \right\|_{\mathcal{L}(L^2(\mathbb{R}^n))} = O(h^\infty).$$

Moreover the estimate remains true for the limit  $(H_h - (\lambda + i0))^{-1}$ ,  $\lambda \in J$ .

This theorem is proved in [RT89] for the self-adjoint case and extended in [Roy10, Roy11] for our non-selfadjoint setting. Before giving an estimate of  $u_h$  in an incoming region, we recall that it concentrates on the hypersurface of energy  $E_0$ . For  $h \in ]0, h_0]$ ,  $t \geq 0$  and  $z \in \mathbb{C}$  we set

$$U_h(t, z) = e^{-\frac{it}{h}(H_h - z)}.$$

**Proposition 2.6.** *Let  $q \in C_b^\infty(\mathbb{R}^{2n})$  be a symbol which vanishes on  $p^{-1}(I)$  for some neighborhood  $I \subset J$  of  $E_0$ . Let  $T \geq 0$ . Then there exists  $C \geq 0$  such that for  $h \in ]0, 1]$ ,  $z \in \mathbb{C}_{I,+}$  and  $\chi \in C_0^\infty(\mathbb{R}, [0, 1])$  non-increasing and supported in  $[0, T + 1]$  we have*

$$\left\| \frac{i}{h} \int_0^\infty \chi(t) \text{Op}_h^w(q) U_h(t, z) dt \right\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C$$

and

$$\left\| \text{Op}_h^w(q)(H_h - z)^{-1} \right\|_{\mathcal{L}(L^{2,\delta}(\mathbb{R}^n), L^{2,-\delta}(\mathbb{R}^n))} \leq C.$$

In particular if  $q \in C_0^\infty(\mathbb{R}^{2n})$  is supported outside  $p^{-1}(\{E_0\})$  we have

$$\langle \text{Op}_h^w(q) u_h, u_h \rangle \xrightarrow{h \rightarrow 0} 0.$$

This is Proposition 2.11 in [Roy10]. Note that the same holds if  $\text{Op}_h^w(q)$  is on the right of the propagator or the resolvent.

**Proposition 2.7.** *Let  $\sigma \in ]\sigma_1, 1[$ . Then there exists  $R \geq 0$  such that for  $q \in C_0^\infty(\mathbb{R}^{2n})$  supported in  $\mathcal{Z}_-(R, 0, -\sigma)$  we have*

$$\langle \text{Op}_h^w(q) u_h, u_h \rangle \xrightarrow{h \rightarrow 0} 0.$$

*Proof.* Let  $\sigma_2, \sigma_3$  be such that  $\sigma_1 < \sigma_2 < \sigma_3 < \sigma$ ,  $R_2 > R_1$  and  $\nu_0 \in ]0, (\inf J)/3[$ . Let  $\omega_- \in \mathcal{S}_0(\mathbb{R}^{2n})$  be supported in  $\mathcal{Z}_-(R_2, \nu_0, -\sigma_2)$  and equal to 1 on  $\mathcal{Z}_-(2R_2, 2\nu_0, -\sigma_3)$ . Let  $R > 2R_2$  be chosen large enough and consider  $q_-, \tilde{q}_- \in C_0^\infty(\mathbb{R}^{2n})$  supported in  $\mathcal{Z}_-(R, 0, -\sigma) \cap p^{-1}(J)$  and such that  $\tilde{q}_- = 1$  on a neighborhood of  $\text{supp } q_-$ . If  $R$  is large enough we have  $\mathcal{Z}_-(R, 0, -\sigma) \cap p^{-1}(J) \subset \mathcal{Z}_-(R, 3\nu_0, -\sigma)$ , so according to Propositions 2.3 and 2.5 we have

$$\begin{aligned} &\left\| \text{Op}_h^w(q_-)(H_h - (E_h + i0))^{-1} S_h \right\|_{L^2(\mathbb{R}^n)} \\ &\leq \left\| \text{Op}_h^w(q_-)(H_h - (E_h + i0))^{-1} \right\|_{\mathcal{L}(L^{2,\delta}(\mathbb{R}^n), L^2(\mathbb{R}^n))} \left\| \text{Op}_h(\omega_-) S_h \right\|_{L^{2,\delta}(\mathbb{R}^n)} \\ &\quad + \left\| \text{Op}_h^w(q_-)(H_h - (E_h + i0))^{-1} (1 - \text{Op}_h(\omega_-)) \langle x \rangle^{-\delta} \right\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \left\| S_h \right\|_{L^{2,\delta}(\mathbb{R}^n)} \\ &= O(\sqrt{h}). \end{aligned}$$

The same applies to  $\tilde{q}_-$  and this finally gives

$$|\langle \text{Op}_h^w(q_-) u_h, u_h \rangle| \leq \left\| \text{Op}_h^w(q_-) u_h \right\| \left\| \text{Op}_h^w(\tilde{q}_-) u_h \right\| + O(h^\infty) \xrightarrow{h \rightarrow 0} 0. \quad \square$$

### 3. CONTROL OF LARGE TIMES AND OF THE SOURCE TERM FAR FROM THE ORIGIN

As mentioned in introduction, we expect that the semiclassical measure of  $(u_h)_{h \in ]0, h_0]}$  on some bounded subset of  $\mathbb{R}^{2n}$  does not depend on the values of the amplitude  $A(z)$  for large  $|z|$ . When restricting our attention to finite times, this is a consequence of Egorov's Theorem and the following proposition, proved in [Roy11] (Proposition 2.1):

**Proposition 3.1.** *Let  $E_2 \geq E_1 > 0$ ,  $J \subset [E_1, E_2]$  and  $\sigma_3 \in [0, 1[$  such that  $\sigma_3^2 E_2 < E_1$ . Then there exist  $\mathcal{R} > 0$  and  $c_0 > 0$  such that*

$$\forall t \geq 0, \forall (x, \xi) \in \mathcal{Z}_\pm(\mathcal{R}, 0, \mp \sigma_3) \cap p^{-1}(J), \quad |X(\pm t, x, \xi)| \geq c_0(t + |x|).$$

For  $r > 0$  we set

$$B_x(r) = \{(x, \xi) : |x| < r\} \subset \mathbb{R}^{2n}.$$

With Proposition 3.1 we can check that on a bounded subset of  $\mathbb{R}^{2n}$  we can ignore the contribution of the source far from the origin:

**Proposition 3.2.** *Let  $r > 0$ . There exists  $R_0 > 0$  such that for  $T \geq 0$ ,  $R \geq R_0$ ,  $\text{Im } z \geq 0$  and  $q \in C_0^\infty(\mathbb{R}^{2n}, [0, 1])$  supported in  $B_x(r)$  we have*

$$\left\| \frac{i}{h} \int_0^\infty \chi_{T,R}(t) \text{Op}_h^w(q) U_h(t, z) (S_h - S_h^R) dt \right\|_{L^2(\mathbb{R}^n)} = O(\sqrt{h}),$$

where the size of the rest depends on  $T$  but not on  $z$ ,  $q$  or  $R \geq R_0$ .

We recall that  $\chi_{T,R} \in C^\infty(\mathbb{R}, [0, 1])$  is non-increasing, equal to 1 on  $] -\infty, T]$  and equal to 0 on  $[T + 1, +\infty[$ .

*Proof.* Let  $\mathcal{R}$  and  $c_0$  be given by Proposition 3.1 applied with  $\sigma_3 = (1 + \sigma_1)/2$  (which is allowed according to Assumption (1.7)). Let  $\tilde{\mathcal{R}} > \max(\mathcal{R}, 2r/c_0, R_1)$  be so large that

$$\nu_0^2 := \inf J - \sup_{|x| \geq \tilde{\mathcal{R}}} |V_1(x)| > 0.$$

Let  $\sigma_2 \in ]\sigma_1, \sigma_3[$ . We consider  $\omega \in C_b^\infty(\mathbb{R}^{2n})$  supported in  $\mathcal{Z}_+(\tilde{\mathcal{R}}, 0, -\sigma_3)$  and equal to 1 in a neighborhood of  $\mathcal{Z}_+(2\tilde{\mathcal{R}}, 0, -\sigma_2)$ . Given  $\theta \in C_0^\infty(\mathbb{R})$  supported in  $J$  and equal to 1 on a neighborhood of  $E_0$  we prove that

$$\sum_{j=1}^3 \left\| \frac{i}{h} \int_0^\infty \chi_{T,R}(t) \text{Op}_h^w(q) U_h(t, z) B_j(h) (S_h - S_h^R) dt \right\|_{L^2(\mathbb{R}^n)} = O(\sqrt{h}),$$

where

$$B_1(h) = \text{Op}_h^w(1 - \theta \circ p), \quad B_2(h) = \text{Op}_h^w(\theta \circ p) \text{Op}_h^w(\omega), \quad B_3(h) = \text{Op}_h^w(\theta \circ p) \text{Op}_h^w(1 - \omega).$$

The first term is estimated with Propositions 2.6 and 2.2. The function  $\chi_{T,R}$  depends on  $R$ , but since it is non-increasing and always vanishes on  $[T + 1, +\infty[$ , we can check that the estimate is actually uniform in  $R > 0$ . According to Proposition 3.1 and Egorov's Theorem (see Theorem 7.2 in [Roy11]), the second term is of size  $O(h^\infty)$ , again uniformly in  $R \geq 0$ . For the third term we set  $R_0 = 3\tilde{\mathcal{R}}$  and choose  $R \geq R_0$ . The symbol  $(\theta \circ p)(1 - \omega)$  is supported in

$$B_x(2\tilde{\mathcal{R}}) \cup \mathcal{Z}_-(2\tilde{\mathcal{R}}, 0, -\sigma_2) \cap p^{-1}(J) \subset B_x(2\tilde{\mathcal{R}}) \cup \mathcal{Z}_-(2\tilde{\mathcal{R}}, \nu_0, -\sigma_2).$$

Since  $A - A_R$  is supported outside  $B(0, 3\tilde{\mathcal{R}})$  and according to Propositions 2.3 and 2.4 (applied with  $A - A_R$  instead of  $A$ ) we obtain that  $\|\text{Op}_h^w(1 - \omega)(S_h - S_h^R)\|_{L^2(\mathbb{R}^n)} = O(h^{\frac{3}{2}})$  uniformly in  $R$ .  $\square$

We know that (1.15) holds for some measure  $\mu_T^R$  when  $u_h$  is replaced by  $u_h^{T,R}$  (see Theorem 4.3 in [Roy10]). We now check that  $u_h^{T,R}$  is actually a good approximation of  $u_h$  (in some sense) when  $h > 0$  is small and  $T, R$  are large enough:

**Proposition 3.3.** *Let  $r > 0$  and  $R_0 > 0$  given by Proposition 3.2. Let  $K$  be a compact subset of  $B_x(r) \cap p^{-1}(J)$  and  $\varepsilon > 0$ . Then there exists  $T_0 \geq 0$  such that for  $q \in C_0^\infty(\mathbb{R}^{2n})$  supported in  $K$ ,  $T \geq T_0$  and  $R \geq R_0$  we have*

$$\limsup_{h \rightarrow 0} \left| \langle \text{Op}_h^w(q) u_h, u_h \rangle - \langle \text{Op}_h^w(q) u_h^{T,R}, u_h^{T,R} \rangle \right| \leq \varepsilon \|q\|_\infty.$$

This proposition relies on the following consequence of the non-selfadjoint version of Egorov's Theorem (see [Roy11, Prop.7.3]):

**Proposition 3.4.** *Let  $J$  be a neighborhood of  $E$  such that Assumption (1.5) holds for all  $\lambda \in J$ . Let  $K_1$  and  $K_2$  be compact subsets of  $p^{-1}(J)$ . Let  $\varepsilon > 0$ . Then there exists  $T_0 \geq 0$  such that for  $T \geq T_0$  and  $q_1, q_2 \in C_0^\infty(\mathbb{R}^{2n})$  respectively supported in  $K_1$  and  $K_2$  we have*

$$\limsup_{h \rightarrow 0} \|\text{Op}_h^w(q_1)U_h(T)\text{Op}_h^w(q_2)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq \varepsilon \|q_1\|_\infty \|q_2\|_\infty.$$

We also need the following result about the classical flow:

**Proposition 3.5.** *Let  $E_1, E_2 \in \mathbb{R}_+^*$  be such that  $E_1 \leq E_2$ , and  $\sigma \in [0, 1[$ . If  $\mathcal{R}$  is chosen large enough then for any compact subset  $K$  of  $p^{-1}([E_1, E_2])$  there exists  $T_0 \geq 0$  such that*

$$\forall w \in K, \forall t \geq T_0, \quad \phi^{\pm t}(w) \in B_x(\mathcal{R}) \cup \mathcal{Z}_\pm(\mathcal{R}, 0, \pm\sigma).$$

This is slightly more general than Lemma 5.2 in [Roy10]. We recall the idea of the proof:

*Proof.* We consider  $\mathcal{R}_0$  such that

$$\forall x \in \mathbb{R}^n, \quad |x| \geq \mathcal{R}_0 \quad \implies \quad |V_1(x)| + |x| |\nabla V_1(x)| \leq \frac{E_1}{3}(1 - \sigma^2).$$

Let  $\tau > 0$  be such that  $\int_0^\tau \frac{(1-\sigma^2)\sqrt{E_1}}{\sqrt{3}(\mathcal{R}_0 + 4s\sqrt{E_2})} ds > \sigma$ . We set  $\mathcal{R} = \mathcal{R}_0 + 4\tau\sqrt{E_2}$  and  $\mathcal{U}_\pm = B_x(\mathcal{R}_0) \cup \mathcal{Z}_\pm(\mathcal{R}, 0, \pm\sigma)$ . We first prove that for any  $w \in K$ , if  $\phi^{\pm t}(w) \in \mathcal{U}_\pm$  for some  $t_w \geq 0$ , then  $\phi^{\pm t}(w) \in B_x(\mathcal{R}) \cup \mathcal{Z}_\pm(\mathcal{R}, 0, \pm\sigma)$  for all  $t \geq t_w$ . Since  $\phi^{\pm(t-t_w)}$  maps  $\mathcal{Z}_\pm(\mathcal{R}, 0, \pm\sigma)$  into itself for all  $t \geq t_w$ , we can assume that  $|X(\pm t_w, w)| = \mathcal{R}_0$ ,  $|X(\pm t, w)| = \mathcal{R}$  and  $|X(\pm s, w)| \in [\mathcal{R}, \mathcal{R}_0]$  for  $s \in [t_w, t]$ . Assume by contradiction that  $\phi^{\pm s}(w) \notin \mathcal{Z}_\pm(\mathcal{R}, 0, \pm\sigma)$  when  $s \in [t_w, t]$ . Then we can check that

$$\pm \frac{\partial}{\partial s} \frac{X(\pm s, w) \cdot \Xi(\pm s, w)}{|X(\pm s, w)| |\Xi(\pm s, w)|} \geq \frac{(1 - \sigma^2)\sqrt{E_1}}{\sqrt{3}(\mathcal{R}_0 + 4(s - t_0)\sqrt{E_2})},$$

which gives a contradiction. Then it only remains to check that if  $T_0$  is chosen large enough, then for all  $w \in K$  we can find  $t_w \in [0, T_0]$  such that  $\phi^{\pm t_w}(w) \in \mathcal{U}_\pm$ . For this we use compactness of  $K$  and the fact that any trajectory has a limit point in  $\Omega_b([E_1, E_2]) \subset B_x(\mathcal{R}_0)$  or goes to infinity and meets  $\mathcal{Z}_\pm(\mathcal{R}, 0, \pm\sigma)$  when  $t$  is large enough.  $\square$

Let  $\sigma_2 < \sigma_3 < \sigma_4 < \sigma_5 \in ]\sigma_1, 1[$  and  $\nu_0 \in ]0, \sqrt{\inf J}/4[$ . Let  $\mathcal{R}$  be given by Proposition 2.5 applied with  $(\tilde{R}, \tilde{\nu}, \tilde{\sigma}) = (3R_1, 2\nu_0, \sigma_3)$  and  $(\nu, \sigma) = (3\nu_0, \sigma_4)$ . Choosing  $\mathcal{R}$  larger if necessary, we can assume that  $|\xi| \geq 4\nu_0$  if  $p(x, \xi) \in J$  and  $|x| \geq \mathcal{R}$ . We can also assume that  $2\mathcal{R}$  satisfies the conclusion of Proposition 3.5 applied with  $[E_1, E_2] \supset J$ .

**Lemma 3.6.** *Let  $r \geq 4\mathcal{R}$  and  $R_0 > 0$  given by Proposition 3.2. Let  $Q \in C_0^\infty(\mathbb{R}^{2n}, [0, 1])$  be supported in  $B_x(r) \cap p^{-1}(J)$  and equal to 1 in a neighborhood of  $B_x(3\mathcal{R}) \cap p^{-1}(\bar{I})$  for some open neighborhood  $I$  of  $E_0$ . Let  $K$  be a compact subset of  $B_x(r) \cap p^{-1}(J)$  and  $\delta > 0$ . Then there exists  $T_0 \geq 0$  such that for  $T \geq T_0$ ,  $R \geq R_0$  and  $q \in C_0^\infty(\mathbb{R}^{2n})$  supported in  $K$  we have*

$$\limsup_{h \rightarrow 0} \left\| \text{Op}_h^w(q) \left( u_h - u_h^{T,R} - A_T^\delta(h) \text{Op}_h^w(Q) u_h \right) \right\|_{L^2(\mathbb{R}^n)} \leq \delta \|q\|_\infty,$$

where  $A_T^\delta(h)$  is a bounded operator such that

$$\forall T \geq T_0, \quad \limsup_{h \rightarrow 0} \|A_T^\delta(h)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq \delta.$$

For the proof we follow the same general idea as in [Roy10]:

*Proof.* • We consider  $\tilde{q} \in C_0^\infty(\mathbb{R}^{2n})$  supported in  $B_x(r) \cap p^{-1}(J)$  and equal to 1 on a neighborhood of  $K$ . Let  $\theta \in C_0^\infty(\mathbb{R}^n)$  be supported in  $B(0, 3\mathcal{R})$  and equal to 1 on  $B(0, 2\mathcal{R})$ . Let  $w_- \in \mathcal{S}_0(\mathbb{R}^{2n})$  be equal to 1 on  $\mathcal{Z}_-(2\mathcal{R}, 4\nu_0, -\sigma_5)$  and supported in  $\mathcal{Z}_-(\mathcal{R}, 3\nu_0, -\sigma_4)$ . Let  $q \in C_0^\infty(\mathbb{R}^{2n})$  be supported in  $K$ ,  $h \in ]0, h_0]$ ,  $T \geq 0$  and  $z \in \mathbb{C}_{I,+}$  ( $h_0 > 0$  was fixed small enough in the introduction). Since  $z$  is not in the spectrum of  $H_h$  we can consider  $(H_h - z)^{-1}S_h \in H^2(\mathbb{R}^n)$

and write:

$$\begin{aligned}
& \text{Op}_h^w(q)(H_h - z)^{-1}S_h - \text{Op}_h^w(q)U_h(T, z)(H_h - z)^{-1}S_h \\
&= \frac{i}{h} \int_0^\infty \text{Op}_h^w(q)(\chi_{T,R}(t) - \chi_{0,R}(t)U_h(T, z))U_h(t, z)S_h dt \\
&= \frac{i}{h} \int_0^\infty \text{Op}_h^w(q)(\chi_{T,R}(t) - \chi_{0,R}(t)U_h(T, z))U_h(t, z)S_h^R dt + \underset{h \rightarrow 0}{O}(\sqrt{h}),
\end{aligned} \tag{3.1}$$

where the rest is estimated in  $L^2(\mathbb{R}^n)$  uniformly in  $R \geq R_0$  but not in  $T$  (see Proposition 3.2).

• We have

$$\begin{aligned}
\text{Op}_h^w(q)U_h(T, z)(H_h - z)^{-1}S_h &= \text{Op}_h^w(q)U_h(T, z)\text{Op}_h^w(Q)(H_h - z)^{-1}S_h \\
&\quad + \text{Op}_h^w(q)U_h(T, z)\text{Op}_h^w(1 - Q)\theta(x)(H_h - z)^{-1}S_h \\
&\quad + \text{Op}_h^w(q)U_h(T, z)\text{Op}_h^w(1 - Q)(1 - \theta(x))\text{Op}_h(\omega_-)(H_h - z)^{-1}S_h \\
&\quad + \text{Op}_h^w(q)U_h(T, z)\text{Op}_h^w(1 - Q)(1 - \theta(x))\text{Op}_h(1 - \omega_-)(H_h - z)^{-1}S_h.
\end{aligned}$$

The second term of the right-hand side is of size  $O(\sqrt{h})$  uniformly in  $z \in \mathbb{C}_{I,+}$  according to Proposition 2.6. Let  $\omega \in \mathcal{S}_0(\mathbb{R}^{2n})$  be supported in  $\mathcal{Z}_-(2R_1, \nu_0, -\sigma_2)$  and equal to 1 in  $\mathcal{Z}_-(3R_1, 2\nu_0, -\sigma_3)$ . According to Propositions 2.3, 2.4 and 2.5 we have

$$\begin{aligned}
& \left\| \text{Op}_h^w(q)U_h(T, z)\text{Op}_h^w(1 - Q)(1 - \theta(x))\text{Op}_h(\omega_-)(H_h - z)^{-1}S_h \right\| \\
& \leq c_{q,T} \left\| \text{Op}_h(\omega_-)(H_h - z)^{-1}(1 - \text{Op}_h(\omega))S_h \right\|_{L^{2,-\delta}(\mathbb{R}^n)} + \underset{h \rightarrow 0}{O}(\sqrt{h}) \\
& = \underset{h \rightarrow 0}{O}(\sqrt{h}),
\end{aligned}$$

uniformly in  $z \in \mathbb{C}_{J,+}$  ( $\|\text{Op}_h^w(q)U_h(T, z)\langle x \rangle^\delta\| = O(1)$  uniformly in  $z \in \mathbb{C}_{J,+}$ ) but not in  $T$ . Finally the last term is of size  $O(h^\infty)$  according to Egorov's Theorem and Proposition 3.5 if  $T \geq T_0$ ,  $T_0$  being given by Proposition 3.5 applied with  $\sigma_5$ . We consider  $\tilde{q}, \tilde{Q} \in C_0^\infty(\mathbb{R}^{2n}, [0, 1])$  supported in  $p^{-1}(J)$  and equal to 1 respectively on a neighborhood of  $\text{supp } q$  and  $\text{supp } Q$ , and we set

$$A_T^\delta(z, h) = \text{Op}_h^w(\tilde{q})U_h(T, z)\text{Op}_h^w(\tilde{Q}).$$

According to Proposition 3.4 we have

$$\limsup_{h \rightarrow 0} \sup_{z \in \mathbb{C}_{I,+}} \|A_T^\delta(z, h)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq \delta$$

when  $T \geq T_0$ , if  $T_0$  was chosen large enough. We finally obtain

$$\text{Op}_h^w(q)U_h(T, z)(H_h - z)^{-1}S_h = \text{Op}_h^w(q)A_T^\delta(z, h)\text{Op}_h^w(Q)(H_h - z)^{-1}S_h + \underset{h \rightarrow 0}{O}(\sqrt{h})$$

in  $L^2(\mathbb{R}^n)$  where the size of the rest is uniform in  $z \in \mathbb{C}_{I,+}$ .

**3.** For  $h \in ]0, h_0]$  and  $T \geq T_0$ , we can take the limit  $z \rightarrow E_h$  in (3.1) ( $E_h \in \overline{\mathbb{C}_{I,+}}$  if  $h_0$  is small enough). This gives in  $L^2(\mathbb{R}^n)$ :

$$\text{Op}_h^w(q)u_h = \text{Op}_h^w(q)u_h^{T,R} - \text{Op}_h^w(q)U_h^E(T)u_h^{0,R} + \text{Op}_h^w(q)A_T^\delta(E_h, h)\text{Op}_h^w(Q)u_h + \underset{h \rightarrow 0}{O}(\sqrt{h}).$$

Let  $q_1 \in C_0^\infty(\mathbb{R}^{2n}, [0, 1])$  be equal to 1 on a neighborhood of  $\{\phi^t(z, \xi), t \in [0, 1], (z, \xi) \in N_E^{R_0}\Gamma\}$ . Using the results about the contribution of small times (see in particular Corollary 4.4 in [Roy10]), we know that

$$\begin{aligned}
\limsup_{h \rightarrow 0} \left\| \text{Op}_h^w(q)U_h^E(T)u_h^{0,R_0} \right\|_{L^2(\mathbb{R}^n)} &= \limsup_{h \rightarrow 0} \left\| \text{Op}_h^w(q)U_h^E(T)\text{Op}_h^w(q_1^2)u_h^{0,R_0} \right\|_{L^2(\mathbb{R}^n)} \\
&\leq C \limsup_{h \rightarrow 0} \left\| \text{Op}_h^w(q_1)U_h^E(T)\text{Op}_h^w(q_1) \right\|_{\mathcal{L}(L^2(\mathbb{R}^n))},
\end{aligned}$$

for some constant  $C$  which does not depend on  $h, T$  or  $R$ . According to Proposition 3.4, this limit is less than  $\delta \|q\|_\infty$  for any  $T \geq T_0$  if  $T_0$  was chosen large enough. Since for any  $T \geq T_0$  we have

$$\phi^{-T}(\text{supp } q) \cap \left\{ \phi^t(z, \xi), t \in [0, 1], (z, \xi) \in N_E^R\Gamma \setminus N_E^{R_0}\Gamma \right\} = \emptyset$$

Egorov's Theorem also gives

$$\limsup_{h \rightarrow 0} \left\| \text{Op}_h^w(q) U_h^E(T) (u_h^{0,R} - u_h^{0,R_0}) \right\|_{L^2(\mathbb{R}^n)} = 0,$$

which concludes the proof.  $\square$

Then Proposition 3.3 is proved exactly as in [Roy10] (see Proposition 5.5), and we can show existence of a semiclassical measure:

**Proposition 3.7.** *There exists a non-negative Radon measure  $\mu$  on  $\mathbb{R}^{2n}$  such that for  $q \in C_0^\infty(\mathbb{R}^{2n})$  we have*

$$\int_{\mathbb{R}^{2n}} q d\mu = \lim_{T, R \rightarrow +\infty} \int_{\mathbb{R}^{2n}} q d\mu_T^R \quad (3.2)$$

and

$$\langle \text{Op}_h^w(q) u_h, u_h \rangle \xrightarrow{h \rightarrow 0} \int_{\mathbb{R}^{2n}} q d\mu.$$

*Proof.* The result is clear outside  $p^{-1}(\{E_0\})$ , so we focus on symbols supported in  $p^{-1}(J)$ . Let  $K$  be a compact subset of  $p^{-1}(J)$  and  $\varepsilon > 0$ . Let  $T_0$  and  $R_0$  given by Proposition 3.3. For  $T_1, T_2 \geq T_0$ ,  $R_1, R_2 \geq R_0$  and  $q \in C_0^\infty(\mathbb{R}^{2n})$  supported in  $K$  we have

$$\begin{aligned} & \left| \int_{\mathbb{R}^{2n}} q d\mu_{T_1}^{R_1} - \int_{\mathbb{R}^{2n}} q d\mu_{T_2}^{R_2} \right| \\ &= \lim_{h \rightarrow 0} \left| \left\langle \text{Op}_h^w(q) u_h^{T_1, R_1}, u_h^{T_1, R_1} \right\rangle - \left\langle \text{Op}_h^w(q) u_h^{T_2, R_2}, u_h^{T_2, R_2} \right\rangle \right| \leq 2\varepsilon \|q\|_\infty. \end{aligned}$$

This proves that  $(T, R) \mapsto \int q d\mu_T^R$  has a limit at infinity, which we denote by  $L(q)$ . The map  $q \mapsto \int q d\mu_T^R$  is a nonnegative linear form on  $C_0^\infty(\mathbb{R}^{2n})$  for all  $T, R > 0$ , and hence so is  $q \mapsto L(q)$ . Let  $T_0$  be as above for  $\varepsilon = 1$  and  $C_K$  be a constant such that for all  $q \in C_0^\infty(\mathbb{R}^{2n})$  we have

$$\left| \int_{\mathbb{R}^{2n}} q d\mu_{T_0}^{R_0} \right| \leq C_K \|q\|_\infty.$$

Then we have

$$\begin{aligned} |L(q)| &\leq \left| L(q) - \int_{\mathbb{R}^{2n}} q d\mu_{T_0}^{R_0} \right| + \left| \int_{\mathbb{R}^{2n}} q d\mu_{T_0}^{R_0} \right| \\ &\leq \lim_{T, R \rightarrow +\infty} \left| \int_{\mathbb{R}^{2n}} q d\mu_T^R - \int_{\mathbb{R}^{2n}} q d\mu_{T_0}^{R_0} \right| + C_K \|q\|_\infty \\ &\leq (2 + C_K) \|q\|_\infty, \end{aligned}$$

which proves that the linear form  $L$  on  $C_0^\infty(\mathbb{R}^{2n})$  can be extended as a continuous linear form on the space of continuous and compactly supported functions on  $\mathbb{R}^{2n}$ . The first assertion is now a consequence of Riesz' Lemma. And the second can be proved as in [Roy10], using the fact that  $\langle \text{Op}_h^w(q) u_h, u_h \rangle$  is close to  $\langle \text{Op}_h^w(q) u_h^{T, R}, u_h^{T, R} \rangle$  in the sense of Proposition 3.3, and  $\langle \text{Op}_h^w(q) u_h^{T, R}, u_h^{T, R} \rangle$  goes to  $\int q d\mu_T^R$  as  $h$  goes to 0.  $\square$

It is now easy to prove the remark about the measures  $\tilde{\mu}_R$  mentioned in introduction:

**Proposition 3.8.** *Let  $r > 0$ ,  $R_0$  given by Proposition 3.2 and  $R \geq R_0$ . Then the measures  $\mu$  and  $\tilde{\mu}_R$  coincide on  $B_x(r)$ .*

To prove this assertion, we only have to apply Proposition 3.3 with  $u_h$  and  $\tilde{u}_h^R$ . Let  $q \in C_0^\infty(\mathbb{R}^{2n})$  be supported in  $B_x(r) \cap p^{-1}(J)$ . Since for large  $T$  and small  $h$  the quantity  $\langle \text{Op}_h^w(q) u_h^{T, R}, u_h^{T, R} \rangle$  is a good approximation both for  $\langle \text{Op}_h^w(q) u_h, u_h \rangle$  and  $\langle \text{Op}_h^w(q) \tilde{u}_h^R, \tilde{u}_h^R \rangle$ , these two quantities have the same limit as  $h$  goes to 0.

It only remains to prove the properties given in Theorem 1.1:

*Proof.* The first two properties are direct consequences of Propositions 2.6 and 2.7. Let  $r > 0$  and  $R_0$  given by Proposition 3.2. Property (c) is already known for the compactly supported amplitude  $A_{R_0}$ , and hence for any  $q \in C_0^\infty(\mathbb{R}^{2n})$  we have

$$\int_{\mathbb{R}^{2n}} (-H_p + 2 \operatorname{Im} \tilde{E} + 2V_2)q \, d\tilde{\mu}_{R_0} = \pi(2\pi)^{d-n} \int_{N_E\Gamma} q(z, \xi) |A_{R_0}(z)|^2 |\xi|^{-1} \left| \hat{S}(\xi) \right|^2 d\sigma_{N_E\Gamma}(z, \xi).$$

Now assume that  $q$ , and hence  $(H_p + 2 \operatorname{Im} \tilde{E} + 2V_2)q$ , are supported in  $B_x(r)$ . According to Proposition 3.8 we have

$$\int_{\mathbb{R}^{2n}} (-H_p + 2 \operatorname{Im} \tilde{E} + 2V_2)q \, d\tilde{\mu}_{R_0} = \int_{\mathbb{R}^{2n}} (-H_p + 2 \operatorname{Im} \tilde{E} + 2V_2)q \, d\mu.$$

On the other hand, according to Proposition 3.1 trajectories coming from the points of  $N_E\Gamma \setminus N_E^{R_0}\Gamma$  never reach  $B_x(r)$ , so we also have

$$\pi(2\pi)^{d-n} \int_{N_E\Gamma} q(z, \xi) (|A(z)|^2 - |A_{R_0}(z)|^2) |\xi|^{-1} \left| \hat{S}(\xi) \right|^2 d\sigma_{N_E\Gamma}(z, \xi) = 0.$$

This proves that Property (c) holds when  $q \in C_0^\infty(\mathbb{R}^{2n})$  is supported in  $B_x(r)$ . Since this holds for any  $r > 0$ , the theorem is proved.  $\square$

#### APPENDIX A. SHORT REVIEW ABOUT DIFFERENTIAL GEOMETRY

We briefly recall in this appendix the basic results of differential geometry we have used. Detailed expositions can be found for instance in [dC92] and [Spi99].

Let  $M$  be a differential manifold. We denote by  $\mathcal{X}(M)$  the set of vector fields on  $M$ . An affine connection on  $M$  is a mapping

$$\nabla : \begin{cases} \mathcal{X}(M) \times \mathcal{X}(M) & \rightarrow \mathcal{X}(M) \\ (X, Y) & \mapsto \nabla_X Y \end{cases}$$

which satisfies the following properties (for  $X, Y, Z \in \mathcal{X}(M)$  and  $f, g \in C^\infty(M)$ ):

- (i)  $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$ ,
- (ii)  $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$ ,
- (iii)  $\nabla_X(fY) = f\nabla_X Y + (X \cdot f)Y$ .

The Levi-Civita connection on  $M$  is the unique connection  $\nabla$  on  $M$  which is

- (i) symmetric:

$$\forall X, Y \in \mathcal{X}(M), \quad \nabla_X Y - \nabla_Y X = [X, Y] = XY - YX,$$

- (ii) and compatible with the Riemannian metric:

$$\forall X, Y, Z \in \mathcal{X}(M), \quad X \cdot \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle_M + \langle Y, \nabla_X Z \rangle_M. \quad (\text{A.1})$$

The Levi-Civita connection on  $\mathbb{R}^n$  endowed with the canonical metric is the usual differential. Now let  $\Gamma$  be a submanifold of  $\mathbb{R}^n$ , endowed by the Riemannian structure defined by restriction of the scalar product of  $\mathbb{R}^n$ . For  $X, Y \in \mathcal{X}(\Gamma)$ ,  $z \in \Gamma$ , and  $\bar{X}, \bar{Y} \in \mathcal{X}(\mathbb{R}^n)$  such that  $X = \bar{X}$  and  $Y = \bar{Y}$  in a neighborhood of  $z$  in  $\Gamma$  we set

$$\nabla_X^\Gamma Y(z) = \left( \nabla_{\bar{X}}^{\mathbb{R}^n} \bar{Y}(z) \right)_z^T.$$

This definition does not depend on the choice of  $\bar{X}$  or  $\bar{Y}$  and defines the Levi-Civita connexion on  $\Gamma$ .

Let  $X \in \mathcal{X}(\Gamma)$ . The divergence  $\operatorname{div} X(z)$  at point  $z \in \Gamma$  is defined as the trace of the linear map  $Y \mapsto \nabla_Y^\Gamma X(z)$  on  $T_z\Gamma$ . If  $X \in \mathcal{X}(\Gamma)$  and  $f \in C^\infty(\Gamma)$  then we have

$$\operatorname{div}(fX) = X \cdot f + f \operatorname{div} X. \quad (\text{A.2})$$

The main theorem we have used in Section 2 is the following:



**Theorem A.1** (Green's Theorem). *If  $M$  is an oriented Riemannian manifold and  $X \in \mathcal{X}(M)$  is compactly supported, then*

$$\int_M \operatorname{div} X \, dV_M = 0,$$

where  $dV_m$  denotes the volume element on  $M$ .

We finally recall the basic properties of the second fundamental form on  $\Gamma$ . Given  $X, Y \in \mathcal{X}(\Gamma)$  and  $z \in \Gamma$  we set

$$\Pi_z(X, Y) = \nabla_{\overline{X}}^{\mathbb{R}^n} \overline{Y}(z) - \nabla_X^\Gamma Y(z) = \left( \nabla_{\overline{X}}^{\mathbb{R}^n} \overline{Y}(z) \right)_z^N \in N_z \Gamma,$$

where  $\overline{X}$  and  $\overline{Y}$  are extensions of  $X$  and  $Y$  on a neighborhood of  $z$  in  $\mathbb{R}^n$ . We can check that  $\Pi_z(X, Y)$  is well-defined and actually only depends on  $X(z)$  and  $Y(z)$ . Moreover the bilinear form  $\Pi_z$  is symmetric.

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