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**ABOUT BREZIS-MERLE PROBLEM WITH HOLDERIAN CONDITION: THE
CASE OF ONE OR TWO BLOW-UP POINTS.**

SAMY SKANDER BAHOURA

ABSTRACT. We consider the following problem on open set Ω of \mathbb{R}^2 :

$$\begin{cases} -\Delta u_i = V_i e^{u_i} & \text{in } \Omega \subset \mathbb{R}^2, \\ u_i = 0 & \text{in } \partial\Omega. \end{cases}$$

We give a quantization analysis of the previous problem under the conditions:

$$0 \leq V_i \leq b < +\infty, \text{ and } \int_{\Omega} e^{u_i} dy \leq C.$$

On the other hand, if we assume that

$$\int_{\Omega} V_i e^{u_i} dy \leq 4\pi,$$

or, V_i s -holderian with $1/2 < s \leq 1$, and,

$$\int_{\Omega} V_i e^{u_i} dy \leq 24\pi - \epsilon, \quad \epsilon > 0$$

then we have a compactness result, namely:

$$\sup_{\Omega} u_i \leq c = c(b, C, A, s, \epsilon, \Omega).$$

where A is the holderian constant of V_i .

1. INTRODUCTION AND MAIN RESULTS

We set $\Delta = \partial_{11} + \partial_{22}$ on open set Ω of \mathbb{R}^2 with a smooth boundary.

We consider the following problem on $\Omega \subset \mathbb{R}^2$:

$$(P) \begin{cases} -\Delta u_i = V_i e^{u_i} & \text{in } \Omega \subset \mathbb{R}^2, \\ u_i = 0 & \text{in } \partial\Omega. \end{cases}$$

We assume that,

$$\int_{\Omega} e^{u_i} dy \leq C,$$

and,

$$0 \leq V_i \leq b < +\infty$$

The above equation is called, the Prescribed Scalar Curvature equation, in relation with conformal change of metrics. The function V_i is the prescribed curvature.

Here, we try to find some a priori estimates for sequences of the previous problem.

Equations of this type were studied by many authors, see [1-26]. One can see in [8] different results for the solutions of those type of equations with or without boundaries conditions and, with minimal conditions on V , for example we suppose $V_i \geq 0$ and $V_i \in L^p(\Omega)$ or $V_i e^{u_i} \in L^p(\Omega)$ with $p \in [1, +\infty]$.

Among other results, we can see in [8], the following important Theorem,

Theorem A(Brezis-Merle [8]). *If $(u_i)_i$ and $(V_i)_i$ are two sequences of functions relatively to the previous problem (P) with, $0 < a \leq V_i \leq b < +\infty$, then, for all compact set K of Ω ,*

$$\sup_K u_i \leq c = c(a, b, K, \Omega).$$

We can find in [8] an interior estimate if we assume $a = 0$, but we need an assumption on the integral of e^{u_i} . We have in [8]:

Theorem B (Brezis-Merle [8]). *If $(u_i)_i$ and $(V_i)_i$ are two sequences of functions relatively to the previous problem (P) with, $0 \leq V_i \leq b < +\infty$, and,*

$$\int_{\Omega} e^{u_i} dy \leq C,$$

then, for all compact set K of Ω ,

$$\sup_K u_i \leq c = c(b, C, K, \Omega).$$

If, we assume V with more regularity, we can have another type of estimates, $\sup + \inf$. It was proved, by Shafrir, see [23], that, if $(u_i)_i, (V_i)_i$ are two sequences of functions solutions of the previous equation without assumption on the boundary and, $0 < a \leq V_i \leq b < +\infty$, then we have the following interior estimate:

$$C \left(\frac{a}{b} \right) \sup_K u_i + \inf_{\Omega} u_i \leq c = c(a, b, K, \Omega).$$

One can see in [12], an explicit value of $C \left(\frac{a}{b} \right) = \sqrt{\frac{a}{b}}$. In his proof, Shafrir has used a blow-up function, the Stokes formula and an isoperimetric inequality, see [6]. For Chen-Lin, they have used the blow-up analysis combined with some geometric type inequality for the integral curvature.

Now, if we suppose $(V_i)_i$ uniformly Lipschitzian with A the Lipschitz constant, then, $C(a/b) = 1$ and $c = c(a, b, A, K, \Omega)$, see Brezis-Li-Shafrir [7]. This result was extended for Hölderian sequences $(V_i)_i$ by Chen-Lin, see [12]. Also, we have in [18] an extension of the Brezis-Li-Shafrir result to compact Riemann surface without boundary. We have in [19] explicit form, $(8\pi m, m \in \mathbb{N}^*$ exactly), for the numbers in front of the Dirac masses, when the solutions blow-up. Here, the notion of isolated blow-up point is used. Also, we have in [13] and [26] refined estimates near the isolated blow-up points and the bubbling behavior of the blow-up sequences.

In [8], Brezis and Merle proposed the following Problem:

Problem (Brezis-Merle [8]). *If $(u_i)_i$ and $(V_i)_i$ are two sequences of functions relatively to the previous problem (P) with,*

$$0 \leq V_i \rightarrow V \text{ in } C^0(\bar{\Omega}).$$

$$\int_{\Omega} e^{u_i} dy \leq C,$$

Is it possible to prove that:

$$\sup_{\Omega} u_i \leq c = c(C, V, \Omega) ?$$

Here, we assume more regularity on V_i , we suppose that $V_i \geq 0$ is C^s (s -holderian) $1/2 < s \leq 1$. We give the answer where $bC < 24\pi$.

On other hand, in our work we give a complete caracterization of the blow-up analysis on the boundary.

In the similar way, we have in dimension $n \geq 3$, with different methods, some a priori estimates of the type $\sup \times \inf$ for equation of the type:

$$-\Delta u + \frac{n-2}{4(n-1)} R_g(x) u = V(x) u^{(n+2)/(n-2)} \text{ on } M.$$

where R_g is the scalar curvature of a riemannian manifold M , and V is a function. The operator $\Delta = \nabla^i(\nabla_i)$ is the Laplace-Beltrami operator on M .

When $V \equiv 1$ and M compact, the previous equation is the Yamabe equation. T. Aubin and R. Schoen solved the Yamabe problem, see for example [1]. Also, we can have an idea on the Yamabe Problem in [16]. If V is not a constant function, the previous equation is called a prescribing curvature equation, we have many existence results see also [1].

Now, if we look at the problem of a priori bound for the previous equation, we can see in [2, 3, 4, 5, 11, 17, 21] some results concerning the $\sup \times \inf$ type of inequalities when the manifold M is the sphere or more generality a locally conformally flat manifold. For these results, the moving-plane was used, we refer to [9, 14, 20] to have an idea on this method and some applications of this method.

Also, there are similar problems defined on complex manifolds for the Complex Monge-Ampere equation, see [24, 25], with various inequalities of type $\sup + \inf$.

Our main results are:

Theorem 1.1. *Assume $\Omega = B_1(0)$, and,*

$$u_i(x_i) = \sup_{B_1(0)} u_i \rightarrow +\infty.$$

There is a finite number of sequences $(x_i^k)_i, (\delta_i^k), 0 \leq k \leq m$, such that:

$$(x_i^0)_i \equiv (x_i)_i, \delta_i^0 = \delta_i = d(x_i, \partial B_1(0)) \rightarrow 0,$$

*and each δ_i^k is of order $d(x_i^k, \partial B_1(0))$.
and,*

$$u_i(x_i^k) = \sup_{B_1(0) - \cup_{j=0}^{k-1} B(x_i^j, \delta_i^j \epsilon)} u_i \rightarrow +\infty,$$

$$u_i(x_i^k) + 2 \log \delta_i^k \rightarrow +\infty,$$

$$\forall \epsilon > 0, \sup_{B_1(0) - \cup_{j=0}^m B(x_i^j, \delta_i^j \epsilon)} u_i \leq C_\epsilon$$

$$\forall \epsilon > 0, \limsup_{i \rightarrow +\infty} \int_{B(x_i^k, \delta_i^k \epsilon)} V_i e^{u_i} dy \geq 4\pi > 0.$$

If we assume:

$$V_i \rightarrow V \text{ in } C^0(\bar{B}_1(0)),$$

then,

$$\forall \epsilon > 0, \limsup_{i \rightarrow +\infty} \int_{B(x_i^k, \delta_i^k \epsilon)} V_i e^{u_i} dy = 8\pi m_k, m_k \in \mathbb{N}^*.$$

And, thus, we have the following convergence in the sense of distributions:

$$\int_{B_1(0)} V_i e^{u_i} dy \rightarrow \int_{B_1(0)} V e^u dy + \sum_{k=0}^m 8\pi m'_k \delta_{x_0^k}, m'_k \in \mathbb{N}^*, x_0^k \in \partial B_1(0).$$

Theorem 1.2. *Assume that:*

$$\int_{B_1(0)} V_i e^{u_i} dy \leq 4\pi,$$

Then,

$$u_i(x_i) = \sup_{B_1(0)} u_i \leq c = c(b, C),$$

Theorem 1.3. Assume that, V_i is uniformly s -holderian with $1/2 < s \leq 1$, and,

$$\int_{B_1(0)} V_i e^{u_i} dy \leq 24\pi - \epsilon, \quad \epsilon > 0,$$

then we have:

$$\sup_{\Omega} u_i \leq c = c(b, C, A, s, \epsilon, \Omega).$$

where A is the holderian constant of V_i .

Question 1: (a Bartolucci type result; one holderian singularity): with the same technique, assume that:

$$V_i(x) = (1 + x_1^s)W_i(x) \text{ for example and } 0 \in \partial\Omega$$

with W_i uniformly lipschitzian and, $0 < s \leq 1$, can one conclude with the Pohozaev identity that the sequence is compact? here we extend the case $0 < s \leq 1/2$.

Question 2: (the limit case $s = 1/2$) assume that V_i is uniformly $1/2$ -holderian with A_i the holderian constant and suppose that $A_i \rightarrow 0$, can one conclude with the blow-up technique that the sequence of the solutions u_i is compact?

2. PROOFS OF THE RESULTS

Proofs of the theorems:

Without loss of generality, we can assume that $\Omega = B_1(0)$ the unit ball centered on the origin.

We assume that:

$$u_i \in W_0^{1,1}(\Omega).$$

According to the work of Brezis-Merle, $e^{ku_i} \in L^1$ for all $k > 2$ and the elliptic estimates and the Sobolev embedding gives:

$$u_i \in W^{2,k}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega}).$$

Here, G is the Green function of the Laplacian with Dirichlet condition on $B_1(0)$. We have (in complex notation):

$$G(x, y) = \frac{1}{2\pi} \log \frac{|1 - \bar{x}y|}{|x - y|},$$

we can write:

$$v_i(x) = \int_{B_1(0)} G(x, y) V_i(y) e^{u_i(y)} dy,$$

Remark that, we can write:

$$F(x) = \int_{B_1(0)} -\frac{1}{2\pi} \log |x - y| V_i(y) e^{u_i(y)} dy,$$

By a result in Gilbarg-Trudinger (chapter 4, Newtonian potential), we have $F \in C^1(\bar{\Omega})$

Also, we have:

$$G(x) = \int_{B_1(0)} -\frac{1}{2\pi} \log |1 - \bar{x}y| V_i(y) e^{u_i(y)} dy,$$

For x near 0, G is smooth by the usual differentiability theorem. For $x \neq 0$, we can write:

$K(x) = F(1/\bar{x})$ which is C^1 by a result of Gilbarg-Trudinger. Combining the two last results, we have v_i is C^1 and by the maximum principle we have $v_i \equiv u_i$. We use the fact that G is real to write $\partial_x G = \bar{\partial}_x \bar{G} = \bar{\partial}_x G$ to have the derivative of u_i .

$$G(x, y) = \frac{1}{2\pi} \log \frac{|1 - \bar{x}y|}{|x - y|},$$

we can write:

$$u_i(x) = \int_{B_1(0)} G(x, y) V_i(y) e^{u_i(y)} dy,$$

We can compute (in complex notation) $\partial_x G$ and $\partial_x u_i$:

$$\partial_x G(x, y) = \frac{1 - |y|^2}{(x - y)(x\bar{y} - 1)},$$

$$\partial_x u_i(x) = \int_{B_1(0)} \partial_x G(x, y) V_i(y) e^{u_i(y)} dy = \int_{B_1(0)} \frac{1 - |y|^2}{(x - y)(x\bar{y} - 1)} V_i(y) e^{u_i(y)} dy$$

we write,

$$u_i(x_i) = \int_{\Omega} G(x_i, y) V_i(y) e^{u_i(y)} dx = \int_{\Omega - B(x_i, \delta_i/2)} G(x_i, y) V_i e^{u_i(y)} dy + \int_{B(x_i, \delta_i/2)} G(x_i, y) V_i e^{u_i(y)} dy$$

According to the maximum principle, the harmonic function $G(x_i, \cdot)$ on $\Omega - B(x_i, \delta_i/2)$ take its maximum on the boundary of $B(x_i, \delta_i/2)$, we can compute this maximum:

$$G(x_i, y_i) = \frac{1}{2\pi} \log \frac{|1 - \bar{x}_i y_i|}{|x_i - y_i|} = \frac{1}{2\pi} \log \frac{|1 - \bar{x}_i(x_i + \delta_i \theta_i)|}{|\delta_i/2|} = \frac{1}{2\pi} \log 2(|(1 + |x_i|) + \bar{x}_i \theta_i|) < +\infty$$

with $|\theta_i| = 1/2$.

Thus,

$$u_i(x_i) \leq C + \int_{B(x_i, \delta_i/2)} G(x_i, y) V_i e^{u_i(y)} dy \leq C + e^{u_i(x_i)} \int_{B(x_i, \delta_i/2)} G(x_i, y) dy$$

Now, we compute $\int_{B(x_i, \delta_i/2)} G(x_i, y) dy$
we set in polar coordinates,

$$y = x_i + \delta_i t \theta$$

we find:

$$\begin{aligned} \int_{B(x_i, \delta_i/2)} G(x_i, y) dy &= \int_{B(x_i, \delta_i/2)} \frac{1}{2\pi} \log \frac{|1 - \bar{x}_i y|}{|x_i - y|} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{1/2} \delta_i^2 \log \frac{|1 - \bar{x}_i(x_i + \delta_i \theta)|}{(\delta_i t)/2} t dt d\theta = \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{1/2} \delta_i^2 (\log 2(|1 + |x_i| + t \bar{x}_i \theta|) - \log t) t dt d\theta \leq C \delta_i^2. \end{aligned}$$

Thus,

$$u_i(x_i) \leq C + C \delta_i^2 e^{u_i(x_i)},$$

which we can write, because $u_i(x_i) \rightarrow +\infty$,

$$u_i(x_i) \leq C' \delta_i^2 e^{u_i(x_i)},$$

We can conclude that:

$$u_i(x_i) + 2 \log \delta_i \rightarrow +\infty.$$

Now, consider the following function :

$$v_i(y) = u_i(x_i + \delta_i y) + 2 \log \delta_i, \quad y \in B(0, 1/2)$$

The function satisfies all conditions of the Brezis-Merle hypothesis, we can conclude that, on each compact set:

$$v_i \rightarrow -\infty$$

we can assume, without loss of generality that for $1/2 > \epsilon > 0$, we have:

$$v_i \rightarrow -\infty, \quad y \in B(0, 2\epsilon) - B(0, \epsilon),$$

Lemma 2.1. For all $1/4 > \epsilon > 0$, we have:

$$\sup_{B(x_i, (3/2)\delta_i\epsilon) - B(x_i, \delta_i\epsilon)} u_i \leq C_\epsilon.$$

Proof of the lemma

Let t'_i and t_i the points of $B(x_i, 2\delta_i\epsilon) - B(x_i, (1/2)\delta_i\epsilon)$ and $B(x_i, (3/2)\delta_i\epsilon) - B(x_i, \delta_i\epsilon)$ respectively where u_i takes its maximum.

According to the Brezis-Merle work, we have:

$$u_i(t'_i) + 2 \log \delta_i \rightarrow -\infty$$

We write,

$$\begin{aligned} u_i(t_i) &= \int_{\Omega} G(t_i, y) V_i(y) e^{u_i(y)} dx = \int_{\Omega - B(x_i, 2\delta_i\epsilon)} G(t_i, y) V_i e^{u_i(y)} dy + \\ &+ \int_{B(x_i, 2\delta_i\epsilon) - B(x_i, (1/2)\delta_i\epsilon)} G(t_i, y) V_i e^{u_i(y)} dy + \\ &+ \int_{B(x_i, (1/2)\delta_i\epsilon)} G(t_i, y) V_i e^{u_i(y)} dy \end{aligned}$$

But, in the first and the third integrale, the point t_i is far from the singularity x_i and we know that the Green function is bounded. For the second integrale, after a change of variable, we can see that this integrale is bounded by (we take the supremum in the annulus and use Brezis-Merle theorem)

$$\delta_i^2 e^{u_i(t'_i)} \times I_j$$

where I_j is a Jensen integrale (of the form $\int_0^1 \int_0^{2\pi} (\log(|1 + |x_i| + t\theta|) - \log|\theta_i - t\theta|) t dt d\theta$ which is bounded).

we conclude the lemma.

From the lemma, we see that far from the singularity the sequence is bounded, thus if we take the supremum on the set $B_1(0) - B(x_i, \delta_i\epsilon)$ we can see that this supremum is bounded and thus the sequence of functions is uniformly bounded or tends to infinity and we use the same arguments as for x_i to conclude that around this point and far from the singularity, the sequence is bounded.

The process will be finished, because, according to Brezis-Merle estimate, around each supremum constructed and tending to infinity, we have:

$$\forall \epsilon > 0, \quad \limsup_{i \rightarrow +\infty} \int_{B(x_i, \delta_i\epsilon)} V_i e^{u_i} dy \geq 4\pi > 0.$$

Finally, with this construction, we have a finite number of "exterior" blow-up points and outside the singularities the sequence is bounded uniformly, for example, in the case of one "exterior" blow-up point, we have:

$$u_i(x_i) \rightarrow +\infty$$

$$\forall \epsilon > 0, \quad \sup_{B_1(0) - B(x_i, \delta_i\epsilon)} u_i \leq C_\epsilon$$

$$\forall \epsilon > 0, \quad \limsup_{i \rightarrow +\infty} \int_{B(x_i, \delta_i\epsilon)} V_i e^{u_i} dy \geq 4\pi > 0.$$

$$x_i \rightarrow x_0 \in \partial B_1(0).$$

We have the following lemma:

Lemma 2.2. Each δ_i^k is of order $d(x_i^k, \partial B_1(0))$. Namely: there is a positive constant $C > 0$ such that for $\epsilon > 0$ small enough:

$$\delta_i^k \leq d(x_i^k, \partial B_1(0)) \leq (2 + \frac{C}{\epsilon})\delta_i^k.$$

Proof of the lemma

Now, if we suppose that there is another "exterior" blow-up $(t_i)_i$, we have, because $(u_i)_i$ is uniformly bounded in a neighborhood of $\partial B(x_i, \delta_i\epsilon)$, we have :

$$d(t_i, \partial B(x_i, \delta_i\epsilon)) \geq \delta_i\epsilon$$

If we set,

$$\delta'_i = d(t_i, \partial(B_1(0) - B(x_i, \delta_i\epsilon))) = \inf\{d(t_i, \partial B(x_i, \delta_i\epsilon)), d(t_i, \partial(B_1(0)))\}$$

then, δ'_i is of order $d(t_i, \partial B_1(0))$. To see this, we write:

$$d(t_i, \partial B_1(0)) \leq d(t_i, \partial B(x_i, \delta_i\epsilon)) + d(\partial B(x_i, \delta_i\epsilon), x_i) + d(x_i, \partial B_1(0)),$$

Thus,

$$\frac{d(t_i, \partial B_1(0))}{d(t_i, \partial B(x_i, \delta_i\epsilon))} \leq 2 + \frac{1}{\epsilon},$$

Thus,

$$\delta'_i \leq d(t_i, \partial B_1(0)) \leq \delta'_i(2 + \frac{1}{\epsilon}).$$

Now, the general case follow by induction. We use the same argument for three, four,..., n blow-up points.

We have, by induction and, here we use the fact that u_i is uniformly bounded outside a small ball centered at $x_i^j, j = 0, \dots, k-1$:

$$\delta_i^j \leq d(x_i^j, \partial B_1(0)) \leq C_1 \delta_i^j, \quad j = 0, \dots, k-1,$$

$$d(x_i^k, \partial B(x_i^j, \delta_i^j\epsilon/2)) \geq \epsilon \delta_i^j, \quad \epsilon > 0, \quad j = 0, \dots, k-1,$$

and let's consider x_i^k such that:

$$u_i(x_i^k) = \sup_{B_1(0) - \cup_{j=0}^{k-1} B(x_i^j, \delta_i^j\epsilon)} u_i \rightarrow +\infty,$$

take,

$$\delta_i^k = \inf\{d(x_i^k, \partial B_1(0)), d(x_i^k, \partial(B_1(0) - \cup_{j=0}^{k-1} B(x_i^j, \delta_i^j\epsilon/2)))\},$$

if, we have,

$$\delta_i^k = d(x_i^k, \partial B(x_i^j, \delta_i^j\epsilon/2)), \quad j \in \{0, \dots, k-1\}.$$

Then,

$$\begin{aligned} \delta_i^k &\leq d(x_i^k, \partial B_1(0)) \leq \\ &\leq d(x_i^k, \partial B(x_i^j, \delta_i^j\epsilon/2)) + d(\partial B(x_i^j, \delta_i^j\epsilon/2), x_i^j) + d(x_i^j, \partial B_1(0)) \\ &\leq (2 + \frac{C_1}{\epsilon})\delta_i^k. \end{aligned}$$

To apply lemma 2.1 for m blow-up points, we use an induction:

We do directly the same approach for t_i as x_i by using directly the Green function of the unit ball.

If we look to the blow-up points, we can see, with this work that, after finite steps, the sequence will be bounded outside a finite number of balls, because of Brezis-Merle estimate (corollary 4 of Brezis-Merle's paper):

$$\forall \epsilon > 0, \limsup_{i \rightarrow +\infty} \int_{B(x_i^k, \delta_i^k \epsilon)} V_i e^{u_i} dy \geq 4\pi > 0.$$

Here, we can take the functions:

$$u_i^k(y) = u_i(x_i^k + \delta_i^k y) + 2 \log \delta_i^k,$$

(By corollary 4 of Brezis-Merle's paper if $\limsup_{i \rightarrow +\infty} \int_{B(x_i^k, \delta_i^k \epsilon)} V_i e^{u_i} dy \leq 4\pi - \epsilon_0 < 4\pi$, then $(u_i^k)^+$ is locally uniformly bounded, which in contradiction with $u_i^k(0) \rightarrow +\infty$).

Finally, we can say that, there is a finite number of sequences $(x_i^k)_i, (\delta_i^k), 0 \leq k \leq m$, such that:

$$(x_i^0)_i \equiv (x_i)_i, \delta_i^0 = \delta_i = d(x_i, \partial B_1(0)),$$

$$(x_i^1)_i \equiv (t_i)_i, \delta_i^1 = \delta_i' = d(t_i, \partial(B_1(0) - B(x_i, \delta_i \epsilon))),$$

and each δ_i^k is of order $d(x_i^k, \partial B_1(0))$.

and,

$$u_i(x_i^k) = \sup_{B_1(0) - \cup_{j=0}^{k-1} B(x_i^j, \delta_i^j \epsilon)} u_i \rightarrow +\infty,$$

$$u_i(x_i^k) + 2 \log \delta_i^k \rightarrow +\infty,$$

$$\forall \epsilon > 0, \sup_{B_1(0) - \cup_{j=0}^m B(x_i^j, \delta_i^j \epsilon)} u_i \leq C_\epsilon$$

$$\forall \epsilon > 0, \limsup_{i \rightarrow +\infty} \int_{B(x_i^k, \delta_i^k \epsilon)} V_i e^{u_i} dy \geq 4\pi > 0.$$

The work of YY.Li-I.Shafrir

With the previous method, we have a finite number of "exterior" blow-up points (perhaps the same) and the sequences tend to the boundary. With the aid of proposition 1 of the paper of Li-Shafrir, we see that around each exterior blow-up, we have a finite number of "interior" blow-ups. Around, each exterior blow-up, we have after rescaling with δ_i^k , the same situation as around a fixed ball with positive radius. If we assume:

$$V_i \rightarrow V \text{ in } C^0(\bar{B}_1(0)),$$

then,

$$\forall \epsilon > 0, \limsup_{i \rightarrow +\infty} \int_{B(x_i^k, \delta_i^k \epsilon)} V_i e^{u_i} dy = 8\pi m_k, m_k \in \mathbb{N}^*.$$

And, thus, we have the following convergence in the sense of distributions:

$$\int_{B_1(0)} V_i e^{u_i} dy \rightarrow \int_{B_1(0)} V e^u dy + \sum_{k=0}^m 8\pi m'_k \delta_{x_0^k}, m'_k \in \mathbb{N}^*, x_0^k \in \partial B_1(0).$$

Consequence 1: Proof of theorem 2

Assume that:

$$\int_{B_1(0)} V_i e^{u_i} dy \leq 4\pi,$$

Then, if the sequence blow-up, there is one and only one blow-up point and we have:

$$u_i(x_i) = \sup_{B_1(0)} u_i \rightarrow +\infty,$$

$$u_i(x_i) + 2 \log \delta_i \rightarrow +\infty,$$

$$\forall \epsilon > 0, \quad \sup_{B_1(0) - B(x_i, \delta_i \epsilon)} u_i \leq C_\epsilon$$

We set,

$$r_i = e^{-u_i(x_i)/2},$$

The blow-up function is locally bounded thus,

$$r_i^2 e^{u_i} \leq C \text{ on } B(x_i, 2r_i).$$

We write:

$$u_i(x_i) = \int_{\Omega - B(x_i, \delta_i \epsilon)} G(x_i, y) V_i e^{u_i(y)} dy + \int_{B(x_i, \delta_i \epsilon)} G(x_i, y) V_i e^{u_i(y)} dy \leq C_\epsilon + \int_{B(x_i, \delta_i \epsilon)} G(x_i, y) V_i e^{u_i(y)} dy$$

we have:

$$\int_{B(x_i, \delta_i \epsilon)} G(x_i, y) V_i e^{u_i(y)} dy = \int_{B(x_i, \delta_i \epsilon) - B(x_i, 2r_i)} G(x_i, y) V_i e^{u_i(y)} dy + \int_{B(x_i, 2r_i)} G(x_i, y) V_i e^{u_i(y)} dy$$

We use the maximum principle on $B(x_i, \delta_i \epsilon) - B(x_i, 2r_i)$ and the explicit formula of G to prove that:

$$G(x_i, y) \leq C + \frac{1}{2\pi} \log \frac{\delta_i}{r_i} = C + \frac{1}{4\pi} (u_i(x_i) + 2 \log \delta_i).$$

On $B(x_i, 2r_i)$ we use the fact that:

$$r_i^2 e^{u_i} \leq C$$

and the explicit formula for G to have:

$$\int_{B(x_i, 2r_i)} G(x_i, y) V_i e^{u_i(y)} dy \leq C + \frac{1}{2\pi} \log \frac{\delta_i}{r_i} \int_{B(x_i, 2r_i)} V_i e^{u_i(y)} dy.$$

We conclude that:

$$u_i(x_i) \leq C + \frac{1}{2\pi} \log \frac{\delta_i}{r_i} \int_{B(x_i, \delta_i \epsilon)} V_i e^{u_i(y)} dy.$$

which we can write as:

$$u_i(x_i) \leq C + \frac{1}{4\pi} (u_i(x_i) + 2 \log \delta_i) \int_{B(x_i, \delta_i \epsilon)} V_i e^{u_i(y)} dy.$$

Our hypothesis on the integrals of $V_i e^{u_i}$ imply that:

$$\log \delta_i \geq -C,$$

in other words, we have uniformly,

$$d(x_i, \partial B_1(0)) = \delta_i \geq e^{-C} > 0.$$

this contradicts the fact that (x_i) tends to the boundary. The sequence (u_i) is bounded in this case.

We can see that the case:

$$\int_{B_1(0)} V_i e^{u_i} dy \leq 4\pi,$$

is optimal, because Brezis-Merle have proved that, there is a counterexample of blow-up sequence with:

$$\int_{B_1(0)} V_i e^{u_i} dy = 4\pi A > 4\pi.$$

Consequence 2: using a Pohozaev-type identity, proof of theorem 3

By a conformal transformation, we can assume that our domain $\Omega = B^+$ is a half ball centered at the origin, $B^+ = \{x, |x| \leq 1, x_1 \geq 0\}$. In this case the normal at the boundary is $\nu = (-1, 0)$ and $u_i(0, x_2) \equiv 0$. Also, we set x_i the blow-up point and $x_i^2 = (0, x_i^2)$ and $x_i^1 = (x_i^1, 0)$ respectively the second and the first part of x_i . Let ∂B^+ the part of the boundary for which u_i and its derivatives are uniformly bounded and thus converge to the corresponding function.

The case of one blow-up point:

Theorem 2.3. *If V_i is s -Holderian with $1/2 < s \leq 1$ and,*

$$\int_{\Omega} V_i e^{u_i} dy \leq 16\pi - \epsilon, \quad \epsilon > 0,$$

we have :

$$V_i(x_i) \int_{\Omega} e^{u_i} dy - V(0) \int_{\Omega} e^u dy = o(1)$$

which means that there is no blow-up points.

Proof of the theorem

The Pohozaev identity gives us the following formula:

$$\int_{\Omega} \langle (x - x_2^i) | \nabla u_i \rangle (-\Delta u_i) dy = \int_{\Omega} \langle (x - x_2^i) | \nabla u_i \rangle V_i e^{u_i} dy = A_i$$

$$A_i = \int_{\partial B^+} \langle (x - x_2^i) | \nabla u_i \rangle \langle \nu | \nabla u_i \rangle d\sigma + \int_{\partial B^+} \langle (x - x_2^i) | \nu \rangle |\nabla u_i|^2 d\sigma$$

We can write it as:

$$\begin{aligned} \int_{\Omega} \langle (x - x_2^i) | \nabla u_i \rangle (V_i - V_i(x_i)) e^{u_i} dy &= A_i + V_i(x_i) \int_{\Omega} \langle (x - x_2^i) | \nabla u_i \rangle e^{u_i} dy = \\ &= A_i + V_i(x_i) \int_{\Omega} \langle (x - x_2^i) | \nabla (e^{u_i}) \rangle dy \end{aligned}$$

And, if we integrate by part the second term, we have (because $x_1 = 0$ on the boundary and $\nu_2 = 0$):

$$\int_{\Omega} \langle (x - x_2^i) | \nabla u_i \rangle (V_i - V_i(x_i)) e^{u_i} dy = -2V_i(x_i) \int_{\Omega} e^{u_i} dy + B_i$$

where B_i is,

$$B_i = V_i(x_i) \int_{\partial B^+} \langle (x - x_2^i) | \nu \rangle e^{u_i} dy$$

applying the same procedure to u , we can write:

$$\begin{aligned} -2V_i(x_i) \int_{\Omega} e^{u_i} dy + 2V(0) \int_{\Omega} e^u dy &= \int_{\Omega} \langle (x - x_2^i) | \nabla u_i \rangle (V_i - V_i(x_i)) e^{u_i} dy - \int_{\Omega} \langle (x - x_2^i) | \nabla u \rangle (V - V(0)) e^u dy + \\ &+ (A_i - A) + (B_i - B), \end{aligned}$$

where A and B are,

$$A = \int_{\partial B^+} \langle (x - x_2^i) | \nabla u \rangle \langle \nu | \nabla u \rangle d\sigma + \int_{\partial B^+} \langle (x - x_2^i) | \nu \rangle |\nabla u|^2 d\sigma$$

$$B = V(0) \int_{\partial B^+} \langle (x - x_2^i) | \nu \rangle e^u dy$$

and, because of the uniform convergence of u_i and its derivatives on ∂B^+ , we have:

$$A_i - A = o(1) \quad \text{and} \quad B_i - B = o(1)$$

which we can write as:

$$\begin{aligned}
V_i(x_i) \int_{\Omega} e^{u_i} dy - V(0) \int_{\Omega} e^u dy &= \int_{\Omega} \langle (x - x_2^i) | \nabla(u_i - u) \rangle (V_i - V_i(x_i)) e^{u_i} dy + \\
&+ \int_{\Omega} \langle (x - x_2^i) | \nabla u \rangle (V_i - V_i(x_i)) (e^{u_i} - e^u) dy + \\
&+ \int_{\Omega} \langle (x - x_2^i) | \nabla u \rangle (V_i - V_i(x_i) - (V - V(0))) e^u dy + o(1)
\end{aligned}$$

We can write the second term as:

$$\begin{aligned}
\int_{\Omega} \langle (x - x_2^i) | \nabla u \rangle (V_i - V_i(x_i)) (e^{u_i} - e^u) dy &= \int_{\Omega - B(0, \epsilon)} \langle (x - x_2^i) | \nabla u \rangle (V_i - V_i(x_i)) (e^{u_i} - e^u) dy + \\
&+ \int_{B(0, \epsilon)} \langle (x - x_2^i) | \nabla u \rangle (V_i - V_i(x_i)) (e^{u_i} - e^u) dy = o(1),
\end{aligned}$$

because of the uniform convergence of u_i to u outside a region which contain the blow-up and the uniform convergence of V_i . For the third integral we have the same result:

$$\int_{\Omega} \langle (x - x_2^i) | \nabla u \rangle (V_i - V_i(x_i) - (V - V(0))) e^u dy = o(1),$$

because of the uniform convergence of V_i to V .

Now, we look to the first integral:

$$\int_{\Omega} \langle (x - x_2^i) | \nabla(u_i - u) \rangle (V_i - V_i(x_i)) e^{u_i} dy,$$

we can write it as:

$$\begin{aligned}
\int_{\Omega} \langle (x - x_2^i) | \nabla(u_i - u) \rangle (V_i - V_i(x_i)) e^{u_i} dy &= \int_{\Omega} \langle (x - x_i) | \nabla(u_i - u) \rangle (V_i - V_i(x_i)) e^{u_i} dy + \\
&+ \int_{\Omega} \langle x_1^i | \nabla(u_i - u) \rangle (V_i - V_i(x_i)) e^{u_i} dy,
\end{aligned}$$

Thus, we have proved by using the Pohozaev identity the following equality, in the case of one blow-up (the term $o(1)$):

$$\begin{aligned}
&\int_{\Omega} \langle (x - x_i) | \nabla(u_i - u) \rangle (V_i - V_i(x_i)) e^{u_i} dy + \\
&+ \int_{\Omega} \langle x_1^i | \nabla(u_i - u) \rangle (V_i - V_i(x_i)) e^{u_i} dy = \\
&= 2V_i(x_i) \int_{\Omega} e^{u_i} dy - 2V(0) \int_{\Omega} e^u dy + o(1)
\end{aligned}$$

We can see, because of the uniform boundedness of u_i outside $B(x_i, \delta_i \epsilon)$ and the fact that :

$$\|\nabla(u_i - u)\|_1 = o(1),$$

it is sufficient to look to the integral on $B(x_i, \delta_i \epsilon)$.

Assume that we are in the case of one blow-up, it must be (x_i) and isolated, we can write the following inequality as a consequence of YY.Li-I.Shafirir result:

$$u_i(x) + 2 \log |x - x_i| \leq C,$$

We use this fact and the fact that V_i is s-holderian to have that, on $B(x_i, \delta_i \epsilon)$,

$$|(x - x_i)(V_i - V_i(x_i)) e^{u_i}| \leq \frac{C}{|x - x_i|^{1-s}} \in L^{(2-\epsilon')/(1-s)}, \quad \forall \epsilon' > 0,$$

and, we use the fact that:

$$\|\nabla(u_i - u)\|_q = o(1), \quad \forall 1 \leq q < 2$$

to conclude by the Holder inequality that for $0 < s \leq 1$:

$$\int_{B(x_i, \delta_i \epsilon)} \langle (x - x_i) | \nabla(u_i - u) \rangle (V_i - V_i(x_i)) e^{u_i} dy = o(1),$$

For the other integral, namely:

$$\int_{B(x_i, \delta_i \epsilon)} \langle x_1^i | \nabla(u_i - u) \rangle (V_i - V_i(x_i)) e^{u_i} dy,$$

We use the fact that, because our domain is a half ball, and the sup + inf inequality to have:

$$x_1^i = \delta_i,$$

$$u_i(x) + 4 \log \delta_i \leq C$$

and,

$$e^{(s/2)u_i(x)} \leq |x - x_i|^{-s},$$

$$|V_i - V_i(x_i)| \leq |x - x_i|^s,$$

Finally, we have:

$$\left| \int_{B(x_i, \delta_i \epsilon)} \langle x_1^i | \nabla(u_i - u) \rangle (V_i - V_i(x_i)) e^{u_i} dy \right| \leq C \int_{B(x_i, \delta_i \epsilon)} |\nabla(u_i - u)| e^{((3/4)-(s/2))u_i},$$

But in the second member, for $1/2 < s \leq 1$, we have $q_s = 1/(3/4 - s/2) > 2$ and thus $q'_s < 2$ and,

$$e^{((3/4)-(s/2))u_i} \in L^{q_s}$$

$$\|\nabla(u_i - u)\|_{q'_s} = o(1), \quad \forall 1 \leq q'_s < 2,$$

one conclude that:

$$\int_{B(x_i, \delta_i \epsilon)} \langle x_1^i | \nabla(u_i - u) \rangle (V_i - V_i(x_i)) e^{u_i} dy = o(1)$$

Finally, with this method, we conclude that, in the case of one blow-up point and V_i is s -Holderian with $1/2 < s \leq 1$:

$$V_i(x_i) \int_{\Omega} e^{u_i} dy - V(0) \int_{\Omega} e^u dy = o(1)$$

which means that there is no blow-up, which is a contradiction.

Finally, for one blow-up point and V_i is s -Holderian with $1/2 < s \leq 1$, the sequence (u_i) is uniformly bounded on Ω .

The case of two blow-up points:

Theorem 2.4. *If V_i is s -Holderian with $1/2 < s \leq 1$ and,*

$$\int_{\Omega} V_i e^{u_i} dy \leq 24\pi - \epsilon, \quad \epsilon > 0,$$

we have :

$$V_i(x_i) \int_{\Omega} e^{u_i} dy - V(0) \int_{\Omega} e^u dy = o(1)$$

which means that there is no blow-up points.

Proof of the Theorem

The case of two "interior" blow-up points:

As in the previous case, we assume that $\Omega = B^+$ is the half ball. We have two "interior" blow-up points x_i and y_i :

$$|y_i - x_i| \leq \delta_i \epsilon,$$

We use a Pohozaev type identity:

$$\int_{\Omega} \langle (x - x_2^i) |\nabla u_i \rangle (-\Delta u_i) dy = \int_{\Omega} \langle (x - x_2^i) |\nabla u_i \rangle V_i e^{u_i} dy = A_i$$

with A_i the regular part of the identity (on which the uniform convergence holds).

$$A_i = \int_{\partial B^+} \langle (x - x_2^i) |\nabla u_i \rangle \langle \nu | \nabla u_i \rangle d\sigma + \int_{\partial B^+} \langle (x - x_2^i) | \nu \rangle |\nabla u_i|^2 d\sigma$$

We divide our domain in two domain Ω_1^i and Ω_2^i such that:

$$\Omega_1^i = \{x, |x - x_i| \leq |x - y_i|\}, \quad \Omega_2^i = \{x, |x - x_i| \geq |x - y_i|\}.$$

We set,

$$D_i = \{x, |x - x_i| = |x - y_i|\}.$$

We write:

$$\begin{aligned} A_i &= \int_{\Omega_1^i} \langle (x - x_2^i) |\nabla u_i \rangle (V_i - V_i(x_i)) e^{u_i} dy + \int_{\Omega_2^i} \langle (x - x_2^i) |\nabla u_i \rangle (V_i - V_i(y_i)) e^{u_i} dy + \\ &\quad + V_i(x_i) \int_{\Omega_1^i} \langle (x - x_2^i) |\nabla u_i \rangle e^{u_i} dy + V_i(y_i) \int_{\Omega_2^i} \langle (x - x_2^i) |\nabla u_i \rangle e^{u_i} dy. \end{aligned}$$

As for the case of one blow-up point, it is sufficient to consider terms which contain the difference $\nabla(u_i - u)$.

We can write the last addition as (after using $\nabla(u_i - u)$):

$$\begin{aligned} &\left(V_i(x_i) \int_{\Omega} \langle (x - x_2^i) |\nabla u_i \rangle e^{u_i} dy - \int_{\Omega} \langle (x - x_2^i) |\nabla u \rangle e^u dy \right) + \\ &\quad + (V_i(y_i) - V_i(x_i)) \int_{\Omega_2^i} \langle (x - x_2^i) |\nabla(u_i - u) \rangle e^{u_i} dy. \end{aligned}$$

First of all, we consider the term (which equal, after integration by part to):

$$\begin{aligned} &V_i(x_i) \int_{\Omega} \langle (x - x_2^i) |\nabla u_i \rangle e^{u_i} dy - \int_{\Omega} \langle (x - x_2^i) |\nabla u \rangle e^u dy = \\ &\quad = -2V_i(x_i) \int_{\Omega} e^{u_i} dy + 2V(0) \int_{\Omega} e^u dy + (B_i - B) \end{aligned}$$

with the same notation for B_i and B as for the previous case.

Case 1: suppose that, $|x - y_i| \geq |x_i - y_i|$,

thus

$$|V_i(x_i) - V_i(y_i)| \leq |x_i - y_i|^s \leq |x - y_i|^s$$

Thus,

$$\begin{aligned} &|(V_i(y_i) - V_i(x_i)) \int_{\Omega_2^i \cap \{x, |x - x_i| \geq |x - y_i|\}} \langle (x - x_2^i) |\nabla(u_i - u) \rangle e^{u_i} dy| \leq \int_{\Omega_2^i} |x - y_i|^{1+s} |\nabla(u_i - u)| e^{u_i} dy + \\ &\quad + |y_2^i - x_2^i| \int_{\Omega_2^i} |x - y_i|^s |\nabla(u_i - u)| e^{u_i} dy + |y_1^i| \int_{\Omega_2^i} |x - y_i|^s |\nabla(u_i - u)| e^{u_i} dy \end{aligned}$$

But,

$$|y_i - x_i| \leq \delta_i \epsilon,$$

$$x_1^i = \delta_i$$

we use the same method (with the sup + inf inequality) to prove that for $1 \geq s > 1/2$ the two integrals converges to 0.

Case 2: suppose that, $|x - y_i| \leq |x_i - y_i|$,

We do integration by parts, we have one part on D_i and the other one on the circle with center y_i . In fact, we have intersection of convex 2-dimensional domains, which is convex (Lipschitz domain or a domain with finite number of singularity) and thus one can apply the Stokes or Green-Riemann formulas.

$$\begin{aligned} & (V_i(y_i) - V_i(x_i)) \int_{\Omega_2^i \cap \{x, |x-y_i| \leq |x_i-y_i|\}} \langle (x - x_2^i) | \nabla(e^{u_i}) \rangle dy = \\ & = (V_i(y_i) - V_i(x_i)) \int_{D_i \cap \{x, |x-y_i| \leq |x_i-y_i|\}} \langle (x - x_2^i) | \nu \rangle e^{u_i} dy + \\ & + (V_i(y_i) - V_i(x_i)) \int_{\{x, |x-y_i|=|x_i-y_i|\} \cap \{x, |x-y_i| \leq |x-x_i|\}} \langle (x - x_2^i) | \nu \rangle e^{u_i} dy + \\ & + 2(V_i(y_i) - V_i(x_i)) \int_{\{x, |x-y_i| \leq |x_i-y_i|\}} e^{u_i} dy \end{aligned}$$

We set:

$$I_1 = (V_i(y_i) - V_i(x_i)) \int_{D_i \cap \{x, |x-y_i| \leq |x_i-y_i|\}} \langle (x - x_2^i) | \nu \rangle e^{u_i} dy,$$

$$I_2 = (V_i(y_i) - V_i(x_i)) \int_{\{x, |x-x_i|=|x_i-y_i|\} \cap \{x, |x-y_i| \leq |x-x_i|\}} \langle (x - x_2^i) | \nu \rangle e^{u_i} dy$$

Lemma 2.5. *We have:*

$$I_1 = o(1),$$

and,

$$I_2 = o(1).$$

Proof of the lemma

For I_1 , we have:

$$|V_i(x_i) - V_i(y_i)| \leq 2C|x - y_i|^s,$$

$$\begin{aligned} |I_1| & \leq C \int_{D_i \cap \{x, |x-y_i| \leq |x_i-y_i|\}} |\langle (x - y^i) | \nu \rangle| |x - y_i|^s e^{u_i} + \\ & + |x_2^i - y_2^i| \int_{D_i \cap \{x, |x-y_i| \leq |x_i-y_i|\}} |x - y_i|^s e^{u_i} dy + \\ & + |y_1^i| \int_{D_i \cap \{x, |x-y_i| \leq |x_i-y_i|\}} |x - y_i|^s e^{u_i} dy \end{aligned}$$

But,

$$x_1^i = \delta_i,$$

$$|y_i - x_i| \leq \delta_i \epsilon,$$

$$u_i(x) + 4 \log \delta_i \leq C$$

and,

$$e^{(3/4)u_i(x)} \leq |x - y_i|^{-3/2},$$

Thus,

$$\begin{aligned} |I_1| &\leq \int_{D_i \cap \{x, |x-y_i| \leq |x_i-y_i|\}} |x - y_i|^{s-1} + \\ &+ C \int_{D_i \cap \{x, |x-y_i| \leq |x_i-y_i|\}} |x - y_i|^{(-3/2)+s} dy, \end{aligned}$$

If we set $t_0 = (x_i + y_i)/2$, we have on one part of D_i :

$$|x - t_0| \leq |x - y_i| = |x - x_i| \leq |x_i - y_i|,$$

by a change of variable $u = x - t_0$ on the line D_i , we can compute the two last integrals directly, to have, for $1 \geq s > 1/2$:

$$|I_1| \leq C(|x_i - y_i|^s + |x_i - y_i|^{s-(1/2)}) = o(1),$$

For I_2 we have:

$$I_2 = (V_i(y_i) - V_i(x_i)) \int_{\{x, |x-y_i|=|x_i-y_i|\} \cap \{x, |x-x_i| \leq |x_i-y_i|\}} \langle (x - x_i^i) | \nu \rangle e^{u_i} dy$$

and,

$$|V_i(x_i) - V_i(y_i)| \leq 2C|x - y_i|^s,$$

$$\begin{aligned} |I_2| &\leq C \int_{\{x, |x-y_i|=|x_i-y_i|\} \cap \{x, |x-y_i| \leq |x-x_i|\}} \langle (x - y_i) | \nu \rangle |x - y_i|^s e^{u_i} + \\ &+ |x_2^i - y_2^i| \int_{\{x, |x-y_i|=|x_i-y_i|\} \cap \{x, |x-y_i| \leq |x-x_i|\}} |x - y_i|^s e^{u_i} dy + \\ &+ |y_1^i| \int_{\{x, |x-y_i|=|x_i-y_i|\} \cap \{x, |x-x_i| \leq |x-x_i|\}} |x - y_i|^s e^{u_i} dy \end{aligned}$$

with the same method as for I_1 we have:

$$\begin{aligned} |I_2| &\leq C \int_{\{x, |x-y_i|=|x_i-y_i|\} \cap \{x, |x-y_i| \leq |x-x_i|\}} |x - y_i|^{s-1} + \\ &+ \int_{\{x, |x-y_i|=|x_i-y_i|\} \cap \{x, |x-y_i| \leq |x-x_i|\}} |x - y_i|^{(-3/2)+s} dy, \end{aligned}$$

Finally, we have:

$$|I_2| \leq C(|x_i - y_i|^s + |x_i - y_i|^{s-(1/2)}) = o(1),$$

The case of two "exterior" blow-up points:

Let $(x_i)_i$ and $(t_i)_i$ two sequences of "exterior" blow-up points. If $d(x_i, t_i) = O(\delta_i)$ or $d(x_i, t_i) = O(\delta'_i)$ then we use the same technique as for two interior blow-up with the Pohozaev identity. In this case the sup + inf inequality holds, because $d(x_i, t_i)$ is of order δ_i or δ'_i . Assume that:

$$\frac{d(x_i, t_i)}{\delta_i} \rightarrow +\infty \quad \text{and} \quad \frac{d(x_i, t_i)}{\delta'_i} \rightarrow +\infty$$

In this case, we assume that, we are on the half ball. (In fact one consider the intersection of disks and half plane which is convex, and we take its image by the conformal map. In this case we have a domain on which we can apply the Stokes formula). By a conformal transformation, f , we can assume that our two sequences are on the unit ball. First of all, we use the Pohozaev identity on the half ball as for the previous cases, but our domain change, we have one part is vertical, the second part is a part of the boundary of the unit ball, in which the sequences (u_i)

and $(\partial u_i)_i$ are uniformly bounded and converge to the corresponding function, and the third part of boundary, is a regular curve D'_i such that its image by f is the mediatrice D_i of the segment (x_i, t_i) . In the Pohozaev identity, we have a terms of type:

$$\int_{D'_i} \langle (x - x_2^i) | \nabla u_i \rangle \langle \nu | \nabla u_i \rangle d\sigma + \int_{D'_i} \langle (x - x_2^i) | \nu \rangle |\nabla u_i|^2 d\sigma$$

But if we integrate on the rest of the domain and if we use the Pohozaev identity on this second domain and we replace x_2^i by t_2^i , the integral on D'_i is :

$$- \int_{D'_i} \langle (x - t_2^i) | \nabla u_i \rangle \langle \nu | \nabla u_i \rangle d\sigma - \int_{D'_i} \langle (x - t_2^i) | \nu \rangle |\nabla u_i|^2 d\sigma$$

If, we add the two integral, we find:

$$\int_{D'_i} \langle (x_2^i - t_2^i) | \nabla u_i \rangle \langle \nu | \nabla u_i \rangle d\sigma + \int_{D'_i} \langle (x_2^i - t_2^i) | \nu \rangle |\nabla u_i|^2 d\sigma$$

We have the same techniques as for the previous cases ("interior" blow-up), except the fact that here, we use the Pohozaev identity on two different domains which the union is our half ball. (In fact the image by a conformal map of a convex domain, intersection of two disks and a half plane. (Intersection of the unit disk and $D(x_0, \epsilon)$ and the mediatrice of $[x_i, t_i]$, here, x_0 is the "blow-up" point, and we apply the conformal map f)). And apply the Green-Riemann theorem for smooth domains with finite number of singular points on the boundary. Or directly the Stokes theorem with the fact that we have a Lipschitz domain because it is the image of a Lipschitz domain by a conformal map (Hofmann-Mitrea-Taylor).

Remark that, here, because we have the two conditions on $d(x_i, t_i)$ and δ_i, δ'_i , the mediatrice of $[x_i, t_i]$ is close to a fixed segment $[0, x_0]$. (use angles for this fact).

To conclude, we must show that this last integral is close to 0 as i tends to $+\infty$. By a conformal map, it is sufficient to prove that the corresponding integral on the unit ball on D_i tends to 0. Without loss of generality, we can assume here that we work on the unit ball (for this integral).

On the unit ball, with the Dirichlet condition, the Green function is (in complex notation) :

$$G(x, y) = \frac{1}{2\pi} \log \frac{|1 - \bar{x}y|}{|x - y|},$$

we can write:

$$u_i(x) = \int_{B_1(0)} G(x, y) V_i(y) e^{u_i(y)} dy,$$

We can compute (in complex notation) $\partial_x G$ and $\partial_x u_i$:

$$\partial_x G(x, y) = \frac{1 - |y|^2}{(x - y)(x\bar{y} - 1)},$$

$$\partial_x u_i(x) = \int_{B_1(0)} \partial_x G(x, y) V_i(y) e^{u_i(y)} dy = \int_{B_1(0)} \frac{1 - |y|^2}{(x - y)(x\bar{y} - 1)} V_i(y) e^{u_i(y)} dy$$

Let $t_0^i = (x_i + t_i)/2$. We assume that $|x - t_0^i| \leq 1 - \epsilon$ and $|t_0^i| \geq 1 - (\epsilon/2)$.

Proposition 2.6. 1) For $((1/2) + \tilde{\epsilon})|x_i - t_i| \leq |x - t_0^i| \leq 1 - \epsilon$ we have,

$$|\partial_x u_i(x)| \leq C' + C \frac{\delta_i}{|x_i - t_i|} \frac{1}{|x - t_0^i|} = C' + \frac{o(1)}{|x - t_0^i|}.$$

2) For $|x - t_0^i| \leq ((1/2) - \tilde{\epsilon})|x_i - t_i|$ we have,

$$|\partial_x u_i(x)| \leq C' + C \frac{\delta_i}{|x_i - t_i|} \frac{1}{|x_i - t_0^i|} = C' + \frac{o(1)}{|x_i - t_0^i|}.$$

with $o(1) \rightarrow 0$ as $i \rightarrow +\infty$.

3) For $((1/2) - \tilde{\epsilon})|x_i - t_i| \leq |x - t_0^i| \leq ((1/2) + \tilde{\epsilon})|x_i - t_i|$ we have,

$$|x_i - t_i| \|\nabla u_i\|_{L^\infty(D_i \cap \{((1/2) - \tilde{\epsilon})|x_i - t_i| \leq |x - t_0^i| \leq ((1/2) + \tilde{\epsilon})|x_i - t_i\})} \leq C.$$

Proof of the proposition:

To estimate $\partial_x u_i$ on D_i , we divide the last integral in three parts:

$$\begin{aligned} \partial_x u_i(x) &= \int_{B_1(0) - (B(x_i, \delta_i \epsilon) \cup B(t_i, \delta'_i \epsilon))} \frac{1 - |y|^2}{(x - y)(x\bar{y} - 1)} V_i(y) e^{u_i(y)} dy + \\ &+ \int_{B(x_i, \delta_i \epsilon)} \frac{1 - |y|^2}{(x - y)(x\bar{y} - 1)} V_i(y) e^{u_i(y)} dy + \\ &+ \int_{B(t_i, \delta'_i \epsilon)} \frac{1 - |y|^2}{(x - y)(x\bar{y} - 1)} V_i(y) e^{u_i(y)} dy \end{aligned}$$

Let us set:

$$\begin{aligned} I_1 &= \int_{B_1(0) - (B(x_i, \delta_i \epsilon) \cup B(t_i, \delta'_i \epsilon))} \frac{1 - |y|^2}{(x - y)(x\bar{y} - 1)} V_i(y) e^{u_i(y)} dy \\ I_2 &= \int_{B(x_i, \delta_i \epsilon)} \frac{1 - |y|^2}{(x - y)(x\bar{y} - 1)} V_i(y) e^{u_i(y)} dy, \\ I_3 &= \int_{B(t_i, \delta'_i \epsilon)} \frac{1 - |y|^2}{(x - y)(x\bar{y} - 1)} V_i(y) e^{u_i(y)} dy \end{aligned}$$

For the first integral, because $u_i \leq C$ on $B_1(0) - (B(x_i, \delta_i \epsilon) \cup B(t_i, \delta'_i \epsilon))$, we have:

$$|I_1| \leq C \int_{B_1(0)} \frac{1 - |y|^2}{|x - y||x\bar{y} - 1|} dy,$$

But, $1 \geq |x| = |x - t_0^i + t_0^i| \geq |t_0^i| - |x - t_0^i| \geq 1 - (\epsilon/2) - (1 - \epsilon) = \epsilon/2$, thus, we can write:

$$|I_1| \leq C \int_{B_1(0)} \frac{1 - |y|^2}{|x - y||x||\bar{y} - 1/x|} dy,$$

and, we use the fact that:

$$|\bar{y} - 1/x| \geq ||\bar{y}| - 1/|x|| \geq |1/|x| - |y|| \geq (1 - |y|),$$

To have:

$$|\partial_x u_i(x)| \leq |I_2| + |I_3| + C \int_{B_1(0)} \frac{1 + |y|}{|x - y|} dy = |I_2| + |I_3| + C',$$

Now, we look to the second and third integrals, it is sufficient to consider the first one :

$$I_2 = \int_{B(x_i, \delta_i \epsilon)} \frac{1 - |y|^2}{(x - y)(x\bar{y} - 1)} V_i(y) e^{u_i(y)} dy$$

Case 1: $((1/2) + \tilde{\epsilon})|x_i - t_i| < |x - t_0^i| < 1 - \epsilon$:

In this case we have:

$$1 - |y|^2 = 1 - |x_i + \delta_i z|^2 = \delta_i(2 + o(1)),$$

and,

$$|x - y| = |x - t_0^i + t_0^i - y_i - \delta_i z| \geq (\tilde{\epsilon}/2)|x_i - t_i|,$$

and,

$$|x\bar{y} - 1| = |((x - t_0^i + t_0^i - x_i) + x_i)(\bar{x}_i + \delta_i \bar{z}) - 1| \geq (\tilde{\epsilon}/2)|x - t_0^i|,$$

Thus,

$$|\partial_x u_i(x)| \leq C' + C \frac{\delta_i}{|x_i - t_i|} \frac{1}{|x - t_0^i|} = C' + \frac{o(1)}{|x - t_0^i|}.$$

with, $o(1) \rightarrow 0$ as $i \rightarrow +\infty$.

Case 2: $|x - t_0^i| < ((1/2) - \tilde{\epsilon})|x_i - t_i|$:

In this case, we have:

$$1 - |y|^2 = 1 - |x_i + \delta_i z|^2 = \delta_i(2 + o(1)),$$

and,

$$|x - y| = |x - t_0^i + t_0^i - y_i - \delta_i z| \geq (\tilde{\epsilon}/2)|x_i - t_i|,$$

and,

$$|x\bar{y} - 1| = |((x - t_0^i + t_0^i - x_i) + x_i)(\bar{x}_i + \delta_i \bar{z}) - 1| \geq (\tilde{\epsilon}/2)|x_i - t_0^i|,$$

Thus,

$$|\partial_x u_i(x)| \leq C' + C \frac{\delta_i}{|x_i - t_i|} \frac{1}{|x_i - t_0^i|} = C' + \frac{o(1)}{|x_i - t_0^i|}.$$

with, $o(1) \rightarrow 0$ as $i \rightarrow +\infty$.

Case 3: $((1/2) - \tilde{\epsilon})|x_i - t_i| < |x - t_0^i| < ((1/2) + \tilde{\epsilon})|x_i - t_i|$:

Let \tilde{t}_0^i the point of D_i such that $|\tilde{t}_0^i - t_0^i| = 1/2(|x_i - t_i|)$. We use the fact that the function:

$$v_i(t) = u_i(\tilde{t}_0^i + (|x_i - t_0^i|/4)t),$$

is uniformly bounded for $|t| \leq 1$ and is a solution of PDE which is uniformly bounded on $|t| \leq 1$. By the elliptic estimates we have:

$$|x_i - t_i| |\nabla u_i|_{L^\infty(D_i \cap \{|(1/2) - \tilde{\epsilon})|x_i - t_i| \leq |x - t_0^i| \leq ((1/2) + \tilde{\epsilon})|x_i - t_i\})} \leq C.$$

Thus, we use the previous cases to compute the following integral:

$$\int_{D_i} \langle (x_2^i - t_2^i) |\nabla u_i \rangle \langle \nu |\nabla u_i \rangle d\sigma + \int_{D_i} \langle (x_2^i - t_2^i) |\nu \rangle |\nabla u_i|^2 d\sigma = o(1)$$

and, thus,

$$\int_{D_i'} \langle (x_2^i - t_2^i) |\nabla u_i \rangle \langle \nu |\nabla u_i \rangle d\sigma + \int_{D_i'} \langle (x_2^i - t_2^i) |\nu \rangle |\nabla u_i|^2 d\sigma = o(1)$$

here, we used the previous estimates with $i \rightarrow +\infty$ and $\tilde{\epsilon} \rightarrow 0$ (for the previous case 3).

REFERENCES

- [1] T. Aubin. Some Nonlinear Problems in Riemannian Geometry. Springer-Verlag 1998
- [2] S.S Bahoura. Majorations du type $\sup u \times \inf u \leq c$ pour l'équation de la courbure scalaire sur un ouvert de \mathbb{R}^n , $n \geq 3$. J. Math. Pures. Appl.(9) 83 2004 no, 9, 1109-1150.
- [3] S.S. Bahoura. Harnack inequalities for Yamabe type equations. Bull. Sci. Math. 133 (2009), no. 8, 875-892
- [4] S.S. Bahoura. Lower bounds for sup+inf and sup \times inf and an extension of Chen-Lin result in dimension 3. Acta Math. Sci. Ser. B Engl. Ed. 28 (2008), no. 4, 749-758
- [5] S.S. Bahoura. Estimations uniformes pour l'equation de Yamabe en dimensions 5 et 6. J. Funct. Anal. 242 (2007), no. 2, 550-562.
- [6] C. Bandle. Isoperimetric inequalities and Applications. Pitman. 1980.
- [7] H. Brezis, YY. Li, I. Shafrir. A sup+inf inequality for some nonlinear elliptic equations involving exponential nonlinearities. J.Funct.Anal.115 (1993) 344-358.
- [8] H.Brezis and F.Merle, Uniform estimates and blow-up behavior for solutions of $-\Delta u = V e^u$ in two dimensions, Commun Partial Differential Equations 16 (1991), 1223-1253.
- [9] L. Caffarelli, B. Gidas, J. Spruck. Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth. Comm. Pure Appl. Math. 37 (1984) 369-402.
- [10] W. Chen, C. Li. A priori Estimates for solutions to Nonlinear Elliptic Equations. Arch. Rational. Mech. Anal. 122 (1993) 145-157.
- [11] C-C.Chen, C-S. Lin. Estimates of the conformal scalar curvature equation via the method of moving planes. Comm. Pure Appl. Math. L(1997) 0971-1017.

- [12] C-C.Chen, C-S. Lin. A sharp sup+inf inequality for a nonlinear elliptic equation in \mathbb{R}^2 . Commun. Anal. Geom. 6, No.1, 1-19 (1998).
- [13] C-C.Chen, C-S. Lin. Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces. Comm. Pure Appl. Math. 55 (2002), no. 6, 728-771
- [14] B. Gidas, W-Y. Ni, L. Nirenberg. Symmetry and related properties via the maximum principle. Comm. Math. Phys. 68 (1979), no. 3, 209-243.
- [15] Hofmann, S. Mitrea, M. Taylor, M. Geometric and transformational properties of Lipschitz domains, Semmes-Kenig-Toro domains, and other classes of finite perimeter domains. J. Geom. Anal. 17 (2007), no. 4, 593-647.
- [16] J.M. Lee, T.H. Parker. The Yamabe problem. Bull.Amer.Math.Soc (N.S) 17 (1987), no.1, 37-91.
- [17] YY. Li. Prescribing scalar curvature on \mathbb{S}_n and related Problems. C.R. Acad. Sci. Paris 317 (1993) 159-164. Part I: J. Differ. Equations 120 (1995) 319-410. Part II: Existence and compactness. Comm. Pure Appl.Math.49 (1996) 541-597.
- [18] YY. Li. Harnack Type Inequality: the Method of Moving Planes. Commun. Math. Phys. 200,421-444 (1999).
- [19] YY. Li, I. Shafrir. Blow-up Analysis for Solutions of $-\Delta u = V e^u$ in Dimension Two. Indiana. Math. J. Vol 3, no 4. (1994). 1255-1270.
- [20] YY. Li, L. Zhang. A Harnack type inequality for the Yamabe equation in low dimensions. Calc. Var. Partial Differential Equations 20 (2004), no. 2, 133-151.
- [21] YY.Li, M. Zhu. Yamabe Type Equations On Three Dimensional Riemannian Manifolds. Commun.Contem.Mathematics, vol 1. No.1 (1999) 1-50.
- [22] L. Ma, J-C. Wei. Convergence for a Liouville equation. Comment. Math. Helv. 76 (2001) 506-514.
- [23] I. Shafrir. A sup+inf inequality for the equation $-\Delta u = V e^u$. C. R. Acad.Sci. Paris Sér. I Math. 315 (1992), no. 2, 159-164.
- [24] Y-T. Siu. The existence of Kahler-Einstein metrics on manifolds with positive anticanonical line bundle and a suitable finite symmetry group. Ann. of Math. (2) 127 (1988), no. 3, 585-627
- [25] G. Tian. A Harnack type inequality for certain complex Monge-Ampre equations. J. Differential Geom. 29 (1989), no. 3, 481-488.
- [26] L. Zhang. Blowup solutions of some nonlinear elliptic equations involving exponential nonlinearities. Comm. Math. Phys. 268 (2006), no. 1, 105-133.

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