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Borel summation of the small time expansion of the heat kernel with a vector potential*†

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Abstract

Let p_t be the heat kernel associated to the Laplacian with a vector potential. We prove under rather strong assumptions on this potential that the small time expansion of p_t is Borel summable. An explicit formula for p_t plays a central role. In the periodic case, a Poisson's formula is introduced.

1 Introduction

Let $\nu \geqslant 1$. Let $c_0, \ldots, c_\nu$ be regular square matrix-valued functions on R^ν. Let H be the operator defined by

$$H := -(\partial_{x_1}^2 + \cdots + \partial_{x_\nu}^2) + 2\big(c_1(x)\partial_{x_1} + \cdots + c_\nu(x)\partial_{x_\nu}\big) + (\partial_{x_1} c_1 + \cdots + \partial_{x_\nu} c_\nu) - c_0(x). \tag{1.1}$$

Let $p_t(x,y)$ be the heat kernel associated to this operator. Let $p_t^{\mathrm{conj}}(x,y)$ be the conjugate heat kernel given by

$$p_t(x,y) = (4\pi t)^{-\frac{\nu}{2}} \exp\Big(-\frac{(x-y)^2}{4t}\Big) p_t^{\mathrm{conj}}(x,y).$$

In a previous work about the scalar potential case ($c_1 = 0, \ldots, c_\nu = 0$), we found conditions on c_0 providing Borel summation of the small time expansion of the conjugate heat kernel. We also studied the partition function on the torus. In this paper, similar questions are considered in the vector potential case. A general motivation for this work is a question asked by Balian and Bloch in [B-B]. Can quantum quantities be exactly recovered with the help of classical quantities? Indeed, coefficients of the small time expansion are

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classical quantities and then Borel summation gives a positive answer to this question for the quantum quantity $\langle y|e^{-tH}|x\rangle = p_t(x,y)$. See [Ha4] for partial comments about these questions and references therein. Another motivation is the introduction and the use of a so-called deformation formula which gives an "explicit" expression of the conjugate heat kernel (see Proposition 3.2). Note also that we always work in a complex setting ($t \in \mathrm{C}$, $x, y \in \mathrm{C}^\nu$).

Let us now introduce more precisely our results. The following asymptotic expansion (Minakshisundaram-Pleijel expansion) is well known

$$p_t^{\mathrm{conj}}(x,y) = a_0(x,y) + a_1(x,y)t + \cdots + a_{r-1}(x,y)t^{r-1} + t^r \mathcal{O}_{t\to 0^+}(1).$$

We shall prove, under rather strong assumptions on $c_0, \ldots, c_\nu$, that this expansion is Borel summable and that its Borel sum is equal to $p_t^{\mathrm{conj}}(x,y)$.

Assume now that $c_0, \ldots, c_\nu$ are defined on the torus $(\mathrm{R}/\mathrm{Z})^\nu$ with values in a space of $d \times d$ matrices and that c_0 is Hermitian whereas $c_1, \ldots, c_\nu$ are anti-Hermitian. Let $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots, \lambda_n \to +\infty$ be the eigenvalues of the operator H acting on periodic C^d-valued functions. We shall prove the Poisson formula: for small $t \in \mathrm{C}$, $\mathcal{R}et > 0$,

$$\sum_{n=1}^{+\infty} e^{-\lambda_n t} = (4\pi t)^{-\frac{\nu}{2}} \sum_{q \in \mathrm{Z}^\nu} e^{-\frac{q^2}{4t}} u_q(t), \tag{1.2}$$

where, for $q \in \mathrm{Z}^\nu$

$$u_q(t) = a_{0,q} + a_{1,q}t + \cdots + a_{r-1,q}t^{r-1} + \cdots \tag{1.3}$$

In (1.3), each expansion is Borel summable and u_q denotes the Borel sum of such an expansion.

Let us compare the scalar and the vector potential cases. We consider in [Ha4] perturbations of $-(\partial_{x_1}^2 + \cdots + \partial_{x_\nu}^2) + \alpha(x_1^2 + \cdots + x_\nu^2)$ with $\alpha \in \mathrm{R}$. For the vector potential case, we prefer to consider only perturbations of $-(\partial_{x_1}^2 + \cdots + \partial_{x_\nu}^2)$, and then focus on the perturbation term. This allows us to work easily with the Borel transform of the conjugate heat kernel. The proofs are then simplified as already remarked in [Ha4]. In what follows, we prove that this Borel transform is analytic on the complex plane and is exponentially dominated by the Borel variable on parabolic domains which are symmetric with respect to the positive real axis. One can expect that this result can be improved (in the scalar case, the Borel transform is exponentially dominated by the square root of Borel variable on the same domains).

Our choice of notation in (1.1) is imposed by our deformation formula without any reference to a Hermitian product. Now, let $\sigma_1, \ldots, \sigma_\nu$ be Dirac matrices ($\sigma_j\sigma_k + \sigma_k\sigma_j = \delta_{jk}$). Then a Halmitonian like $\big(\sigma \cdot \partial_x - ia(x)\big)^2$ is covered by our results both in an Abelian setting (electromagnetic field) and a non-Abelian setting (Yang-Mills field).

In the scalar potential case, the deformation formula is considered by E. Onifri [On] in a heuristic way and can be obtained with the help of Wiener's

(or Feynman's) formula and Wick's theorem [Ha4, Appendix]. In the vectorial potential case, this connection also exists but it seems more difficult to exhibit it. In [F-H-S-S], an algorithm, using Wick's theorem, is proposed for computing, in a covariant invariant way, the coefficients of the small time expansion of $p_t(x,x)$.

2 Notation and main results

For $z = |z|e^{i\theta} \in \mathrm{C}$, $\theta \in]-\pi,\pi]$, let $z^{1/2} := |z|^{1/2}e^{i\theta/2}$. Let $T > 0$. Let $\mathrm{C}^+ := \{z \in \mathrm{C} | \mathcal{R}e(z) > 0\}$, $D_T := \{z \in \mathrm{C} | |z| < T\}$, $D_T^+ := D_T \cap \mathrm{C}^+$ and $\tilde{D}_T := \left\{z \in \mathrm{C} | \mathcal{R}e(\frac{1}{z}) > \frac{1}{T}\right\}$. $\tilde{D}_T$ is the open disk of center $\frac{T}{2}$ and radius $\frac{T}{2}$. Let $\kappa > 0$. Let $\tilde{S}_\kappa := \{z \in \mathrm{C} | d(z,[0,+\infty[) < \kappa\}$ and

$$S_\kappa := \{z \in \mathrm{C} | |\mathcal{I}m z^{1/2}|^2 < \kappa\} = \{z \in \mathrm{C} | \mathcal{R}ez > \frac{1}{4\kappa}\mathcal{I}m^2 z - \kappa\}.$$

S_κ is the interior of a parabola which contains $\tilde{S}_\kappa$ (see figures 2.1 and 2.2 in [Ha4]).

We work with finite dimensional spaces of square matrices. We always consider multiplicative norms on these spaces ($|AB| \leqslant |A||B|$, for A and B square matrices) and we assume that $|1| = 1$. For $A = (a_{i,j})_{1\leqslant i,j\leqslant d}$ with $a_{i,j} \in \mathrm{C}$, we set $A^* = (\bar{a}_{j,i})_{1\leqslant i,j\leqslant d}$. For $\lambda, \mu \in \mathrm{C}^\nu$, we denote $\lambda \cdot \mu := \lambda_1\mu_1 + \cdots + \lambda_\nu\mu_\nu$, $\bar{\lambda} := (\bar{\lambda}_1, \ldots, \bar{\lambda}_\nu)$, $\mathcal{I}m\lambda := (\mathcal{I}m\lambda_1, \ldots, \mathcal{I}m\lambda_\nu)$, $\lambda^2 := \lambda \cdot \lambda$, $|\lambda| := (\lambda \cdot \bar{\lambda})^{1/2}$ (if $\lambda \in \mathrm{R}^\nu$, $|\lambda| = \sqrt{\lambda^2}$). Later, we shall extend these notations to operators and measures.

Let Ω be an open domain in C^m and let F be a complex finite dimensional space. We denote by $\mathcal{A}(\Omega)$ the space of F-valued analytic functions on Ω, if there is no ambiguity on F.

Let $\mathfrak{B}$ denote the collection of all Borel sets on R^m. An F-valued measure μ on R^m is an F-valued function on $\mathfrak{B}$ satisfying the classical countable additivity property (cf. [Ru]). Let $|\cdot|$ be a norm on F. We denote by $|\mu|$ the positive measure defined by

$$|\mu|(E) = \sup \sum_{j=1}^{\infty} |\mu(E_j)| (E \in \mathfrak{B}),$$

the supremum being taken over all partitions $\{E_j\}$ of E. In particular, $|\mu|(\mathrm{R}^m) < \infty$. Note that $d\mu = hd|\mu|$ where h is some F-valued function satisfying $|h| = 1$ $|\mu|$-a.e. In the sequel, we shall consider vector spaces F^ν and $F^{\nu+1}$ with the following norms (depending on the norm of F). For $f = (f_0, \ldots, f_\nu) \in F^{\nu+1}$, we set $\boldsymbol{f} := (f_1, \ldots, f_\nu)$ and $|f|^\star := \max(|\boldsymbol{f}|, |f_0|)$. If $f = (f_0, \ldots, f_\nu)$ is an $F^{\nu+1}$-valued function on R^m and θ is a positive measure such that

$$\int_{\mathrm{R}^m} |f_0| d\theta < \infty, \ldots, \int_{\mathrm{R}^m} |f_\nu| d\theta < \infty,$$

one can define an F^ν-valued (respectively $F^{\nu+1}$-valued) measure $\boldsymbol{\lambda}$ (respectively λ) by setting $d\boldsymbol{\lambda} = \boldsymbol{f} d\theta$ (respectively $d\lambda = f d\theta$). Then $d|\boldsymbol{\lambda}| = |\boldsymbol{f}| d\theta$ and

$d|\lambda|^\star = |f|^\star d\theta$. Let θ be a positive measure, w be a positive function and μ be a normed vector space valued measure. The notation $|d\mu| \leqslant w d\theta$ means that $|\mu(E)| \leqslant \int_E w d\theta$ (or $|\mu|(E) \leqslant \int_E w d\theta$) for every $E \in \mathfrak{B}$. Equivalently, there exists a vector-valued function h such that $d\mu = h d\theta$ and $|h| \leqslant w$ θ-a.e. For instance, let $u_1, \ldots, u_\nu$ be measurable C-valued functions and $\lambda_1, \ldots, \lambda_\nu$ be F-valued measures. Then $|\boldsymbol{u} \cdot d\boldsymbol{\lambda}| \leqslant |\boldsymbol{u}| d|\boldsymbol{\lambda}|$ (where $\boldsymbol{u} \cdot d\boldsymbol{\lambda} := u_1 d\lambda_1 + \cdots + u_\nu d\lambda_\nu$).

We refer to [Ha4] for a rigorous definition of Borel and Laplace transform. Roughly speaking, assuming that f (respectively $\hat{f}$) is a function of a complex variable t (respectively τ), f is the Laplace transform of $\hat{f}$ if

$$f(t) = \int_0^{+\infty} \hat{f}(\tau) e^{-\frac{\tau}{t}} \frac{d\tau}{t} \tag{2.1}$$

whereas $\hat{f}$ is the Borel transform of $f = a_0 + a_1 t + \cdots + a_n t^n + \cdots$ if

$$\hat{f}(\tau) = \sum_{r=0}^{\infty} \frac{a_r}{r!} \tau^r.$$

With suitable assumptions, these two transforms are inverse each to other. In the whole paper, sums indexed by an empty set are, by convention, equal to zero.

Theorem 2.1 *Let $\varepsilon > 0$. Let $\lambda_0, \ldots, \lambda_\nu$ be measures on R^ν with values in a complex finite dimensional space of square matrices verifying for $q \in \{0, \ldots, \nu\}$*

$$\int_{\mathrm{R}^\nu} \exp(\varepsilon \xi^2) d|\lambda_q|(\xi) < \infty. \tag{2.2}$$

Let

$$c_q(x) := \int \exp(ix \cdot \xi) d\lambda_q(\xi).$$

Let $\mathbf{c}(x) := \big(c_1(x), \ldots, c_\nu(x)\big)$ and $\partial_x \cdot \mathbf{c} := \partial_{x_1} c_1 + \cdots + \partial_{x_\nu} c_\nu$. Let u[1] be the solution of

$$\begin{cases} \partial_t u = \partial_x^2 u - 2\mathbf{c}(x) \cdot \partial_x u + \big(c_0(x) - \partial_x \cdot \mathbf{c}\big) u \\ u|_{t=0^+} = \delta_{x=y} 1 \end{cases} . \tag{2.3}$$

Let v be defined by $u = (4\pi t)^{-\nu/2} e^{-\frac{(x-y)^2}{4t}} v$. Then v admits a Borel transform $\hat{v}$ (with respect to t) which is analytic on $\mathrm{C}^{1+2\nu}$. Let $\kappa, R > 0$. Let C be defined by

$$C := \int_{\mathrm{R}^\nu} \exp\Big(\frac{\varepsilon}{2}\xi^2 + (1+R)|\xi| + \frac{2\kappa}{\varepsilon}\Big) d|\lambda|^\star(\xi).$$

Then, for every $(\tau, x, y) \in S_\kappa \times \mathrm{C}^{2\nu}$ such that $|\mathcal{I}mx| < R$ and $|\mathcal{I}my| < R$,

$$|\hat{v}(\tau, x, y)| \leqslant e^{C|x-y|} e^{C(|\tau| + 2|\tau|^{1/2})}. \tag{2.4}$$

[1]Since $c_0, \ldots, c_\nu$ are analytic and bounded on R^ν, (2.3) admits a unique analytic solution on $\mathrm{R}^+ \times \mathrm{R}^{2\nu}$.

Remark 2.2 *By (2.4) and (2.1), the solution u of (2.3) through $\hat{v}$ is meaningful for $t \in \mathrm{C}, \mathcal{R}e\big(\frac{1}{t}\big) > \frac{1}{C}$. Since S_κ contains $\tilde{S}_\kappa$, (2.4) implies that the small time expansion of v is Borel summable and that v is equal to the Borel sum of this expansion (cf. [Ha4]).*

Remark 2.3 *One can expect that (2.4) can be improved. Let us assume that $\nu = 1$, that the measure λ_1 takes its values in C and satisfies (2.2) where ε is replaced by 2ε. Let*

$$\tilde{c}(x) := \int \exp(ix \cdot \xi) d\tilde{\lambda}(\xi)$$

where $\tilde{\lambda}$ is a C-valued measure on R satisfying (2.2). Let us choose $c_0 := c_1^2 + \tilde{c}$. Then c_0 is the Fourier transform of a measure satisfying (2.2) and the system (2.3) is equivalent to

$$\begin{cases} \partial_t u = \big(\partial_x - c_1(x)\big)^2 u + \tilde{c}(x) u \\ u|_{t=0^+} = \delta_{x=y} \end{cases} .$$

Performing the substitution

$$u = \exp\Big(\int_y^x c_1(z) dz\Big) \tilde{u} \tag{2.5}$$

in the previous system allows one to use the scalar case Borel summation result [Ha4, Theorem 3.1]. Therefore (2.4) can be improved as follows:

$$|\hat{v}(\tau, x, y)| \leqslant e^{C_1 |x-y|} e^{\tilde{C} |\tau|^{1/2}}$$

where

$$C_1 := \int_{\mathrm{R}} e^{R|\xi|} d|\lambda_1|(\xi),$$

$$\tilde{C} := 2\Big(\int_{\mathrm{R}^\nu} \exp\big(\frac{2\kappa}{\varepsilon} + \frac{\varepsilon}{2}\xi^2 + R|\xi|\big) d|\tilde{\lambda}|(\xi)\Big)^{1/2}.$$

In particular, unlike in the general case (see Remark 2.2), the solution u of (2.3) through $\hat{v}$ is meaningful for $t \in \mathrm{C}^+$, which is more natural. Actually the proof of Theorem 2.1 uses a deformation formula which does no take into account the fondamental notion of magnetic field (or curvature). The substitution (2.5) is efficient because the magnetic field vanishes in the one-dimensional case.

Corollary 2.4 *Let $\varepsilon > 0$ and $d \in \mathrm{N}^*$. For every $(j, q) \in \{0, \ldots, \nu\} \times \mathrm{Z}^\nu$, let $c_{j,q}$ be square matrices acting on C^d such that $c_{0,-q} = c_{0,q}^*$ and $c_{j,-q} = -c_{j,q}^*$ for every $j \in \{1, \ldots, \nu\}$. Assume that*

$$\sum_{q \in \mathrm{Z}^\nu} e^{4\pi^2 \varepsilon q^2} |c_{j,q}| < \infty$$

for every $j \in \{0, \dots, \nu\}$. Let $c_j(x) = \sum_{q \in \mathrm{Z}^\nu} c_{j,q} e^{2i\pi q \cdot x}$. Let $\Lambda_1 \leqslant \Lambda_2 \leqslant \cdots$ be the eigenvalues of the operator

$$H := -\partial_x^2 + 2\boldsymbol{c}(x) \cdot \partial_x + \big(\partial_x \cdot \boldsymbol{c} - c_0(x)\big)$$

acting on C^d-valued functions defined on the torus $(\mathrm{R}/\mathrm{Z})^\nu$. For each $q \in \mathrm{Z}^\nu$, there exists a function $\hat{w}(q, .)$ analytic on C satisfying

- *For every $\kappa > 0$, there exists $T, K > 0$ such that, for every $q \in \mathrm{Z}^\nu$ and $\tau \in S_\kappa$,*

$$|\hat{w}(q, \tau)| \leqslant K e^{\frac{|q|}{T}} e^{\frac{|\tau|}{T}}.$$

- *There exists $\tilde{T}$ such that for every $t \in \tilde{D}_{\tilde{T}}$*

$$\sum_{n=1}^{+\infty} e^{-\Lambda_n t} = (4\pi t)^{-\nu/2} \sum_{q \in \mathrm{Z}^\nu} e^{-\frac{q^2}{4t}} \int_0^{+\infty} e^{-\frac{\tau}{t}} \hat{w}(q, \tau) \frac{d\tau}{t}. \tag{2.6}$$

Remark 2.5 *Let us consider the usual Hermitian product on $L^2\big((\mathrm{R}/\mathrm{Z})^\nu, \mathrm{C}^d\big)$. By our assumptions on the coefficients $c_{j,q}$, H can be viewed as a self-adjoint operator with compact resolvent. The proof of Corollary 2.4 is similar to the proof of Corollary 3.3 in [Ha4] and is therefore omitted.*

Remark 2.6

- *Formula (2.6) can be viewed as a Poisson formula (see also [Ha4]): for each $q \in \mathrm{Z}^\nu$, there are numbers $a_{0,q}, a_{1,q}, \dots \in \mathrm{C}$ and functions $R_{0,q}, R_{1,q}, \dots \in \mathcal{A}(\tilde{D}_{\tilde{T}})$ such that, for every $r \geqslant 0$ and $t \in \tilde{D}_{\tilde{T}}$*

$$\sum_{n=1}^{+\infty} e^{-\Lambda_n t} = (4\pi t)^{-\nu/2} \sum_{q \in \mathrm{Z}^\nu} e^{-\frac{q^2}{4t}} \big(a_{0,q} + a_{1,q} t + \cdots + a_{r-1,q} t^{r-1} + R_{r,q}(t)\big), \tag{2.7}$$

  and for κ small enough, there exist $M > 0$ such that

$$|R_{r,q}(t)| \leqslant M \frac{r!}{\kappa^r} |t|^r,$$

  for every $r \geqslant 0, q \in \mathrm{Z}^\nu, t \in \tilde{D}_{\tilde{T}}$.

- *One can easily check, using the deformation formula (3.3), that*

$$a_{0,q} = \int_{[0,1]^\nu} \mathrm{Tr}\Big(\mathrm{Texp}\Big(\int_0^1 q \cdot c(x + sq) ds\Big)\Big) dx.$$

  with the following definition for the ordered exponential

$$1 + \int_0^1 q \cdot c(x + s_1 q) ds_1 + \int_{0 < s_1 < s_2 < 1} q \cdot c(x + s_2 q) q \cdot c(x + s_1 q) ds_1 ds_2 + \cdots$$

3 Proof of the theorems

Our proofs use the following deformation formula which gives a representation of the heat kernel. First, we need

Notation 3.1 *Let $\lambda_0, \dots, \lambda_\nu$ be measures on R^ν with values in a complex finite dimensional space of square matrices. Let us assume that for every $R > 0$ and $q \in \{0, \dots, \nu\}$*

$$\int_{\mathrm{R}^\nu} \exp(R|\xi|) d|\lambda_q|(\xi) < +\infty. \tag{3.1}$$

Let $\boldsymbol{c}$ and c_0 as in Theorem 2.1. For $t, x_1, \dots, x_n, y_1, \dots y_n \in \mathrm{C}^{2n+1}$ and $0 < s_1 < \dots < s_n < 1$, let

$$V_n(t, x_1, y_1, \dots) := \Big[(x_n - y_n) \cdot \boldsymbol{c}\big(y_n + s_n(x_n - y_n)\big) + tc_0\big(y_n + s_n(x_n - y_n)\big)\Big] \cdots$$

$$\Big[(x_1 - y_1) \cdot \boldsymbol{c}\big(y_1 + s_1(x_1 - y_1)\big) + tc_0\big(y_1 + s_1(x_1 - y_1)\big)\Big].$$

Let P_n be the operator acting on $\mathcal{A}(\mathrm{C}^{2\nu n})$ defined by

$$P_n := \sum_{j,k=1}^{n} \partial_{x_{j \wedge k}} \cdot \partial_{y_{j \vee k}},$$

where $j \wedge k := \min(j, k)$ and $j \vee k := \max(j, k)$. Finally, for $\xi = (\xi_1, \dots, \xi_n) \in \mathrm{R}^{\nu n}$, let

$$|\xi|_1 := |\xi_1| + \dots + |\xi_n|,$$

and let $\|\lambda\|$ be the measure on $\mathrm{R}^{\nu n}$ defined by

$$d^{\nu n}||\lambda||(\xi) = d|\lambda|^\star(\xi_n) \cdots d|\lambda|^\star(\xi_1).$$

Proposition 3.2 *Let $\lambda_0, \dots, \lambda_\nu$ be as in Notation 3.1. Let v be defined by*

$$v = 1 + \sum_{n \geqslant 1} v_n \tag{3.2}$$

where

$$v_n(t, x, y) := \int_{0 < s_1 < \dots < s_n < 1} \big[\exp(tP_n) V_n(t, x_1, y_1, \dots)\big]\Big|_{\substack{x_1 = x, y_1 = y \\ \dots \\ x_n = x, y_n = y}} d^n s. \tag{3.3}$$

Let $R > 0$ and let $\Upsilon_R := \{(x, y) \in \mathrm{C}^{2\nu} || \mathcal{I}mx| < R, |\mathcal{I}my| < R\}$. Let

$$T_R := \left(\int_{\mathrm{R}^\nu} \exp\big((1 + R)|\xi|\big) d|\lambda|^\star(\xi)\right)^{-1}.$$

Then $v \in \mathcal{A}(D^+_{T_R} \times \Upsilon_R)$. The function $u := (4\pi t)^{-\nu/2} e^{-\frac{(x-y)^2}{4t}} v$ is a solution of (2.3).

Let us give another useful expression of the function v. Let $\bar{z} = (\bar{z}_1, \dots, \bar{z}_n) \in \mathrm{C}^{\nu n}$ and $\xi = (\xi_1, \dots, \xi_n) \in \mathrm{R}^{\nu n}$. Let $\mu_{\bar{z}}$ be the measure defined on $\mathrm{R}^{\nu n}$ by

$$d^{\nu n}\mu_{\bar{z}}(\xi) = \big(\bar{z}_n \cdot d\boldsymbol{\lambda}(\xi_n) + td\lambda_0(\xi_n)\big) \cdots \big(\bar{z}_1 \cdot d\boldsymbol{\lambda}(\xi_1) + td\lambda_0(\xi_1)\big).$$

Let $s = (s_1, \dots, s_n)$ such that $0 < s_1 < \cdots < s_n < 1$. Let $\tilde{P}_n$ be the operator defined by

$$\tilde{P}_n := i \sum_{j,k=1}^{n} (1 - s_{j\vee k})\xi_{j\vee k} \cdot \partial_{\bar{z}_{j\wedge k}} - s_{j\wedge k}\xi_{j\wedge k} \cdot \partial_{\bar{z}_{j\vee k}} - 2 \sum_{1\leqslant j<k\leqslant n} \partial_{\bar{z}_j} \cdot \partial_{\bar{z}_k}.$$

Let

$$s(1-s) \cdot_n \xi \otimes \xi := \sum_{j,k=1}^{n} s_{j\wedge k}(1 - s_{j\vee k})\xi_j \cdot \xi_k,$$

$$\big(y + s(x-y)\big) \cdot \xi := \big(y + s_1(x-y)\big) \cdot \xi_1 + \cdots + \big(y + s_n(x-y)\big) \cdot \xi_n.$$

Remark 3.3 *The following identity holds*

$$v_n = \int_{0<s_1<\cdots<s_n<1} \int_{\mathrm{R}^{\nu n}} e^{i\left(y+s(x-y)\right)\cdot\xi} \times$$

$$e^{-ts(1-s)\cdot_n \xi\otimes\xi} \left[e^{t\tilde{P}_n} d^{\nu n}\mu_{\bar{z}}(\xi)\right]\Big|_{\substack{\bar{z}_1 = x-y \\ \cdots \\ \bar{z}_n = x-y}} d^n s. \tag{3.4}$$

Remark 3.3 and Proposition 3.2 will be proved together; we shall need the following remarks and lemma. Remark 3.4 will partially explain the shape of the operator P_n in (3.3). Remark 3.5 (cf. Lemma 4.1 in [Ha4]) is crucial for the proof of the convergence of the integrals in (3.4) and for Theorem 2.1.

Remark 3.4

$$\Big[\partial_t + \sum_{j=1}^{n} \frac{x_j - y_j}{t} \cdot \partial_{x_j}, tP_n\Big] = \Big(\sum_{j=1}^{n} \partial_{x_j}\Big) \cdot \Big(\sum_{j=1}^{n} \partial_{x_j}\Big).$$

Remark 3.5 *Let $\xi \in \mathrm{R}^{\nu n}$ and $0 < s_1 < \cdots < s_n < 1$. Then*

$$0 \leqslant s(1-s) \cdot_n \xi \otimes \xi \leqslant n(\xi_1^2 + \cdots + \xi_n^2). \tag{3.5}$$

Lemma 3.6 *Let $\lambda_0, \dots, \lambda_\nu$ be as in Notation 3.1. Then, for $m, n \in \mathrm{N}$, $m \leqslant n$, there exist measures $\mu_{n,m}$ defined on $\mathrm{R}^{\nu n}$ which are polynomial with respect to (x, y) and which do not depend on t such that*

$$\left[e^{t\tilde{P}_n} d^{\nu n}\mu_{\bar{z}}(\xi)\right]\Big|_{\substack{\bar{z}_1 = x-y \\ \cdots \\ \bar{z}_n = x-y}} = \sum_{m\leqslant n} t^m d^{\nu n}\mu_{n,m}(\xi) \tag{3.6}$$

where

$$|d^{\nu n}\mu_{n,m}(\xi)| \leqslant n! e^{|\xi|_1} \times \sum_{\substack{p+q=m \\ 2p+q \leqslant n \\ p,q \geqslant 0}} \frac{1}{p!} \frac{|x-y|^{n-2p-q}}{(n-2p-q)!} d^{\nu n}\|\lambda\|(\xi). \tag{3.7}$$

Proof Let

$$b_\gamma(\xi) := -i \sum_{1 \leqslant j \leqslant \gamma} s_j \xi_j + i \sum_{\gamma \leqslant j \leqslant n} (1-s_j)\xi_j.$$

Then

$$\tilde{P}_n = \sum_{\gamma=1}^{n} b_\gamma(\xi) \cdot \partial_{\bar{z}_\gamma} - 2 \sum_{1 \leqslant \alpha < \beta \leqslant n} \partial_{\bar{z}_\alpha} \cdot \partial_{\bar{z}_\beta}.$$

For $\alpha, \beta, \gamma = 1, \ldots, n$, $\alpha < \beta$, let $a_\gamma := b_\gamma(\xi) \cdot \partial_{\bar{z}_\gamma}$ and $A_{\alpha,\beta} := \partial_{\bar{z}_\alpha} \cdot \partial_{\bar{z}_\beta}$ be the operators acting on linear combinations of monomials such as $\prod_{j \in J} \bar{z}_j$ where $J \subset \{1, \ldots, n\}$. Since $a_\gamma^2 = A_{\alpha,\beta}^2 = 0$, $a_\gamma A_{\alpha,\beta} = 0$ if $\gamma \in \{\alpha, \beta\}$ and $A_{\alpha,\beta} A_{\alpha',\beta'} = 0$ if $\{\alpha, \beta\} \cap \{\alpha', \beta'\} \neq \varnothing$, one gets

$$\frac{1}{r!}\Big(\sum_{\gamma=1}^{n} a_\gamma - 2 \sum_{1 \leqslant \alpha < \beta \leqslant n} A_{\alpha,\beta}\Big)^r = \sum_{I_1, \ldots, I_p, J} (-2)^p \prod_{\gamma \in J} a_\gamma \prod_{k=1}^{p} A_{\min(I_k), \max(I_k)}. \tag{3.8}$$

Here, $p \geqslant 0$ and the sum runs over all collections of pairwise disjoint subsets $I_1, \ldots, I_p, J$ of $\{1, \ldots, n\}$, such that $|I_1| = \cdots = |I_p| = 2$ and $|J| + p = r$, without ordering on $I_1, \ldots, I_p$.

Expanding $e^{t\tilde{P}_n}$ using (3.8) implies that the left hand side of (3.6) is equal to

$$\sum_{I_1, \ldots, I_p, J, K} (-2t)^p t^{|J|+|K|} \mathcal{P}_{\{I\},J,K} \tag{3.9}$$

where

$$\mathcal{P}_{\{I\},J,K} := \prod_{\{\alpha,\beta\},\gamma,\delta,\varepsilon}$$

$$\Big\{\big(d\boldsymbol{\lambda}(\xi_\alpha) \cdot d\boldsymbol{\lambda}(\xi_\beta)\big)\big(b_\gamma(\xi) \cdot d\boldsymbol{\lambda}(\xi_\gamma)\big)\big((x-y) \cdot d\boldsymbol{\lambda}(\xi_\delta)\big) d\lambda_0(\xi_\varepsilon)\Big\}_{>}. \tag{3.10}$$

Here we use the following convention:

First, $p \in \mathrm{N}$, $I_1, \ldots, I_p, J, K$ are subsets of $\{1, \ldots, n\}$ such that $|I_1| = \cdots = |I_p| = 2$ and $I_1, \ldots, I_p, J, K$ are pairwise disjoint (hence $2p + |J| + |K| \leqslant n$). The sum (3.9) runs over all such subsets without ordering as far as $I_1, \ldots, I_p$ are concerned.

Second, if $I_1, \ldots, I_p, J, K$ satisfy the previous assumptions, the product in (3.10) runs over all $\{\alpha, \beta\} = I_1, \ldots, I_p$, $\gamma \in J$, $\varepsilon \in K$ and δ lying in the complementary of $I_1 \cup \cdots \cup I_p \cup J \cup K$.

Third, the symbol $\{\cdot\}_>$ means that the terms of the product are ordered. For instance, if $n = 6$, $p = 1$, $I_1 = \{1, 5\}$, $J = \{3\}$ and $K = \{2, 4\}$,

$$\mathcal{P}_{\{I\},J,K} = \sum_{\ell=1}^{\nu} (x-y) \cdot d\boldsymbol{\lambda}(\xi_6) d\lambda_\ell(\xi_5) d\lambda_0(\xi_4) b_3(\xi) \cdot d\boldsymbol{\lambda}(\xi_3) d\lambda_0(\xi_2) d\lambda_\ell(\xi_1).$$

Then (3.6) holds with

$$d^{\nu n}\mu_{n,m}(\xi) := \sum_{\substack{I_1,\dots,I_p,\,J,\,K\\ p+|J|+|K|=m}} (-2)^p \mathcal{P}_{\{I\},J,K}.$$

Let $p,j,k \in \mathrm{N}$ such that $2p+j+k \leqslant n$. There are $\frac{n!}{2^p p!j!k!(n-2p-j-k)!}$ subsets $I_1,\dots,I_p,J,K$ such that $|J|=j$ and $|K|=k$ in (3.9). Moreover

$$|b_\gamma(\xi)\cdot d\boldsymbol{\lambda}(\xi_\gamma)| \leqslant |\xi|_1 d|\boldsymbol{\lambda}|(\xi_\gamma).$$

Then

$$|d^{\nu n}\mu_{n,m}(\xi)| \leqslant \sum_{\substack{p+j+k=m\\ 2p+j+k\leqslant n}} \frac{n!}{p!j!k!(n-2p-j-k)!}|\xi|_1^j|x-y|^{n-2p-j-k}d^{\nu n}||\lambda||(\xi).$$

Since, for $q \geqslant 0$,

$$\sum_{j+k=q}\frac{1}{k!}\frac{|\xi|_1^j}{j!} \leqslant e^{|\xi|_1}, \tag{3.11}$$

one gets (3.7). □

We can now prove Proposition 3.2.

Proof We claim that the right hand side of (3.4) is well defined and analytic on $\mathrm{C}^+\times\mathrm{C}^{2\nu}$. Let $|\lambda|$ as in Lemma 3.6. For $n\geqslant 1$ and $(t,x,y)\in\mathrm{C}^+\times\mathrm{C}^{2\nu}$, let

$$d^{\nu n}W_n(t,x,y) := e^{i\left(y+s(x-y)\right)\cdot\xi}e^{-ts(1-s)\cdot_n\xi\otimes\xi}\left[e^{t\tilde{P}_n}d^{\nu n}\mu_{\tilde{z}}(\xi)\right]\Big|_{\substack{\tilde{z}_1=x-y\\ \cdots\\ \tilde{z}_n=x-y}}$$

and

$$\tilde{v}_n(t,x,y) := \int_{0<s_1<\cdots<s_n<1}\int_{\mathrm{R}^{\nu n}} d^{\nu n}W_n(t,x,y). \tag{3.12}$$

Let $R>0$ and suppose now that $(x,y)\in\Upsilon_R$. By (3.5) and Lemma 3.6

$$|d^{\nu n}W_n(t,x,y)| \leqslant n!e^{(R+1)|\xi|_1}\sum_{\substack{2p+q\leqslant n\\ p,q\geqslant 0}}\frac{1}{p!}\frac{|x-y|^{n-2p-q}}{(n-2p-q)!}|t|^{p+q}d^{\nu n}\|\lambda\|(\xi).$$

Let

$$A := \int_{\mathrm{R}^\nu}\exp\big((1+R)|\xi|\big)d|\lambda|^\star(\xi).$$

Then

$$\int_{0<s_1<\cdots<s_n<1}\int_{\mathrm{R}^{\nu n}}|d^{\nu n}W_n(t,x,y)| \leqslant A^n\sum_{2p+q\leqslant n}\frac{1}{p!}\frac{|x-y|^{n-2p-q}}{(n-2p-q)!}|t|^{p+q}.$$

Therefore

$$\begin{aligned}\mathcal{Q} := \quad & 1+\sum_{n\geqslant 1}\int_{0<s_1<\cdots<s_n<1}\int_{\mathrm{R}^{\nu n}}|d^{\nu n}W_n(t,x,y)| \\ \leqslant \quad & e^{A|x-y|}\sum_{p,q\geqslant 0}A^{2p+q}\frac{|t|^{p+q}}{p!}\end{aligned}$$

since, for $p,q\in\mathrm{N}$,

$$\sum_{n\geqslant 2p+q}A^{n-2p-q}\frac{|x-y|^{n-2p-q}}{(n-2p-q)!}=e^{A|x-y|}.$$

Hence $\mathcal{Q}<+\infty$ if $A|t|<1$. Set $T_R:=\frac{1}{A}$. Then $\tilde{v}_n$ and hence

$$\tilde{v}:=1+\sum_{n\geqslant 1}\tilde{v}_n$$

are well defined on $D^+_{T_R}\times\Upsilon_R$. By dominated convergence theorem, one can also check that $\tilde{v}$ is analytic on $D^+_{T_R}\times\Upsilon_R$.

Let us now prove that $\tilde{v}_n=v_n$. One has

$$V_n(t,x_1,y_1,\ldots)=\int_{\mathrm{R}^{\nu n}}\exp\Big(i\big(y_1+s_1(x_1-y_1)\big)\cdot\xi_1+\cdots+i\big(y_n+s_n(x_n-y_n)\big)\cdot\xi_n\Big)\times$$
$$\big[(x_n-y_n)\cdot d\boldsymbol{\lambda}(\xi_n)+td\lambda_0(\xi_n)\big]\cdots\big[(x_n-y_n)\cdot d\boldsymbol{\lambda}(\xi_n)+td\lambda_0(\xi_n)\big].$$

Then

$$\partial_{x_\alpha}\cdot\partial_{y_\beta}V_n(t,x_1,y_1,\ldots)=\int_{\mathrm{R}^{\nu n}}\exp\Big(i\big(y_1+s_1(x_1-y_1)\big)\cdot\xi_1+\cdots\Big)\times$$
$$\big[(I+II+III)d^{\nu n}\mu_{\bar{z}}(\xi)\big]\Big|_{\substack{\bar{z}_1=x-y\\ \cdots\\ \bar{z}_n=x-y}}.$$

where

$$I:=-s_\alpha(1-s_\beta)\xi_\alpha\cdot\xi_\beta, II:=-\partial_{\bar{z}_\alpha}\cdot\partial_{\bar{z}_\beta},$$
$$III:=i\big((1-s_\beta)\xi_\beta\cdot\partial_{\bar{z}_\alpha}-s_\alpha\xi_\alpha\cdot\partial_{\bar{z}_\beta}\big).$$

This proves that $\tilde{v}_n=v_n$ and therefore Remark 3.3. Then $v=\tilde{v}$ and $v\in\mathcal{A}(D^+_{T_R}\times\Upsilon_R)$. We claim that the function

$$u=(4\pi t)^{-\nu/2}e^{-\frac{(x-y)^2}{4t}}v, \tag{3.13}$$

where v is given by (3.2), is a solution of (2.3). Indeed, any solution u of (2.3) is obtained, using (3.13), from a solution v of the conjugate equation

$$\begin{cases}\big(\partial_t+\frac{x-y}{t}\cdot\partial_x\big)v=\partial_x^2v+\frac{(x-y)\cdot\mathbf{c}(x)}{t}v+\big(c_0(x)-\partial_x\cdot\mathbf{c}(x)\big)v-2\mathbf{c}(x)\cdot\partial_x v\\ v|_{t=0^+,x=y}=1\end{cases}.$$

Let $v_0 = 1$. It is then sufficient to verify that v_n satisfies

$$\begin{cases} \big(\partial_t + \frac{x-y}{t}\cdot\partial_x\big)v_n = \partial_x^2 v_n + \frac{(x-y)\cdot\boldsymbol{c}}{t}v_{n-1} + (c_0 - \partial_x\cdot\boldsymbol{c})v_{n-1} - 2\boldsymbol{c}\cdot\partial_x v_{n-1} \\ v_n|_{t=0^+,x=y} = 0 \end{cases} \tag{3.14}$$

for $(t,x,y) \in D^+_{T_R} \times \Upsilon_R$, $n \geqslant 1$. It suffices to check (3.14) for $(t,x,y) \in]0,T_R[\times\Upsilon_R$. By (3.3)

$$v_n(t,x,y) = \int_{0<s_1<\cdots<s_n<t} \big[\exp(tP_n)V_n^\natural(t,x_1,y_1,\ldots)\big]\Big|_{\substack{x_1=x,\,y_1=y\\ \cdots\\ x_n=x,\,y_n=y}} d^n s \tag{3.15}$$

where

$$V_n^\natural(t,x_1,y_1,\ldots) := \Big[\frac{x_n-y_n}{t}\cdot\boldsymbol{c}(y_n + s_n\frac{x_n-y_n}{t}) + c_0(y_n + s_n\frac{x_n-y_n}{t})\Big]\cdots$$
$$\Big[\frac{x_1-y_1}{t}\cdot\boldsymbol{c}(y_1 + s_1\frac{x_1-y_1}{t}) + c_0(y_1 + s_1\frac{x_1-y_1}{t})\Big].$$

One has $\big(\partial_t + \frac{x-y}{t}\cdot\partial_x\big)v_n = I + II$ where I is obtained by differentiating the domain of the integral in (3.15) whereas II is obtained by differentiating the integrand. Then

$$I = \int_{0<s_1<\cdots<s_{n-1}<t} \big[\exp(tP_n)V_n^\natural(t,x_1,y_1,\ldots)\big]\Big|_{\substack{s_n=t\\ x_1=x,\,y_1=y\\ \cdots\\ x_n=x,\,y_n=y}} d^{n-1}s.$$

Note that

$$V_n^\natural(t,x_1,y_1,\ldots)|_{s_n=t} = \Big(\frac{x_n-y_n}{t}\cdot\boldsymbol{c}(x_n) + c_0(x_n)\Big)V_{n-1}^\natural(t,x_1,y_1,\ldots).$$

Since

$$P_n = \partial_{x_n}\cdot\partial_{y_n} + 2(\sum_{j=1}^{n-1}\partial_{x_j})\cdot\partial_{y_n} + P_{n-1},$$

one gets

$$I = \Big(\frac{(x-y)\cdot\boldsymbol{c}(x)}{t} + c_0(x)\Big)v_{n-1} - \partial_x\cdot\boldsymbol{c}(x)v_{n-1} - 2\boldsymbol{c}(x)\cdot\partial_x v_{n-1}. \tag{3.16}$$

Let us evaluate II. By Remark 3.4

$$\big(\partial_t + \sum_{j=1}^n \frac{x_j-y_j}{t}\cdot\partial_{x_j}\big)e^{tP_n} = \big(\sum_{j=1}^n\partial_{x_j}\big)^2 e^{tP_n} + e^{tP_n}\big(\partial_t + \sum_{j=1}^n\frac{x_j-y_j}{t}\cdot\partial_{x_j}\big). \tag{3.17}$$

Let $j \in \{1,\ldots,n\}$ and $\alpha \in \{1,\ldots,\nu\}$. Denoting by $x_{j,\alpha}$ the α-coordinate of $x_j \in \mathrm{C}^\nu$, one has

$$\big(\partial_t + \frac{x_j-y_j}{t}\cdot\partial_{x_j}\big)\Big(\frac{x_{j,\alpha}-y_{j,\alpha}}{t}\Big) = 0. \tag{3.18}$$

Then

$$\big(\partial_t + \sum_{j=1}^{n} \frac{x_j - y_j}{t} \cdot \partial_{x_j}\big) V_n^\natural(t, x_1, y_1, \ldots) = 0. \tag{3.19}$$

Hence, by (3.17) and (3.19)

$$\big(\partial_t + \frac{x-y}{t} \cdot \partial_x - \partial_x^2\big) \big[\exp(tP_n) V_n^\natural(t, x_1, y_1, \ldots)\big] \big|_{\substack{x_1 = x,\, y_1 = y \\ \cdots \\ x_n = x,\, y_n = y}} = 0.$$

Then

$$II = \partial_x^2 v_n. \tag{3.20}$$

Hence, by (3.16) and (3.20), we have checked (3.14) (the second line is trivial). □

For the proof of Theorem 2.1, we need [Ha4, Lemma 4.9]:

Lemma 3.7 *Let $m \geqslant 1$. For $B \in \mathrm{C}$, let $\tau \longrightarrow K_m(B, \tau)$ be the Borel transform of the function $t \longrightarrow t^m \exp(-Bt)$. Then, for $\tau \in \mathrm{C}$ and $B \geqslant 0$*

$$|K_m(B, \tau)| \leqslant \frac{|\tau|^m}{m!} \exp\Big(2\sqrt{B} |\mathcal{I}m(\tau^{1/2})|\Big). \tag{3.21}$$

Let us now prove Theorem 2.1.

Proof First, let us define $\hat{v}$. Remark 3.3 and Lemma 3.6 suggest the following construction. Let $d^{\nu n}\hat{F}_n, \hat{v}_n, \hat{v}$ be defined by

$$d^{\nu n}\hat{F}_n := \sum_{m \leqslant n} \exp\Big(i\big(y + s(x-y)\big) \cdot \xi\Big) K_m\big(s(1-s) \cdot_n \xi \otimes \xi, \tau\big) d^{\nu n}\mu_{n,m}(\xi), \tag{3.22}$$

$$\hat{v}_n(\tau, x, y) := \int_{0 < s_1 < \cdots < s_n < 1} \int_{\mathrm{R}^{\nu n}} d^{\nu n}\hat{F}_n d^n s, \tag{3.23}$$

$$\hat{v} := 1 + \sum_{n \geqslant 1} \hat{v}_n\,. \tag{3.24}$$

Let us check that $\hat{v}_n$ and $\hat{v}$ are well defined. Let $\kappa, R > 0$. Let $(x, y) \in \Upsilon_R$ and $\tau \in S_\kappa$. By (3.7) and (3.21)

$$|d^{\nu n}\hat{F}_n| \leqslant e^{(1+R)|\xi|_1} n! \sum_{2p+q \leqslant n} \frac{|\tau|^{p+q} |x-y|^{n-2p-q}}{p!(p+q)!(n-2p-q)!} \times$$

$$\exp\big(2\kappa^{1/2} \sqrt{s(1-s) \cdot_n \xi \otimes \xi}\big) d^{\nu n} \|\lambda\|(\xi).$$

By (3.5)

$$\begin{aligned}
2\kappa^{1/2}\sqrt{s(1-s)\cdot_n \xi\otimes\xi} \leqslant \quad & 2\kappa^{1/2}\sqrt{n(\xi_1^2+\cdots+\xi_n^2)} \\
\leqslant \quad & 2\times\Big(\frac{2\kappa n}{\varepsilon}\Big)^{1/2}\times\sqrt{\frac{\varepsilon}{2}(\xi_1^2+\cdots+\xi_n^2)} \\
\leqslant \quad & \frac{2\kappa n}{\varepsilon}+\frac{\varepsilon}{2}(\xi_1^2+\cdots+\xi_n^2).
\end{aligned}$$

Hence

$$|d^{\nu n}\hat{F}_n| \leqslant \exp\Big((1+R)|\xi|_1+\frac{\varepsilon}{2}(\xi_1^2+\cdots+\xi_n^2)\varepsilon+\frac{2\kappa n}{\varepsilon}\Big)\times$$
$$n!\sum_{2p+q\leqslant n}\frac{|\tau|^{p+q}|x-y|^{n-2p-q}}{p!(p+q)!(n-2p-q)!}d^{\nu n}\|\lambda\|(\xi).$$

Let

$$C:=\int_{\mathrm{R}^\nu}\exp\Big(\frac{2\kappa}{\varepsilon}+\frac{\varepsilon}{2}\xi^2+(1+R)|\xi|\Big)d|\lambda|^\star(\xi).$$

Then

$$\int_{0<s_1<\cdots<s_n<1}\int_{\mathrm{R}^{\nu n}}|d^{\nu n}\hat{F}_n|d^n s\leqslant C^n\sum_{2p+q\leqslant n}\frac{|\tau|^{p+q}|x-y|^{n-2p-q}}{p!(p+q)!(n-2p-q)!}.$$

Then

$$\begin{aligned}
\mathcal{Q}:= \quad & 1+\sum_{n\geqslant 1}\int_{0<s_1<\cdots<s_n<1}\int_{\mathrm{R}^{\nu n}}|d^{\nu n}\hat{F}_n|d^n s \\
\leqslant \quad & e^{C|x-y|}\sum_{p,q\geqslant 0}\frac{|\tau|^{p+q}}{p!(p+q)!}C^{2p+q}
\end{aligned}$$

since, for $p,q\in\mathrm{N}$,

$$\sum_{n\geqslant 2p+q}C^{n-2p-q}\frac{|x-y|^{n-2p-q}}{(n-2p-q)!}=e^{C|x-y|}.$$

Hence $\mathcal{Q}<\infty$. This proves that $\hat{v}_n$ and $\hat{v}$ are well defined on $\mathrm{C}^{1+2\nu}$ since κ and R are arbitrary. By dominated convergence theorem, one can also check that $\hat{v}_n$, hence $\hat{v}$, are analytic on $\mathrm{C}^{1+2\nu}$.

Since $p!(p+q)!\geqslant p!p!q!\geqslant\frac{(2p)!q!}{2^{2p}}$

$$\mathcal{Q}\leqslant e^{C|x-y|}e^{2C|\tau|^{1/2}}e^{C|\tau|}.$$

Then $\hat{v}$ satisfies (2.4). By (3.12), (3.23) and the definition of K_m (Lemma 3.7), $\hat{v}_n$ is the Laplace transform of $v_n=\tilde{v}_n$. Then v defined by (3.2) is the Laplace transform of $\hat{v}$. Hence Proposition 3.2 implies Theorem 2.1. □

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