



Deformations of third order Peregrine breather solutions of the NLS equation with four parameters

Pierre Gaillard

► To cite this version:

Pierre Gaillard. Deformations of third order Peregrine breather solutions of the NLS equation with four parameters. 2013. hal-00783882

HAL Id: hal-00783882

<https://hal.science/hal-00783882v1>

Preprint submitted on 1 Feb 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Deformations of third order Peregrine breather solutions of the NLS equation with four parameters.

⁺Pierre Gaillard, ⁺ Université de Bourgogne, Dijon, France :
e-mail: Pierre.Gaillard@u-bourgogne.fr,

February 1, 2013

Abstract

In this paper, we give new solutions of the focusing NLS equation as a quotient of two determinants. This formulation gives in the case of the order 3, new deformations of the Peregrine breather with four parameters.

This gives a very efficient procedure to construct families of quasi-rational solutions of the NLS equation and to describe the apparition of multi rogue waves. With this method, we construct the analytical expressions of deformations of the Peregrine breather of order $N = 3$ depending on 4 real parameters and plot different types of rogue waves.

1 Introduction

The first results concerning the nonlinear Schrödinger equation (NLS) date from the Seventies. Precisely, in 1972 Zakharov and Shabat solved it using the the inverse scattering method [21, 22]. The first quasi-rational solutions of NLS equation were constructed in 1983 by Peregrine [20]. In 1986 Eleon-ski, Akhmediev and Kulagin obtained the two-phase almost periodic solution to the NLS equation and obtained the first higher order analogue of the Peregrine breather[3]. Other families of higher order were constructed in a series of articles by Akhmediev et al. [1, 2] using Darboux transformations. It has been shown in [8] in 2010, that rational solutions of the NLS equation

can be written as a quotient of two Wronskians. Recently, it has been constructed in [11] a new representation of the solutions of the NLS equation in terms of a ratio of two Wronskians determinants of even order $2N$ composed of elementary functions; the related solutions of NLS are called of order N . Quasi-rational solutions of the NLS equation were obtained by the passage to the limit when some parameter tended to 0.

These results can be compared with those obtain recently by Akhmediev et al. in [5] with Darboux dressing method and numerical approach.

Recently, an other representation of the solutions of the focusing NLS equation, as a ratio of two determinants has been given in [14] using generalized Darboux transform.

A new approach has been done in [19] which gives a determinant representation of solutions of the focusing NLS equation, obtained from Hirota bilinear method, derived by reduction of the Gram determinant representation for Davey-Stewartson system.

A little later, in 2012 one obtained a representation in terms of determinants which does not involve limits [13]. But, these first two formulations given in [11, 13] did depend in fact only on two parameters; this remark was first made by V.B. Matveev. Then, in the middle of the year 2012, one determined multi-parametric families of quasi rational solutions of NLS in terms of determinants of order N (determinants of order $2N$) dependent on $2N-2$ real parameters. They are similar to those previously explicitly found by V.B Matveev and P. Dubard by the method given in [2], for the first time in the case $N = 3$.

With this method, we obtain news deformations at order 3 with 4 real parameters. With this representation, one finds at the same time the circular forms well-known, but also the triangular forms put recently in obviousness by Ohta and Yang [19], and also by Akhmediev et al. [17].

The following orders will be the object of other publications.

2 Wronskian representation of solutions of NLS equation and quasi-rational limit

2.1 Solutions of NLS equation in terms of Wronskian determinant

We consider the focusing NLS equation

$$iv_t + v_{xx} + 2|v|^2v = 0. \quad (1)$$

From the works [11, 10], the solution of the NLS equation can be written in the form

$$v(x, t) = \frac{\det(I + A_3(x, t))}{\det(I + A_1(x, t))} \exp(2it - i\varphi). \quad (2)$$

In (2), the matrix $A_r = (a_{\nu\mu})_{1 \leq \nu, \mu \leq 2N}$ ($r = 3, 1$) is defined by

$$a_{\nu\mu} = (-1)^{\epsilon_\nu} \prod_{\lambda \neq \mu} \left| \frac{\gamma_\lambda + \gamma_\nu}{\gamma_\lambda - \gamma_\mu} \right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} + e_\nu). \quad (3)$$

κ_ν , δ_ν , γ_ν are functions of the parameters λ_ν , $\nu = 1, \dots, 2N$ satisfying the relations

$$0 < \lambda_j < 1, \quad \lambda_{N+j} = -\lambda_j, \quad 1 \leq j \leq N. \quad (4)$$

They are given by the following equations,

$$\kappa_\nu = 2\sqrt{1 - \lambda_\nu^2}, \quad \delta_\nu = \kappa_\nu \lambda_\nu, \quad \gamma_\nu = \sqrt{\frac{1 - \lambda_\nu}{1 + \lambda_\nu}}, \quad (5)$$

and

$$\kappa_{N+j} = \kappa_j, \quad \delta_{N+j} = -\delta_j, \quad \gamma_{N+j} = 1/\gamma_j, \quad j = 1 \dots N. \quad (6)$$

The terms $x_{r,\nu}$ ($r = 3, 1$) are defined by

$$x_{r,\nu} = (r - 1) \ln \frac{\gamma_\nu - i}{\gamma_\nu + i}, \quad 1 \leq j \leq 2N. \quad (7)$$

The parameters e_ν are defined by

$$e_j = ia_j - b_j, \quad e_{N+j} = ia_j + b_j, \quad 1 \leq j \leq N, \quad (8)$$

where a_j and b_j , for $1 \leq j \leq N$ are arbitrary real numbers.
The terms ϵ_ν are defined by :

$$\begin{aligned}\epsilon_\nu &= 0, & 1 \leq \nu \leq N \\ \epsilon_\nu &= 1, & N+1 \leq \nu \leq 2N.\end{aligned}$$

We use the following notations :

$$\Theta_{r,\nu} = \kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu, \quad 1 \leq \nu \leq 2N.$$

We consider the functions

$$\begin{aligned}\phi_{r,\nu}(y) &= \sin \Theta_{r,\nu}, & 1 \leq \nu \leq N, \\ \phi_{r,\nu}(y) &= \cos \Theta_{r,\nu}, & N+1 \leq \nu \leq 2N.\end{aligned}\tag{9}$$

$W_r(y) = W(\phi_{r,1}, \dots, \phi_{r,2N})$ is the Wronskian

$$W_r(y) = \det[(\partial_y^{\mu-1} \phi_{r,\nu})_{\nu, \mu \in [1, \dots, 2N]}].\tag{10}$$

Then we get the following link between Fredholm and Wronskian determinants [12]

Theorem 2.1

$$\det(I + A_r) = k_r(0) \times W_r(\phi_{r,1}, \dots, \phi_{r,2N})(0),\tag{11}$$

where

$$k_r(y) = \frac{2^{2N} \exp(i \sum_{\nu=1}^{2N} \Theta_{r,\nu})}{\prod_{\nu=2}^{2N} \prod_{\mu=1}^{\nu-1} (\gamma_\nu - \gamma_\mu)}.$$

In (11), the matrix A_r is defined by (3).

It can be deduced the following result :

Theorem 2.2 *The function v defined by*

$$v(x, t) = \frac{W_3(0)}{W_1(0)} \exp(2it - i\varphi).\tag{12}$$

is solution of the NLS equation (1)

$$iv_t + v_{xx} + 2|v|^2 v = 0.$$

2.2 Quasi-rational solutions of NLS equation

In the following, we take the limit when the parameters $\lambda_j \rightarrow 1$ for $1 \leq j \leq N$ and $\lambda_j \rightarrow -1$ for $N+1 \leq j \leq 2N$.

We consider the parameter λ_j written in the form

$$\lambda_j = 1 - 2j^2\epsilon^2, \quad 1 \leq j \leq N. \quad (13)$$

When ϵ goes to 0, we realize limited expansions at order p , for $1 \leq j \leq N$, of the terms

$$\begin{aligned} \kappa_j &= 4j\epsilon(1 - \epsilon^2j^2)^{1/2}, \quad \delta_j = 4j\epsilon(1 - 2\epsilon^2j^2)(1 - \epsilon^2j^2)^{1/2}, \\ \gamma_j &= j\epsilon(1 - \epsilon^2j^2)^{-1/2}, \quad x_{r,j} = (r-1) \ln \frac{1+i\epsilon j(1-\epsilon^2j^2)^{-1/2}}{1-i\epsilon j(1-\epsilon^2j^2)^{-1/2}}, \\ \kappa_{N+j} &= 4j\epsilon(1 - \epsilon^2j^2)^{1/2}, \quad \delta_{N+j} = -4j\epsilon(1 - 2\epsilon^2j^2)(1 - \epsilon^2j^2)^{1/2}, \\ \gamma_{N+j} &= 1/(j\epsilon)(1 - \epsilon^2j^2)^{1/2}, \quad x_{r,N+j} = (r-1) \ln \frac{1-i\epsilon j(1-\epsilon^2j^2)^{-1/2}}{1+i\epsilon j(1-\epsilon^2j^2)^{-1/2}}. \end{aligned}$$

Then we get quasi-rational solutions of the NLS equation given by :

Theorem 2.3 *With the parameters λ_j defined by (13), a_j and b_j chosen as in (??), for $1 \leq j \leq N$, the function v defined by*

$$v(x, t) = \exp(2it - i\varphi) \lim_{\epsilon \rightarrow 0} \frac{W_3(0)}{W_1(0)}, \quad (14)$$

is a quasi-rational solution of the NLS equation (1)

$$iv_t + v_{xx} + 2|v|^2v = 0.$$

3 Expression of solutions of NLS equation in terms of a ratio of two determinants

We construct here solutions of the NLS equation which is expressed as a quotient of two determinants.

For this we need the following notations :

$$\begin{aligned} A_\nu &= \kappa_\nu x/2 + i\delta_\nu t - ix_{3,\nu}/2 - ie_\nu/2, \\ B_\nu &= \kappa_\nu x/2 + i\delta_\nu t - ix_{1,\nu}/2 - ie_\nu/2, \end{aligned}$$

for $1 \leq \nu \leq 2N$, with κ_ν , δ_ν , $x_{r,\nu}$ defined in (5), (6) and (7).

The parameters e_ν are defined by (8).

With particular special choices of the parameters a_j and b_j , for $1 \leq N$, we get new deformations depending on four parameters. Below we use the following functions :

$$\begin{aligned} f_{4j+1,k} &= \gamma_k^{4j-1} \sin A_k, & f_{4j+2,k} &= \gamma_k^{4j} \cos A_k, \\ f_{4j+3,k} &= -\gamma_k^{4j+1} \sin A_k, & f_{4j+4,k} &= -\gamma_k^{4j+2} \cos A_k, \end{aligned} \quad (15)$$

for $1 \leq k \leq N$, and

$$\begin{aligned} f_{4j+1,k} &= \gamma_k^{2N-4j-2} \cos A_k, & f_{4j+2,k} &= -\gamma_k^{2N-4j-3} \sin A_k, \\ f_{4j+3,k} &= -\gamma_k^{2N-4j-4} \cos A_k, & f_{4j+4,k} &= \gamma_k^{2N-4j-5} \sin A_k, \end{aligned} \quad (16)$$

for $N+1 \leq k \leq 2N$.

We define the functions $g_{j,k}$ for $1 \leq j \leq 2N$, $1 \leq k \leq 2N$ in the same way, we replace only the term A_k by B_k .

$$\begin{aligned} g_{4j+1,k} &= \gamma_k^{4j-1} \sin B_k, & g_{4j+2,k} &= \gamma_k^{4j} \cos B_k, \\ g_{4j+3,k} &= -\gamma_k^{4j+1} \sin B_k, & g_{4j+4,k} &= -\gamma_k^{4j+2} \cos B_k, \end{aligned} \quad (17)$$

for $1 \leq k \leq N$, and

$$\begin{aligned} g_{4j+1,k} &= \gamma_k^{2N-4j-2} \cos B_k, & g_{4j+2,k} &= -\gamma_k^{2N-4j-3} \sin B_k, \\ g_{4j+3,k} &= -\gamma_k^{2N-4j-4} \cos B_k, & g_{4j+4,k} &= \gamma_k^{2N-4j-5} \sin B_k, \end{aligned} \quad (18)$$

for $N+1 \leq k \leq 2N$.

Then we get the following result :

Theorem 3.1 *The function v defined by*

$$v(x, t) = \exp(2it - i\varphi) \times \frac{\det((n_{jk})_{j,k \in [1, 2N]})}{\det((d_{jk})_{j,k \in [1, 2N]})} \quad (19)$$

is a quasi-rational solution of the NLS equation (1)

$$iv_t + v_{xx} + 2|v|^2v = 0,$$

where

$$n_{j1} = f_{j,1}(x, t, 0), \quad 1 \leq j \leq 2N \quad n_{jk} = \frac{\partial^{2k-2} f_{j,1}}{\partial \epsilon^{2k-2}}(x, t, 0), \quad 2 \leq k \leq N, \quad 1 \leq j \leq 2N$$

$$n_{jN+1} = f_{j,N+1}(x, t, 0), \quad 1 \leq j \leq 2N \quad n_{jN+k} = \frac{\partial^{2k-2} f_{j,N+1}}{\partial \epsilon^{2k-2}}(x, t, 0), \quad 2 \leq k \leq N, \quad 1 \leq j \leq 2N$$

$$d_{j1} = g_{j,1}(x, t, 0), \quad 1 \leq j \leq 2N \quad d_{jk} = \frac{\partial^{2k-2} g_{j,1}}{\partial \epsilon^{2k-2}}(x, t, 0), \quad 2 \leq k \leq N, \quad 1 \leq j \leq 2N$$

$$d_{jN+1} = g_{j,N+1}(x, t, 0), \quad 1 \leq j \leq 2N \quad d_{jN+k} = \frac{\partial^{2k-2} g_{j,N+1}}{\partial \epsilon^{2k-2}}(x, t, 0), \quad 2 \leq k \leq N, \quad 1 \leq j \leq 2N$$

The functions f and g are defined in (15), (16), (17), (18).

We don't give here the proof of this result in order to not weight down the text. We postpone the redaction of the proof to a next publication.

The solutions of the NLS equation can also be written in the form :

$$v(x, t) = \exp(2it - i\varphi) \times Q(x, t)$$

where $Q(x, t)$ is defined by :

$$Q(x, t) := \frac{\begin{vmatrix} f_{1,1}[0] & \dots & f_{1,1}[N-1] & f_{1,N+1}[0] & \dots & f_{1,N+1}[N-1] \\ f_{2,1}[0] & \dots & f_{2,1}[N-1] & f_{2,N+1}[0] & \dots & f_{2,N+1}[N-1] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{2N,1}[0] & \dots & f_{2N,1}[N-1] & f_{2N,N+1}[0] & \dots & f_{2N,N+1}[N-1] \end{vmatrix}}{\begin{vmatrix} g_{1,1}[0] & \dots & g_{1,1}[N-1] & g_{1,N+1}[0] & \dots & g_{1,N+1}[N-1] \\ g_{2,1}[0] & \dots & g_{2,1}[N-1] & g_{2,N+1}[0] & \dots & g_{2,N+1}[N-1] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{2N,1}[0] & \dots & g_{2N,1}[N-1] & g_{2N,N+1}[0] & \dots & g_{2N,N+1}[N-1] \end{vmatrix}} \quad (20)$$

4 Quasi-rational solutions of order 3 with four parameters

We have already constructed in [11] solutions for the cases $N = 1, 2, 3$, and in [13] with two parameters.

4.1 Analytical expressions of the solutions of NLS equation with Four parameters

Here, we give the expression v of the solution of NLS equation with four parameters; it is defined by

$$v_3(x, t, a, b) = \frac{n(x, t)}{d(x, t)} \exp(2it) = (1 - 4 \frac{G_3(2x, 4t) + iH_3(2x, 4t)}{Q_3(2x, 4t)}) e^{2it}$$

with

$$G_3(X, T) = \sum_{k=0}^{12} g_k(T) X^k, \quad H_3(X, T) = \sum_{k=0}^{12} h_k(T) X^k, \quad Q_3(X, T) = \sum_{k=0}^{12} q_k(T) X^k.$$

$$\begin{aligned} \mathbf{g}_{12} &= 0, \quad \mathbf{g}_{11} = 0, \quad \mathbf{g}_{10} = 6, \quad \mathbf{g}_9 = 0, \quad \mathbf{g}_8 = 90 T^2 + 90, \quad \mathbf{g}_7 = 0, \\ \mathbf{g}_6 &= 300 T^4 - 360 T^2 - 960 T b_1 + 1260, \quad \mathbf{g}_5 = -1440 T^2 a_1 - 1440 a_1 + 18 a_2, \\ \mathbf{g}_4 &= 420 T^6 - 900 T^4 - 2400 T^3 b_1 + 2700 T^2 + 3600 a_1^2 - 1200 b_1^2 + (-2490 b_1 \\ &\quad - 30 b_2) T - 2700, \quad \mathbf{g}_3 = -4800 T^4 a_1 - 19200 T a_1 b_1 + (-28800 a_1 + 60 a_2) T^2 \\ &\quad + 14400 a_1 + 60 a_2, \quad \mathbf{g}_2 = 270 T^8 + 2520 T^6 + 40500 T^4 + (57060 b_1 - 180 b_2) T^3 \\ &\quad + (-7200 a_1^2 + 21600 b_1^2 - 81000) T^2 - 7200 a_1^2 - 7200 b_1^2 + (-58140 b_1 - 180 b_2) T \\ &\quad - 4050, \quad \mathbf{g}_1 = -3360 T^6 a_1 + 19200 T^3 a_1 b_1 + (7200 a_1 - 150 a_2) T^4 + 9600 a_1^3 + 9600 a_1 b_1^2 \\ &\quad + (-21600 a_1 - 540 a_2) T^2 + (19920 a_1 b_1 + 240 a_1 b_2 - 240 a_2 b_1) T + 21600 a_1 + 90 a_2, \\ \mathbf{g}_0 &= 66 T^{10} + 2970 T^8 + 1440 T^7 b_1 + 13140 T^6 + (17886 b_1 + 42 b_2) T^5 + (8400 a_1^2 \\ &\quad + 3600 b_1^2 - 45900) T^4 + (-44340 b_1 + 420 b_2) T^3 + (7200 a_1^2 + 240 a_1 a_2 \\ &\quad - 30480 b_1^2 + 240 b_1 b_2 - 12150) T^2 + 18000 a_1^2 + 240 a_1 a_2 + a_2^2 + 10489 b_1^2 \\ &\quad + 166 b_1 b_2 + b_2^2 + (-9600 a_1^2 b_1 - 9600 b_1^3 + 7470 b_1 + 90 b_2) T + 4050 \end{aligned}$$

$$\begin{aligned} \mathbf{h}_{12} &= 0, \quad \mathbf{h}_{11} = 0, \quad \mathbf{h}_{10} = 6 T, \quad \mathbf{h}_9 = 0, \quad \mathbf{h}_8 = 30 T^3 - 90 T - 60 b_1, \quad \mathbf{h}_7 = 0, \\ \mathbf{h}_6 &= 60 T^5 - 840 T^3 - 480 T^2 b_1 - 900 T - 305 b_1 + 5 b_2, \quad \mathbf{h}_5 = -480 T^3 a_1 + 960 a_1 b_1 \\ &\quad + (1440 a_1 + 18 a_2) T, \quad \mathbf{h}_4 = 60 T^7 - 1260 T^5 - 600 T^4 b_1 - 2700 T^3 + (-1245 b_1 - 15 b_2) T^2 \\ &\quad + (3600 a_1^2 - 1200 b_1^2 - 8100) T - 555 b_1 + 15 b_2, \quad \mathbf{h}_3 = -960 T^5 a_1 - 9600 T^2 a_1 b_1 \\ &\quad + (-9600 a_1 + 20 a_2) T^3 + 3320 a_1 b_1 + 40 a_1 b_2 - 40 a_2 b_1 + (14400 a_1 - 60 a_2) T, \\ \mathbf{h}_2 &= 30 T^9 - 360 T^7 + 10260 T^5 + (21465 b_1 - 45 b_2) T^4 + (-2400 a_1^2 + 7200 b_1^2 \\ &\quad - 37800) T^3 + 4800 a_1^2 b_1 + 4800 b_1^3 + (-14130 b_1 + 90 b_2) T^2 + (7200 a_1^2 + 7200 b_1^2 \\ &\quad + 28350) T + 22005 b_1 + 135 b_2, \quad \mathbf{h}_1 = -480 T^7 a_1 + 4800 T^4 a_1 b_1 + (10080 a_1 - 30 a_2) T^5 \\ &\quad + (21600 a_1 - 60 a_2) T^3 + (9960 a_1 b_1 + 120 a_1 b_2 - 120 a_2 b_1) T^2 + 4440 a_1 b_1 - 120 a_1 b_2 \\ &\quad + 120 a_2 b_1 + (9600 a_1^3 + 9600 a_1 b_1^2 + 64800 a_1 + 450 a_2) T, \quad \mathbf{h}_0 = 6 T^{11} + 150 T^9 + 180 T^8 b_1 \\ &\quad - 5220 T^7 + (101 b_1 + 7 b_2) T^6 + (1680 a_1^2 + 720 b_1^2 - 57780) T^5 + (-63975 b_1 + 75 b_2) T^4 \\ &\quad + (-12000 a_1^2 + 80 a_1 a_2 - 24560 b_1^2 + 80 b_1 b_2 - 14850) T^3 - 7760 a_1^2 b_1 + 80 a_1^2 b_2 \\ &\quad - 160 a_1 a_2 b_1 - 1840 b_1^3 - 80 b_1^2 b_2 + (-4800 a_1^2 b_1 - 4800 b_1^3 - 41085 b_1 - 495 b_2) T^2 \\ &\quad + (-25200 a_1^2 - 240 a_1 a_2 + a_2^2 - 14951 b_1^2 - 314 b_1 b_2 + b_2^2 + 28350) T + 11835 b_1 + 45 b_2 \end{aligned}$$

$$\begin{aligned}
& \mathbf{q}_{12} = 1, \quad \mathbf{q}_{11} = 0, \quad \mathbf{q}_{10} = 6T^2 + 6, \quad \mathbf{q}_9 = 40a_1, \quad \mathbf{q}_8 = 15T^4 - 90T^2 - 120Tb_1 \\
& + 135, \mathbf{q}_7 = -2a_2, \mathbf{q}_6 = 20T^6 - 180T^4 - 320T^3b_1 + 540T^2 + 240a_1^2 + 560b_1^2 \\
& + (350b_1 + 10b_2)T + 2340, \quad \mathbf{q}_5 = -240T^4a_1 + 1920Ta_1b_1 + (1440a_1 + 18a_2)T^2 \\
& - 2160a_1 + 18a_2, \mathbf{q}_4 = 15T^8 + 60T^6 - 240T^5b_1 - 1350T^4 + (-5630b_1 - 10b_2)T^3 \\
& + (3600a_1^2 - 1200b_1^2 + 13500)T^2 + 3600a_1^2 - 40a_1a_2 + 280b_1^2 - 40b_1b_2 \\
& + (3330b_1 - 90b_2)T + 3375, \quad \mathbf{q}_3 = -320T^6a_1 - 6400T^3a_1b_1 + (-14400a_1 \\
& + 10a_2)T^4 - 3200a_1^3 - 3200a_1b_1^2 + (-43200a_1 - 60a_2)T^2 + (-31760a_1b_1 \\
& + 80a_1b_2 - 80a_2b_1)T + 14400a_1 + 90a_2, \quad \mathbf{q}_2 = 6T^{10} + 270T^8 + 13500T^6 \\
& + (11466b_1 - 18b_2)T^5 + (-1200a_1^2 + 3600b_1^2 + 78300)T^4 + (114660b_1 \\
& - 180b_2)T^3 + (7200a_1^2 + 64800b_1^2 - 36450)T^2 - 10800a_1^2 + a_2^2 - 9431b_1^2 \\
& - 74b_1b_2 + b_2^2 + (9600a_1^2b_1 + 9600b_1^3 - 58950b_1 - 450b_2)T + 12150, \\
& \mathbf{q}_1 = -120T^8a_1 + 1920T^5a_1b_1 + (-480a_1 - 10a_2)T^6 + (10800a_1 - 270a_2)T^4 \\
& + (45040a_1b_1 + 80a_1b_2 - 80a_2b_1)T^3 + 9600a_1^3 + 160a_1^2a_2 - 2240a_1b_1^2 \\
& + 320a_1b_1b_2 - 160a_2b_1^2 + (9600a_1^3 + 9600a_1b_1^2 - 108000a_1 - 990a_2)T^2 \\
& + (-26640a_1b_1 + 720a_1b_2 - 720a_2b_1)T - 27000a_1 - 90a_2, \quad \mathbf{q}_0 = T^{12} + 126T^{10} \\
& + 40T^9b_1 + 3735T^8 + (2086b_1 + 2b_2)T^7 + (560a_1^2 + 240b_1^2 + 15300)T^6 \\
& + (-5214b_1 + 102b_2)T^5 + (3600a_1^2 + 40a_1a_2 - 12280b_1^2 + 40b_1b_2 + 143775)T^4 \\
& + 6400a_1^4 + 12800a_1^2b_1^2 + 6400b_1^4 + (-3200a_1^2b_1 - 3200b_1^3 + 179730b_1 \\
& - 90b_2)T^3 + (32400a_1^2 - 240a_1a_2 + a_2^2 + 100249b_1^2 - 314b_1b_2 + b_2^2 \\
& + 93150)T^2 + 39600a_1^2 + 360a_1a_2 + a_2^2 + 27649b_1^2 + 286b_1b_2 + b_2^2 + (22880a_1^2b_1 \\
& + 160a_1^2b_2 - 320a_1a_2b_1 + 34720b_1^3 - 160b_1^2b_2 + 96750b_1 + 450b_2)T + 2025
\end{aligned}$$

4.2 Plots of the solutions of NLS equation with four parameters

Conversely to the study with two parameters given in preceding works [10, 11, 13], we get other type of symmetries in the plots in the (x, t) plane, in particular we obtain beside already known circular shapes, triangular configurations. We give some examples of this fact in the following.

4.2.1 Peregrine breather of order 3

If we choose $\tilde{a}_1 = \tilde{b}_1 = \tilde{a}_2 = \tilde{b}_2 = 0$, we obtain the classical Peregrine breather

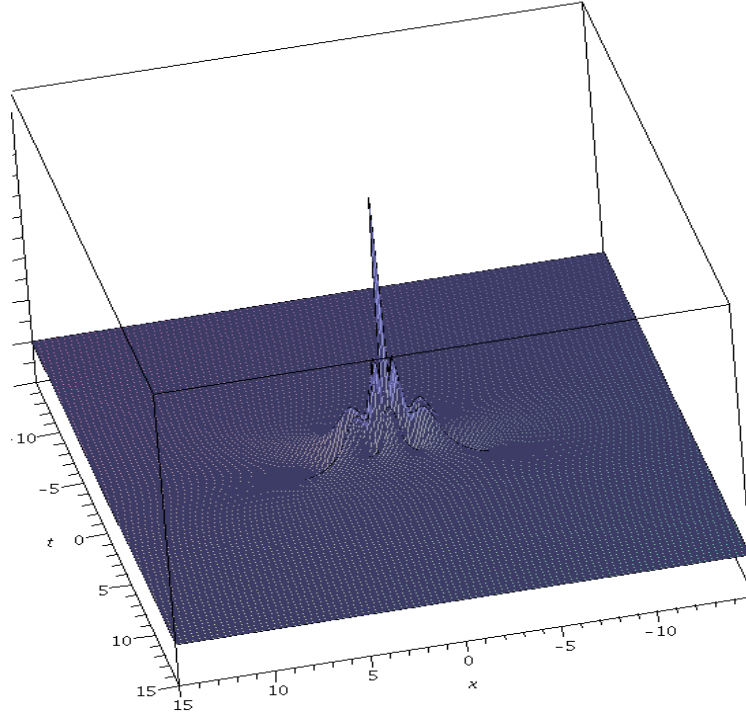


Figure 1: Solution of NLS, $N=3$, $\tilde{a}_1 = \tilde{b}_1 = \tilde{a}_2 = \tilde{b}_2 = 0$.

With other choices of parameters, we obtain all types of configurations : triangular with 6 peaks, circular with 6 peak, different cases with 1 until 6 peaks.

4.2.2 Variation of one parameter

If we choose $\tilde{a}_1 = 10^9$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 0$, $\tilde{b}_2 = 0$, we obtain :

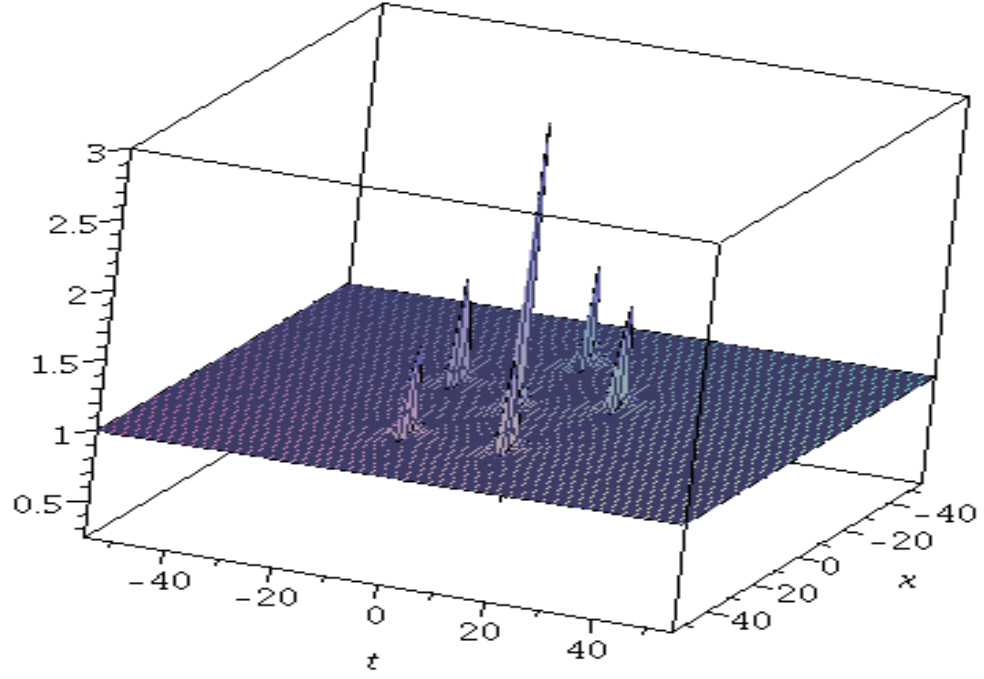


Figure 2: Solution of NLS, $N=3$, $\tilde{a}_1 = 10^9$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 0$, $\tilde{b}_2 = 0$.

If we choose $\tilde{a}_1 = 0$, $\tilde{b}_1 = 10^6$, $\tilde{a}_2 = 0$, $\tilde{b}_2 = 0$, we obtain :

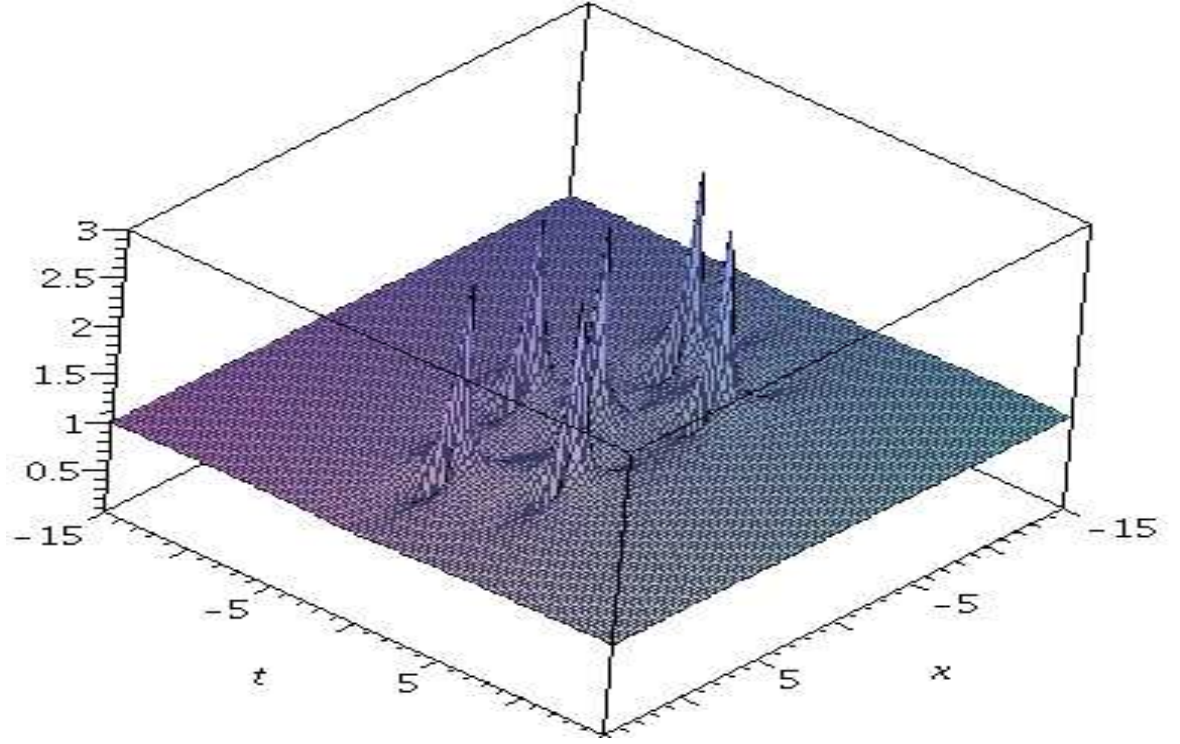


Figure 3: Solution of NLS, $N=3$, $\tilde{a}_1 = 0$, $\tilde{b}_1 = 10^6$, $\tilde{a}_2 = 0$, $\tilde{b}_2 = 0$.

If we choose $\tilde{a}_1 = 0$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 10^4$, $\tilde{b}_2 = 0$, we obtain :

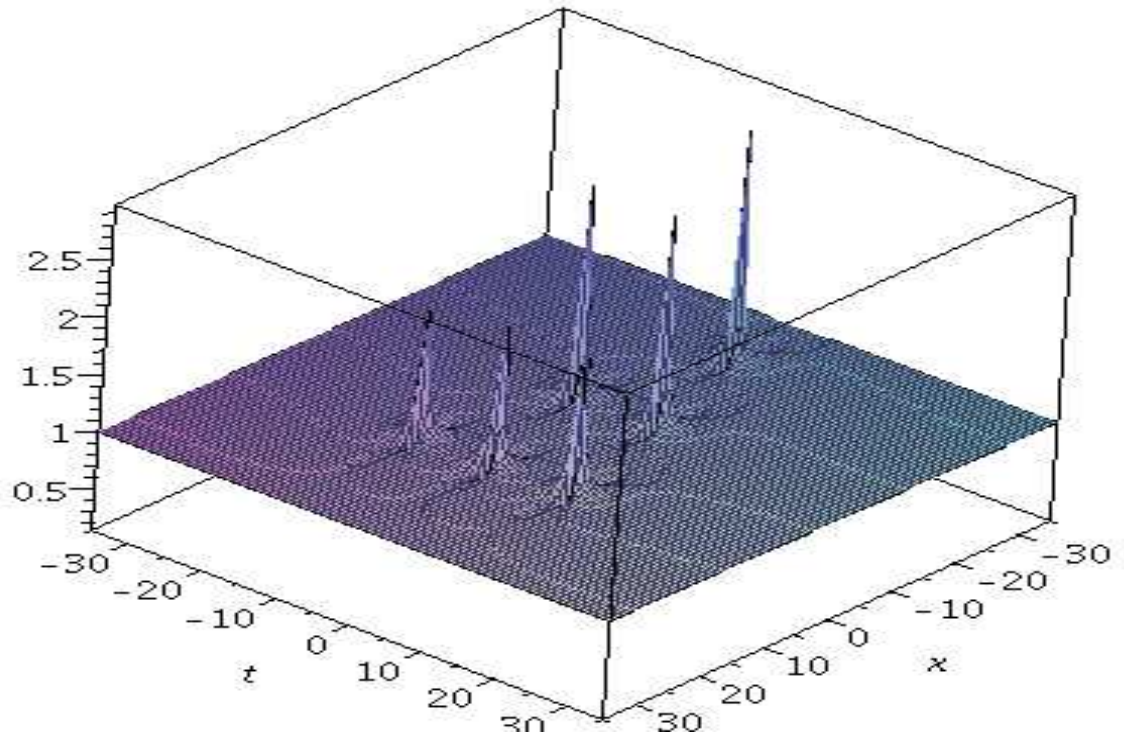


Figure 4: Solution of NLS, $N=3$, $\tilde{a}_1 = 0$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 10^4$, $\tilde{b}_2 = 0$, .

If we choose $\tilde{a}_1 = 0$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 0$, $\tilde{b}_2 = 10^5$, we obtain :

We obtain circular shapes in the case $a_2 = b_2 = 0$, and triangular configurations for $a_1 = b_1 = 0$. In the following we present the apparition of different configurations with 1 until 6 peaks.

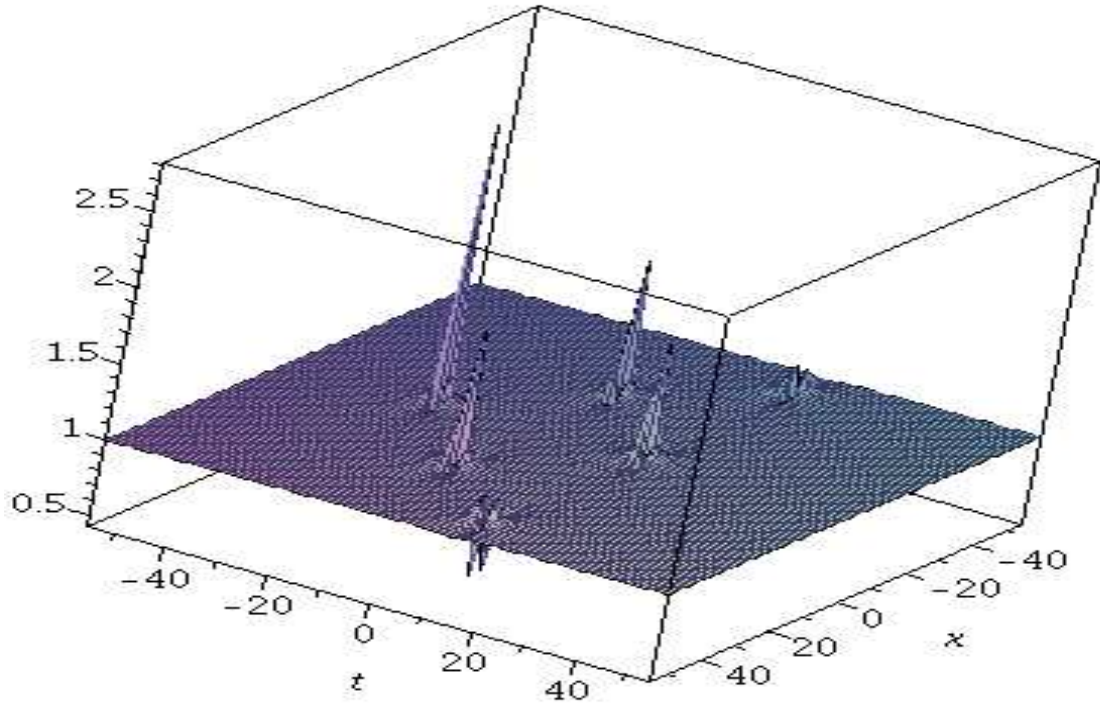


Figure 5: Solution of NLS, $N=3$, $\tilde{a}_1 = 0$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 10^5$.

4.2.3 Apparition of 1 until 6 peaks

If we choose $\tilde{a}_1 = 0$, $\tilde{b}_1 = 10^7$, $\tilde{a}_2 = 0$, $\tilde{b}_2 = 10^7$, we obtain :

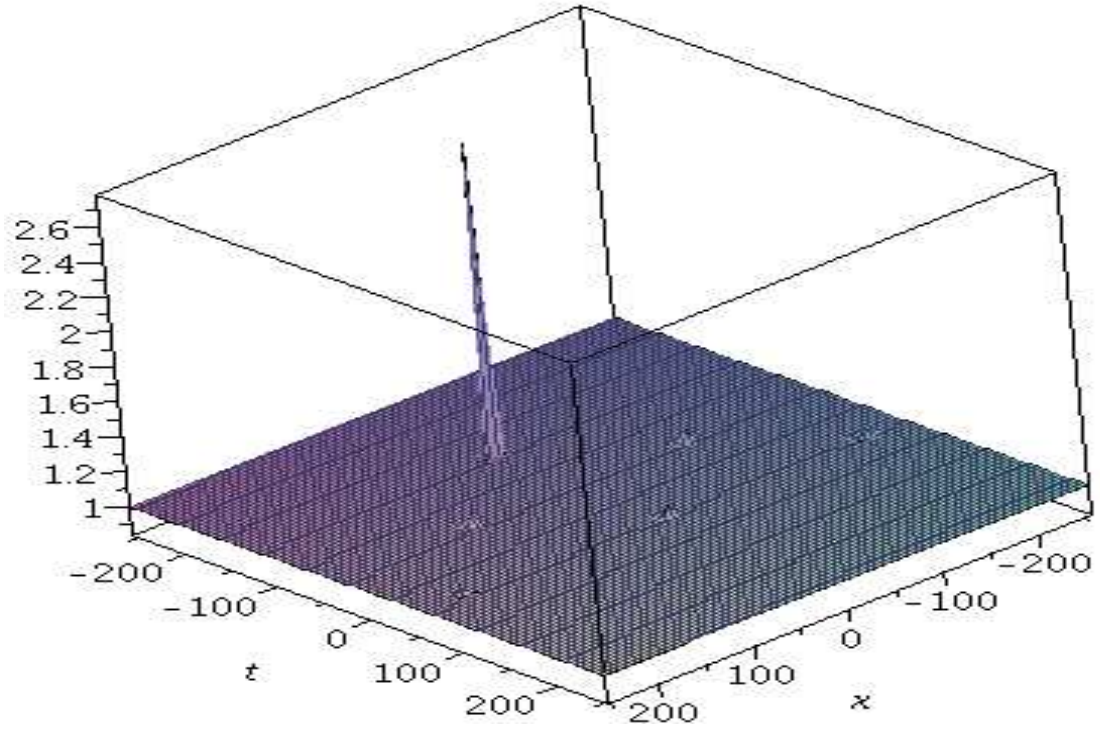


Figure 6: Solution of NLS, $N=3$, $\tilde{a}_1 = 0$, $\tilde{b}_1 = 10^7$, $\tilde{a}_2 = 0$, $\tilde{b}_2 = 10^7$.

If we choose $\tilde{a}_1 = 10^5$, $\tilde{b}_1 = 10^5$, $\tilde{a}_2 = 10^5$, $\tilde{b}_2 = 0$, we obtain :

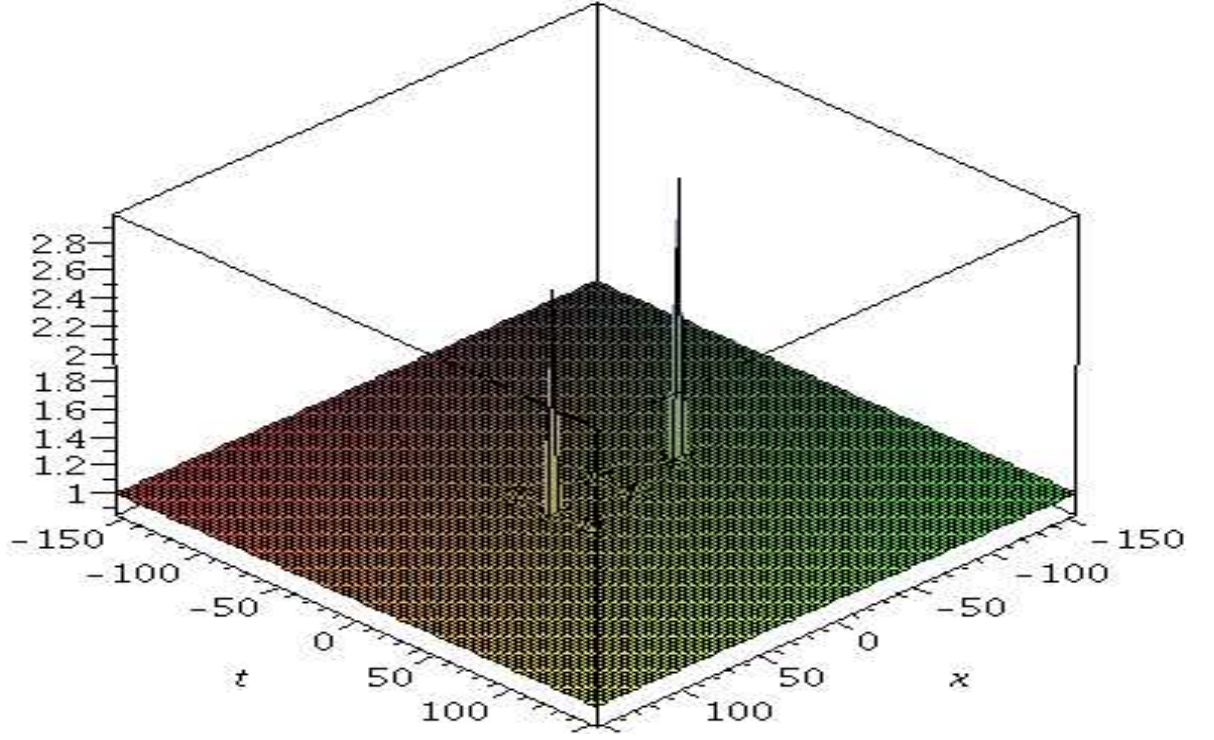


Figure 7: Solution of NLS, $N=3$, $\tilde{a}_1 = 10^5$, $\tilde{b}_1 = 10^5$, $\tilde{a}_2 = 10^5$, $\tilde{b}_2 = 0$.

If we choose $\tilde{a}_1 = 0$, $\tilde{b}_1 = 10^6$, $\tilde{a}_2 = 10^6$, $\tilde{b}_2 = 0$, we obtain :

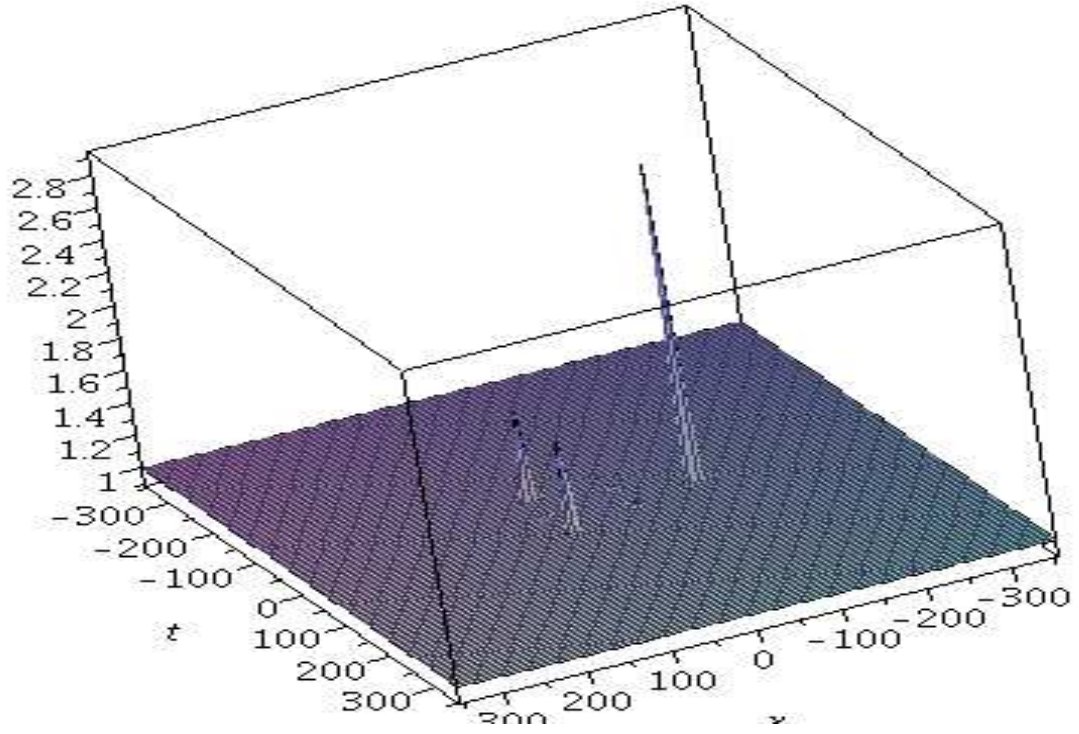


Figure 8: Solution of NLS, $N=3$, $\tilde{a}_1 = 0$, $\tilde{b}_1 = 10^6$, $\tilde{a}_2 = 10^6$, $\tilde{b}_2 = 0$.

If we choose $\tilde{a}_1 = 0$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 10^5$, $\tilde{b}_2 = 0$, we obtain :

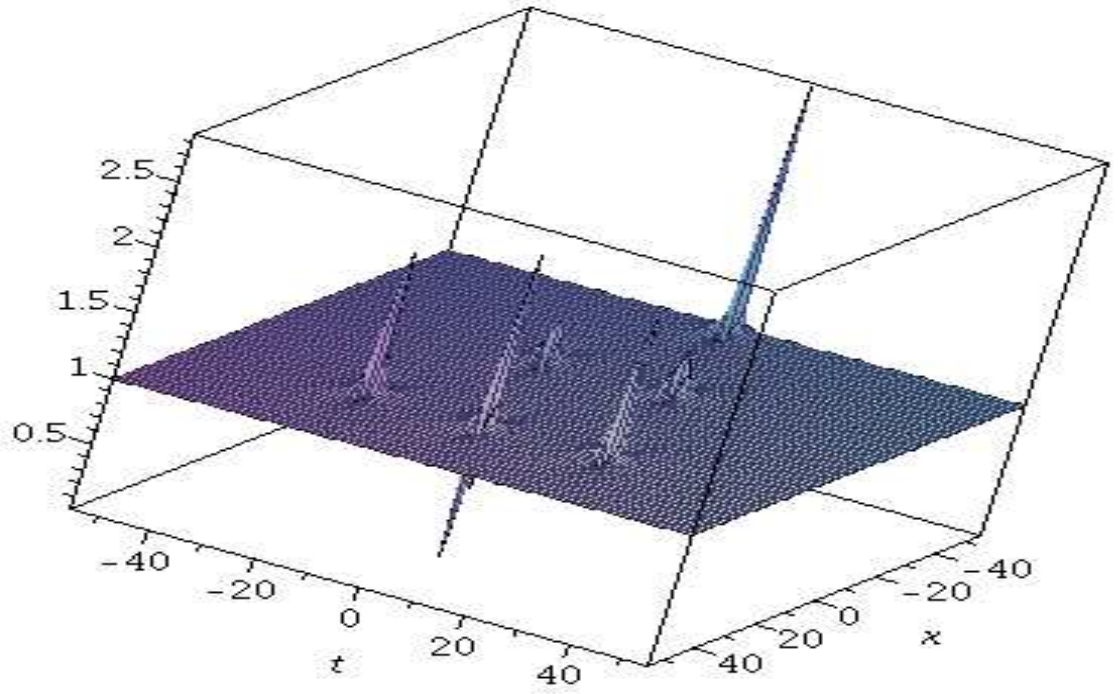


Figure 9: Solution of NLS, $N=3$, $\tilde{a}_1 = 0$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 10^5$, $\tilde{b}_2 = 0$.

If we choose $\tilde{a}_1 = 10^4$, $\tilde{b}_1 = 10^4$, $\tilde{a}_2 = 10^4$, $\tilde{b}_2 = 10^4$, we obtain :

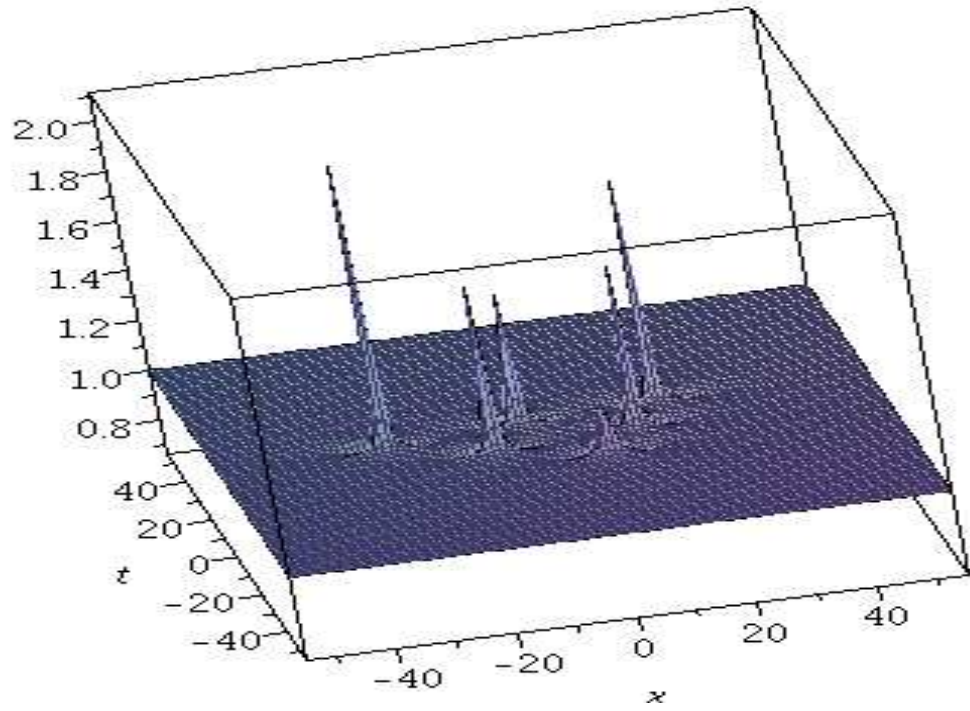


Figure 10: Solution of NLS, $N=3$, $\tilde{a}_1 = 10^4$, $\tilde{b}_1 = 10^4$, $\tilde{a}_2 = 10^4$, $\tilde{b}_2 = 10^4$.

If we choose $\tilde{a}_1 = 0$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 10^4$, $\tilde{b}_2 = 0$, we obtain :

4.2.4 Circular and triangular configurations

In general we obtain generically circular shapes in the case $a_2 = b_2 = 0$, and triangular configurations in the case $a_1 = b_1 = 0$. We present here some examples.

If we choose $\tilde{a}_1 = 0$, $\tilde{b}_1 = 10^7$, $\tilde{a}_2 = 0$, $\tilde{b}_2 = 0$, we obtain :

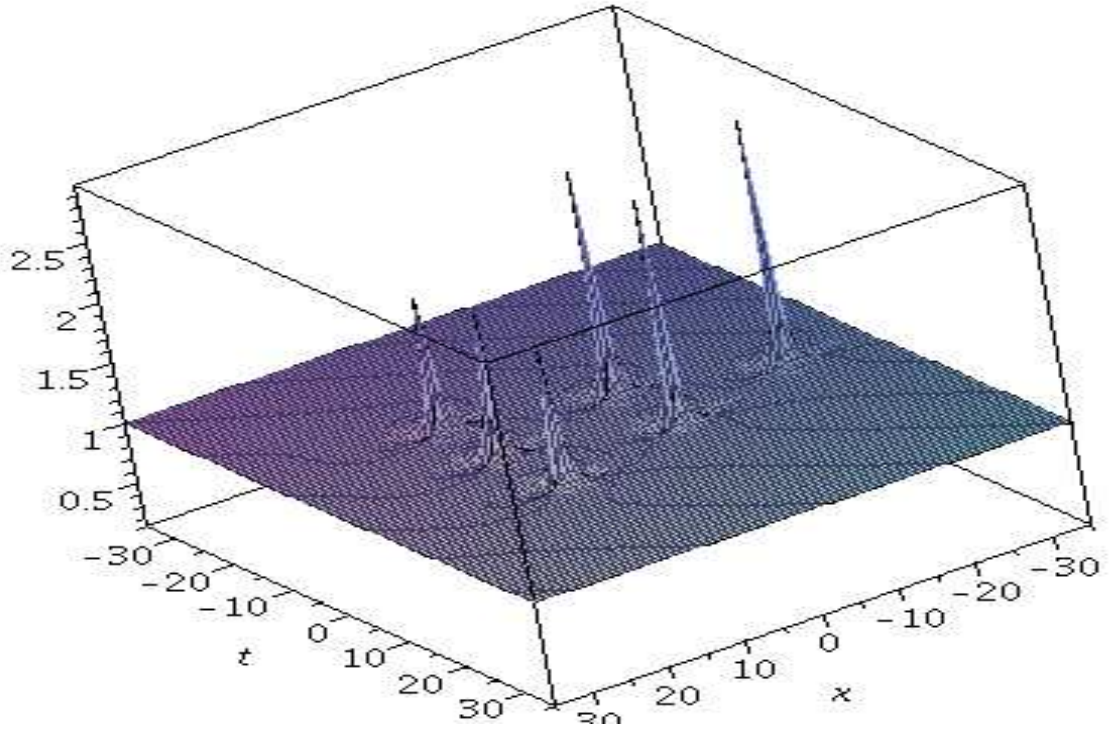


Figure 11: Solution of NLS, $N=3$, $\tilde{a}_1 = 0$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 10^4$, $\tilde{b}_2 = 0$.

If we choose $\tilde{a}_1 = 0$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 0$, $\tilde{b}_2 = 10^3$, we obtain :

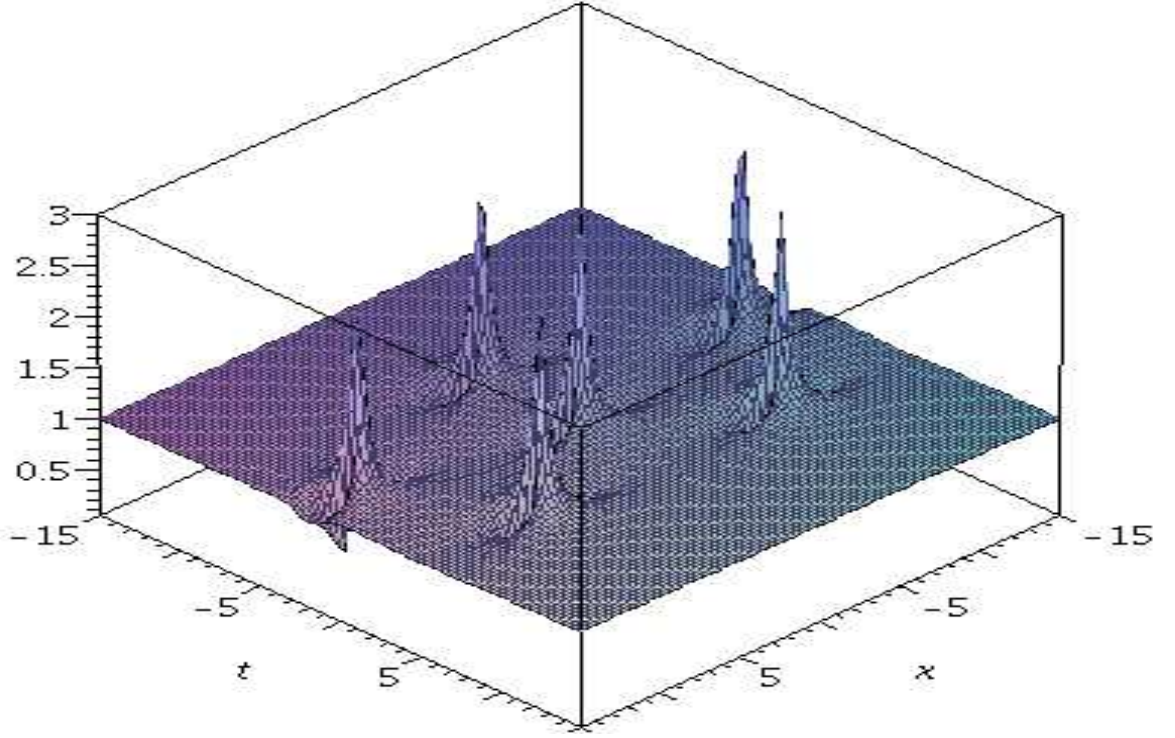


Figure 12: Solution of NLS, $N=3$, $\tilde{a}_1 = 0$, $\tilde{b}_1 = 10^7$, $\tilde{a}_2 = 0$, $\tilde{b}_2 = 0$.

5 Conclusion

The method described in the present paper provides a new tool to get explicitly solutions of the NLS equation.

The introduction of new parameters gives the appearance of new forms in conformity with those presented by Ohta and Yang [19], and also by Akhmediev et al. [17]. As it was already noted in previous studies with two parameters, one finds Peregrine breathers in the case where all the parameters are equal to 0.

We chose to present here the solutions of the NLS equation in the cases $N = 3$ only in order not to weigh down the text. We postpone the presentation of the higher orders in another publications.

Acknowledgments

I will never thank enough V.B. Matveev for having been introduced into the

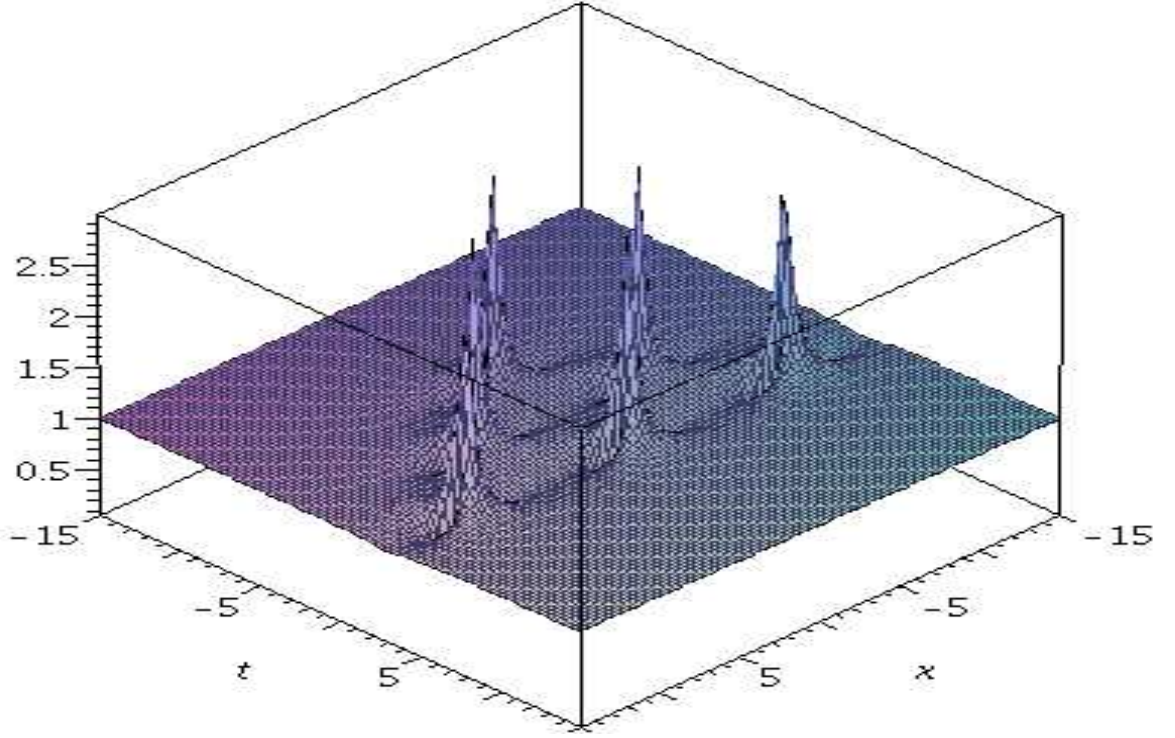


Figure 13: Solution of NLS, $N=3$, $\tilde{a}_1 = 0$, $\tilde{b}_1 = 0$, $\tilde{a}_2 = 0$, $\tilde{b}_2 = 10^3$.

universe of the nonlinear Schrödinger equation. I am very grateful to him for long fruitful discussions which we could have.

References

- [1] N. Akhmediev, A. Ankiewicz, J.M. Soto-Crespo, Rogue waves and rational solutions of nonlinear Schrödinger equation, *Physical Review E*, V. **80**, N. 026601, (2009).
- [2] N. Akhmediev, V. Eleonskii, N. Kulagin, Exact first order solutions of the nonlinear Schrödinger equation, *Th. Math. Phys.*, V. **72**, N. 2, 183-196, (1987).

- [3] N. Akhmediev, V. Eleonsky, N. Kulagin, Generation of periodic trains of picosecond pulses in an optical fiber : exact solutions, Sov. Phys. J.E.T.P., V. **62**, 894-899, (1985).
- [4] N. Akhmediev, A. Ankiewicz, P.A. Clarkson, Rogue waves, rational solutions, the patterns of their zeros and integral relations, J. Phys. A : Math. Theor., V. **43**, 122002, 1-9, (2010).
- [5] N. Akhmediev, A. Ankiewicz, D. J. Kedziora, Circular rogue wave clusters, Phys. Review E, V. **84**, 1-7, 2011
- [6] E.D. Belokolos, A.i. Bobenko, A.R. Its, V.Z. Enolskij and V.B. Matveev, Algebro-geometric approach to nonlinear integrable equations, Springer series in nonlinear dynamics, Springer Verlag, 1-360, (1994).
- [7] A. Chabchoub, H. Hoffmann, M. Onorato, N. Akhmediev, Super rogue waves : observation of a higher-order breather in water waves, Phys. Review X, V. **2**, 1-6, (2012).
- [8] P. Dubard, P. Gaillard, C. Klein, V.B. Matveev, On multi-rogue waves solutions of the NLS equation and positon solutions of the KdV equation, Eur. Phys. J. Special Topics, V. **185**, 247-258, (2010).
- [9] V. Eleonskii, I. Krichever, N. Kulagin, Rational multisoliton solutions to the NLS equation, Soviet Doklady 1986 sect. Math. Phys., V. **287**, 606-610, (1986).
- [10] P. Gaillard, Higher order Peregrine breathers and multi-rogue waves solutions of the NLS equation, halshs-00589556, 2011
- [11] P. Gaillard, Families of quasi-rational solutions of the NLS equation and multi-rogue waves, J. Phys. A : Meth. Theor., V. **44**, 1-15, 2011
- [12] P. Gaillard, Wronskian representation of solutions of the NLS equation and higher Peregrine breathers, Scientific Advances, V. **13**, N. 2, 71-153, 2012
- [13] P. Gaillard, Degenerate determinant representation of solution of the NLS equation, higher Peregrine breathers and multi-rogue waves, Jour. Of Math. Phys., V. **54**, 013504-1-32, 2013

- [14] B. Guo, L. Ling, Q.P. Liu, Nonlinear Schrodinger equation: Generalized Darboux transformation and rogue wave solutions, Phys. Rev. E, V. **85**, 026607, (2012).
- [15] A.R. Its, A.V. Rybin, M.A. Salle, Exact integration of nonlinear Schrödinger equation, Teore. i Mat. Fiz., V. **74.**, N. 1, 29-45, (1988).
- [16] A.R. Its, V.P. Kotlyarov, Explicit expressions for the solutions of nonlinear Schrödinger equation, Dokl. Akad. Nauk. SSSR, S. A, V. **965.**, N. 11, (1976).
- [17] D.J. Kedziora, A. Ankiewicz, N. Akhmediev, Triangular rogue waves, Phys. Review E, V. 86, 056602-1-9, 2012.
- [18] V.B. Matveev, M.A. Salle, Darboux transformations and solitons, Series in Nonlinear Dynamics, Springer Verlag, Berlin, (1991).
- [19] Y Ohta, J. Yang, General high-order rogue waves and their dynamics in the nonlinear Schrödinger equation, arXiv : 1110.5873 [nlin.SI] 26 Oct. 2011.
- [20] D. Peregrine, Water waves, nonlinear Schrödinger equations and their solutions, J. Austral. Math. Soc. Ser. B, V. **25**, 16-43, (1983).
- [21] V. E. Zakharov, Stability of periodic waves of finite amplitude on a surface of a deep fluid, J. Appl. Tech. Phys, V. **9**, 86-94, (1968)
- [22] V. E. Zakharov, A.B. Shabat Exact theory of two dimensional self focusing and one dimensional self modulation of waves in nonlinear media, Sov. Phys. JETP, V. **34**, 62-69, (1972)