

An adaptation of Nitsche's method to the Tresca friction problem

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Abstract

We propose a simple adaptation to the Tresca friction case of the Nitsche-based finite element method given in [Chouly-Hild, 2012, Chouly-Hild-Renard, 2013] for frictionless unilateral contact. Both cases of unilateral and bilateral contact with friction are taken into account, with emphasis on frictional unilateral contact for the numerical analysis. We manage to prove theoretically the fully optimal convergence rate of the method in the $H^1(\Omega)$ -norm which is $O(h^{\frac{1}{2}+\nu})$ when the solution lies in $H^{\frac{3}{2}+\nu}(\Omega)$, $0 < \nu \leq 1/2$, in two dimensions and three dimensions, for Lagrange piecewise linear and quadratic finite elements. No additional assumption on the friction set is needed to obtain this proof.

Keywords: Tresca friction problem, finite elements, Nitsche's method.

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1. Introduction

Contact problems with friction are widespread in engineering applications and are generally solved with the Finite Element Method (FEM) which approximates the corresponding variational inequalities (see e.g. [30, 21, 26, 22, 32, 40, 39]). In this paper we are interested in the Tresca problem, which involves a simplified friction law with a given friction threshold [16, 20, 30, 26]. This model problem is a first step towards more realistic situations involving Coulomb's friction, and serves for instance in fixed-point arguments when Coulomb's friction is studied. For the Tresca problem the obtention of optimal error estimates with the weakest additional assumptions in standard numerical approximations with FEM has revealed itself to be a challenging issue (see e.g. [6, 39]). Namely the Tresca problem can be recasted as a variational inequality of the second kind with non-linear non-differentiable integral terms on the friction interface (see the formulation (6)). These terms introduce in the error estimates some additional quantities that are difficult to bound optimally unless additional assumptions are stated (see e.g. [26, 39]).

To our knowledge the first estimates for this problem have been obtained by Haslinger and Hlaváček [25]: in 2D and for a mixed $\mathbb{P}_1/\mathbb{P}_0$ FEM approximation, with the assumption that the solution has a regularity $H^{1+\nu}(\Omega)$ (and the normal stress a regularity $L^2(\Gamma_C)$), they obtain a rate $O(h^{\min(\frac{1}{4}, \nu)})$ [25, Theorem 4.1]. For a regularity $H^2(\Omega)$ and with the additional assumption that

the set of transition points on the contact/friction boundary is finite, they manage to increase this rate to $O(h^{\frac{1}{2}})$ [25, Theorem 4.2] (see also [26]). Later, still in 2D, a rate of $O(h^{\frac{3}{4}})$ for a regularity $H^2(\Omega)$ is obtained by Hild in [29] for a mortar FEM. The same rate is given by Baillet and Sassi in [5] for a mixed $\mathbb{P}_1/\mathbb{P}_1$ (and $\mathbb{P}_1/\mathbb{P}_0$) FEM, under the same assumption of regularity $H^2(\Omega)$. The first results in the 3D case, still with a mixed $\mathbb{P}_1/\mathbb{P}_0$ FEM, are established by Haslinger and Sassi in [27]: a rate of $O(h^{\frac{1}{2}})$ is obtained for a regularity $H^2(\Omega)$. This rate was improved to $O(h^{\frac{3}{4}})$ with the additional assumption that the friction threshold is a constant [27, Remark 5.1]. Later in [6], a rate of $O(h^{\frac{3}{4}})$ is still mentioned by Baillet and Sassi for a mixed $\mathbb{P}_1/\mathbb{P}_1$ FEM in the 2D case [6, Theorem 4.4], but this result is improved to the quasi-optimal rate of $O(h|\log(h)|^{\frac{1}{4}})$ with additional regularity assumptions on the tangential displacement, the tangential Lagrange multiplier and the friction threshold, and also with the assumption that the number of transition points on the contact/friction zone is finite [6, Theorem 4.5]. The rate of $O(h^\nu)$ for a regularity $H^{1+\nu}(\Omega)$ ($0 \leq \nu \leq 1$) is obtained by Wohlmuth in the 2D case for a mixed low-order FEM, under technical assumptions on the contact/friction set [39, Theorem 4.9] (this is the first optimal bound to the best of our knowledge). In the 3D case the rate $O(h^{\min(\nu, \frac{1}{2})})$ is also given, without additional assumption [39, Theorem 4.10]. For the penalty method, the rate of $O(h^{\frac{1}{2} + \frac{\nu}{2} + \nu^2})$ for a regularity $H^{\frac{3}{2} + \nu}(\Omega)$ ($0 < \nu < \frac{1}{2}$) and the quasi-optimal rate of $O(h|\log h|^{\frac{1}{2}})$ for a regularity $H^2(\Omega)$ are established by Chouly and Hild in [11], without additional assumptions on the contact/friction set.

In previous works [10, 12] a new FEM based on Nitsche's formulation has been proposed and analysed in the case of frictionless unilateral contact. An extension to Coulomb's friction has been formulated in [36] and tested numerically using a generalized Newton algorithm. In this paper we detail how this Nitsche-based method can in fact be extended to the Tresca friction case. We also underline that the same attractive property of optimal convergence that arised in the frictionless case still holds here. More precisely we are able to prove the optimal convergence of this method in the $H^1(\Omega)$ -norm, of order $O(h^{\frac{1}{2} + \nu})$ provided the solution has a regularity $H^{\frac{3}{2} + \nu}(\Omega)$, $0 < \nu \leq 1/2$. To this purpose we do not need any additional assumption on the contact/friction zone, such as an increased regularity of the tangential components (displacement and stress) or a finite number of transitions points between contact and non-contact. The proof applies in two-dimensional and three-dimensional cases, and for piecewise linear and quadratic finite elements. The model problem we focus on is unilateral contact with Tresca friction but the results we obtain can be straightforwardly applied to bilateral (persistent) contact with Tresca friction.

The Nitsche method orginally proposed in [35, 3] aims at treating the boundary or interface conditions in a weak sense, thanks to a consistent penalty term. It differs in this aspect from standard penalization techniques which are generally non-consistent [30]. Moreover, no additional unknown (Lagrange multiplier) is needed and no discrete inf-condition must be fulfilled, contrarily to mixed methods (see, e.g., [26]). Most of the applications of Nitsche's method during the last decades have been focused on linear conditions on the boundary of a domain or at the interface between sub-domains: see, e.g., [37] for the Dirichlet problem, [7] for do-

main decomposition with non-matching meshes and [24] for a global review. In some recent works [23, 28] it has been adapted for bilateral (persistent) contact, which still includes linear boundary conditions on the contact zone. Remark in particular that an algorithm for unilateral contact which makes use of Nitsche's method in its original form is given and implemented in [23]. Furthermore an extension to large strain bilateral contact has been performed in [41]. Our paper is outlined as follows. In Section 2 we recall the formulation for the Tresca friction problem and introduce our Nitsche-based FEM. In Section 3 we carry out the numerical analysis of this method and we prove in particular its optimal convergence. Conclusion and perspectives are drawn in Section 4.

As usual, we denote by $(H^s(\cdot))^d$, $s \in \mathbb{R}$, $d \in \mathbb{N}^*$ the Sobolev spaces in d space dimensions (see [1]). The standard norm of $(H^s(D))^d$ is denoted by $\|\cdot\|_{s,D}$ and we keep the same notation for all the values of d . The letter C stands for a generic constant, independent of the discretization parameters.

2. Setting

2.1. The Tresca friction problem

We consider an elastic body whose reference configuration is represented by the domain Ω in \mathbb{R}^d with $d = 2$ or $d = 3$. Small strain assumption is made, as well as plane strain when $d = 2$. The boundary $\partial\Omega$ of Ω is polygonal or polyhedral and we partition $\partial\Omega$ in three nonoverlapping parts Γ_D , Γ_N and the contact/friction boundary Γ_C , with $\text{meas}(\Gamma_D) > 0$ and $\text{meas}(\Gamma_C) > 0$. The contact/friction boundary is supposed to be a straight line segment when $d = 2$ or a polygon when $d = 3$ to simplify. The normal unit outward vector on $\partial\Omega$ is denoted \mathbf{n} . The body is clamped on Γ_D for the sake of simplicity. It is subjected to volume forces $\mathbf{f} \in (L^2(\Omega))^d$ and to surface loads $\mathbf{g} \in (L^2(\Gamma_N))^d$.

The Tresca friction problem with unilateral contact consists in finding the displacement field $\mathbf{u} : \Omega \rightarrow \mathbb{R}^d$ verifying the equations and conditions (1)–(2)–(3):

$$\begin{aligned} \mathbf{div} \boldsymbol{\sigma}(\mathbf{u}) + \mathbf{f} &= \mathbf{0} && \text{in } \Omega, && \boldsymbol{\sigma}(\mathbf{u}) &= \mathbf{A} \boldsymbol{\varepsilon}(\mathbf{u}) && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{on } \Gamma_D, && \boldsymbol{\sigma}(\mathbf{u})\mathbf{n} &= \mathbf{g} && \text{on } \Gamma_N, \end{aligned} \quad (1)$$

where $\boldsymbol{\sigma} = (\sigma_{ij})$, $1 \leq i, j \leq d$, stands for the stress tensor field and \mathbf{div} denotes the divergence operator of tensor valued functions. The notation $\boldsymbol{\varepsilon}(\mathbf{v}) = (\nabla\mathbf{v} + \nabla\mathbf{v}^T)/2$ represents the linearized strain tensor field and \mathbf{A} is the fourth order symmetric elasticity tensor having the usual uniform ellipticity and boundedness property. For any displacement field \mathbf{v} and for any density of surface forces $\boldsymbol{\sigma}(\mathbf{v})\mathbf{n}$ defined on $\partial\Omega$ we adopt the following decomposition into normal and tangential components:

$$\mathbf{v} = v_n \mathbf{n} + \mathbf{v}_t \quad \text{and} \quad \boldsymbol{\sigma}(\mathbf{v})\mathbf{n} = \sigma_n(\mathbf{v})\mathbf{n} + \boldsymbol{\sigma}_t(\mathbf{v}).$$

The unilateral contact conditions on Γ_C are formulated as follows:

$$u_n \leq 0 \quad (i) \quad \sigma_n(\mathbf{u}) \leq 0 \quad (ii) \quad \sigma_n(\mathbf{u}) u_n = 0 \quad (iii) \quad (2)$$

Let $g \in L^2(\Gamma_C)$, $g \geq 0$ be a given threshold. The Tresca friction conditions on Γ_C read:

$$\begin{cases} |\boldsymbol{\sigma}_t(\mathbf{u})| \leq g, & \text{if } \mathbf{u}_t = \mathbf{0}, \quad (i) \\ \boldsymbol{\sigma}_t(\mathbf{u}) = -g \frac{\mathbf{u}_t}{|\mathbf{u}_t|} & \text{otherwise,} \quad (ii) \end{cases} \quad (3)$$

where $|\cdot|$ stands for the euclidean norm in \mathbb{R}^{d-1} . Note that conditions (i) and (ii) imply that $|\boldsymbol{\sigma}_t(\mathbf{u})| \leq g$ in all cases, and that if $|\boldsymbol{\sigma}_t(\mathbf{u})| < g$, we must have $\mathbf{u}_t = \mathbf{0}$.

Remark 2.1. *The case of bilateral contact with Tresca friction can also be considered, simply substituting to equations (2) the following one on Γ_C :*

$$u_n = 0. \quad (4)$$

Remark 2.2. *The conditions of Coulomb friction can be written similarly as:*

$$\begin{cases} |\boldsymbol{\sigma}_t(\mathbf{u})| \leq \nu_F |\sigma_n(\mathbf{u})|, & \text{if } \mathbf{u}_t = \mathbf{0}, \quad (i) \\ \boldsymbol{\sigma}_t(\mathbf{u}) = -\nu_F |\sigma_n(\mathbf{u})| \frac{\mathbf{u}_t}{|\mathbf{u}_t|} & \text{otherwise,} \quad (ii) \end{cases} \quad (5)$$

where ν_F is the friction coefficient. In the Tresca friction model is made the additional assumption that the amplitude of normal stress is known ($\nu_T |\sigma_n(\mathbf{u})| = g$) [30].

We introduce the Hilbert space \mathbf{V} and the convex cone \mathbf{K} of admissible displacements which satisfy the noninterpenetration on the contact zone Γ_C :

$$\mathbf{V} := \left\{ \mathbf{v} \in (H^1(\Omega))^d : \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D \right\}, \quad \mathbf{K} := \left\{ \mathbf{v} \in \mathbf{V} : v_n = \mathbf{v} \cdot \mathbf{n} \leq 0 \text{ on } \Gamma_C \right\}.$$

Define

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &:= \int_{\Omega} \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) \, d\Omega, & L(\mathbf{v}) &:= \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\Omega + \int_{\Gamma_N} \mathbf{g} \cdot \mathbf{v} \, d\Gamma, \\ j(\mathbf{v}) &:= \int_{\Gamma_C} g |\mathbf{v}_t| \, d\Gamma \end{aligned}$$

for any \mathbf{u} and \mathbf{v} in \mathbf{V} .

The weak formulation of Problem (1)-(3) as a variational inequality of the second kind is:

$$\begin{cases} \text{Find } \mathbf{u} \in \mathbf{K} \text{ such that:} \\ a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + j(\mathbf{v}) - j(\mathbf{u}) \geq L(\mathbf{v} - \mathbf{u}), \quad \forall \mathbf{v} \in \mathbf{K}. \end{cases} \quad (6)$$

which admits a unique solution according to [34].

Remark 2.3. *In the case of bilateral contact (condition (4)), the same weak formulation (6) holds, replacing simply the convex cone \mathbf{K} by the vectorial space:*

$$\mathbf{K}_b := \left\{ \mathbf{v} \in \mathbf{V} : v_n = 0 \text{ on } \Gamma_C \right\}.$$

2.2. The Nitsche-based finite element method

Let $\mathbf{V}^h \subset \mathbf{V}$ be a family of finite dimensional vector spaces (see [13, 17, 8]) indexed by h coming from a family \mathcal{T}^h of triangulations of the domain Ω ($h = \max_{T \in \mathcal{T}^h} h_T$ where h_T is the diameter of T). We suppose that the family of triangulations is regular, i.e., there exists $\sigma > 0$ such that $\forall T \in \mathcal{T}^h, h_T/\rho_T \leq \sigma$ where ρ_T denotes the radius of the inscribed ball in T . Furthermore we suppose that this family is conformal to the subdivision of the boundary into Γ_D, Γ_N and Γ_C (i.e., a face of an element $T \in \mathcal{T}^h$ is not allowed to have simultaneous non-empty intersection with more than one part of the subdivision). We choose a standard Lagrange finite element method of degree k with $k = 1$ or $k = 2$, i.e.:

$$\mathbf{V}^h := \left\{ \mathbf{v}^h \in (\mathcal{C}^0(\bar{\Omega}))^d : \mathbf{v}^h|_T \in (\mathbb{P}_k(T))^d, \forall T \in \mathcal{T}^h, \mathbf{v}^h = \mathbf{0} \text{ on } \Gamma_D \right\}. \quad (7)$$

However, the analysis would be similar for any \mathcal{C}^0 -conforming finite element method.

For any $\alpha \in \mathbb{R}^+$, we introduce the notation $[\cdot]_\alpha$ for the orthogonal projection onto $\mathcal{B}(0, \alpha) \subset \mathbb{R}^{d-1}$, where $\mathcal{B}(\mathbf{0}, \alpha)$ is the closed ball centered at the origin $\mathbf{0}$ and of radius α . This operation can be defined analytically, for $\mathbf{x} \in \mathbb{R}^{d-1}$ by:

$$[\mathbf{x}]_\alpha = \begin{cases} \mathbf{x} & \text{if } |\mathbf{x}| \leq \alpha, \\ \alpha \frac{\mathbf{x}}{|\mathbf{x}|} & \text{otherwise.} \end{cases}$$

We recall the following property, which is classical for projections, and that will be of fundamental importance in the sequel, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{d-1}$:

$$(\mathbf{y} - \mathbf{x}) \cdot ([\mathbf{y}]_\alpha - [\mathbf{x}]_\alpha) \geq |[\mathbf{y}]_\alpha - [\mathbf{x}]_\alpha|^2, \quad (8)$$

where \cdot is the euclidean scalar product in \mathbb{R}^{d-1} .

Let γ be a positive fonction defined on Γ_C . Observe now that the Tresca friction conditions (3) can be reformulated as follows (see, e.g., [2], or Appendix A):

$$\boldsymbol{\sigma}_t(\mathbf{u}) = -\frac{1}{\gamma} [\mathbf{u}_t - \gamma \boldsymbol{\sigma}_t(\mathbf{u})]_{\gamma g}. \quad (9)$$

As in [10, 12] (see also, e.g., [2]) we reformulate the contact conditions (2) as :

$$\sigma_n(\mathbf{u}) = -\frac{1}{\gamma} [u_n - \gamma \sigma_n(\mathbf{u})]_+, \quad (10)$$

where the notation $[\cdot]_+$ stands for the positive part of a scalar quantity ($[x]_+ = \frac{1}{2}(x + |x|)$ for $x \in \mathbb{R}$). Note that we have also, for $x, y \in \mathbb{R}$:

$$(y - x)([y]_+ - [x]_+) \geq ([y]_+ - [x]_+)^2. \quad (11)$$

We consider in what follows that γ is a positive piecewise constant function on the contact/friction interface Γ_C : for any $x \in \Gamma_C$, let T be an element such that $x \in T$ and set $\gamma(x) = \gamma_0 h_T$ where

γ_0 is a positive constant. Let now $\theta \in \mathbb{R}$ be a fixed parameter. Let us introduce the discrete linear operators

$$\mathbf{P}_\gamma^{\mathbf{t}} : \begin{array}{l} \mathbf{V}^h \rightarrow (L^2(\Gamma_C))^{d-1} \\ \mathbf{v}^h \mapsto \mathbf{v}_t^h - \gamma \boldsymbol{\sigma}_t(\mathbf{v}^h) \end{array}, \quad \text{and} \quad \mathbf{P}_\gamma^{\mathbf{n}} : \begin{array}{l} \mathbf{V}^h \rightarrow L^2(\Gamma_C) \\ \mathbf{v}^h \mapsto v_n^h - \gamma \sigma_n(\mathbf{v}^h) \end{array}.$$

Define also the bilinear form:

$$A_{\theta\gamma}(\mathbf{u}^h, \mathbf{v}^h) := a(\mathbf{u}^h, \mathbf{v}^h) - \int_{\Gamma_C} \theta\gamma \boldsymbol{\sigma}(\mathbf{u}^h)\mathbf{n} \cdot \boldsymbol{\sigma}(\mathbf{v}^h)\mathbf{n} \, d\Gamma.$$

Our Nitsche-based method for unilateral contact with Tresca friction then reads:

$$\left\{ \begin{array}{l} \text{Find } \mathbf{u}^h \in \mathbf{V}^h \text{ such that:} \\ A_{\theta\gamma}(\mathbf{u}^h, \mathbf{v}^h) + \int_{\Gamma_C} \frac{1}{\gamma} [\mathbf{P}_\gamma^{\mathbf{n}}(\mathbf{u}^h)]_+ \mathbf{P}_{\theta\gamma}^{\mathbf{n}}(\mathbf{v}^h) \, d\Gamma + \int_{\Gamma_C} \frac{1}{\gamma} [\mathbf{P}_\gamma^{\mathbf{t}}(\mathbf{u}^h)]_{\gamma g} \cdot \mathbf{P}_{\theta\gamma}^{\mathbf{t}}(\mathbf{v}^h) \, d\Gamma = L(\mathbf{v}^h), \\ \forall \mathbf{v}^h \in \mathbf{V}^h. \end{array} \right. \quad (12)$$

Remark 2.4. For bilateral contact with friction (equations (1)–(4)–(3)), the Nitsche-based formulation reads:

$$\left\{ \begin{array}{l} \text{Find } \mathbf{u}^h \in \mathbf{V}_b^h \text{ such that:} \\ A_{\theta\gamma}^b(\mathbf{u}^h, \mathbf{v}^h) + \int_{\Gamma_C} \frac{1}{\gamma} [\mathbf{P}_\gamma^{\mathbf{t}}(\mathbf{u}^h)]_{\gamma g} \cdot \mathbf{P}_{\theta\gamma}^{\mathbf{t}}(\mathbf{v}^h) \, d\Gamma = L(\mathbf{v}^h), \\ \forall \mathbf{v}^h \in \mathbf{V}_b^h, \end{array} \right. \quad (13)$$

where $A_{\theta\gamma}^b(\mathbf{u}^h, \mathbf{v}^h) := a(\mathbf{u}^h, \mathbf{v}^h) - \int_{\Gamma_C} \theta\gamma \boldsymbol{\sigma}_t(\mathbf{u}^h) \cdot \boldsymbol{\sigma}_t(\mathbf{v}^h) \, d\Gamma$ and $\mathbf{V}_b^h := \mathbf{V}^h \cap \mathbf{K}_b$.

Remark 2.5. As in [12] the parameter θ can be set to some interesting particular values, namely:

1. for $\theta = 1$ we recover a symmetric method for which the contact term is positive when $\mathbf{v}^h = \mathbf{u}^h$.
2. for $\theta = 0$ we recover a simple method close to penalty, which involves only a few terms and may be of easiest implementation.
3. for $\theta = -1$ the method admits one unique solution and converges optimally irrespectively of the value of $\gamma_0 > 0$.

3. Analysis of the Nitsche-based method

As for frictionless unilateral contact (see [10, 12]), our Nitsche-based formulation (12) for the Tresca problem is consistent. It is a direct consequence of reformulations (9) and (10) followed by integration-by-parts:

Lemma 3.1. *The Nitsche-based method for Tresca friction is consistent: suppose that the solution \mathbf{u} of (1)–(3) is in $(H^{\frac{3}{2}+\nu}(\Omega))^d$, with $\nu > 0$, then \mathbf{u} is also solution of*

$$A_{\theta\gamma}(\mathbf{u}, \mathbf{v}^h) + \int_{\Gamma_C} \frac{1}{\gamma} [\mathbf{P}_\gamma^{\mathbf{n}}(\mathbf{u})]_+ \mathbf{P}_{\theta\gamma}^{\mathbf{n}}(\mathbf{v}^h) d\Gamma + \int_{\Gamma_C} \frac{1}{\gamma} [\mathbf{P}_\gamma^{\mathbf{t}}(\mathbf{u})]_{\gamma g} \cdot \mathbf{P}_{\theta\gamma}^{\mathbf{t}}(\mathbf{v}^h) d\Gamma = L(\mathbf{v}^h), \quad \forall \mathbf{v}^h \in \mathbf{V}^h. \quad (14)$$

Note that for the same reasons the formulation (13) for bilateral contact is also consistent. In the following we establish the well-posedness and the optimal convergence of the method (12) for unilateral contact with friction. The results and the proofs can be adapted without difficulty in the case of bilateral contact with friction.

3.1. Well-posedness

We recall first the following classical property:

Lemma 3.2. *There exists $C > 0$, independent of the parameter γ_0 and of the mesh size h , such that:*

$$\|\gamma^{\frac{1}{2}} \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{v}^h)\|_{0,\Gamma_C}^2 + \|\gamma^{\frac{1}{2}} \sigma_n(\mathbf{v}^h)\|_{0,\Gamma_C}^2 \leq C\gamma_0 \|\mathbf{v}^h\|_{1,\Omega}^2, \quad (15)$$

for all $\mathbf{v}^h \in \mathbf{V}^h$.

Proof: It follows from the definitions of $\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{v}^h)$, $\sigma_n(\mathbf{v}^h)$ and the boundedness of \mathbf{A} that:

$$\|\gamma^{\frac{1}{2}} \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{v}^h)\|_{0,\Gamma_C}^2 + \|\gamma^{\frac{1}{2}} \sigma_n(\mathbf{v}^h)\|_{0,\Gamma_C}^2 \leq \gamma_0 h (\|\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{v}^h)\|_{0,\Gamma_C}^2 + \|\sigma_n(\mathbf{v}^h)\|_{0,\Gamma_C}^2) \leq C\gamma_0 h \|\nabla \mathbf{v}^h\|_{0,\Gamma_C}^2.$$

Then estimation (15) is obtained using a scaling argument: see [38, Lemma 2.1, p.24] for a detailed proof in the general case (for an arbitrary degree k and any dimension d). \square

To show that Problem (12) is well-posed we use an argument by Brezis for M-type and pseudo-monotone operators [9] (see also [33] and [31]).

Theorem 3.3. *Suppose that one of the two following assumptions hold:*

1. $\theta \neq -1$ and $\gamma_0 > 0$ is sufficiently small,
2. $\theta = -1$ and $\gamma_0 > 0$.

Then Problem (12) admits one unique solution \mathbf{u}^h in \mathbf{V}^h .

Proof: With help of the Riesz representation theorem, we introduce a (non-linear) operator $\mathbf{B}^h : \mathbf{V}^h \rightarrow \mathbf{V}^h$, by means of the formula:

$$(\mathbf{B}^h \mathbf{v}^h, \mathbf{w}^h)_{1,\Omega} := A_{\theta\gamma}(\mathbf{v}^h, \mathbf{w}^h) + \int_{\Gamma_C} \frac{1}{\gamma} [\mathbf{P}_\gamma^{\mathbf{n}}(\mathbf{v}^h)]_+ \mathbf{P}_{\theta\gamma}^{\mathbf{n}}(\mathbf{w}^h) d\Gamma + \int_{\Gamma_C} \frac{1}{\gamma} \left[\mathbf{P}_\gamma^{\mathbf{t}}(\mathbf{v}^h) \right]_{\gamma g} \cdot \mathbf{P}_{\theta\gamma}^{\mathbf{t}}(\mathbf{w}^h) d\Gamma,$$

for all $\mathbf{v}^h, \mathbf{w}^h \in \mathbf{V}^h$, and where $(\cdot, \cdot)_{1,\Omega}$ stands for the scalar product in $(H^1(\Omega))^d$. Then Problem (12) is well-posed if and only if \mathbf{B}^h is a bijective operator.

For $\mathbf{v}^h, \mathbf{w}^h \in \mathbf{V}^h$ we first use the decomposition $\mathbf{P}_{\theta\gamma}^{\mathbf{n}}(\cdot) = \mathbf{P}_\gamma^{\mathbf{n}}(\cdot) + (1 - \theta)\sigma_n(\cdot)$ (and the same for $\mathbf{P}_{\theta\gamma}^{\mathbf{t}}$) so that:

$$\begin{aligned} & (\mathbf{B}^h \mathbf{v}^h - \mathbf{B}^h \mathbf{w}^h, \mathbf{v}^h - \mathbf{w}^h)_{1,\Omega} \\ &= a(\mathbf{v}^h - \mathbf{w}^h, \mathbf{v}^h - \mathbf{w}^h) - \theta \|\gamma^{\frac{1}{2}} \boldsymbol{\sigma}(\mathbf{v}^h - \mathbf{w}^h) \mathbf{n}\|_{0,\Gamma_C}^2 \\ & \quad + \int_{\Gamma_C} \frac{1}{\gamma} \left([\mathbf{P}_\gamma^{\mathbf{n}}(\mathbf{v}^h)]_+ - [\mathbf{P}_\gamma^{\mathbf{n}}(\mathbf{w}^h)]_+ \right) \mathbf{P}_\gamma^{\mathbf{n}}(\mathbf{v}^h - \mathbf{w}^h) d\Gamma \\ & \quad + \int_{\Gamma_C} \frac{1}{\gamma} \left(\left[\mathbf{P}_\gamma^{\mathbf{t}}(\mathbf{v}^h) \right]_{\gamma g} - \left[\mathbf{P}_\gamma^{\mathbf{t}}(\mathbf{w}^h) \right]_{\gamma g} \right) \cdot \mathbf{P}_\gamma^{\mathbf{t}}(\mathbf{v}^h - \mathbf{w}^h) d\Gamma \tag{16} \\ & \quad + (1 - \theta) \int_{\Gamma_C} \frac{1}{\gamma} \left([\mathbf{P}_\gamma^{\mathbf{n}}(\mathbf{v}^h)]_+ - [\mathbf{P}_\gamma^{\mathbf{n}}(\mathbf{w}^h)]_+ \right) \gamma \sigma_n(\mathbf{v}^h - \mathbf{w}^h) d\Gamma \\ & \quad + (1 - \theta) \int_{\Gamma_C} \frac{1}{\gamma} \left(\left[\mathbf{P}_\gamma^{\mathbf{t}}(\mathbf{v}^h) \right]_{\gamma g} - \left[\mathbf{P}_\gamma^{\mathbf{t}}(\mathbf{w}^h) \right]_{\gamma g} \right) \cdot \gamma \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{v}^h - \mathbf{w}^h) d\Gamma. \end{aligned}$$

With help of inequalities (8) and (11) followed by Cauchy-Schwarz inequality, we get from (16):

$$\begin{aligned} & (\mathbf{B}^h \mathbf{v}^h - \mathbf{B}^h \mathbf{w}^h, \mathbf{v}^h - \mathbf{w}^h)_{1,\Omega} \\ & \geq a(\mathbf{v}^h - \mathbf{w}^h, \mathbf{v}^h - \mathbf{w}^h) - \theta \|\gamma^{\frac{1}{2}} \boldsymbol{\sigma}(\mathbf{v}^h - \mathbf{w}^h) \mathbf{n}\|_{0,\Gamma_C}^2 \\ & \quad + \|\gamma^{-\frac{1}{2}} ([\mathbf{P}_\gamma^{\mathbf{n}}(\mathbf{v}^h)]_+ - [\mathbf{P}_\gamma^{\mathbf{n}}(\mathbf{w}^h)]_+) \|_{0,\Gamma_C}^2 + \|\gamma^{-\frac{1}{2}} (\left[\mathbf{P}_\gamma^{\mathbf{t}}(\mathbf{v}^h) \right]_{\gamma g} - \left[\mathbf{P}_\gamma^{\mathbf{t}}(\mathbf{w}^h) \right]_{\gamma g}) \|_{0,\Gamma_C}^2 \tag{17} \\ & \quad - |1 - \theta| \|\gamma^{-\frac{1}{2}} ([\mathbf{P}_\gamma^{\mathbf{n}}(\mathbf{v}^h)]_+ - [\mathbf{P}_\gamma^{\mathbf{n}}(\mathbf{w}^h)]_+) \|_{0,\Gamma_C} \|\gamma^{\frac{1}{2}} \sigma_n(\mathbf{v}^h - \mathbf{w}^h) \|_{0,\Gamma_C} \\ & \quad - |1 - \theta| \|\gamma^{-\frac{1}{2}} (\left[\mathbf{P}_\gamma^{\mathbf{t}}(\mathbf{v}^h) \right]_{\gamma g} - \left[\mathbf{P}_\gamma^{\mathbf{t}}(\mathbf{w}^h) \right]_{\gamma g}) \|_{0,\Gamma_C} \|\gamma^{\frac{1}{2}} \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{v}^h - \mathbf{w}^h) \|_{0,\Gamma_C}. \end{aligned}$$

If $\theta = 1$ (symmetric case) we use the coercivity of $a(\cdot, \cdot)$ and the property (15) in the previous expression (17). Therefore:

$$\begin{aligned} & (\mathbf{B}^h \mathbf{v}^h - \mathbf{B}^h \mathbf{w}^h, \mathbf{v}^h - \mathbf{w}^h)_{1,\Omega} \\ & \geq a(\mathbf{v}^h - \mathbf{w}^h, \mathbf{v}^h - \mathbf{w}^h) - \|\gamma^{\frac{1}{2}} \sigma_n(\mathbf{v}^h - \mathbf{w}^h) \|_{0,\Gamma_C}^2 - \|\gamma^{\frac{1}{2}} \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{v}^h - \mathbf{w}^h) \|_{0,\Gamma_C}^2 \tag{18} \\ & \geq C \|\mathbf{v}^h - \mathbf{w}^h\|_{1,\Omega}^2, \end{aligned}$$

when γ_0 is chosen sufficiently small.

We now suppose that $\theta \neq 1$ and take $\beta > 0$. Applying Young inequality in (17) yields:

$$\begin{aligned}
& (\mathbf{B}^h \mathbf{v}^h - \mathbf{B}^h \mathbf{w}^h, \mathbf{v}^h - \mathbf{w}^h)_{1,\Omega} \\
& \geq a(\mathbf{v}^h - \mathbf{w}^h, \mathbf{v}^h - \mathbf{w}^h) - \left(\theta + \frac{|1-\theta|\beta}{2} \right) \|\gamma^{\frac{1}{2}} \boldsymbol{\sigma}(\mathbf{v}^h - \mathbf{w}^h) \mathbf{n}\|_{0,\Gamma_C}^2 \\
& \quad + \left(1 - \frac{|1-\theta|}{2\beta} \right) \left(\|\gamma^{-\frac{1}{2}}([\mathbf{P}_\gamma^{\mathbf{n}}(\mathbf{v}^h)]_+ - [\mathbf{P}_\gamma^{\mathbf{n}}(\mathbf{w}^h)]_+)\|_{0,\Gamma_C}^2 \right. \\
& \quad \left. + \|\gamma^{-\frac{1}{2}}([\mathbf{P}_\gamma^{\mathbf{t}}(\mathbf{v}^h)]_{\gamma g} - [\mathbf{P}_\gamma^{\mathbf{t}}(\mathbf{w}^h)]_{\gamma g})\|_{0,\Gamma_C}^2 \right). \tag{19}
\end{aligned}$$

Choosing $\beta = |1 - \theta|/2$ in (19), we get:

$$\begin{aligned}
(\mathbf{B}^h \mathbf{v}^h - \mathbf{B}^h \mathbf{w}^h, \mathbf{v}^h - \mathbf{w}^h)_{1,\Omega} & \geq a(\mathbf{v}^h - \mathbf{w}^h, \mathbf{v}^h - \mathbf{w}^h) - \frac{1}{4} (1 + \theta)^2 \|\gamma^{\frac{1}{2}} \boldsymbol{\sigma}(\mathbf{v}^h - \mathbf{w}^h) \mathbf{n}\|_{0,\Gamma_C}^2 \\
& \geq C \|\mathbf{v}^h - \mathbf{w}^h\|_{1,\Omega}^2, \tag{20}
\end{aligned}$$

either for $\theta \neq -1$ and γ_0 sufficiently small, or when $\theta = -1$ and $\gamma_0 > 0$ (γ_0 does not need to be small in this case).

Next, let us show that \mathbf{B}^h is also hemicontinuous i.e.

$$[0, 1] \ni t \mapsto \varphi(t) := (\mathbf{B}^h(\mathbf{v}^h - t\mathbf{w}^h), \mathbf{w}^h)_{1,\Omega} \in \mathbb{R}$$

is a continuous real function, for all $\mathbf{v}^h, \mathbf{w}^h \in \mathbf{V}^h$ (remark that \mathbf{V}^h is a vector space). Let $s, t \in [0, 1]$, with help of the bounds $|[x]_+ - [y]_+| \leq |x - y|$, for all $x, y \in \mathbb{R}$, $|[\mathbf{x}]_{\gamma g} - [\mathbf{y}]_{\gamma g}| \leq |\mathbf{x} - \mathbf{y}|$, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{d-1}$, and next using the linearity of $\mathbf{P}_\gamma^{\mathbf{n}}$ and $\mathbf{P}_\gamma^{\mathbf{t}}$ we obtain:

$$\begin{aligned}
& |\varphi(t) - \varphi(s)| \\
& \leq |t - s| \left(A_{\theta\gamma}(\mathbf{w}^h, \mathbf{w}^h) + \int_{\Gamma_C} \frac{1}{\gamma} |\mathbf{P}_\gamma^{\mathbf{n}}(\mathbf{w}^h)| |\mathbf{P}_{\theta\gamma}^{\mathbf{n}}(\mathbf{w}^h)| d\Gamma + \int_{\Gamma_C} \frac{1}{\gamma} |\mathbf{P}_\gamma^{\mathbf{t}}(\mathbf{w}^h)| |\mathbf{P}_{\theta\gamma}^{\mathbf{t}}(\mathbf{w}^h)| d\Gamma \right),
\end{aligned}$$

which means that φ is Lipschitz, so that \mathbf{B}^h is hemicontinuous. Since properties (18), (20) also hold, we finally apply the Corollary 15 (p.126) of [9] to conclude that \mathbf{B}^h is a bijective operator. This ends the proof. \square

3.2. Error analysis

First we establish an abstract error estimate.

Theorem 3.4. *Suppose that the solution \mathbf{u} of Problem (6) belongs to $(H^{\frac{3}{2}+\nu}(\Omega))^d$ with $\nu > 0$ and $d = 2$ or $d = 3$.*

1. *Let $\theta \in \mathbb{R}$. Suppose that the parameter $\gamma_0 > 0$ is sufficiently small. Then the solution \mathbf{u}^h of Problem (12) satisfies the following abstract error estimate:*

$$\begin{aligned}
& \|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} + \|\gamma^{\frac{1}{2}}(\sigma_n(\mathbf{u}) + \frac{1}{\gamma}[\mathbf{P}_\gamma^{\mathbf{n}}(\mathbf{u}^h)]_+)\|_{0,\Gamma_C} + \|\gamma^{\frac{1}{2}}(\boldsymbol{\sigma}_t(\mathbf{u}) + \frac{1}{\gamma}[\mathbf{P}_\gamma^{\mathbf{t}}(\mathbf{u}^h)]_{\gamma g})\|_{0,\Gamma_C} \\
& \leq C \inf_{\mathbf{v}^h \in \mathbf{V}^h} \left(\|\mathbf{u} - \mathbf{v}^h\|_{1,\Omega} + \|\gamma^{-\frac{1}{2}}(\mathbf{u} - \mathbf{v}^h)\|_{0,\Gamma_C} + \|\gamma^{\frac{1}{2}} \boldsymbol{\sigma}(\mathbf{u} - \mathbf{v}^h) \mathbf{n}\|_{0,\Gamma_C} \right), \tag{21}
\end{aligned}$$

where C is a positive constant, independent of h , \mathbf{u} and γ_0 .

2. Set $\theta = -1$. Then for all values of $\gamma_0 > 0$, the solution \mathbf{u}^h of Problem (12) satisfies the abstract error estimate (21) where C is a positive constant, dependent of γ_0 but independent of h and \mathbf{u} .

Proof: Let $\mathbf{v}^h \in \mathbf{V}^h$. We first use the \mathbf{V} -ellipticity and the continuity of $a(\cdot, \cdot)$, as well as Young's inequality, to obtain:

$$\frac{\alpha}{2} \|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega}^2 \leq \frac{C^2}{2\alpha} \|\mathbf{u} - \mathbf{v}^h\|_{1,\Omega}^2 + a(\mathbf{u}, \mathbf{v}^h - \mathbf{u}^h) - a(\mathbf{u}^h, \mathbf{v}^h - \mathbf{u}^h),$$

where $\alpha > 0$ is the ellipticity constant of $a(\cdot, \cdot)$. We transform the term $a(\mathbf{u}, \mathbf{v}^h - \mathbf{u}^h) - a(\mathbf{u}^h, \mathbf{v}^h - \mathbf{u}^h)$ since \mathbf{u} solves (6), \mathbf{u}^h solves (12) and using Lemma 3.1, which yields:

$$\begin{aligned} \frac{\alpha}{2} \|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega}^2 &\leq \frac{C^2}{2\alpha} \|\mathbf{u} - \mathbf{v}^h\|_{1,\Omega}^2 - \theta \int_{\Gamma_C} \gamma \boldsymbol{\sigma}(\mathbf{u}^h - \mathbf{u}) \mathbf{n} \cdot \boldsymbol{\sigma}(\mathbf{v}^h - \mathbf{u}^h) \mathbf{n} \, d\Gamma \\ &\quad + \int_{\Gamma_C} \frac{1}{\gamma} \left([\mathbf{P}_\gamma^t(\mathbf{u}^h)]_{\gamma g} - [\mathbf{P}_\gamma^t(\mathbf{u})]_{\gamma g} \right) \cdot \mathbf{P}_{\theta\gamma}^t(\mathbf{v}^h - \mathbf{u}^h) \, d\Gamma \\ &\quad + \int_{\Gamma_C} \frac{1}{\gamma} \left([\mathbf{P}_\gamma^n(\mathbf{u}^h)]_+ - [\mathbf{P}_\gamma^n(\mathbf{u})]_+ \right) \mathbf{P}_{\theta\gamma}^n(\mathbf{v}^h - \mathbf{u}^h) \, d\Gamma. \end{aligned} \quad (22)$$

The first integral term in (22) is bounded as follows:

$$\begin{aligned} & - \theta \int_{\Gamma_C} \gamma \boldsymbol{\sigma}(\mathbf{u}^h - \mathbf{u}) \mathbf{n} \cdot \boldsymbol{\sigma}(\mathbf{v}^h - \mathbf{u}^h) \mathbf{n} \, d\Gamma \\ &= \theta \int_{\Gamma_C} \gamma \boldsymbol{\sigma}(\mathbf{v}^h - \mathbf{u}^h) \mathbf{n} \cdot \boldsymbol{\sigma}(\mathbf{v}^h - \mathbf{u}^h) \mathbf{n} \, d\Gamma - \theta \int_{\Gamma_C} \gamma \boldsymbol{\sigma}(\mathbf{v}^h - \mathbf{u}) \mathbf{n} \cdot \boldsymbol{\sigma}(\mathbf{v}^h - \mathbf{u}^h) \mathbf{n} \, d\Gamma \\ &\leq \theta \|\gamma^{\frac{1}{2}} \boldsymbol{\sigma}(\mathbf{v}^h - \mathbf{u}^h) \mathbf{n}\|_{0,\Gamma_C}^2 + |\theta| \|\gamma^{\frac{1}{2}} \boldsymbol{\sigma}(\mathbf{v}^h - \mathbf{u}) \mathbf{n}\|_{0,\Gamma_C} \|\gamma^{\frac{1}{2}} \boldsymbol{\sigma}(\mathbf{v}^h - \mathbf{u}^h) \mathbf{n}\|_{0,\Gamma_C} \\ &\leq \frac{\beta_1 \theta^2}{2} \|\gamma^{\frac{1}{2}} \boldsymbol{\sigma}(\mathbf{v}^h - \mathbf{u}) \mathbf{n}\|_{0,\Gamma_C}^2 + \left(\frac{1}{2\beta_1} + \theta \right) \|\gamma^{\frac{1}{2}} \boldsymbol{\sigma}(\mathbf{v}^h - \mathbf{u}^h) \mathbf{n}\|_{0,\Gamma_C}^2, \end{aligned} \quad (23)$$

with $\beta_1 > 0$. The second integral term in (22) is considered next:

$$\begin{aligned} & \int_{\Gamma_C} \frac{1}{\gamma} \left([\mathbf{P}_\gamma^t(\mathbf{u}^h)]_{\gamma g} - [\mathbf{P}_\gamma^t(\mathbf{u})]_{\gamma g} \right) \cdot \mathbf{P}_{\theta\gamma}^t(\mathbf{v}^h - \mathbf{u}^h) \, d\Gamma \\ &= \int_{\Gamma_C} \frac{1}{\gamma} \left([\mathbf{P}_\gamma^t(\mathbf{u}^h)]_{\gamma g} - [\mathbf{P}_\gamma^t(\mathbf{u})]_{\gamma g} \right) \mathbf{P}_\gamma^t(\mathbf{v}^h - \mathbf{u}) \, d\Gamma \\ &\quad + \int_{\Gamma_C} \frac{1}{\gamma} \left([\mathbf{P}_\gamma^t(\mathbf{u}^h)]_{\gamma g} - [\mathbf{P}_\gamma^t(\mathbf{u})]_{\gamma g} \right) \mathbf{P}_\gamma^t(\mathbf{u} - \mathbf{u}^h) \, d\Gamma \\ &\quad + \int_{\Gamma_C} \frac{1}{\gamma} \left([\mathbf{P}_\gamma^t(\mathbf{u}^h)]_{\gamma g} - [\mathbf{P}_\gamma^t(\mathbf{u})]_{\gamma g} \right) \cdot \gamma(1 - \theta) \boldsymbol{\sigma}_t(\mathbf{v}^h - \mathbf{u}^h) \, d\Gamma. \end{aligned} \quad (24)$$

With help of the bound (8) and applying two times Cauchy-Schwarz and Young's inequalities in (24) we obtain

$$\begin{aligned}
& \int_{\Gamma_C} \frac{1}{\gamma} \left(\left[\mathbf{P}_\gamma^{\mathbf{t}}(\mathbf{u}^h) \right]_{\gamma g} - \left[\mathbf{P}_\gamma^{\mathbf{t}}(\mathbf{u}) \right]_{\gamma g} \right) \cdot \mathbf{P}_{\theta\gamma}^{\mathbf{t}}(\mathbf{v}^h - \mathbf{u}^h) d\Gamma \\
& \leq \frac{1}{2\beta_2} \|\gamma^{\frac{1}{2}}(\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) + \frac{1}{\gamma} \left[\mathbf{P}_\gamma^{\mathbf{t}}(\mathbf{u}^h) \right]_{\gamma g})\|_{0,\Gamma_C}^2 + \frac{\beta_2}{2} \|\gamma^{-\frac{1}{2}} \mathbf{P}_\gamma^{\mathbf{t}}(\mathbf{v}^h - \mathbf{u})\|_{0,\Gamma_C}^2 \\
& \quad - \|\gamma^{\frac{1}{2}}(\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) + \frac{1}{\gamma} \left[\mathbf{P}_\gamma^{\mathbf{t}}(\mathbf{u}^h) \right]_{\gamma g})\|_{0,\Gamma_C}^2 \\
& \quad + \frac{|1-\theta|}{2\beta_3} \|\gamma^{\frac{1}{2}}(\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) + \frac{1}{\gamma} \left[\mathbf{P}_\gamma^{\mathbf{t}}(\mathbf{u}^h) \right]_{\gamma g})\|_{0,\Gamma_C}^2 + \frac{|1-\theta|\beta_3}{2} \|\gamma^{\frac{1}{2}} \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{v}^h - \mathbf{u}^h)\|_{0,\Gamma_C}^2, \quad (25)
\end{aligned}$$

with $\beta_2 > 0$ and $\beta_3 > 0$. We apply exactly the same treatment for the third (and last) integral term in (22) (see also [12]) and we note that

$$\|\gamma^{-\frac{1}{2}} \mathbf{P}_\gamma^{\mathbf{n}}(\mathbf{v}^h - \mathbf{u})\|_{0,\Gamma_C}^2 + \|\gamma^{-\frac{1}{2}} \mathbf{P}_\gamma^{\mathbf{t}}(\mathbf{v}^h - \mathbf{u})\|_{0,\Gamma_C}^2 \leq 2\|\gamma^{-\frac{1}{2}}(\mathbf{u} - \mathbf{v}^h)\|_{0,\Gamma_C}^2 + 2\|\gamma^{\frac{1}{2}} \boldsymbol{\sigma}(\mathbf{u} - \mathbf{v}^h) \mathbf{n}\|_{0,\Gamma_C}^2,$$

and then putting together estimates (23) and (25) in (22) gives

$$\begin{aligned}
& \frac{\alpha}{2} \|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega}^2 \leq \frac{C^2}{2\alpha} \|\mathbf{u} - \mathbf{v}^h\|_{1,\Omega}^2 \\
& + \left(\frac{\beta_1 \theta^2}{2} + \beta_2 \right) \|\gamma^{\frac{1}{2}} \boldsymbol{\sigma}(\mathbf{u} - \mathbf{v}^h) \mathbf{n}\|_{0,\Gamma_C}^2 + \beta_2 \|\gamma^{-\frac{1}{2}}(\mathbf{u} - \mathbf{v}^h)\|_{0,\Gamma_C}^2 \\
& + \left(-1 + \frac{1}{2\beta_2} + \frac{|1-\theta|}{2\beta_3} \right) \left(\|\gamma^{\frac{1}{2}}(\sigma_n(\mathbf{u}) + \frac{1}{\gamma} [\mathbf{P}_\gamma^{\mathbf{n}}(\mathbf{u}^h)]_+) \|_{0,\Gamma_C}^2 + \|\gamma^{\frac{1}{2}}(\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) + \frac{1}{\gamma} \left[\mathbf{P}_\gamma^{\mathbf{t}}(\mathbf{u}^h) \right]_{\gamma g}) \|_{0,\Gamma_C}^2 \right) \\
& + \left(\frac{1}{2\beta_1} + \theta + \frac{|1-\theta|\beta_3}{2} \right) \|\gamma^{\frac{1}{2}} \boldsymbol{\sigma}(\mathbf{v}^h - \mathbf{u}^h) \mathbf{n}\|_{0,\Gamma_C}^2. \quad (26)
\end{aligned}$$

The norm term on the discrete Cauchy constraints on Γ_C , in (26), is bounded as follows by using (15):

$$\|\gamma^{\frac{1}{2}} \boldsymbol{\sigma}(\mathbf{v}^h - \mathbf{u}^h) \mathbf{n}\|_{0,\Gamma_C} \leq C\gamma_0^{\frac{1}{2}} \|\mathbf{v}^h - \mathbf{u}^h\|_{1,\Omega} \leq C\gamma_0^{\frac{1}{2}} (\|\mathbf{v}^h - \mathbf{u}\|_{1,\Omega} + \|\mathbf{u}^h - \mathbf{u}\|_{1,\Omega}). \quad (27)$$

For a fixed $\theta \in \mathbb{R}$ we then choose β_2 and β_3 large enough such that

$$-1 + \frac{1}{2\beta_2} + \frac{|1-\theta|}{2\beta_3} < -\frac{1}{2}.$$

Choosing then γ_0 small enough in (27) and putting the estimate in (26) establishes the first statement of theorem.

We now consider separately the case $\theta = -1$ for which (26) becomes:

$$\begin{aligned}
& \frac{\alpha}{2} \|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega}^2 \\
& \leq \frac{C^2}{2\alpha} \|\mathbf{u} - \mathbf{v}^h\|_{1,\Omega}^2 + \left(\frac{\beta_1}{2} + \beta_2 \right) \|\gamma^{\frac{1}{2}} \boldsymbol{\sigma}(\mathbf{u} - \mathbf{v}^h) \mathbf{n}\|_{0,\Gamma_C}^2 + \beta_2 \|\gamma^{-\frac{1}{2}}(\mathbf{u} - \mathbf{v}^h)\|_{0,\Gamma_C}^2 \\
& \quad + \left(-1 + \frac{1}{2\beta_2} + \frac{1}{\beta_3} \right) \left(\|\gamma^{\frac{1}{2}}(\sigma_n(\mathbf{u}) + \frac{1}{\gamma} [\mathbf{P}_\gamma^{\mathbf{n}}(\mathbf{u}^h)]_+) \|_{0,\Gamma_C}^2 + \|\gamma^{\frac{1}{2}}(\boldsymbol{\sigma}_t(\mathbf{u}) + \frac{1}{\gamma} [\mathbf{P}_\gamma^{\mathbf{t}}(\mathbf{u}^h)]_{\gamma g}) \|_{0,\Gamma_C}^2 \right) \\
& \quad + \left(\frac{1}{2\beta_1} - 1 + \beta_3 \right) \|\gamma^{\frac{1}{2}} \boldsymbol{\sigma}(\mathbf{v}^h - \mathbf{u}^h) \mathbf{n}\|_{0,\Gamma_C}^2. \tag{28}
\end{aligned}$$

Let be given $\eta > 0$. Set $\beta_1 = 1/(2\eta)$, $\beta_2 = 1 + (1/\eta)$ and $\beta_3 = 1 + \eta$. Therefore (28) becomes:

$$\begin{aligned}
& \frac{\alpha}{2} \|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega}^2 \\
& \leq \frac{C^2}{2\alpha} \|\mathbf{u} - \mathbf{v}^h\|_{1,\Omega}^2 + \left(\frac{5}{4\eta} + 1 \right) \|\gamma^{\frac{1}{2}} \boldsymbol{\sigma}(\mathbf{u} - \mathbf{v}^h) \mathbf{n}\|_{0,\Gamma_C}^2 + \frac{1+\eta}{\eta} \|\gamma^{-\frac{1}{2}}(\mathbf{u} - \mathbf{v}^h)\|_{0,\Gamma_C}^2 \\
& \quad - \frac{\eta}{2(1+\eta)} \left(\|\gamma^{\frac{1}{2}}(\sigma_n(\mathbf{u}) + \frac{1}{\gamma} [\mathbf{P}_\gamma^{\mathbf{n}}(\mathbf{u}^h)]_+) \|_{0,\Gamma_C}^2 + \|\gamma^{\frac{1}{2}}(\boldsymbol{\sigma}_t(\mathbf{u}) + \frac{1}{\gamma} [\mathbf{P}_\gamma^{\mathbf{t}}(\mathbf{u}^h)]_{\gamma g}) \|_{0,\Gamma_C}^2 \right) \\
& \quad + 2\eta \|\gamma^{\frac{1}{2}} \boldsymbol{\sigma}(\mathbf{v}^h - \mathbf{u}^h) \mathbf{n}\|_{0,\Gamma_C}^2.
\end{aligned}$$

Let $\gamma_0 > 0$. Set $\eta = \alpha/(16C^2\gamma_0)$ where C is the constant in (27) to conclude the proof of the theorem. \square

The optimal convergence of the method is stated below.

Theorem 3.5. *Suppose that the solution \mathbf{u} to Problem (6) belongs to $(H^{\frac{3}{2}+\nu}(\Omega))^d$ with $0 < \nu \leq k - \frac{1}{2}$ ($k = 1, 2$ is the degree of the finite element method, given in (7)) and $d = 2, 3$. When $\theta \neq -1$, suppose in addition that the parameter γ_0 is sufficiently small. The solution \mathbf{u}^h of Problem (12) satisfies the following error estimate:*

$$\begin{aligned}
& \|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega} \\
& \quad + \|\gamma^{\frac{1}{2}}(\sigma_n(\mathbf{u}) + \frac{1}{\gamma} [\mathbf{P}_\gamma^{\mathbf{n}}(\mathbf{u}^h)]_+) \|_{0,\Gamma_C} + \|\gamma^{\frac{1}{2}}(\boldsymbol{\sigma}_t(\mathbf{u}) + \frac{1}{\gamma} [\mathbf{P}_\gamma^{\mathbf{t}}(\mathbf{u}^h)]_{\gamma g}) \|_{0,\Gamma_C} \\
& \leq Ch^{\frac{1}{2}+\nu} \|\mathbf{u}\|_{\frac{3}{2}+\nu,\Omega}, \tag{29}
\end{aligned}$$

where C is a positive constant, independent of h and \mathbf{u} .

Proof: We need to bound the right terms in estimate (21) and we choose $\mathbf{v}^h = \mathcal{I}^h \mathbf{u}$ where \mathcal{I}^h stands for the Lagrange interpolation operator mapping onto \mathbf{V}^h . The estimation of the Lagrange interpolation error in H^1 -norm on a domain Ω is classical (see, e.g., [8, 15, 17]):

$$\|\mathbf{u} - \mathcal{I}^h \mathbf{u}\|_{1,\Omega} \leq Ch^{\frac{1}{2}+\nu} \|\mathbf{u}\|_{\frac{3}{2}+\nu,\Omega}, \tag{30}$$

for $-1/2 < \nu \leq k - 1/2$.

Let E in Γ_C be an edge of a triangle $T \in \mathcal{T}^h$:

$$\|\gamma^{-\frac{1}{2}}(\mathbf{u} - (\mathcal{I}^h \mathbf{u}))\|_{0,E} \leq Ch_T^{-\frac{1}{2}} h_T^{1+\nu} \|\mathbf{u}\|_{1+\nu,E} \leq Ch^{\frac{1}{2}+\nu} \|\mathbf{u}\|_{1+\nu,E},$$

(see [17] for instance). A summation on all the edges E together with the trace theorem yield:

$$\|\gamma^{-\frac{1}{2}}(\mathbf{u} - (\mathcal{I}^h \mathbf{u}))\|_{0,\Gamma_C} \leq Ch^{\frac{1}{2}+\nu} \|\mathbf{u}\|_{1+\nu,\Gamma_C} \leq Ch^{\frac{1}{2}+\nu} \|\mathbf{u}\|_{\frac{3}{2}+\nu,\Omega}. \quad (31)$$

From [19] (see also the details in [18], or alternatively in [12, Appendix A]), the following estimate also holds:

$$\|\gamma^{\frac{1}{2}} \boldsymbol{\sigma}(\mathbf{u} - \mathcal{I}^h \mathbf{u}) \mathbf{n}\|_{0,\Gamma_C} \leq Ch^{\frac{1}{2}+\nu} \|\mathbf{u}\|_{\frac{3}{2}+\nu,\Omega}. \quad (32)$$

We conclude by inserting the three estimates (30)–(31)–(32) into (21). \square

We can easily obtain the following error estimate on the Cauchy constraint $\|\gamma^{\frac{1}{2}} \boldsymbol{\sigma}(\mathbf{u} - \mathbf{u}^h) \mathbf{n}\|_{0,\Gamma_C}$ in the weighted $L^2(\Gamma_C)$ -norm (note that $\sigma_n(\mathbf{u}^h) \neq -\frac{1}{\gamma} [\mathbf{P}_\gamma^{\mathbf{n}}(\mathbf{u}^h)]_+$ and $\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}^h) \neq -\frac{1}{\gamma} [\mathbf{P}_\gamma^{\mathbf{t}}(\mathbf{u}^h)]_{\gamma\theta}$ on Γ_C conversely to the continuous case).

Corollary 3.6. *Suppose that the solution \mathbf{u} to Problem (6) belongs to $(H^{\frac{3}{2}+\nu}(\Omega))^d$ with $0 < \nu \leq k - \frac{1}{2}$ and $d = 2, 3$. When $\theta \neq -1$, suppose in addition that the parameter γ_0 is sufficiently small. The solution \mathbf{u}^h of Problem (12) satisfies the following error estimate:*

$$\|\gamma^{\frac{1}{2}} \boldsymbol{\sigma}(\mathbf{u} - \mathbf{u}^h) \mathbf{n}\|_{0,\Gamma_C} \leq Ch^{\frac{1}{2}+\nu} \|\mathbf{u}\|_{\frac{3}{2}+\nu,\Omega}, \quad (33)$$

where C is a positive constant, independent of h and \mathbf{u} .

Proof: We use (32), (15), (30) and (29) to establish the bound:

$$\begin{aligned} \|\gamma^{\frac{1}{2}} \boldsymbol{\sigma}(\mathbf{u} - \mathbf{u}^h) \mathbf{n}\|_{0,\Gamma_C} &\leq \|\gamma^{\frac{1}{2}} \boldsymbol{\sigma}(\mathbf{u} - \mathcal{I}^h \mathbf{u}) \mathbf{n}\|_{0,\Gamma_C} + \|\gamma^{\frac{1}{2}} \boldsymbol{\sigma}(\mathcal{I}^h \mathbf{u} - \mathbf{u}^h) \mathbf{n}\|_{0,\Gamma_C} \\ &\leq Ch^{\frac{1}{2}+\nu} \|\mathbf{u}\|_{\frac{3}{2}+\nu,\Omega} + C\gamma_0^{\frac{1}{2}} (\|\mathcal{I}^h \mathbf{u} - \mathbf{u}\|_{1,\Omega} + \|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega}) \\ &\leq Ch^{\frac{1}{2}+\nu} \|\mathbf{u}\|_{\frac{3}{2}+\nu,\Omega}. \end{aligned}$$

\square

4. Conclusion and perspectives

In this paper we extend the Nitsche-based formulation of [10, 12] to the case of unilateral contact with Tresca friction and we achieve the corresponding numerical analysis. This analysis allows us to obtain the first optimal a priori error estimate under H^s regularity ($3/2 < s \leq 5/2$) on the solution without any additional assumption, in two-dimensions and three-dimensions,

and for piecewise linear and quadratic finite elements (which may be of interest when the solution has an increased regularity). It is remarkable that the method originally proposed for variational inequations arising from frictionless unilateral contact extends naturally to more complex problems involving other variational inequations (here of the second kind), preserving all the interesting features observed initially (well-posedness, optimal convergence, same behavior respectively to the numerical parameters θ and γ_0). This may open possibilities for further extensions (for instance the Stokes problem with Tresca friction [4]). It is also worth-noticing that no special difficulty is encountered in the three-dimensional situation, which is not the case in mixed methods where bounds are still sub-optimal and seem very difficult to improve (see e.g. [27, 39]). In forthcoming studies may be considered the extension to the Coulomb friction problem, numerical experiments and a-posteriori error estimators (as in [14]).

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Appendix A. On reformulation of Tresca conditions

In this Appendix, we give a simple and direct proof of the equivalence between equations (3) and (9) which was pointed out in [2].

- First let us suppose that \mathbf{u}_t and $\boldsymbol{\sigma}_t(\mathbf{u})$ verify equation (3).

Consider the case $\mathbf{u}_t = \mathbf{0}$, then from (3) (i), we have the inequality $|\boldsymbol{\sigma}_t(\mathbf{u})| \leq g$. Due to the property of projection $[\mathbf{x}]_{\gamma g} = \mathbf{x}$ for $\mathbf{x} \in \mathcal{B}(\mathbf{0}, \gamma g)$, it results:

$$-\frac{1}{\gamma} [\mathbf{u}_t - \gamma \boldsymbol{\sigma}_t(\mathbf{u})]_{\gamma g} = -\frac{1}{\gamma} [-\gamma \boldsymbol{\sigma}_t(\mathbf{u})]_{\gamma g} = -\frac{1}{\gamma} (-\gamma \boldsymbol{\sigma}_t(\mathbf{u})) = \boldsymbol{\sigma}_t(\mathbf{u}),$$

so that (9) holds.

In the case $\mathbf{u}_t \neq \mathbf{0}$, from (3) (ii) we obtain:

$$\mathbf{u}_t - \gamma \boldsymbol{\sigma}_t(\mathbf{u}) = \left(1 + \frac{\gamma g}{|\mathbf{u}_t|}\right) \mathbf{u}_t.$$

It results that $|\mathbf{u}_t - \gamma \boldsymbol{\sigma}_t(\mathbf{u})| = |\mathbf{u}_t| + \gamma g \geq \gamma g$, which means that its projection onto $\mathcal{B}(\mathbf{0}, \gamma g)$ is simply

$$[\mathbf{u}_t - \gamma \boldsymbol{\sigma}_t(\mathbf{u})]_{\gamma g} = \gamma g \frac{\mathbf{u}_t}{|\mathbf{u}_t|},$$

noting also that \mathbf{u}_t and $\mathbf{u}_t - \gamma \boldsymbol{\sigma}_t(\mathbf{u})$ have the same orientation since $\gamma > 0$. Finally using again (3) (ii) yields

$$\boldsymbol{\sigma}_t(\mathbf{u}) = -\frac{1}{\gamma}\gamma g \frac{\mathbf{u}_t}{|\mathbf{u}_t|} = -\frac{1}{\gamma} [\mathbf{u}_t - \gamma\boldsymbol{\sigma}_t(\mathbf{u})]_{\gamma g},$$

which is (9).

- Suppose now that the condition (9) is satisfied by \mathbf{u}_t and $\boldsymbol{\sigma}_t(\mathbf{u})$.

We first consider the case $|\mathbf{u}_t - \gamma\boldsymbol{\sigma}_t(\mathbf{u})| \leq \gamma g$, for which $[\mathbf{u}_t - \gamma\boldsymbol{\sigma}_t(\mathbf{u})]_{\gamma g} = \mathbf{u}_t - \gamma\boldsymbol{\sigma}_t(\mathbf{u})$. In this case condition (9) is simply

$$\boldsymbol{\sigma}_t(\mathbf{u}) = -\frac{1}{\gamma}(\mathbf{u}_t - \gamma\boldsymbol{\sigma}_t(\mathbf{u})),$$

so that $\mathbf{u}_t = \mathbf{0}$. Moreover in this case, since $\gamma > 0$,

$$|\boldsymbol{\sigma}_t(\mathbf{u})| = \frac{1}{\gamma}|\mathbf{u}_t - \gamma\boldsymbol{\sigma}_t(\mathbf{u})| \leq g,$$

which means that (3) (i) is satisfied.

Consider then the case $|\mathbf{u}_t - \gamma\boldsymbol{\sigma}_t(\mathbf{u})| > \gamma g$, which means that there is $\lambda \in (0, 1)$ such that $[\mathbf{u}_t - \gamma\boldsymbol{\sigma}_t(\mathbf{u})]_{\gamma g} = \lambda(\mathbf{u}_t - \gamma\boldsymbol{\sigma}_t(\mathbf{u}))$. Relationship (9) can now be written:

$$\boldsymbol{\sigma}_t(\mathbf{u}) = \frac{\lambda}{\gamma(\lambda - 1)}\mathbf{u}_t, \tag{A.1}$$

with the quantity $\frac{\lambda}{\gamma(\lambda-1)} < 0$. Since $[\mathbf{u}_t - \gamma\boldsymbol{\sigma}_t(\mathbf{u})]_{\gamma g} \in \partial\mathcal{B}(\mathbf{0}, \gamma g)$ and owing to relationship (9) we deduce $|\boldsymbol{\sigma}_t(\mathbf{u})| = g$. Taking then the norm of (A.1) yields

$$g = \frac{\lambda}{\gamma(1 - \lambda)}|\mathbf{u}_t|,$$

from which we obtain finally $\mathbf{u}_t \neq \mathbf{0}$ and $\boldsymbol{\sigma}_t(\mathbf{u}) = -g\frac{\mathbf{u}_t}{|\mathbf{u}_t|}$, that is (3) (ii).

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