



Asymptotics of a vanishing period: characterization of semi-simplicity

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Asymptotics of a vanishing period : characterization of semi-simplicity

Daniel Barlet.*

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Abstract

In this paper we introduce the word *fresco* to denote a monogenic geometric (a,b) -module. This "basic object" (generalized Brieskorn module with one generator) corresponds to the formal germ of the minimal filtered (regular) differential equation. Such an equation is satisfied by a relative de Rham cohomology class at a critical value of a holomorphic function on a smooth complex manifold. In [B.09] the first structure theorems are proved. Then in [B.10] we introduced the notion of *theme* which corresponds in the $[\lambda]$ -primitive case to frescos having a unique Jordan-Hölder sequence (a unique Jordan block for the monodromy). Themes correspond to asymptotic expansion of a given vanishing period, so to an image of a fresco in the module of asymptotic expansions. For a fixed relative de Rham cohomology class (for instance given by a smooth differential form d -closed and df -closed) each choice of a vanishing cycle in the spectral eigenspace of the monodromy for the eigenvalue $\exp(2i\pi.\lambda)$ produces a $[\lambda]$ -primitive theme, which is a quotient of the fresco associated to the given relative de Rham class itself.

We show that for any fresco there exists an *unique* Jordan-Hölder sequence, called the *principal J-H. sequence*, with corresponding quotients giving the opposite of the roots of the Bernstein polynomial in increasing order. We study the semi-simple part of a given fresco and we characterize the semi-simplicity of a fresco by the fact for any given order on the roots of its Bernstein polynomial we may find a J-H. sequence making them appear with this order. Then we construct a numerical invariant, called the β -invariant, and we show that it produces numerical criteria in order to give a necessary and sufficient condition on a fresco to be semi-simple.

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We show that these numerical invariants define a natural algebraic stratification on the set of isomorphism classes of fresco with given fundamental invariants (or equivalently with given roots of the Bernstein polynomial).

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Introduction

Let $f : X \rightarrow D$ be an holomorphic function on a connected complex manifold. Assume that $\{df = 0\} \subset \{f = 0\} := X_0$. We consider X as a degenerating family of complex manifolds parametrized by $D^* := D \setminus \{0\}$ with a singular member X_0 at the origin of D . Let ω be a smooth $(p+1)$ -differential form on X satisfying $d\omega = 0 = df \wedge \omega$. Then in many interesting cases (see for instance [Br.70], [M.74], [S.89] for the isolated singularity case, [B.I], [B.II], [B.III] for the case of a function with 1-dimensional singular set and [B.12] for the general proper case) the relative family of de Rham cohomology classes induced on the fibers $(X_s)_{s \in D^*}$ of f by ω/df is solution of a minimal filtered differential equation defined from the Gauss-Manin connection of f . This object, called a **fresco**, is a monogenic¹ regular (a,b)-module satisfying an extra condition, called "geometric", which encodes simultaneously the

¹i.e. a left $\tilde{\mathcal{A}}$ -module generated by one element.

regularity at 0 of the Gauss-Manin connection, the monodromy theorem and B. Malgrange's positivity theorem (see [M.75], the result [K.76] on the roots of the Bernstein polynomial or [Bj.93]).

After some easy preliminaries, we prove in section 1 that a fresco E admits an *unique* Jordan-Hölder sequence, called the *principal* J-H. sequence, in which the opposite of the roots of the Bernstein polynomial of E appears in the increasing order². This uniqueness result is important because it implies, for instance, that the isomorphism class of each quotient of two terms of the principal J-H. sequence only depends on the isomorphism class of E .

In section 2 we study semi-simple regular (a,b)-modules and the corresponding semi-simple filtration. In the case of a fresco we prove that semi-simplicity is characterized by the fact that we may find a J-H. sequence in which the opposite of the roots of the Bernstein polynomial appears in strictly decreasing order (but also in any given order). We conclude this section by an example of fresco such that the semi-simple part of E and of its dual E^* have not the same rank.

In section 3 we answer to the following question : How to recognize from the principal J-H. sequence of a $[\lambda]$ -primitive fresco if it is semi-simple? Of course the answer is easy if some rank 2 sub-quotient theme appears from this sequence. But when it is not the case, such a rank 2 sub-quotient theme may appear after "commuting" some terms in the J-H. sequence. The simplest example is when

$$E := \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a - \lambda_1.b)(1 + \alpha.b^{p_1+p_2})^{-1}.(a - \lambda_2.b).(a - \lambda_3.b)$$

with $\lambda_{i+1} = \lambda_i + p_i - 1$ for $i = 1, 2$ with $p_i \in \mathbb{N}^*, i = 1, 2$ and $\alpha \in \mathbb{C}^*$. Using the identity in $\tilde{\mathcal{A}}$

$$(a - \lambda_2.b).(a - \lambda_3.b) = (a - (\lambda_3 + 1).b).(a - (\lambda_2 - 1).b)$$

it is easy to see that $(a - (\lambda_2 - 1).b).[1]$ generates a (normal) rank 2 theme in E , because we assume $\alpha \neq 0$.

We solve this question introducing for a $[\lambda]$ -primitive rank $k \geq 3$ fresco E such that $F_{k-1}(E)$ and $E/F_1(E)$ are semi-simple, where $(F_j(E))_{j \in [1, k]}$ is the principal J-H. sequence of E , the β -invariant $\beta(E)$. This complex number only depends on the isomorphism class of E and is zero if and only if E is semi-simple. We also prove that when $\beta(E)$ is not zero it determines the isomorphism class of any normal rank 2 sub-theme of E . This easily implies that the polynomial β is quasi-invariant by change of variables. We use this β -invariant as a main tool to describe, in the set $\mathcal{F}(\lambda_1, \dots, \lambda_k)$ of isomorphism classes of rank k frescos with fundamental invariants $\lambda_1, \dots, \lambda_k$, an *algebraic*³ stratification ending in the subset of semi-simple frescos. This stratification is also invariant by any change of variable.

²We fix an order on \mathbb{Q} deduced from a total order on \mathbb{Q}/\mathbb{Z} and compatible with the usual order on each class modulo \mathbb{Z} .

³notion defined in the section 3.

1 Preliminaries.

1.1 Definitions and characterization as $\tilde{\mathcal{A}}$ -modules.

We are interested in "standard" formal asymptotic expansions of the following type

$$\sum_{q=1}^k \sum_{j=0}^N \mathbb{C}[[s]].s^{\lambda_q-1}.(\text{Log } s)^j$$

where $\lambda_1, \dots, \lambda_k$ are positive rational numbers, and in fact in vector valued such expansions. The two basic operations on such expansions are

- the multiplication by s that we shall denote a ,
- and the primitive in s without constant that we shall denote b .

This leads to consider on the set of such expansions a left module structure on the non commutative \mathbb{C} -algebra

$$\tilde{\mathcal{A}} := \left\{ \sum_{\nu=0}^{\infty} P_{\nu}(a).b^{\nu}, \quad \text{where } P_{\nu} \in \mathbb{C}[x] \right\}$$

defined by the following conditions

- The commutation relation $a.b - b.a = b^2$ which is the translation of the Leibnitz rule ;
- The continuity for the b -adic filtration of $\tilde{\mathcal{A}}$ of the left and right multiplications by a .

Define now for λ in $\mathbb{Q} \cap]0, 1]$ and $N \in \mathbb{N}$ the left $\tilde{\mathcal{A}}$ -module

$$\Xi_{\lambda}^{(N)} := \oplus_{j=0}^N \mathbb{C}[[s]].s^{\lambda-1}.(\text{Log } s)^j = \oplus_{j=0}^N \mathbb{C}[[a]].s^{\lambda-1}.(\text{Log } s)^j = \oplus_{j=0}^N \mathbb{C}[[b]].s^{\lambda-1}.(\text{Log } s)^j.$$

Of course we let a and b act on $\Xi_{\lambda}^{(N)}$ as explained above.

Define also, when Λ is a finite subset in $\mathbb{Q} \cap]0, 1]$, the $\tilde{\mathcal{A}}$ -module

$$\Xi_{\Lambda}^{(N)} := \oplus_{\lambda \in \Lambda} \Xi_{\lambda}^{(N)}.$$

More generally, if V is a finite dimensional complex vector space we shall put a structure of left $\tilde{\mathcal{A}}$ -module on $\Xi_{\Lambda}^{(N)} \otimes_{\mathbb{C}} V$ with the following rules :

$$a.(\varphi \otimes v) = (a.\varphi) \otimes v \quad \text{and} \quad b.(\varphi \otimes v) = (b.\varphi) \otimes v$$

for any $\varphi \in \Xi_{\Lambda}^{(N)}$ and any $v \in V$. It will be convenient to denote Ξ_{λ} the $\tilde{\mathcal{A}}$ -module $\sum_{N \in \mathbb{N}} \Xi_{\lambda}^{(N)}$.

Definition 1.1.1 A $\tilde{\mathcal{A}}$ -module is called a **fresco** when it is isomorphic to a submodule $\tilde{\mathcal{A}}.\varphi \subset \Xi_{\Lambda}^{(N)} \otimes V$ where φ is any element in $\Xi_{\Lambda}^{(N)} \otimes V$, for some choice of Λ, N and V as above.

A fresco is a **theme** when we may choose $V := \mathbb{C}$ in the preceding choice.

For the motivation of these definition see the fundamental example below.

Now the characterization of frescos among all left $\tilde{\mathcal{A}}$ -modules is not so obvious. The following theorem is proved in [B.09].

Theorem 1.1.2 A left $\tilde{\mathcal{A}}$ -module E is a fresco if and only if it is a geometric (a, b) -module which is generated (as a $\tilde{\mathcal{A}}$ -module) by one element. Moreover the annihilator in $\tilde{\mathcal{A}}$ of a generator of E is a left ideal of the form $\tilde{\mathcal{A}}.P$ where P may be written as follows :

$$P = (a - \lambda_1.b).S_1^{-1}.(a - \lambda_2.b).S_2^{-1} \dots (a - \lambda_k.b).S_k^{-1}, \quad k := \dim_{\mathbb{C}}(E/b.E)$$

where λ_j are rational numbers such that $\lambda_j + j > k$ for $j \in [1, k]$ and where S_1, \dots, S_k are invertible elements in the sub-algebra $\mathbb{C}[[b]]$ of $\tilde{\mathcal{A}}$.

Conversely, for such a $P \in \tilde{\mathcal{A}}$ the left $\tilde{\mathcal{A}}$ -module $E := \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.P$ is a fresco and it is a free rank k module on $\mathbb{C}[[b]]$.

Let me recall briefly for the convenience of the reader the definitions of the notions involved in the previous statement.

- A **(a,b)-module** E is a free finite rank $\mathbb{C}[[b]]$ module endowed with an \mathbb{C} -linear endomorphism a such that $a.S = S.a + b^2.S'$ for $S \in \mathbb{C}[[b]]$; or, in an equivalent way, a $\tilde{\mathcal{A}}$ -module which is free and finite type over the subalgebra $\mathbb{C}[[b]] \subset \tilde{\mathcal{A}}$.

It has a **simple pole** when a satisfies $a.E \subset b.E$. In this case the **Bernstein polynomial** B_E of E is defined as the *minimal* polynomial of $-b^{-1}.a$ acting on $E/b.E$.

- A (a, b) -module E is **regular** when it may be embedded in a simple pole (a, b) -module. In this case there is a minimal such embedding which is the inclusion of E in its saturation E^\sharp by $b^{-1}.a$. The **Bernstein polynomial** of a regular (a, b) -module is, by definition, the Bernstein polynomial of its saturation E^\sharp .
- A regular (a, b) -module E is called **geometric** when all roots of its Bernstein polynomial are rational and strictly negative (compare with [K.76]).

Note that the formal completion in b of the Brieskorn module of a function with an isolated singularity is a geometric (a, b) -module (see [Br.70], [M.74]). In [B.I], [B.II] and [B.III] we have used this notion to study holomorphic germ in \mathbb{C}^{n+1} with a 1-dimensional singularity. The result in [B.12] shows that this structure appears in a rather systematic way in the study of the Gauss-Manin connection of a proper holomorphic function on a complex manifold.

FUNDAMENTAL EXAMPLE. Let E be the formal completion of the Brieskorn module of a holomorphic germ $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ with an isolated singularity. Then for any $(n+1)$ -holomorphic form ω we have a fresco $\tilde{\mathcal{A}}[\omega] \subset E$. Then for each γ , a n -homology class of the Milnor fiber of f , we have a $\tilde{\mathcal{A}}$ -linear map $\varphi_\gamma : E \rightarrow \Xi$ given by $\varphi_\gamma(\omega) := \int_{\gamma_s} \omega/df$, where Ξ is the $\tilde{\mathcal{A}}$ -module of asymptotic expansions and $(\gamma_s)_{s \in D^*}$ the horizontal multivalued family of n -cycles in the fibers of f associated to γ . Then for a given class $[\omega] \in E$ the image by φ_γ of the fresco $\tilde{\mathcal{A}}[\omega]$ is a theme. If we choose γ in the spectral subspace of the the monodromy for the eigenvalue $\exp(2i\pi\lambda)$, this theme will be $[\lambda]$ -primitive (see the definition above). In this case the rank of such a theme is bounded by the nilpotency order of the monodromy acting on γ . \square

Definition 1.1.3 A submodule F in E is **normal** when $F \cap b.E = b.F$.

For any sub-module F of a (a,b) -module E there exists a minimal normal sub-module \tilde{F} of E containing F . We shall call it the **normalization** of F . It is easy to see that \tilde{F} is the pull-back by the quotient map $E \rightarrow E/F$ of the b -torsion of E/F . Note that F is always a finite codimensional complex vector space of its normalization. In particular F and \tilde{F} have the same rank as $\mathbb{C}[[b]]$ -modules.

When F is normal the quotient E/F is again a (a,b) -module. Note that a sub-module of a regular (resp. geometric) (a,b) -module is regular (resp. geometric) and when F is normal E/F is also regular (resp. geometric).

Lemma 1.1.4 Let E be a fresco and F be a normal sub-module in E . Then F is a fresco and also the quotient E/F .

PROOF. The only point to prove, as we already know that F and E/F are geometric (a,b) -modules thanks to [B.09], is the fact that F and E/F are generated as $\tilde{\mathcal{A}}$ -modules by one element. This is obvious for E/F , but not for F . We shall use the theorem 1.1.2 for E/F to prove that F is generated by one element. Let e be a generator of E and let P as in the theorem 1.1.2 which generates the annihilator ideal of the image of e in E/F . Then $P.e$ is in F . We shall prove that $P.e$ generates F as a $\tilde{\mathcal{A}}$ -module. Let y be an element in F and write $y = u.e$ where u is in $\tilde{\mathcal{A}}$. As P is, up to an invertible element in $\mathbb{C}[[b]]$, a monic polynomial in a with coefficients in $\mathbb{C}[[b]]$, we may write $u = Q.P + R$ where Q and R are in $\tilde{\mathcal{A}}$ and R is a polynomial in a with coefficient in $\mathbb{C}[[b]]$ of degree $r < \deg(P) = \text{rank}(E/F)$. Now, as y is in F , the image in E/F of $u.e$ is 0, and this implies that R annihilates the image of e in E/F . So R lies in $\tilde{\mathcal{A}}.P$ and so $R = 0$. Then we have $u = Q.P$ and $y = Q.P.e$ proving our claim. \blacksquare

The following lemma is useful to recognize a generator of a fresco.

Lemma 1.1.5 *Let E be a non zero geometric (a,b) -module. Then E is a fresco if and only if the vector space $E/a.E + b.E$ has dimension 1. In this case any element in $E \setminus a.E + b.E$ is a generator of E as a $\tilde{\mathcal{A}}$ -module.*

PROOF. If E is a rank $k \geq 1$ fresco then a acts on the k -dimensional vector space $E/b.E$ as a principal nilpotent endomorphism (i. e. its minimal polynomial is $P(x) = x^k$). So for any $e \notin a.E + b.E$ the set $(e, a.e, \dots, a^{k-1}.e)$, induces a basis of $E/b.E$ and then, it is a $\mathbb{C}[[b]]$ -basis of E . Such an element e is a generator of E as a left $\tilde{\mathcal{A}}$ -module.

Conversely, assume that, for the rank k geometric (a,b) -module E , the vector space $E/a.E + b.E$ is 1-dimensional; then a acts as a principal nilpotent endomorphism on $E/b.E$, and for any $e \in E \setminus a.E + b.E$ the set $(e, a.e, \dots, a^{k-1}.e)$ is a $\mathbb{C}[[b]]$ -basis of E . So such an element generates E as a left $\tilde{\mathcal{A}}$ -module. ■

In the case of a fresco the Bernstein polynomial is more easy to describe, thanks to the following proposition proved in [B.09].

Proposition 1.1.6 *Soit $E = \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.P$ be a rank k fresco as described in the previous theorem. The Bernstein polynomial of E is the **characteristic** polynomial of $-b^{-1}.a$ acting on $E^\# / b.E^\#$. And the **Bernstein element** P_E of E , which is the element in $\tilde{\mathcal{A}}$ defined by the Bernstein polynomial B_E of E by the following formula*

$$P_E := (-b)^k . B_E(-b^{-1}.a)$$

is equal to $(a - \lambda_1.b) \dots (a - \lambda_k.b)$ for any such a choice of presentation of E as in the theorem 1.1.2

As an easy consequence, the k roots of the Bernstein polynomial of E are the opposite of the numbers $\lambda_1 + 1 - k, \dots, \lambda_k + k - k$. So the Bernstein polynomial is readable on the element P : the initial form of P in (a,b) has degree k and is the Bernstein element P_E of E .

REMARK. If we have an exact sequence of (a,b) -modules

$$0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$$

where E is a fresco, then F and G are frescos and the Bernstein elements satisfy the equality $P_E = P_F.P_G$ in the algebra $\tilde{\mathcal{A}}$ (see [B.09] proposition 3.4.4.). □

EXAMPLE. The $\tilde{\mathcal{A}}$ -module $\Xi_\lambda^{(N)}$ is a simple pole (a,b) -module with rank $N+1$. Its Bernstein polynomial is equal to $(x + \lambda)^{N+1}$.

The theme $\tilde{\mathcal{A}}.\varphi \subset \Xi_\lambda^{(N)}$ where $\varphi = s^{\lambda-1}.(\text{Log } s)^N$ has rank $N+1$ and its Bernstein element is $(a - (\lambda + N).b)(a - (\lambda + N - 1).b) \dots (a - \lambda.b)$. Its saturation by $b^{-1}.a$ is $\Xi_\lambda^{(N)}$. □

The dual of a regular (a,b) -module is regular, but duality does not preserve the property of being geometric because duality changes the sign of the roots of the Bernstein polynomial. As duality preserves regularity (see [B.95]), to find again a geometric (a,b) -module it is sufficient to make the tensor product by a rank 1 (a,b) -module E_δ for δ a large enough rational number⁴. The next lemma states that this "twist duality" preserves the notion of fresco.

Lemma 1.1.7 *Let E be a fresco and let δ be a rational number such that $E^* \otimes E_\delta$ is geometric. Then $E^* \otimes E_\delta$ is a fresco.*

The proof is obvious. ■

The following definition is useful when we want to consider only the part of asymptotic expansions corresponding to prescribe eigenvalues of the monodromy.

Definition 1.1.8 *Let Λ be a subset of $\mathbb{Q} \cap]0, 1]$. We say that a regular (a,b) -module E is $[\Lambda]$ -**primitive** when all roots of its Bernstein polynomial are in $-\Lambda + \mathbb{Z}$.*

The following easy proposition is proved in [B.09]

Proposition 1.1.9 *Let E be a regular (a,b) -module and Λ a subset of $\mathbb{Q} \cap]0, 1]$. Then there exists a maximal submodule $E_{[\Lambda]}$ in E which is $[\Lambda]$ -primitive. Moreover the quotient $E/E_{[\Lambda]}$ is a $[\Lambda^c]$ -primitive (a,b) -module, where we denote $\Lambda^c := \mathbb{Q} \cap]0, 1] \setminus \Lambda$.*

We shall mainly consider the case where Λ is a single element. Note that the $[\lambda]$ -primitive part of an (a,b) -module $E \subset \Xi_\Lambda^{(N)} \otimes V$ corresponds to its intersection with $\Xi_\lambda^{(N)} \otimes V$.

1.2 The principal Jordan-Hölder sequence.

The classification of rank 1 regular (a,b) -modules is very simple : each isomorphy class is given by a complex number and to $\lambda \in \mathbb{C}$ corresponds the isomorphy class of $E_\lambda := \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a - \lambda.b)$. Then a **Jordan-Hölder sequence** for a regular (a,b) -module E is a sequence

$$0 = F_0 \subset F_1 \subset \dots \subset F_k = E$$

of normal sub-modules such that for each $j \in [1, k]$ the quotient F_j/F_{j-1} has rank 1. Then to each J-H. sequence we may associate an ordered sequence of complex numbers $\lambda_1, \dots, \lambda_k$ such that $F_j/F_{j-1} \simeq E_{\lambda_j}$.

⁴to tensor by E_δ is the same that to replace a by $a + \delta.b$; see [B.I] for the general definition of the tensor product.

EXAMPLE. A regular rank 1 (a,b) -module is a fresco if and only if it is isomorphic to E_λ for some $\lambda \in \mathbb{Q}^{+*}$. All rank 1 frescos are themes. The classification of rank 2 regular (a,b) -modules obtained in [B.93] gives the list of $[\lambda]$ -primitive rank 2 frescos which is the following, where $\lambda_1 > 1$ is a rational number :

$$E = E \simeq \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a - \lambda_1.b).(a - (\lambda_1 - 1).b) \quad (1)$$

$$E \simeq \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a - \lambda_1.b).(1 + \alpha.b^p)^{-1}.(a - (\lambda_1 + p - 1).b) \quad (2)$$

where $p \in \mathbb{N} \setminus \{0\}$ and $\alpha \in \mathbb{C}$.

The themes in this list are these in (1) and these in (2) with $\alpha \neq 0$. For a $[\lambda]$ -primitive fresco in case (2) the number α will be called the **parameter** of the rank 2 fresco. By convention we shall define $\alpha := 1$ in the case (1). \square

The following existence result is proved in [B.93]

Proposition 1.2.1 *For any regular (a,b) -module E of rank k there exists a J -H. sequence. The numbers $\exp(2i\pi.\lambda_j)$ are independent of the J -H. sequence, up to permutation. Moreover the number $\mu(E) := \sum_{j=1}^k \lambda_j$ is also independent of the choice of the J -H. sequence of E .*

EXERCICE. Let E be a regular (a,b) -module and $E' \subset E$ be a sub- (a,b) -module with the same rank than E . Show that E' has finite \mathbb{C} -codimension in E given by

$$\dim_{\mathbb{C}} E/E' = \mu(E') - \mu(E).$$

hint : make an induction on the rank of E . \square

For a $[\lambda]$ -primitive fresco a more precise result is proved in [B.09]. The following generalization to the case of any fresco is an easy application of the proposition 1.1.9.

Proposition 1.2.2 *For any J -H. sequence of a rank k fresco E the numbers $\lambda_j + j - k, j \in [1, k]$ are the opposite of the roots (with multiplicities) of the Bernstein polynomial of E . So, up to a permutation, they are independent of the choice of the J -H. sequence. Moreover, when E is $[\lambda]$ -primitive, there always exists a J -H. sequence such that the associated sequence $\lambda_j + j$ is increasing.*

For a $[\lambda]$ -primitive theme the situation is extremely rigid (see [B.09]) :

Proposition 1.2.3 *Let E a rank k $[\lambda]$ -primitive theme. Then, for each j in $[0, k]$, there exists an **unique** normal rank j submodule F_j . The numbers associated to the unique J -H. sequence satisfy the condition that $\lambda_j + j, j \in [1, k]$ is a increasing sequence.*

ORDER ON \mathbb{Q} . We shall fix a total order on \mathbb{Q}/\mathbb{Z} , for instance the order induced by the inclusion $\mathbb{Q}/\mathbb{Z} \subset]0, 1]$. But any other total order would be also convenient. Then we shall use the corresponding order on \mathbb{Q} defined as follows :

- i) If λ, μ are in \mathbb{Q} such $[\lambda] < [\mu]$ in \mathbb{Q}/\mathbb{Z} we put $\lambda \tilde{<} \mu$.
- ii) If we have $[\lambda] = [\mu]$ in \mathbb{Q} we put $\lambda \tilde{<} \mu$ when we have $\lambda < \mu$ for the usual order of \mathbb{Q} .

Remark that with such an order, a increasing sequence $\lambda_1, \dots, \lambda_k$ is such that theses numbers appears successively in the different classes modulo \mathbb{Z} in increasing order in \mathbb{Q}/\mathbb{Z} and in each class they are increasing in the usual sens.

Note that using the proposition 1.1.9 it is easy to see that any rank k fresco admits a J.H. sequence such corresponding sequence of rational numbers $\lambda_1, \dots, \lambda_k$ satisfies the condition that the sequence $[\lambda_1], \dots, [\lambda_k]$ is increasing in \mathbb{Q}/\mathbb{Z} .

Definition 1.2.4 Let E be a fresco of rank k and let

$$0 = F_0 \subset F_1 \subset \dots \subset F_k = E$$

be a J-H. sequence of E . Then for each $j \in [1, k]$ we have $F_j/F_{j-1} \simeq E_{\lambda_j}$, where $\lambda_1, \dots, \lambda_k$ are in \mathbb{Q}^{+*} . We shall say that such a J-H. sequence is **principal** when the sequence $[1, k] \ni j \mapsto \lambda_j + j$ is increasing for the order of \mathbb{Q} fixed above.

Theorem 1.2.5 Let E be a fresco. Then E admits an **unique** principal J-H. sequence.

PROOF. The existence is an easy consequence of the propositions 1.1.9 and 1.2.2. As we know the uniqueness of the $[\lambda]$ -primitive part for any regular (a,b)-module, and that for a fresco E the $[\lambda]$ -primitive part is again a fresco (and normal so that the quotient is again a fresco), it is enough to prove the uniqueness in the $[\lambda]$ -primitive case. We shall prove the uniqueness by induction on the rank k of the $[\lambda]$ -primitive fresco E .

We begin by the case of rank 2.

Lemma 1.2.6 Let E be a rank 2 $[\lambda]$ -primitive fresco and let λ_1, λ_2 the numbers corresponding to a principal J-H. sequence of E (so $\lambda_1 + 1 \leq \lambda_2 + 2$). Then the normal rank 1 submodule of E isomorphic to E_{λ_1} is unique. Moreover, if there exists a $\mu \neq \lambda_1$ and a rank 1 normal submodule isomorphic to E_μ then $\mu = \lambda_2 + 1$. In this case there exists infinitely many different normal rank 1 submodules isomorphic to E_{λ_2+1} .

PROOF. The case $\lambda_1 + 1 = \lambda_2 + 2$ is obvious because then E is a $[\lambda]$ -primitive theme (see the example at the beginning of the paragraph 1.2 or [B.10] corollary 2.1.7). So we may assume that $\lambda_2 = \lambda_1 + p_1 - 1$ with $p_1 \geq 1$ and that E is the quotient $E \simeq \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a - \lambda_1.b).(a - \lambda_2.b)$ (see the classification of rank 2 frescos in 2.2), because the result is clear when E is a theme. We shall use the $\mathbb{C}[[b]]$ -basis e_1, e_2 of E where a is defined by the relations

$$(a - \lambda_2.b).e_2 = e_1 \quad \text{and} \quad (a - \lambda_1.b).e_1 = 0.$$

This basis comes from the isomorphism $E \simeq \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a - \lambda_1.b).(a - \lambda_2.b)$ deduced from the classification of rank 2 frescos with $e_2 = [1]$ and $e_1 = (a - \lambda_2.b).e_2$.

Let look for $x := U.e_2 + V.e_1$ such that $(a - \mu.b).x = 0$. Then we obtain

$$b^2.U'.e_2 + U.(a - \lambda_2.b).e_2 + (\lambda_2 - \mu).b.U.e_2 + b^2.V'.e_1 + (\lambda_1 - \mu).b.V.e_1 = 0$$

which is equivalent to the two equations :

$$b^2.U' + (\lambda_2 - \mu).b.U = 0 \quad \text{and} \quad U + b^2.V' + (\lambda_1 - \mu).b.V = 0.$$

The first equation gives $U = 0$ for $\mu \notin \lambda_2 + \mathbb{N}$. As the case $U = 0$ will give (as we want also that $x \notin b.E$) that x is equal to e_1 , up to a non zero multiplicative constant, we may assume that $\mu = \lambda_2 + q$ for some $q \in \mathbb{N}$. Moreover, as the second equation implies $U(0) = 0$, we may assume that $q \geq 1$. This already shows that $\mu \neq \lambda_1$ ($\lambda_2 + 1 = \lambda_1 + p_1 > \lambda_1$) and this proves the first assertion. Now to finish our computation of normal rank 1 sub-modules, we have $U = \rho.b^q$. Then the solutions in $\mathbb{C}[[b]]$ of the equation

$$\rho.b^{q-1} + b.V' - (p_1 + q - 1).V = 0$$

are given by : $V = -(\rho/p).b^{q-1} + \sigma.b^{p_1+q-1}$ and the condition $x \notin b.E$ implies now $q = 1$. So we obtain $\mu = \lambda_2 + 1$ and for each $\tau \in \mathbb{C}$ the element $x = (1 - p_1.\tau.b^{p_1}).e_1 - p_1.b.e_2$ generates a normal rank 1 sub-module isomorphic to E_{λ_2+1} . And with the unique sub-module isomorphic to E_{λ_1} they are all the normal rank 1 sub-modules in such an E . ■

END OF THE PROOF OF THEOREM 1.2.5. As the result is obvious for $k = 1$, we may assume $k \geq 2$ and the result proved in rank $\leq k - 1$. Let $F_j, j \in [1, k]$ and $G_j, j \in [1, k]$ two J-H. principal sequences for E . As the sequences $\lambda_j + j$ and $\mu_j + j$ coïncide up to the order (they are of the form $-x_j + k$ where the $(x_j)_{j \in [1, k]}$ are the roots of the Bernstein polynomial, counting multiplicities) and are both increasing, they coïncide. Now let j_0 be the first integer in $[1, k]$ such that $F_{j_0} \neq G_{j_0}$. If $j_0 \geq 2$ applying the induction hypothesis to E/F_{j_0-1} gives $F_{j_0}/F_{j_0-1} = G_{j_0}/F_{j_0-1}$ and so $F_{j_0} = G_{j_0}$.

So we may assume that $j_0 = 1$. Let H be the normalization of $F_1 + G_1$. As F_1 and G_1 are normal rank 1 and distinct, then H is a rank 2 normal sub-module.

It is a $[\lambda]$ -primitive fresco of rank 2 with two different normal rank 1 sub-modules which are isomorphic as $\lambda_1 = \mu_1$. Moreover the principal J-H. sequence of H begins by a normal submodule isomorphic to E_{λ_1} . This contradicts the previous lemma and so $F_1 = G_1$. So for any $j \in [1, k]$ we have $F_j = G_j$. ■

NOTATION. For a fresco E we shall denote $F_j(E)$ the j -th term of the principal J.H. sequence of E . Remark that the isomorphism class of the fresco $F_j(E)$ depends only on the isomorphism class of E , thanks to the previous theorem : it implies that any isomorphism $f : E \rightarrow E'$ of frescos sends $F_j(E)$ on $F_j(E')$.

Remark also that for any change of variable⁵ $\theta \in \mathbb{C}[[a]]$ we have a natural isomorphism $\theta_*(F_j(E)) \simeq F_j(\theta_*(E))$, again as a consequence of the previous theorem and the invariance of the Bernstein polynomial by changes of variable.

Definition 1.2.7 Let E be a fresco and consider its principal J-H. sequence $F_j, j \in [1, k]$. Put $F_j/F_{j-1} \simeq E_{\lambda_j}$ for $j \in [1, k]$ (with $F_0 = \{0\}$). We shall call the **fundamental invariants** of E the ordered k -tuple $(\lambda_1, \dots, \lambda_k)$.

Of course, if we have any J-H. sequence for a fresco E with rank 1 quotients associated to the rational numbers μ_1, \dots, μ_k , it is easy to recover the fundamental invariants of E because the numbers $\mu_j + j, j \in [1, k]$ are the same than the numbers $\lambda_j + j$ up to a permutation. But as the sequence $\lambda_j + j$ is increasing with j for the fixed order on \mathbb{Q} , it is enough to put the $\mu_j + j$ in the increasing order with j to recover the fundamental invariants of E .

Corollary 1.2.8 Let $\lambda_1, \dots, \lambda_k$ the fundamental invariants of a rank k fresco E and assume that we have an isomorphism

$$E \simeq \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.P \quad \text{where} \quad P := (a - \lambda_1.b).S_1^{-1} \dots S_{k-1}^{-1}.(a - \lambda_k.b).S_k^{-1}$$

where S_1, \dots, S_k are invertible elements in $\mathbb{C}[[b]]$, by sending the generator $e \in E$ to $1 \in \tilde{\mathcal{A}}$. Then for each $j \in [1, k-1]$ the element

$$e_j := S_j^{-1}.(a - \lambda_{j+1}.b) \dots S_{k-1}^{-1}.(a - \lambda_k.b).S_k^{-1}.e$$

is a generator of the fresco F_j which is the j -th term in the principal J.H. sequence of E .

PROOF. By an easy induction, it is enough to prove that $S_{k-1}.e_{k-1} = (a - \lambda_k.b).e_k$ is a generator of the fresco F_{k-1} . So define $G := \tilde{\mathcal{A}}.(a - \lambda_k.b).e_k$. Let first prove that G is normal in E . So assume that for some $x \in E$ we have $b.x \in G$. We may write

$$x = u.e_k \quad \text{and} \quad b.x = v.(a - \lambda_k.b).e_k$$

⁵see [B.11].

for some $u, v \in \tilde{\mathcal{A}}$. So we have $b.u - v.(a - \lambda_k.b) \in \tilde{\mathcal{A}}.P.S_k$. This implies that $b.u$ is in $\tilde{\mathcal{A}}.(a - \lambda_k.b)$. But $\tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a - \lambda_k.b) \simeq E_{\lambda_k}$ has no b -torsion. Then we have $u = w.(a - \lambda_k.b)$ and $x = w.(a - \lambda_k.b).e_k$. So G is normal and E/G is isomorphic to E_{λ_k} . Then the uniqueness of the principal J.H. sequence of E implies the equality $G = F_{k-1}$. \blacksquare

So in the situation of the previous corollary we have a natural $\mathbb{C}[[b]]$ -basis e_1, \dots, e_k of E associated to a generator e of E . Note that the uniqueness of F_{k-1} implies the uniqueness of S_k up to a non zero constant and then we have the uniqueness of S_{k-1} again up to a non zero constant etc. . . . So P is unique up to a non zero constant also (and we may normalize it asking that the coefficient of a^k is 1). This means that the elements e_1, \dots, e_k are well determined by the choice of e up to a diagonal invertible matrix with coefficients in \mathbb{C} . And for each $j \in [1, k]$ we have $F_j = \tilde{\mathcal{A}}.e_j$.

Let $\lambda_1, \dots, \lambda_k$ be the fundamental invariants of a rank k fresco E_0 , and note $\mathcal{F}(\lambda_1, \dots, \lambda_k)$ the set of isomorphism classes of frescos with these fundamental invariants. The uniqueness of the principal J-H. sequence of a fresco allows to define for each (i, j) $1 \leq i < j \leq k$ a map

$$q_{i,j} : \mathcal{F}(\lambda_1, \dots, \lambda_k) \rightarrow \mathcal{F}(\lambda_i, \dots, \lambda_j)$$

defined by $q_{i,j}([E]) = [F_j/F_{i-1}]$ where $(F_h)_{h \in [0,k]}$ is the principal J-H. sequence of E . This makes sense because we know that any isomorphism $\varphi : E^1 \rightarrow E^2$ between two frescos induces isomorphisms between each term of the corresponding principal J-H. sequences.

Now from the corollary above it is clear that such an application is algebraic (see the definition in section 3).

For instance the classification of rank 2 $[\lambda]$ -primitive frescos gives (see example before proposition 1.2.1) for any rational number $\lambda_1 > 1$ and $p_1 \in \mathbb{N}$ the following description :

- for $p_1 = 0$ $\mathcal{F}(\lambda_1, \lambda_1 - 1) \simeq \{pt\}$.
- for $p_1 \geq 1$ $\alpha : \mathcal{F}(\lambda_1, \lambda_1 + p_1 - 1) \xrightarrow{\simeq} \mathbb{C}$.

where $\{pt\}$ is given by the isomorphism class of $\tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a - \lambda_1.b).(a - (\lambda_1 - 1).b)$ and where the isomorphism class associated to $\alpha^{-1}(x)$ for $x \in \mathbb{C}$ in the case $p_1 \geq 1$ is given by $\tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a - \lambda_1.b).(1 + x.b^{p_1})^{-1}.(a - (\lambda_1 + p_1 - 1).b)$.

This implies that the function $\alpha_j : \mathcal{F}(\lambda_1, \dots, \lambda_k) \rightarrow \mathbb{C}$ given by sending $[E]$ to the parameter $\alpha([F_{j+1}/F_{j-1}])$ for $j \in [1, k-1]$ is a polynomial (see section 3 for the definition). Remark that this function is not identically 0 or 1 if and only if $\lambda_{j+1} - \lambda_j + 1$ is a positive integer.

2 Semi-simplicity.

2.1 Semi-simple regular (a,b)-modules.

Definition 2.1.1 *Let E be a regular (a,b)-module. We say that E is **semi-simple** if it is a sub-module of a finite direct sum of rank 1 regular (a,b)-modules.*

Note that if E is a sub-module of a regular (a,b)-module it is necessary regular. As a direct sum of regular (a,b)-modules is regular, our assumption that E is regular is superfluous.

It is clear from this definition that a sub-module of a semi-simple (a,b)-module is semi-simple and that a (finite) direct sum of semi-simple (a,b)-modules is again semi-simple.

REMARK. A rather easy consequence of the classification of rank 2 (a,b)-modules given in [B.93] is that the rank 2 simple pole (a,b)-modules defined in the $\mathbb{C}[[b]]$ -basis x, y by the relations :

$$(a - (\lambda + p).b).x = b^{p+1}.y \quad \text{and} \quad (a - \lambda).y = 0$$

for any $\lambda \in \mathbb{C}$ and any $p \in \mathbb{N}$ are not semi-simple. We leave the verification of this point to the reader. \square

Let us begin by a characterization of the semi-simple (a,b)-modules which have a simple pole. First we shall prove that a quotient of a semi-simple (a,b)-module is semi-simple. This will be deduced from the following lemma.

Lemma 2.1.2 *Let E be a (a,b)-module which is direct sum of regular rank 1 (a,b)-modules, and let $F \subset E$ be a rank 1 normal sub-module. Then F is a direct factor of E .*

Corollary 2.1.3 *If E is a semi-simple regular (a,b)-module and F a normal sub-module of E , the quotient E/F is a (regular) semi-simple (a,b)-module.*

PROOF OF THE LEMMA. Let $E = \bigoplus_{j=1}^k E_{\lambda_j}$ and assume that $F \simeq E_\mu$. Let e_j be a standard generator of E_{λ_j} and e a standard generator of E_μ . Write

$$e = \sum_{j=1}^k S_j(b).e_j$$

and compute $(a - \mu.b).e = 0$ using the fact that $e_j, j \in [1, k]$ is a $\mathbb{C}[[b]]$ -basis of E and the relations $(a - \lambda_j.b).e_j = 0$ for each j . We obtain for each $j \in [1, k]$ the relation

$$b.S'_j - (\mu - \lambda_j).S_j = 0.$$

So, if $\mu - \lambda_j$ is not in \mathbb{N} , we have $S_j = 0$. When $\mu = \lambda_j + p_j$ with $p_j \in \mathbb{N}$ we obtain $S_j(b) = \rho_j \cdot b^{p_j}$ for some $\rho_j \in \mathbb{C}$. As we assume that e is not in $b \cdot E$, there exists at least one $j_0 \in [1, k]$ such that $p_{j_0} = 0$ and $\rho_{j_0} \neq 0$. Then it is clear that we have

$$E = F \oplus \left(\bigoplus_{j \neq j_0} E_{\lambda_j} \right)$$

concluding the proof. ■

PROOF OF THE COROLLARY. We argue by induction on the rank of F . In the rank 1 case, we have $F \subset E \subset \mathcal{E} := \bigoplus_{j=1}^k E_{\lambda_j}$. Let \tilde{F} the normalization of F in $\bigoplus_{j=1}^k E_{\lambda_j}$. Then the lemma shows that there exists a $j_0 \in [1, k]$ such that

$$\mathcal{E} = \tilde{F} \oplus \left(\bigoplus_{j \neq j_0} E_{\lambda_j} \right).$$

Then, as $\tilde{F} \cap E = F$, the quotient map $\mathcal{E} \rightarrow \mathcal{E}/\tilde{F} \simeq \bigoplus_{j \neq j_0} E_{\lambda_j}$ induces an injection of E/F in a direct sum of regular rank 1 (a, b) -modules. So E/F is semi-simple. Assume now that the result is proved for F with rank $\leq d-1$ and assume that F has rank d . Then using a rank 1 normal sub-module G in F , we obtain that F/G is a normal rank $d-1$ sub-module of E/G . Using the rank 1 case we know that E/G is semi-simple, and the induction hypothesis gives that

$$E/F = (E/G) / (F/G)$$

is semi-simple. ■

Proposition 2.1.4 *Let E be a simple pole semi-simple (a, b) -module. Then E is a direct sum of regular rank 1 (a, b) -modules.*

PROOF. We shall prove the proposition by induction on the rank k of E . As the rank 1 case is obvious, assume the proposition proved for $k-1$ and that E is a simple pole rank $k \geq 2$ semi-simple (a, b) -module. From the existence of J-H. sequence, we may find an exact sequence

$$0 \rightarrow F \rightarrow E \rightarrow E_\lambda \rightarrow 0$$

where F has rank $k-1$. By definition F is semi-simple, but it also has a simple pole because $a \cdot F \subset a \cdot E \subset b \cdot E$ implies that $a \cdot F \subset F \cap b \cdot E = b \cdot F$ as F is normal in E . So by the induction hypothesis F is a direct sum of regular rank 1 (a, b) -modules.

Let $e \in E$ such that its image in E_λ is e_λ a standard generator of E_λ satisfying $a \cdot e_\lambda = \lambda \cdot b \cdot e_\lambda$. Then we have $(a - \lambda \cdot b) \cdot e \in F$.

We shall first look at the case $k = 2$.

So F is a rank 1 and we have $F \simeq E_\mu$ for some $\mu \in \mathbb{C}$. Let e_μ be a standard generator in F and put

$$(a - \lambda \cdot b) \cdot e = S(b) \cdot e_\mu.$$

Our simple pole assumption on E implies $S(0) = 0$ and we may write $S(b) = b.\tilde{S}(b)$. We look for $T \in \mathbb{C}[[b]]$ such that $\varepsilon := e + T(b).e_\mu$ satisfies

$$(a - \lambda.b).\varepsilon = 0.$$

If such a T exists, then we would have $E = E_\mu \oplus E_\lambda$ where E_λ is the submodule generated by ε , because it is clear that we have $E = F \oplus \mathbb{C}[[b]].e$ as a $\mathbb{C}[[b]]$ -module.

To find T we have to solve in $\mathbb{C}[[b]]$ the differential equation

$$b.\tilde{S}(b) + (\mu - \lambda).b.T(b) + b^2.T'(b) = 0.$$

If $\lambda - \mu$ is not a non negative integer, such a T exists and is unique. But when $\lambda = \mu + p$ with $p \in \mathbb{N}$, the solution exists if and only if the coefficient of b^p in \tilde{S} is zero. If it is not the case, define \tilde{T} as the solution of the differential equation

$$\tilde{S}(b) - \alpha.b^p + b.\tilde{T}'(b) - p.\tilde{T}(b) = 0$$

where $\alpha \neq 0$ is the coefficient of b^p in \tilde{S} and where we choose \tilde{T} by asking that it has no b^p term. Then $\varepsilon_1 := e + \tilde{T}(b).e_{\lambda-p}$ satisfies

$$(a - (\lambda - p).b).\varepsilon_1 = \alpha.b^{p+1}.e_{\lambda-p}.$$

Then, after changing $\lambda - p$ in λ and $e_{\lambda-p}$ in $\alpha.e_{\lambda-p}$, we recognize one of the rank 2 modules which appears in the remark following the definition 2.1.1, and which is not semi-simple. So we have a contradiction. This concludes the rank 2 case.

Now consider the case $k \geq 3$ and using the induction hypothesis write

$$F \simeq \bigoplus_{j=1}^{k-1} E_{\mu_j}$$

and denote e_j a standard generator for E_{μ_j} . Write

$$(a - \lambda.b).e = \sum_{j=1}^{k-1} S_j(b).e_j.$$

The simple pole assumption again gives $S_j(0) = 0$ for each j ; now we look for T_1, \dots, T_{k-1} in $\mathbb{C}[[b]]$ such that, defining $\varepsilon := e + \sum_{j=1}^{k-1} T_j(b).e_j$, we have

$$(a - \lambda.b).\varepsilon = 0.$$

For each j this leads to the differential equation

$$b.\tilde{S}_j(b) + (\mu_j - \lambda).b.T_j(b) + b^2.T_j'(b) = 0$$

where we put $S_j(b) = b.\tilde{S}_j(b)$. We are back, for each given $j \in [1, k-1]$, to the previous problem in the rank 2 case. So if for each $j \in [1, k-1]$ we have a solution $T_j \in \mathbb{C}[[b]]$ it is easy to conclude that

$$E \subset F \oplus \mathbb{C}[[b]].\varepsilon \simeq F \oplus E_\lambda.$$

If for some j_0 there is no solution, the coefficient of b^{j_0} in \tilde{S}_{j_0} does not vanish and then the image of E in $E / \bigoplus_{j \neq j_0} E_{\mu_j}$ is a rank 2 not semi-simple (a,b)-module. This contradicts the corollary 2.1.3 and concludes the proof. ■

REMARK. Let E be a semi-simple rank k (a,b) -module and let E^\sharp its saturation by $b^{-1}.a$, then E^\sharp is semi-simple because there exists $N \in \mathbb{N}$ with an inclusion $E^\sharp \subset b^{-N}.E$. Also E^\flat , the maximal simple pole sub-module of E , is semi-simple. Then we have

$$E^\flat \simeq \bigoplus_{j=1}^k E_{\lambda_j+d_j} \subset E \subset E^\sharp \simeq \bigoplus_{j=1}^k E_{\lambda_j}$$

where d_1, \dots, d_k are non negative integers.

Corollary 2.1.5 *Let E be a semi-simple (a,b) -module. Then its dual E^* is semi-simple.*

PROOF. For a regular (a,b) -module E we have $(E^\flat)^* = (E^*)^\sharp$. But the dual of E_λ is $E_{-\lambda}$ and the duality commutes with direct sums. So we conclude that $E^* \subset (E^\flat)^*$ and so E^* is a semi-simple. ■

REMARK. The tensor product of two semi-simple (a,b) -modules is semi-simple as a consequence of the fact that $E_\lambda \otimes E_\mu \simeq E_{\lambda+\mu}$.

2.2 The semi-simple filtration.

Definition 2.2.1 *Let E be a regular (a,b) -module and x an element in E . We shall say that x is **semi-simple** if $\tilde{\mathcal{A}}.x$ is a semi-simple (a,b) -module.*

It is clear that any element in a semi-simple (a,b) -module is semi-simple. The next lemma shows that the converse is true.

Lemma 2.2.2 *Let E be a regular (a,b) -module such that any $x \in E$ is semi-simple. Then E is semi-simple.*

PROOF. Let e_1, \dots, e_k be a $\mathbb{C}[[b]]$ -basis of E . Then each $\tilde{\mathcal{A}}.e_j$ is semi-simple, and the map $\bigoplus_{j=1}^k \tilde{\mathcal{A}}.e_j \rightarrow E$ is surjective. So E is semi-simple thanks to the corollary 2.1.3 and the comment following the definition 2.1.1. ■

Lemma 2.2.3 *Let E be a regular (a,b) -module. The subset $S_1(E)$ of semi-simple elements in E is a normal submodule in E .*

PROOF. As a direct sum and quotient of semi-simple (a,b) -modules are semi-simple, it is clear that for x and y semi-simple the sum $\tilde{\mathcal{A}}.x + \tilde{\mathcal{A}}.y$ is semi-simple. So $x + y$ is semi-simple. This implies that $S_1(E)$ is a sub-module of E . If $b.x$ is in $S_1(E)$, then $\tilde{\mathcal{A}}.b.x$ is semi-simple. Then $\tilde{\mathcal{A}}.x \simeq \tilde{\mathcal{A}}.b.x \otimes E_{-1}$ is also semi-simple, and so $S_1(E)$ is normal in E . ■

Definition 2.2.4 Let E be a regular (a,b) -module. Then the sub-module $S_1(E)$ of semi-simple elements in E will be called the **semi-simple part** of E . Defining $S_j(E)$ as the pull-back on E of the semi-simple part of $E/S_{j-1}(E)$ for $j \geq 1$ with the initial condition $S_0(E) = \{0\}$, we define a sequence of normal sub-modules in E such that $S_j(E)/S_{j-1}(E) = S_1(E/S_{j-1}(E))$. We shall call it the **semi-simple filtration** of E . The smallest integer d such we have $S_d(E) = E$ will be called the **semi-simple depth** of E and we shall denote it $d(E)$.

REMARKS.

- i) As $S_1(E)$ is the maximal semi-simple sub-module of E it contains any rank 1 sub-module of E . So $S_1(E) = \{0\}$ happens if and only if $E = \{0\}$.
- ii) Then the ss-filtration of E is strictly increasing for $0 \leq j \leq d(E)$.
- iii) It is easy to see that for any submodule F in E we have $S_j(F) = S_j(E) \cap F$. So $S_j(E)$ is the subset of $x \in E$ such that $d(x) := d(\tilde{\mathcal{A}}.x) \leq j$ and $d(E)$ is the supremum of $d(x)$ for $x \in E$. \square

Lemma 2.2.5 Let $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ be an exact sequence of regular (a,b) -modules such that F and G are semi-simple and such that for any roots λ, μ of the Bernstein polynomial of F and G respectively we have $\lambda - \mu \notin \mathbb{Z}$. Then E is semi-simple.

PROOF. By an easy induction we may assume that F is rank 1 and also that G is rank 1. Then the conclusion follows from the classification of the rank 2 regular (a,b) -module. \blacksquare

APPLICATION. A regular (a,b) -module E is semi-simple if and only if for each $\lambda \in \mathbb{C}$ its primitive part $E[\lambda]$ is semi-simple.

The following easy facts are left as exercices for the reader.

1. Let $0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$ be a short exact sequence of regular (a,b) -modules. Then we have the inequalities

$$\sup\{d(F), d(G)\} \leq d(E) \leq d(F) + d(G).$$

2. Let $E \subset \Xi_\lambda^{(N)} \otimes V$ be a sub-module where V is a finite dimensional complex space. Then for each $j \in [1, N+1]$ we have

$$S_j(E) = E \cap \left(\Xi_\lambda^{(j-1)} \otimes V \right).$$

3. In the same situation as above, the minimal N for such an embedding of a geometric $[\lambda]$ -primitive E is equal to $d(E) - 1$.
4. In the same situation as above, the equality $S_1(E) = E \cap \Xi_\lambda^{(0)} \otimes V$ implies $\text{rank}(S_1(E)) \leq \dim_{\mathbb{C}} V$. It is easy to see that any $[\lambda]$ -primitive semi-simple geometric (a,b) -module F may be embedded in $\Xi_\lambda^{(0)} \otimes V$ for some V of dimension $\text{rank}(F)$. But it is also easy to prove that when $F = S_1(E)$ with E as above, such an embedding may be extended to an embedding of E into $\Xi_\lambda^{(d(E)-1)} \otimes V$. \square

2.3 The co-semi-simple filtration.

We shall discuss briefly the filtration deduced by duality from the semi-simple filtration of a regular (a,b) -module.

Lemma 2.3.1 *For any regular (a,b) -module E there exists a maximal co-semi-simple normal sub-module, noted $\Sigma_1(E)$. That is to say that any normal sub-module F of E such that E/F is semi-simple contains $\Sigma_1(E)$, and that the quotient $E/\Sigma_1(E)$ is semi-simple.*

REMARK. We have $\Sigma_1(E) = 0$ if and only if E is semi-simple. Note also that for $E \neq 0$ the inclusion $\Sigma_1(E) \subset E$ is strict because E always has a co-rank 1 normal sub-module, and any rank 1 regular (a,b) -module is semi-simple. \square

PROOF. We shall prove that we have $\Sigma_1(E) = (E^*/S_1(E^*))^*$. Dualizing the exact sequence of (a,b) -modules

$$0 \rightarrow S_1(E^*) \rightarrow E^* \rightarrow E^*/S_1(E^*) \rightarrow 0$$

we see that $G := (E^*/S_1(E^*))^*$ is a normal submodule of E and that E/G is semi-simple because its dual $S_1(E^*)^*$ is semi-simple.

Consider now a normal sub-module H in E such that E/H is semi-simple. Then we have an exact sequence

$$0 \rightarrow (E/H)^* \rightarrow E^* \rightarrow H^* \rightarrow 0$$

where the sub-module $(E/H)^*$ is semi-simple. So it is contained in $S_1(E^*)$. The surjective map $H^* \simeq E^*/(E/H)^* \rightarrow G^*$ implies that $G \subset H$ proving the minimality of G . \blacksquare

Definition 2.3.2 *Let E a regular (a,b) -module. Define inductively, for any $h \geq 2$, the normal sub-modules $\Sigma_h(E) := \Sigma_1(\Sigma_{h-1}(E))$. We shall call this filtration of E the **co-semi-simple filtration** of E .*

Corollary 2.3.3 *For any regular (a,b) -module E and any $h \in [1, d(E^*)]$ we have a natural isomorphism*

$$\Sigma_h(E) \simeq (E^*/S_h(E^*))^*.$$

PROOF. As the case $h = 1$ has been proved in the previous lemma, assume $h \geq 2$ and the isomorphism obtained for $h - 1$. Note $F := \Sigma_{h-1}(E)$; the case $h = 1$ applied to F gives $\Sigma_1(F) \simeq (F^*/S_1(F^*))^*$. The induction hypothesis implies

$$F^* \simeq E^*/S_{h-1}(E^*)$$

so we have $\Sigma_1(F)^* \simeq E^*/S_{h-1}(E^*)/S_1(E^*/S_{h-1}(E^*)) \simeq E^*/S_h(E^*)$ and we conclude by dualizing this isomorphism. \blacksquare

An immediate consequence of this corollary is that we have $\Sigma_{d-1}(E) \neq 0$ and $\Sigma_d(E) = 0$ for $d = d(E^*)$.

Lemma 2.3.4 *Let E a regular (a,b) -module. For any $h \geq 1$ we have*

$$\Sigma_h(E) \subset S_{d-h}(E)$$

with $d := d(E)$.

PROOF. We shall make an induction on $h \geq 1$. As $E/S_{d-1}(E)$ is semi-simple, we have $\Sigma_1(E) \subset S_{d-1}(E)$, by definition of $\Sigma_1(E)$. Assume the inclusion proved for $h - 1 \geq 1$. Then as $S_{d-h+1}(E)/S_{d-h}(E) = S_1(E/S_{d-h}(E))$ is semi-simple, the quotient

$$\Sigma_{h-1}(E)/\Sigma_{h-1}(E) \cap S_{d-h}(E) \hookrightarrow S_{d-h+1}(E)/S_{d-h}(E)$$

is also semi-simple, and this implies that

$$\Sigma_h(E) = \Sigma_1(\Sigma_{h-1}(E)) \subset \Sigma_{h-1}(E) \cap S_{d-h}(E) \subset S_{d-h}(E)$$

which gives the result. \blacksquare

Corollary 2.3.5 *For any regular (a,b) -module we have $d(E^*) = d(E)$.*

PROOF. As $S_1(E)$ is semi-simple, $\Sigma_{d(E)-1}(E)$ is semi-simple, and then $\Sigma_{d(E)}(E) = 0$. Using the remark following the corollary 2.3.3, we deduce that $d(E) \geq d(E^*)$. By duality, we obtain the equality. \blacksquare

2.4 The case of frescos.

In the case of a fresco, we shall give a simple characterization of the semi-simple filtration. Let me begin by a simple lemma.

- Lemma 2.4.1** • *If $T \subset E$ is a $[\lambda]$ -primitive theme in a fresco E , its normalization is also a $[\lambda]$ -primitive theme (of same rank than T).*
- *If $S \subset E$ is a semi-simple fresco in a fresco E , its normalization is also a semi-simple fresco.*

PROOF. Note first that a normal sub-module of a fresco is a fresco. So the normalization \tilde{T} will be a $[\lambda]$ -primitive theme if we can show that it contains a unique rank 1 normal submodule⁶. But as T has an unique normal rank 1 submodule L , it is clear that the only rank 1 normal sub-module of \tilde{T} is the normalization \tilde{L} of L .

If now $S \subset E$ is a semi-simple fresco, then if a $\tilde{\mathcal{A}}$ -linear map $\varphi : \tilde{S} \rightarrow \Xi_\lambda$ has rank ≥ 2 , then its restriction to S has the same rank as S has finite codimension in its normalization \tilde{S} and this contradicts the semi-simplicity of S . ■.

For a fresco E we have the following characterization for the semi-simple (and co-semi-simple) filtration of E .

Proposition 2.4.2 *Let E be a fresco. Then we have the following properties :*

- i) *Any $[\lambda]$ -primitive sub-theme T in E of rank j is contained in $S_j(E)$.*
- ii) *Any $[\lambda]$ -primitive quotient theme T of $S_j(E)$ has rank $\leq j$.*
- iii) *For any $j \in \mathbb{N}$ we have*

$$S_j(E) = \cap_{\lambda} \cap_{\varphi \in \text{Hom}_{\tilde{\mathcal{A}}}(E, \Xi_\lambda)} [\varphi^{-1}(F_j(\varphi))]$$

where $F_j(\varphi)$ is the normal sub-module of rank j of the $[\lambda]$ -primitive theme $\varphi(E)$, with the convention that $F_j(\varphi) = \varphi(E)$ when the rank of φ is $\leq j$.

- iv) *The ss-depth of E is equal to d if and only if d is the maximal rank of a $[\lambda]$ -primitive quotient theme of E .*
- v) *The ss-depth of E is equal to d if and only if d is the maximal rank of a normal $[\lambda]$ -primitive sub-theme of E .*

⁶see [B. 10].

PROOF OF PROPOSITION 2.4.2. Let us prove i) by induction on j . As the case $j = 1$ is obvious, let us assume that $j \geq 2$ and that the result is proved for $j - 1$. Let T a $[\lambda]$ -primitive theme in E , and let $F_{j-1}(T)$ be its normal submodule of rank $j - 1$ (equal to T if the rank of T is less than $j - 1$). Then by the induction hypothesis, we have $F_{j-1}(T) \subset S_{j-1}(E)$. Then we have a $\tilde{\mathcal{A}}$ -linear map $T/F_{j-1}(T) \rightarrow E/S_{j-1}(E)$. If the rank of T is at most j , then $T/F_{j-1}(T)$ has rank at most 1 and its image is in $S_1(E/S_{j-1}(E))$. So $T \subset S_j(E)$.

To prove ii) we also make an induction on j . The case $j = 1$ is obvious. So we may assume $j \geq 2$ and the result proved for $j - 1$. Let $\varphi : S_j(E) \rightarrow T$ a surjective map on a $[\lambda]$ -primitive theme T . By the inductive hypothesis we have $\varphi(S_{j-1}(E)) \subset F_{j-1}(T)$. So we have an induced surjective map

$$\tilde{\varphi} : S_j(E)/S_{j-1}(E) \rightarrow T/F_{j-1}(T).$$

As $S_j(E)/S_{j-1}(E)$ is semi-simple, the image of $\tilde{\varphi}$ has rank ≤ 1 . It shows that T has rank $\leq j$.

To prove iii) consider first a $\tilde{\mathcal{A}}$ -linear map $\varphi : E \rightarrow \Xi_\lambda$. As $\varphi(E)$ is a $[\lambda]$ -primitive theme, $\varphi(S_j(E))$ is a $[\lambda]$ -primitive theme quotient of $S_j(E)$. So its rank is $\leq j$ and we have $\varphi(S_j(E)) \subset F_j(\varphi)$.

Conversely, for any $\tilde{\mathcal{A}}$ -linear map $\varphi : E \rightarrow \Xi_\lambda$, the image $\varphi(S_j(E))$ is a $[\lambda]$ -primitive quotient theme of $S_j(E)$. So its rank is $\leq j$ and it is contained in $F_j(\varphi)$.

Let us prove iv). If $S_d(E) = E$ then any $[\lambda]$ -primitive sub-theme in E has rank $\leq d$ thanks to ii). Conversely, assume that for any $[\lambda]$ any $[\lambda]$ -primitive sub-theme of E has rank $\leq d - 1$ and $S_{d-1}(E) \neq E$. Then choose a $\tilde{\mathcal{A}}$ -linear map $\varphi : E \rightarrow \Xi_\lambda$ such that $\varphi^{-1}(F_{d-1}(\varphi)) \neq E$. Then $\varphi(E)$ is a $[\lambda]$ -primitive theme of rank d which is a quotient of E , thanks to the following lemma 2.4.3.

The point v) is an easy consequence of the fact that $d(E) = d(E \otimes E_\lambda)$ for any $\lambda \in \mathbb{C}$ and any regular (a,b)-module E and the fact that for a fresco E and an integer N large enough, $E^* \otimes E_N$ is again a fresco. So $d(E) = d(E^* \otimes E_N)$ is the maximal rank of a $[N - \lambda]$ -primitive quotient theme T of $E^* \otimes E_N$; for such T , $T^* \otimes E_N$ is a $[\lambda]$ -primitive sub-theme of E of rank $d(E)$. The point i) allows to conclude. ■

Lemma 2.4.3 *Let E be a rank k $[\lambda]$ -primitive theme and denote by F_j its normal rank j submodule. Let $x \in E \setminus F_{k-1}$. Then the (a,b)-module $\tilde{\mathcal{A}}.x \subset E$ is a rank k theme.*

PROOF. We may assume $E \subset \Xi_\lambda^{(k-1)}$ and then (see [B.10]) we have the equality $F_{k-1} = E \cap \Xi_\lambda^{(k-2)}$. So x contains a non zero term with $(Log s)^{k-1}$ and then the result is clear. ■

We conclude this paragraph showing that any geometric (a,b)-module contains a fresco with finite \mathbb{C} -codimension.

Lemma 2.4.4 *Let E be a geometric (a,b) -module of rank k . There exists a fresco $E' \subset E$ with rank k ; so the quotient E/E' is a finite dimensional complex vector space.*

PROOF. We shall prove this fact by induction on the rank k of E . As the statement is obvious for $k \leq 1$, assume $k \geq 2$ and the result proved in rank $k-1$. As there exists a normal rank 1 submodule of E , consider an exact sequence

$$0 \rightarrow E_\lambda \rightarrow E \xrightarrow{\pi} F \rightarrow 0$$

where F is a rank $k-1$ geometric (a,b) -module. Let $x \in E$ such that $\pi(x)$ generates a rank $k-1$ fresco in F . Let $P \in \tilde{\mathcal{A}}$ be a monic degree $k-1$ polynomial in a with coefficients in $\mathbb{C}[[b]]$, which generates the annihilator of $\pi(x)$ in F . Then $P.x$ is in E_λ . We may assume that there exists an invertible element $S \in \mathbb{C}[[b]]$ and an integer m such that $S.P.x = b^m.e_\lambda$, where e_λ is a standard generator of E_λ , because, if we have $P.x = 0$ we may replace x by $x + b^N.e_\lambda$, and for $N \gg 1$ we have $P.b^N.e_\lambda \neq 0$. Then we have $\tilde{\mathcal{A}}.x \cap E_\lambda = b^m.E_\lambda$ and so the exact sequence

$$0 \rightarrow E_\lambda/b^m.E_\lambda \rightarrow E/\tilde{\mathcal{A}}.x \rightarrow F/\tilde{\mathcal{A}}.\pi(x) \rightarrow 0$$

gives the finiteness of the complex vector space $E/\tilde{\mathcal{A}}.x$. ■

2.5 A characterization of semi-simple frescos.

We begin by a simple remark : A $[\lambda]$ -primitive theme is semi-simple if and only if it has rank ≤ 1 . This is an easy consequence of the fact that the saturation of a rank 2 $[\lambda]$ -primitive theme is one of the rank 2 (a,b) -modules considered in the remark following the definition 2.1.1 which are not semi-simple

Lemma 2.5.1 *A geometric (a,b) -module E is **semi-simple** if and only if any quotient of E which is a $[\lambda]$ -primitive theme for some $[\lambda] \in \mathbb{Q}/\mathbb{Z}$ is of rank ≤ 1 .*

PROOF. The condition is clearly necessary as a quotient of a semi-simple (a,b) -module is semi-simple (see corollary 2.1.3), thanks to the remark above. Using the application of the lemma 2.2.5, it is enough to consider the case of a $[\lambda]$ -primitive E to prove that the condition is sufficient. Let $\varphi : E \rightarrow \Xi_\lambda^{(N)} \otimes V$ be an embedding of E which exists thanks to the embedding theorem 4.2.1. of [B.09], we obtain that each component of this map in a basis v_1, \dots, v_p of V has rank at most 1 as its image is a $[\lambda]$ -primitive theme. Then each of these images is isomorphic to some $E_{\lambda+q}$ for some integer q . So we have in fact an embedding of E in a direct sum of $E_{\lambda+q_i}$ and E is semi-simple. ■

REMARK. The preceeding proof shows that a geometric (a,b)-module is a non zero $[\lambda]$ -primitive theme for some $[\lambda] \in \mathbb{Q}/\mathbb{Z}$ if and only if $S_1(E)$ has rank 1 (compare with the theorem 3.1.7 of [B.10]). \square

Lemma 2.5.2 *Let E be a semi-simple fresco with rank k and let $\lambda_1, \dots, \lambda_k$ be the numbers associated to a J-H. sequence of E . Let μ_1, \dots, μ_k be a twisted permutation⁷ of $\lambda_1, \dots, \lambda_k$. Then there exists a J-H. sequence for E with quotients corresponding to μ_1, \dots, μ_k .*

PROOF. As the symmetric group \mathfrak{S}_k is generated by the transpositions $t_{j,j+1}$ for $j \in [1, k-1]$, it is enough to show that, if E has a J-H. sequence with quotients given by the numbers $\lambda_1, \dots, \lambda_k$, then there exists a J-H. sequence for E with quotients $\lambda_1, \dots, \lambda_{j-1}, \lambda_{j+1} + 1, \lambda_j - 1, \lambda_{j+2}, \dots, \lambda_k$ for $j \in [1, k-1]$. But $G := F_{j+1}/F_{j-1}$ is a rank 2 sub-quotient of E with an exact sequence

$$0 \rightarrow E_{\lambda_j} \rightarrow G \rightarrow E_{\lambda_{j+1}} \rightarrow 0.$$

As G is a rank 2 semi-simple fresco, it admits also an exact sequence

$$0 \rightarrow G_1 \rightarrow G \rightarrow G/G_1 \rightarrow 0$$

with $G_1 \simeq E_{\lambda_{j+1}+1}$ and $G/G_1 \simeq E_{\lambda_j-1}$. Let $q : F_{j+1} \rightarrow G$ be the quotient map. Now the J-H. sequence for E given by

$$F_1, \dots, F_{j-1}, q^{-1}(G_1), F_{j+1}, \dots, F_k = E$$

satisfies our requirement. \blacksquare

REMARK. If E is a semi-simple geometric (a,b)-module, we may have $G \simeq E_{\lambda_j} \oplus E_{\lambda_{j+1}}$ in the proof above, and then the conclusion does not hold. \square

Proposition 2.5.3 *Let E be a $[\lambda]$ -primitive fresco. A necessary and sufficient condition in order that E is semi-simple is that it admits a J-H. sequence with quotient corresponding to μ_1, \dots, μ_k such that the sequence $\mu_j + j$ is strictly decreasing.*

REMARKS.

1. As a fresco is semi-simple if and only if for each $[\lambda]$ its $[\lambda]$ -primitive part is semi-simple, this proposition gives also a criterium to semi-simplicity for any fresco.
2. This criterium is a very efficient tool to produce easily examples of semi-simple frescos.

⁷This means that the sequence $\mu_j + j, j \in [1, k]$ is a permutation (in the usual sens) of $\lambda_j + j, j \in [1, k]$.

PROOF. Remark first that if we have, for a fresco E , a J-H. sequence $F_j, j \in [1, k]$ such that $\lambda_j + j = \lambda_{j+1} + j + 1$ for some $j \in [1, k-1]$, then F_{j+1}/F_{j-1} is a sub-quotient of E which is a $[\lambda]$ -primitive theme of rank 2. So E is not semi-simple. As a consequence, when a fresco E is semi-simple the principal J-H. sequence corresponds to a strictly increasing sequence $\lambda_j + j$. Now, thanks to the previous lemma we may find a J-H. sequence for E corresponding to the strictly decreasing order for the sequence $\lambda_j + j$.

No let us prove the converse. We shall use the following lemma.

Lemma 2.5.4 *Let F be a rank k semi-simple $[\lambda]$ -primitive fresco and let $\lambda_j + j$ the strictly increasing sequence corresponding to its principal J-H. sequence. Let $\mu \in [\lambda]$ such that $0 < \mu + k < \lambda_1 + 1$. Then any fresco E in an exact sequence*

$$0 \rightarrow F \rightarrow E \rightarrow E_\mu \rightarrow 0$$

is semi-simple (and $[\lambda]$ -primitive).

PROOF. The case $k = 1$ is clear from the classification of rank 2 frescos; so assume that $k \geq 2$ and that we have a rank 2 quotient $\varphi : E \rightarrow T$ where T is a $[\lambda]$ -primitive theme. Then $\text{Ker } \varphi \cap F$ is a normal sub-module of F of rank $k-2$ or $k-3$ (for $k \geq 3$) in E . If $\text{Ker } \varphi \cap F$ is of rank $k-3$, the rank of $F/(\text{Ker } \varphi \cap F)$ is 2 and it injects in T via φ . So $F/(\text{Ker } \varphi \cap F)$ is a rank 2 $[\lambda]$ -primitive theme. As it is semi-simple, because F is semi-simple, we get a contradiction.

So the rank of $F/(\text{Ker } \varphi \cap F)$ is 1 and we have an exact sequence

$$0 \rightarrow F/(\text{Ker } \varphi \cap F) \rightarrow T \rightarrow E/F \rightarrow 0.$$

Put $F/(\text{Ker } \varphi \cap F) \simeq F_1(T) \simeq E_\nu$. Because T is a $[\lambda]$ -primitive theme, we have the inequality $\nu + 1 \leq \mu + 2$. Looking at a J-H. sequence of E ending by

$$\cdots \subset \text{Ker } \varphi \cap F \subset \varphi^{-1}(F_1(T)) = F \subset E$$

we see that $\nu + k - 1$ is in the set $\{\lambda_j + j, j \in [1, k]\}$ and, as $\lambda_1 + 1$ is the infimum of this set, we obtain $\lambda_1 + 1 \leq \nu + k - 1 \leq \mu + k$ contradicting our assumption. ■

END OF PROOF OF THE PROPOSITION 2.5.3. Now we shall prove by induction on the rank of a $[\lambda]$ -primitive fresco E that if it admits a J-H. sequence corresponding to a strictly decreasing sequence $\mu_j + j$, it is semi-simple. As the result is obvious in rank 1, we may assume $k \geq 1$ and the result proved for k . So let E be a fresco of rank $k+1$ and let $F_j, j \in [1, k+1]$ a J-H. sequence for E corresponding to the strictly decreasing sequence $\mu_j + j, j \in [1, k+1]$. Put $F_j/F_{j-1} \simeq E_{\mu_j}$ for all $j \in [1, k+1]$, define $F := F_k$ and $\mu := \mu_{k+1}$; then the induction hypothesis gives that F is semi-simple and we apply the previous lemma to conclude. ■

The following interesting corollary is an obvious consequence of the previous proposition.

Corollary 2.5.5 *Let E be a fresco and let $\lambda_1, \dots, \lambda_k$ be the numbers associated to any J-H. sequence of E . Let μ_1, \dots, μ_d be the numbers associated to any J-H. sequence of $S_1(E)$. Then, for $j \in [1, k]$, there exists a rank 1 normal sub-module of E isomorphic to E_{λ_j+j-1} if and only if there exists $i \in [1, d]$ such that we have $\lambda_j + j - 1 = \mu_i + i - 1$.*

Of course, this gives the list of all isomorphy classes of rank 1 normal sub-modules contained in E . So, using shifted duality, we get also the list of all isomorphy classes of rank 1 quotients of E . It is interesting to note that this gives the list of the possible initial exponents for maximal logarithmic terms which appear in the asymptotic expansion of a given relative de Rham cohomology class after integration on any vanishing cycle in the spectral subspace of the monodromy associated to the eigenvalue $\exp(2i\pi.\lambda)$.

2.6 An interesting example.

Our aim is to produce a rank 4 $[\lambda]$ -primitive fresco E with the following properties:

- i) Its semi-simple part $S_1(E)$ has rank 2 and equal to $F_2(E)$.
- ii) Its co-semi-simple part $\Sigma_1(E)$ has rank 1 and equal to $F_1(E)$.
- iii) There exists two normal rank 2 sub-themes in E with different fundamental invariants (but, of course, with $F_1(T) = \Sigma_1(E)$).

Of course the dual of E has a rank 3 semi-simple part and a rank 2 co-semi-simple part. So the rank of $S_1(E^*)$ is different from the rank of $S_1(E)$.

This example shows that the semi-simple and co-semi-simple filtrations of a geometric (a,b)-module do not behave as in the case of the semi-simple and co-semi-simple filtrations of a complex vector space associated to an endomorphism.

Fix $\lambda_1 > 3$ a rational number and $p_i, i = 1, 2, 3$ natural integers at least equal to 2 and define

$$\lambda_2 := \lambda_1 + p_1 - 1 \quad \lambda_3 := \lambda_2 + p_2 - 1 \quad \lambda_4 := \lambda_3 + p_3 - 1.$$

Fix also two non zero complex numbers α, β and define

$$E := \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.P \quad \text{with} \quad P := (a - \lambda_1.b).S^{-1}.(a - \lambda_2.b).(a - \lambda_3.b).(a - \lambda_4.b)$$

where $S := 1 + \alpha.b^{p_1+p_2} + \beta.b^{p_1+p_2+p_3}$.

PROOF. First remark that E is not semi-simple because the identity

$$(a - \lambda_2.b).(a - \lambda_3.b) = (a - (\lambda_3 + 1).b).(a - (\lambda_2 - 1).b)$$

and the fact that $\alpha \neq 0$, show that we have a rank 2 (normal) sub-theme in E with fundamental invariants $(\lambda_1, \lambda_3 + 1)$ and parameter $\alpha \neq 0$.

As the quotient $E/F_1(E)$ is semi-simple, we conclude that $\Sigma_1(E) = F_1(E)$.

To prove that $S_1(E) = F_2(E)$, we first remark that the inclusion $F_2(E) \subset S_1(E)$ is clear as F_2 is semi-simple. To prove the opposite inclusion we shall show that any rank 1 sub-module in E is contained in $F_2(E)$. This is enough to conclude because if $S_1(E)$ has rank 3, it has a rank 1 sub-module⁸ L with $L \cap F_2 = 0$. For that purpose consider the standard basis e_1, e_2, e_3, e_4 of E associated to its principal J.H. sequence. It satisfies

$$(a - \lambda_4.b).e_4 = e_3, \quad (a - \lambda_3.b).e_3 = e_2, \quad (a - \lambda_2.b).e_2 = S.e_1, \quad (a - \lambda_1.b).e_1 = 0.$$

Consider now an element in E , write $x := U_4.e_4 + U_3.e_3 + U_2.e_2 + U_1.e_1$ and assume that it satisfies : $(a - \mu.b).x = 0$ for some complex number μ . This equality is equivalent to the system of equations

$$\begin{aligned} b^2.U'_4 + (\lambda_4 - \mu).b.U_4 &= 0 \\ U_4 + b^2.U'_3 + (\lambda_3 - \mu).b.U_3 &= 0 \\ U_3 + b^2.U'_2 + (\lambda_2 - \mu).b.U_2 &= 0 \\ S.U_2 + b^2.U'_1 + (\lambda_1 - \mu).b.U_1 &= 0 \end{aligned}$$

If we have $\mu - \lambda_4 \notin \mathbb{N}$, the first equation gives $U_4 = 0$. If $\mu = \lambda_4 + q$, for some natural integer q , we find $U_4 = \rho.b^q$ for some complex number ρ . In the case $U \neq 0$ the second equation forces $q \geq 1$ and becomes $\rho.b^{q-1} + b.U'_3 - (p_3 + q - 1).U_3 = 0$ and so we have

$$U_3 = \sigma.b^{q-1} + \tau.b^{p_3+q-1} \quad \text{with} \quad \sigma = \rho/p_3.$$

Now the third equation forces $q \geq 2$ and becomes

$$(\sigma.b^{q-2} + \tau.b^{p_3+q-2}) + b.U'_2 - (p_2 + p_3 + q - 2).U_2 = 0.$$

Then U_2 is given by

$$U_2 = \sigma'.b^{q-2} + \tau'.b^{p_3+q-2} + \eta.b^{p_2+p_3+q-2}$$

where the complex number η is arbitrary and we have

$$\sigma' = \sigma/(p_2 + p_3) \quad \tau' = \tau/p_2.$$

⁸for instance the maximal simple pole sub-module $S_1(E)^b$ of $S_1(E)$ has a decomposition $(F_2 \cap S_1(E)^b) \oplus L$ so $L \cap F_2 = 0$.

Then the last equation will have a solution if and only if $U_2(0) = 0$, so $q \geq 3$, and there is no term in $b^{p_1+p_2+p_3+q-2}$ in $S.U_2$. As we assume that $\alpha.\beta \neq 0$ this is the case only when $\sigma' = \tau' = 0$, and so when $\sigma = \tau = 0$. Then we have $U_4 = U_3 = 0$ and x is in F_2 .

We come back to the case $U_4 = 0$. Then if $\mu - \lambda_3 \notin \mathbb{N}$ the second equation implies $U_3 = 0$, which conclude this case.

So we may assume $\mu = \lambda_3 + q$. Then we have $U_3 = \rho.b^q$. Again the third equation forces $q \geq 1$ and becomes

$$\rho.b^{q-1} + b.U_2' - (p_2 + q - 1).U_2 = 0.$$

So we have $U_2 = \rho'.b^{q-1} + \sigma.b^{p_2+q-1}$ with $\rho' = \rho/p_2$. Then, to solve the last equation, it is necessary that $S.U_2$ has no constant term (so $q \geq 2$) and no term in $b^{p_1+p_2+q-1}$; this implies $\rho' = \rho = 0$ as we assume $\alpha \neq 0$. So we must have $U_3 = 0$ and x is in F_2 in all cases. \blacksquare

3 Numerical criteria for semi-simplicity.

3.1 Polynomial dependance.

All \mathbb{C} -algebras have a unit.

When we consider a sequence of algebraically independent variables $\rho := (\rho_i)_{i \in \mathbb{N}}$ we shall denote $\mathbb{C}[\rho]$ the \mathbb{C} -algebra generated by these variables. Then $\mathbb{C}[\rho][[b]]$ will be the commutative \mathbb{C} -algebra of formal power series

$$\sum_{\nu=0}^{\infty} P_{\nu}(\rho).b^{\nu}$$

where $P_{\nu}(\rho)$ is an element in $\mathbb{C}[\rho]$ so a polynomial in $\rho_0, \dots, \rho_{N(\nu)}$ where $N(\nu)$ is an integer depending on ν . So each coefficient in the formal power series in b depends only on a finite number of the variables ρ_i .

Definition 3.1.1 *Let E be a (a, b) -module and let $e(\rho)$ be a family of elements in E depending on a family of variables $(\rho_i)_{i \in \mathbb{N}}$. We say that $e(\rho)$ **depends polynomially on** ρ if there exists a fixed $\mathbb{C}[[b]]$ -basis e_1, \dots, e_k of E such that*

$$e(\rho) = \sum_{j=1}^k S_j(\rho).e_j$$

where S_j is for each $j \in [1, k]$ an element in the algebra $\mathbb{C}[\rho][[b]]$.

REMARKS.

1. It is important to note that when $e(\rho)$ depends polynomially of ρ , then $a.e(\rho)$ also. Then for any $u \in \tilde{\mathcal{A}}$, again $u.e(\rho)$ depends polynomially on ρ .

2. It is easy to see that we obtain an equivalent condition on the family $e(\rho)$ by asking that the coefficient of $e(\rho)$ are in $\mathbb{C}[\rho][[b]]$ in a $\mathbb{C}[[b]]$ -basis of E whose elements depend polynomially of ρ .
3. The invertible elements in the algebra $\mathbb{C}[\rho][[b]]$ are exactly those elements with a constant term in b invertible in the algebra $\mathbb{C}[\rho]$. As we assume that the variables $(\rho)_{i \in \mathbb{N}}$ are algebraically independent, the invertible elements are those with a constant term in b in \mathbb{C}^* . \square

Proposition 3.1.2 *Let E be a rank k fresco and let $\lambda_1, \dots, \lambda_k$ its fundamental invariants. Let $e(\rho)$ be a family of generators of E depending polynomially on a family of algebraically independent variables $(\rho_i)_{i \in \mathbb{N}}$. Then there exists S_1, \dots, S_k in $\mathbb{C}[\rho][[b]]$ such that $S_j(\rho)[0] \equiv 1$ for each $j \in [1, k]$ and such that the annihilator of $e(\rho)$ in E is generated by the element of $\tilde{\mathcal{A}}$*

$$P(\rho) := (a - \lambda_1.b).S_1(\rho)^{-1} \dots S_{k-1}(\rho)^{-1}.(a - \lambda_k.b).S_k(\rho)^{-1}.$$

PROOF. The key result to prove this proposition is the rank 1 case. In this case we may consider a standard generator e_1 of E which is a $\mathbb{C}[[b]]$ -basis of E and satisfies

$$(a - \lambda_1.b).e_1 = 0.$$

Then, by definition, we may write

$$e(\rho) = S_1(\rho).e_1$$

where S_1 is in $\mathbb{C}[\rho][[b]]$ is invertible in this algebra, so has a constant term in \mathbb{C}^* . Up to normalizing e_1 , we may assume that $S_1(\rho)[0] \equiv 1$ and then define

$$P(\rho) := (a - \lambda_1.b).S_1(\rho)^{-1}.$$

It clearly generates the annihilator of $e(\rho)$ for each ρ .

Assume now that the result is already proved for the rank $k - 1 \geq 1$. Then consider the family $[e(\rho)]$ in the quotient E/F_{k-1} where F_{k-1} is the rank $k - 1$ sub-module of E in its principal J-H. sequence. Remark first that $[e(\rho)]$ is a family of generators of E/F_{k-1} which depends polynomially on ρ . This is a trivial consequence of the fact that we may choose a $\mathbb{C}[[b]]$ -basis e_1, \dots, e_k in E such that e_1, \dots, e_{k-1} is a $\mathbb{C}[[b]]$ basis of F_{k-1} and e_k maps to a standard generator of $E/F_{k-1} \simeq E_{\lambda_k}$. Then the rank 1 case gives $S_k \in \mathbb{C}[\rho][[b]]$ with $S_k(\rho)[0] = 1$ and such that $(a - \lambda_k.b).S_k(\rho)^{-1}.e(\rho)$ is in F_{k-1} for each ρ . But then, thanks to the corollary 1.2.8, it is a family of generators of F_{k-1} which depends polynomially on ρ and the inductive assumption allows to conclude. \blacksquare

Fix now the fundamental invariants $\lambda_1, \dots, \lambda_k$ of some fresco.

Definition 3.1.3 Consider a complex valued function f defined on a subset \mathcal{F}_0 of the isomorphism classes $\mathcal{F}(\lambda_1, \dots, \lambda_k)$ of frescos with fundamental invariants $\lambda_1, \dots, \lambda_k$. We shall say that f **depends polynomially on the isomorphism class** $[E] \in \mathcal{F}_0$ **of the fresco** E (or simply that f is a polynomial on \mathcal{F}_0) if the following condition is satisfied :

Let s be the collection of algebraically independent variables corresponding to the non constant coefficients of k éléments S_1, \dots, S_k in $\mathbb{C}[[b]]$ such that $S_j(0) = 1 \quad \forall j \in [1, k]$ and consider for each value of s the rank k fresco $E(s) := \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.P(s)$ where

$$P(s) := (a - \lambda_1.b)S_1(s)^{-1} \dots (a - \lambda_k.b).S_k(s)^{-1}$$

where $S_1(s), \dots, S_k(s)$ correspond to the given values for s .

Then there exists a polynomial $F \in \mathbb{C}[s]$ such that for each value of s such that $[E(s)]$ is in \mathcal{F}_0 , the value of $F(s)$ is equal to $f([E(s)])$.

Definition 3.1.4 We shall say that a subset in $\mathcal{F}(\lambda_1, \dots, \lambda_k)$ is **algebraic** if it is the common zero set of a finite set of polynomial functions on $\mathcal{F}(\lambda_1, \dots, \lambda_k)$. An algebraic stratification of $\mathcal{F}(\lambda_1, \dots, \lambda_k)$ is a finite family of algebraic subsets

$$\emptyset \subset S_N \subset \dots \subset S_0 = \mathcal{F}(\lambda_1, \dots, \lambda_k).$$

REMARK. Let \mathcal{G} be an algebraic subset in $\mathcal{F}(\lambda_1, \dots, \lambda_k)$ and $f : \mathcal{G} \rightarrow \mathbb{C}$ a polynomial, then $\{f = 0\} \subset \mathcal{G}$ is an algebraic subset of $\mathcal{F}(\lambda_1, \dots, \lambda_k)$. \square

Definition 3.1.5 Let $f : \mathcal{F}(\lambda_1, \dots, \lambda_k) \longrightarrow \mathcal{F}(\mu_1, \dots, \mu_l)$ an application. We shall say that f is algebraic if for each polynomial $p : \mathcal{F}(\mu_1, \dots, \mu_l) \rightarrow \mathbb{C}$ the composition $p \circ f$ is a polynomial.

REMARK. In the situation of the previous definition, let s the collection of algebraically independent variables corresponding to the non constant coefficients of k éléments S_1, \dots, S_k in $\mathbb{C}[[b]]$ satisfying $S_j(0) = 1 \quad \forall j \in [1, k]$; assume that there exist $T_1, \dots, T_l \in \mathbb{C}[s][[b]]$ such that for each s we have

$$f([E(s)]) = [\tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a - \mu_1.b).T_1(s)^{-1} \dots T_{l-1}(s)^{-1}.(a - \mu_l.b).T_l(s)^{-1}]$$

where we define $E(s) := \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.P(s)$, then the map f is algebraic.

EXAMPLES.

1. Let $(s_i)_{i \in \mathbb{N}^*}$ a family of algebraically independent variables and define $S(s) := 1 + \sum_{i=1}^{\infty} s_i.b^i \in \mathbb{C}[s][[b]]$. Let

$$E(s) := \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a - \lambda_1.b).S(s)^{-1}.(a - \lambda_2.b)$$

where $\lambda_1 > 1$ is rational and $\lambda_2 := \lambda_1 + p_1 - 1$. For $p_1 \in \mathbb{N}^*$ define $\alpha(s) := s_{p_1}$. For $p_1 = 0$ define $\alpha(s) \equiv 1$ and for $p_1 \notin \mathbb{N}$ define $\alpha(s) \equiv 0$. Then the number $\alpha(s)$ depends only of the isomorphism class of the fresco $E(s)$ and defines a polynomial on $\mathcal{F}(\lambda_1, \lambda_2)$.

2. Fix $\lambda_1, \dots, \lambda_k$ and (i, h) such that $[i, i+h] \subset [1, k]$ and consider the map

$$\pi : \mathcal{F}(\lambda_1, \dots, \lambda_k) \longrightarrow \mathcal{F}(\lambda_i, \dots, \lambda_{i+h})$$

given by $\pi([E]) = [F_{i+h}/F_{i-1}]$ where $(F_j)_{j \in [0, k]}$ is the principal J.H. sequence of E . Then this map is obviously algebraic, thanks to the previous remark.

3. Combining the two examples above, we see that for each $j \in [1, k-1]$ the function

$$\alpha_j : [E] \mapsto \alpha([F_{j+1}/F_{j-1}])$$

is a polynomial on $\mathcal{F}(\lambda_1, \dots, \lambda_k)$.

4. We shall define

$$\mathcal{F}_2(\lambda_1, \dots, \lambda_k) := (\alpha_1, \dots, \alpha_{k-1})^{-1}(0).$$

This is an algebraic subset of $\mathcal{F}(\lambda_1, \dots, \lambda_k)$. Note that if there exists at least one $j \in [1, k-1]$ with $p_j := \lambda_{j+1} - \lambda_j + 1 = 0$, then the subset $\mathcal{F}_2(\lambda_1, \dots, \lambda_k)$ is empty. Note also that in this case there is no semi-simple fresco with fundamental invariants $\lambda_1, \dots, \lambda_k$ as F_{j+1}/F_{j-1} is a theme.

Conversely, it is easy to see (for instance using the proposition 2.5.3 and the lemma 2.2.5) that if we have $p_j \neq 0$ for all $j \in [1, k-1]$ there exists a semi-simple rank k fresco with fundamental invariants $\lambda_1, \dots, \lambda_k$.

Remark that a class $[E]$ is in the subset $\mathcal{F}_2(\lambda_1, \dots, \lambda_k)$ if and only if any rank 2 sub-quotient of the principal J.H. sequence of E is semi-simple. \square

3.2 The β -invariant.

In this section we shall fix $k \geq 3$ and $\lambda_1, \dots, \lambda_k$ the fundamental invariants of a rank k $[\lambda]$ -primitive semi-simple fresco. So we have $p_j \geq 1$ for all $j \in [1, k-1]$ and $\lambda_1 > k-1$ is a rational number.

Let $\mathcal{F}_{k-1}(\lambda_1, \dots, \lambda_k)$ the subset of $\mathcal{F}(\lambda_1, \dots, \lambda_k)$ of isomorphism class of rank k $[\lambda]$ -primitive fresco E with invariants $\lambda_1, \dots, \lambda_k$ such that F_{k-1} and E/F_1 are semi-simple where $(F_j)_{j \in [1, k]}$ is the principal J.H. sequence of E .

Fix E with $[E]$ in $\mathcal{F}_{k-1}(\lambda_1, \dots, \lambda_k)$. Then E admits a generator e such that $(a - \lambda_{k-1}.b).(a - \lambda_k.b).e$ lies in F_{k-2} . This is a consequence of the fact that E/F_{k-2} is semi-simple.

Lemma 3.2.1 *In the above situation, let e be a generator of E such that*

$$(a - \lambda_{k-1}.b).(a - \lambda_k.b).e$$

lies in F_{k-2} . Then the sub-module G_{k-1} of E generated by the element $(a - (\lambda_{k-1} - 1).b).e$ is a normal rank $k-1$ sub-module of E which contains F_{k-2} and which is in $\mathcal{F}_{k-2}(\lambda_1, \dots, \lambda_{k-2}, \lambda_k + 1)$.

PROOF. Using the identity in $\tilde{\mathcal{A}}$:

$$(a - \lambda_{k-1}.b).(a - \lambda_k.b) = (a - (\lambda_k + 1).b).(a - (\lambda_{k-1} - 1).b)$$

we see that G_{k-1} contains F_{k-2} (because $(a - \lambda_{k-1}.b).(a - \lambda_k.b).e$ is a generator of F_{k-2} thanks to the corollary 1.2.8), that $G_{k-1}/F_{k-2} \simeq E_{\lambda_k+1}$ and so G_{k-1} admits

$$F_1 \subset \cdots \subset F_{k-2} \subset G_{k-1}$$

as principal J-H. sequence. Then the fundamental invariants for G_{k-1} are equal to $\lambda_1, \dots, \lambda_{k-2}, \lambda_k + 1$, and the corank 1 term F_{k-2} is semi-simple. As G_{k-1}/F_1 is a sub-module of E/F_1 which is semi-simple by assumption, it is semi-simple and we have proved that $[G_{k-1}]$ is in $\mathcal{F}_{k-2}(\lambda_1, \dots, \lambda_{k-2}, \lambda_k + 1)$. \blacksquare

Lemma 3.2.2 *If ε is another generator of E such that $(a - \lambda_{k-1}.b).(a - \lambda_k.b).\varepsilon$ is in F_{k-2} , we have*

$$\varepsilon = \rho.e + \sigma.b^{p_{k-1}-1}.(a - \lambda_k.b).e \quad \text{modulo } F_{k-2}$$

where (ρ, σ) is in $\mathbb{C}^* \times \mathbb{C}$. And conversely, for any (ρ, σ) in $\mathbb{C}^* \times \mathbb{C}$, an element $x = \rho.e + \sigma.b^{p_{k-1}-1}.(a - \lambda_k.b).e$ modulo F_{k-2} is a generator of E and satisfies $(a - \lambda_{k-1}.b).(a - \lambda_k.b).x$ is in F_{k-2} .

PROOF. Write $\varepsilon = U.e_k + V.e_{k-1}$ modulo F_{k-2} where we defined $e_k := e$ and $e_{k-1} := (a - \lambda_k.b).e$ and where U, V are in $\mathbb{C}[[b]]$. Now

$$(a - \lambda_k.b).\varepsilon = b^2.U'.e_k + U.e_{k-1} + (\lambda_{k-1} - \lambda_k).b.V.e_{k-1} + b^2.V'.e_{k-1} \quad \text{modulo } F_{k-2}$$

and, as $(a - \lambda_{k-1}.b).b^2.U'.e_k \in F_{k-1}$ implies $U' = 0$ (because $p_{k-1} + 1 > 0$) we have $U = \rho \in \mathbb{C}^*$, and we obtain

$$(a - \lambda_k.b).\varepsilon = [\rho + b^2.V' - (p_{k-1} - 1).b.V].e_{k-1} \quad \text{modulo } F_{k-2}.$$

If $T := \rho + b^2.V' - (p_{k-1} - 1).b.V$ we have

$$(a - \lambda_{k-1}.b).(a - \lambda_k.b).\varepsilon = b^2.T'.e_{k-1} \quad \text{modulo } F_{k-2}$$

and so T is a constant and equal to ρ . Then $V = \sigma.b^{p_{k-1}-1}$ for some complex number σ .

The converse is easy, thanks to the lemma 1.1.5 and the previous computation. \blacksquare

Corollary 3.2.3 *Let $\varepsilon = \rho.e + \sigma.b^{p_{k-1}-1}.(a - \lambda_k.b).e$ modulo F_{k-2} be a generator of E where (ρ, σ) is in $\mathbb{C}^* \times \mathbb{C}$. Then the sub-module $G := \tilde{\mathcal{A}}.(a - (\lambda_{k-1} - 1).b).\varepsilon$ is equal to the sub-module $G^\tau := \tilde{\mathcal{A}}.(a - (\lambda_{k-1} - 1).b).\varepsilon(\tau)$ where we define $\varepsilon(\tau) := e_k + \tau.b^{p_{k-1}-1}.e_{k-1}$ and $\tau := \sigma/\rho$.*

PROOF. Using the lemma 1.1.5, we see that ε and $\varepsilon(\tau)$ are generators of E such that their images by $(a - \lambda_{k-1}.b).(a - \lambda_k.b)$ are in F_{k-2} . As the sub-module G contains F_{k-2} it contains $\varepsilon(\tau)$ and so G contains G^τ . But they are normal and have same rank ; so they are equal. \blacksquare

Proposition 3.2.4 *Assume that we already have proved that for any fixed fundamental invariants of rank $l \leq (k-1)$ frescos the following assertion :*

- *There exists a polynomial $\beta : \mathcal{F}_{l-1} \rightarrow \mathbb{C}$ such that $[G] \in \mathcal{F}_{l-1}$ is semi-simple if and only if $\beta([G]) = 0$.*

Then for a given $[E] \in \mathcal{F}_{k-1}(\lambda_1, \dots, \lambda_k)$ the value of $\beta(G^\tau)$ is independent of the choice of τ and defines a polynomial on $\mathcal{F}_{k-1}(\lambda_1, \dots, \lambda_k)$ such that a class $[E] \in \mathcal{F}_{k-1}(\lambda_1, \dots, \lambda_k)$ is semi-simple if and only if $\beta([E]) = 0$.

PROOF. Remark that for $k = 2$ we have $\mathcal{F}_1(\lambda_1, \lambda_2) = \mathcal{F}(\lambda_1, \lambda_2)$ and that the polynomial β given by the parameter of the rank 2 fresco satisfies our requirement; explicitly this means that :

1. When $\lambda_2 - \lambda_1 + 1 \notin \mathbb{N}$ define $\beta \equiv 0$.
2. When $\lambda_2 - \lambda_1 + 1 = p \in \mathbb{N}$ define $\beta([E]) := \alpha(E)$ (recall that $\alpha(E)$ is the parameter of E ; see example 1 at the end of paragraph 3.1.)

So we may assume now that $k \geq 3$. We shall prove that the function $\tau \mapsto \beta(G^\tau)$ is a polynomial, using the induction hypothesis. For that purpose, it is enough to show that there exists, for a given $[E] \in \mathcal{F}_{k-1}(\lambda_1, \dots, \lambda_k)$, elements T_1, \dots, T_{k-2} in $\mathbb{C}[\tau][[b]]$ such that

$$T_j(0) = 1 \quad \forall j \in [1, k-2] \quad \text{and} \quad G^\tau \simeq \tilde{A}/\tilde{A}.Q(\tau)$$

where

$$Q_0(\tau) := (a - \lambda_1.b).T_1(\tau)^{-1}.(a - \lambda_2.b) \dots (a - \lambda_{k-2}.b).T_{k-2}(\tau)^{-1}$$

and $Q(\tau) := Q_0(\tau).(a - (\lambda_k + 1).b)$.

Now, as $\varepsilon(\tau)$ depends polynomially on τ , the element of F_{k-2}

$$(a - \lambda_{k-1}.b).(a - \lambda_k.b).\varepsilon(\tau)$$

which is a generator of F_{k-2} thanks to the corollary 1.2.8, depends polynomially on τ . Thanks to the proposition 3.1.2 we may find $T_1, \dots, T_{k-2} \in \mathbb{C}[\tau][[b]]$ such that the annihilator of $(a - \lambda_{k-1}.b).(a - \lambda_k.b).\varepsilon(\tau)$ is equal to $\tilde{A}.Q_0(\tau)$. Then we have for each τ an isomorphism $G^\tau \simeq \tilde{A}/\tilde{A}.Q(\tau)$, where $Q(\tau) := Q_0(\tau).(a - (\lambda_k + 1).b)$. Assume that for some value of τ we have $\beta(G^\tau) = 0$. Then G^τ is semi-simple, and then $F_{k-1} + G^\tau$ is semi-simple. So its normalization in E is semi-simple. But

this normalization is E , and so any sub-module of E is semi-simple. This implies that either we have $\beta(G^\tau) \neq 0$ for each τ or $\beta(G^\tau) = 0$ for each τ . This means that the polynomial $\tau \mapsto \beta(G^\tau)$ is constant, and that this constant is 0 if and only if E is semi-simple. To conclude, we define the function $\beta : \mathcal{F}_{k-1}(\lambda_1, \dots, \lambda_k) \rightarrow \mathbb{C}$ by $\beta(E) := \beta(G^0)$. This is a polynomial because the application $[E] \mapsto [G^0]$ is algebraic with value in $\mathcal{F}_{k-2}(\lambda_1, \dots, \lambda_{k-2}, \lambda_k + 1)$ and thanks to our inductive assumption. \blacksquare

Let me give in rank 3 a polynomial $F \in \mathbb{C}[s]$ such that we have $F(s) = \beta([E(s)])$ for those s for which $E(s)$ is in $\mathcal{F}_2(\lambda_1, \lambda_2, \lambda_3)$.

Assume $p_1 \geq 1$ and $p_2 \geq 1$. Then the necessary and sufficient condition to be in $\mathcal{F}_2(\lambda_1, \lambda_2, \lambda_3)$ is $s_{p_1}^1 = s_{p_2}^2 = 0$.

Lemma 3.2.5 *Assume that $p_1 \geq 1$ and $p_2 \geq 1$ and let*

$$E := \tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a - \lambda_1.b).S_1^{-1}.(a - \lambda_2.b).S_2^{-1}.(a - \lambda_3.b)$$

with $s_{p_1}^1 = s_{p_2}^2 = 0$; the complex number $\beta(E)$ is the coefficient of $b^{p_1+p_2}$ in $V.S_1$ where $V \in \mathbb{C}[[b]]$ is any solution of the differential equation

$$b.V' = p_2.(V - S_2).$$

The proof is left to the reader.

This gives the following formula for β :

$$\beta(s) = p_2 \cdot \sum_{j \neq p_2, j=0}^{p_1+p_2} s_{p_1+p_2-j}^1 \cdot \frac{s_j^2}{p_2 - j}.$$

Remark that for $S_2 = 1$ we find that $F(s)$ reduces to $s_{p_1+p_2}^1$. Using the commutation relation $(a - \lambda_2.b).(a - \lambda_3.b) = (a - (\lambda_3 + 1).b).(a - (\lambda_2 - 1).b)$ we find in this case that $E(s)$ has a normal rank 2 sub-module with fundamental invariants $\lambda_1, \lambda_3 + 1$ and parameter $s_{p_1+p_2}^1$ which is precisely $\alpha(E(s))$.

Now we shall show that in the previous situation when the β -invariant of a fresco $E \in \mathcal{F}_{k-1}(\lambda_1, \dots, \lambda_k)$ is not zero it is the parameter of any normal rank 2 sub-theme of E . Although such a rank 2 normal sub-theme is not unique (in general), its isomorphism class is uniquely determined from the fundamental invariants of E and from the β -invariant of E . It will be enough to consider the $[\lambda]$ -primitive case thanks to the following remark.

REMARK. If $E \in \mathcal{F}_{k-1}(\lambda_1, \dots, \lambda_k)$ satisfies $\beta(E) \neq 0$ then E is $[\lambda]$ -primitive: if $\lambda_1 \neq \lambda_k \text{ modulo } \mathbb{Z}$, let

$$i := \sup\{j \in [1, k] / \lambda_j = \lambda_1 \text{ modulo } \mathbb{Z}\} < k.$$

Then the lemma 1.2.8 applied to the exact sequence of frescos

$$0 \rightarrow F_i \rightarrow E \rightarrow E/F_i \rightarrow 0$$

shows that E is semi-simple, contradicting our assertion $\beta(E) \neq 0$. So $i = k$ and E is $[\lambda]$ -primitive. \square

Proposition 3.2.6 *Let E be a rank $k \geq 2$ fresco such that F_{k-1} and E/F_1 are semi-simple and with $\beta(E) \neq 0$. Note $\lambda_1, \dots, \lambda_k$ the fundamental invariants⁹ of E and $p(E) := \sum_{j=1}^{p-1} p_j$. Then there exists at least one rank 2 normal sub-theme in E and each rank 2 normal sub-theme of E is isomorphic to*

$$\tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a - \lambda_1.b).(1 + \beta(E).b^{p(E)})^{-1}.(a - (\lambda_k + k - 2).b). \quad (@)$$

PROOF. The case $k = 2$ is clear, so we may assume that $k \geq 3$ and we shall prove the proposition by induction on k . So assume that the proposition is known for the rank $k-1 \geq 2$ and let E be a rank k ($[\lambda]$ -primitive) fresco satisfying our assumptions. The fact that there exists a rank 2 normal sub-theme is consequence of the induction hypothesis as G^τ for any $\tau \in \mathbb{C}$ is normal rank $k-1$ in E and satisfies again our assumptions.

Recall that the fundamental invariants of G^τ are μ_1, \dots, μ_{k-1} with $\mu_j = \lambda_j$ for $j \in [1, k-2]$ and $\mu_{k-1} = \lambda_k + 1$. We have also $\beta(G^\tau) = \beta(E)$ for each τ , thanks to the proof of the proposition 3.2.4. As we have $p(G^\tau) = p(E)$ for each τ , and $\mu_{k-1} = \mu_1 + p(E) - 1$ the inductive hypothesis implies that any rank 2 normal sub-theme of any G^τ is isomorphic to (@). Then, to complete the proof, it is enough to show that any rank 2 normal sub-theme of E is contained in some G^τ .

Let T be a rank 2 normal sub-theme of E . We shall first prove that its fundamental invariants are equal to $\lambda_1, \lambda_k + k - 2$. As E/F_1 is semi-simple, the image of T by the quotient map $E \rightarrow E/F_1$ has rank 1 and this implies that $F_1 \cap T$ is a rank 1 normal sub-module. Then this implies that F_1 is the unique normal rank 1 submodule of T , proving that $\lambda_1(T) = \lambda_1$. We shall prove now that $\lambda_2(T) = \lambda_k + k - 2$.

First remark that the uniqueness of the principal J-H. sequence of E implies the uniqueness of the quotient map $E \rightarrow E/F_{k-1} \simeq E_{\lambda_k}$ because any surjective map $q : E \rightarrow E_{\lambda_k}$ will produce a principal a J-H. sequence for E by adjoining a principal J-H. sequence for $\text{Ker } q$. So a quotient E/H admits a surjective map on E_{λ_k} if and only if $H \subset F_{k-1}$. So H has to be semi-simple. Then the quotient E/T has no surjective map on E_{λ_k} .

The exact sequence of frescos

$$0 \rightarrow T/F_1 \rightarrow E/F_1 \rightarrow E/T \rightarrow 0$$

⁹Our hypothesis implies that E is $[\lambda]$ -primitive, thanks to the remark above, so $p_j := \lambda_{j+1} - \lambda_j + 1$ is a positive integer for each $j \in [1, k-1]$.

gives the equality in $\tilde{\mathcal{A}}$, thanks to the remark following the proposition 1.1.6 :

$$P_{T/F_1} \cdot P_{E/T} = P_{E/F_1} = (a - \lambda_2.b) \dots (a - \lambda_k.b).$$

But as E/T is semi-simple and does not have a quotient isomorphic to E_{λ_k} this implies $T/F_1 \simeq E_{\lambda_k+k-2}$, proving our claim.

So T is isomorphic to $\tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a - \lambda_1.b).(1 + \beta.b^{p(E)})^{-1}.(a - (\lambda_k + k - 2).b)$ for some $\beta \in \mathbb{C}^*$. Let $x \in T$ be a generator of T which is annihilated by

$$(a - \lambda_1.b).(1 + \beta.b^{p(E)})^{-1}.(a - (\lambda_k + k - 2).b).$$

It satisfies

$$(a - (\lambda_k + k - 2).b).x \in F_1 \subset F_{k-2}. \quad (*)$$

We shall determine all $x \in E$ which satisfies the condition $(*)$ modulo F_{k-2} . Fix a generator $e := e_k$ of E such that $e_{k-1} := (a - \lambda_k.b).e_k$ is in F_{k-1} and satisfies $(a - \lambda_{k-1}.b).e_{k-1} \in F_{k-2}$. Then we look for $U, V \in \mathbb{C}[[b]]$ such that

$$x := U.e_k + V.e_{k-1} \quad \text{satisfies} \quad (a - (\lambda_k + k - 2).b).x \in F_{k-2}.$$

This leads to the equations

$$\begin{aligned} b.U' - (k - 2).b.U &= 0 \\ U + b^2.V' - (p_{k-1} + k - 3).b.V &= 0 \end{aligned}$$

and so we get

$$U = \rho.b^{k-2} \quad \text{and} \quad V = \frac{\rho}{p_{k-1}}.b^{k-3} + \sigma.b^{p_{k-1}+k-3}$$

Note that for $\rho = 0$ we would have $x \in F_{k-1}$ and this is not possible for the generator of a rank 2 theme as F_{k-1} is semi-simple. Then assuming $\rho \neq 0$ we may write

$$x = \frac{\rho.b^{k-3}}{p_{k-1}} \cdot \left[p_{k-1}.b.e_k + e_{k-1} + \frac{p_{k-1}.\sigma}{\rho}.b^{p_{k-1}}.e_{k-1} \right] \quad \text{modulo } F_{k-2}.$$

Now recall that the generator of G^τ is given by

$$(a - (\lambda_{k-1} - 1).b).\varepsilon(\tau)$$

where $\varepsilon(\tau) = e_k + \tau.b^{p_{k-1}-1}.e_{k-1}$. A simple computation gives

$$(a - (\lambda_{k-1} - 1).b).\varepsilon(\tau) = p_{k-1}.b.e_k + e_{k-1} + p_{k-1}.\tau.b^{p_{k-1}}.e_{k-1} \quad \text{modulo } F_{k-2}.$$

As we know that each G^τ contains F_{k-2} , we conclude that any such x is in G^τ for $\tau = \sigma/\rho$, concluding the proof, thanks to the induction hypothesis and the equalities $\beta(E) = \beta(G^\tau)$ and $p(E) = p(G^\tau)$. \blacksquare

Using duality we may deduce from this result the following corollary.

Corollary 3.2.7 *In the situation of the previous proposition E has a rank 2 quotient theme and any such rank 2 quotient theme is isomorphic to*

$$\tilde{\mathcal{A}}/\tilde{\mathcal{A}}.(a - (\lambda_1 - k + 2).b).(1 + \beta^*(E).b^{p(E)})^{-1}.(a - \lambda_k.b)$$

where

$$\beta^*(E) := (-1)^k \frac{p_1 \cdot (p_1 + p_2) \cdots (p_1 + \cdots + p_{k-2})}{p_{k-1} \cdot (p_{k-2} + p_{k-1}) \cdots (p_2 + \cdots + p_{k-1})} \beta(E)$$

The proof is left as an exercise for the reader who may use the following remark.

REMARK. Let E be a rank 2 $[\lambda]$ -primitive theme with fundamental invariants λ_1, λ_2 and parameter $\alpha(E)$. For $\delta \in \mathbb{N}, \delta \gg 1$ $E^* \otimes E_\delta$ is a 2 $[\delta - \lambda]$ -primitive theme with fundamental invariants $(\delta - \lambda_2, \delta - \lambda_1)$ and parameter $-\alpha(E)$. Then in the situation of the proposition 3.2.6 $E^* \otimes E_\delta, \delta \gg 1$ satisfies again our assumptions and we have $\beta(E^* \otimes E_\delta) = \beta^*(E)$. \square

Using the β -invariant constructed above, we obtain the following result.

Theorem 3.2.8 *Fix a positive integer k and let $\lambda_1, \dots, \lambda_k$ the fundamental invariants of some rank k fresco. Then there exists a natural algebraic stratification $\mathcal{F}_h, h \in [1, k]$ of the subset $\mathcal{F}(\lambda_1, \dots, \lambda_k)$ with the following properties:*

- i) *A class $[E] \in \mathcal{F}(\lambda_1, \dots, \lambda_k)$ is in \mathcal{F}_h if and only if for any $i \in [0, k - h]$ the rank h sub-quotient $F_{i+h}(E)/F_i(E)$ is semi-simple.
So $\mathcal{F}_1 := \mathcal{F}(\lambda_1, \dots, \lambda_k)$ and \mathcal{F}_k is the subset of isomorphism classes of semi-simple frescos with fundamental invariants $\lambda_1, \dots, \lambda_k$.*
- ii) *This stratification is invariant by any change of variable.*

PROOF. We shall prove the algebraicity of this stratification and properties i) by induction on $h \geq 1$. For $h = 1$ the assertions are clear. Assume $h \geq 2$ and that we have already proved algebraicity and assertions i) for $l \leq h - 1$. For each $i \in [0, k - h]$ we have algebraic maps

$$\pi_i : \mathcal{F}_{h-1}(\lambda_1, \dots, \lambda_k) \rightarrow \mathcal{F}_{h-1}(\lambda_{i+1}, \dots, \lambda_{i+h})$$

given by $[E] \mapsto [F_{i+h}(E)/F_i(E)]$. Now for each $i \in [0, k - h]$ we have a polynomial on $\mathcal{F}_{h-1}(\lambda_{i+1}, \dots, \lambda_{i+h})$ given by the β -invariant constructed in the proposition 3.2.4. Composed with π_i it gives a polynomial $\beta_i : \mathcal{F}_{h-1}(\lambda_1, \dots, \lambda_k) \rightarrow \mathbb{C}$, and, thanks to the induction hypothesis, we have

$$\mathcal{F}_h(\lambda_1, \dots, \lambda_k) = \cap_{i \in [0, k-h+1]} \beta_i^{-1}(0) \subset \mathcal{F}_{h-1}(\lambda_1, \dots, \lambda_k).$$

So we obtain the algebraicity of $\mathcal{F}_h(\lambda_1, \dots, \lambda_k)$ defined by the condition i). The invariance by change of variable (see [B.11]) is then a direct consequence of the invariance of the principal J.H. sequence with change of variable and the fact that a change of variable preserves the semi-simplicity. \blacksquare

REMARK. The polynomial $\beta : \mathcal{F}_{k-1}(\lambda_1, \dots, \lambda_k) \rightarrow \mathbb{C}$ is quasi-invariant by changes of variable. In the $[\lambda]$ -primitive case, its weight is equal to $w := p_1 + \dots + p_{k-1}$. This means explicitly that, for any change of variable $\theta \in a.\mathbb{C}[[a]]$ we have, with $\theta'(0) = \chi(\theta) \neq 0$ $\beta(\theta_*(E)) = \chi(\theta)^w \cdot \beta(E)$. \square

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