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Generalized Gradient on Vector Bundle - Application to Image Denoising

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Abstract. We introduce a gradient operator that generalizes the Euclidean and Riemannian gradients. This operator acts on sections of vector bundles and is determined by three geometric data: a Riemannian metric on the base manifold, a Riemannian metric and a covariant derivative on the vector bundle. Under the assumption that the covariant derivative is compatible with the metric of the vector bundle, we consider the problems of minimizing the L2 and L1 norms of the gradient. In the L2 case, the gradient descent for reaching the solutions is a heat equation of a differential operator of order two called connection Laplacian. We present an application to color image denoising by replacing the regularizing term in the Rudin-Osher-Fatemi (ROF) denoising model by the L1 norm of a generalized gradient associated with a well-chosen covariant derivative. Experiments are validated by computations of the PSNR and Q-index.

Keywords: Generalized gradient, Riemannian manifold, Vector bundle, Total variation, Color image denoising, Rudin-Osher-Fatemi model.

1 Introduction

Total variation regularization methods have been widely used for image denoising tasks. Given an image $I_0: \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R} \in BV(\Omega)$ corrupted by additive white Gaussian noise of standard deviation σ , the seminal model of Rudin-Osher-Fatemi (ROF) [19] estimates the denoised image as the solution of the following variational problem

$$\arg \min_{I \in BV(\Omega)} \int_{\Omega} \frac{1}{2} \lambda (I - I_0)^2 + \|\nabla I\| \, d\Omega \quad (1)$$

where λ is a tuning parameter. The first term in formula (1) is the attached data term and the second one is the regularizing term. Since then, this model has been extended in several ways (see e.g. [4],[10],[15],[16],[17],[18],[24] for local methods based on a modification of the regularizing term, and [9],[11] for nonlocal methods). In this paper, we construct a new regularizing term by the introduction of a generalization of the gradient operator.

The gradient operator we introduce generalizes the Euclidean and Riemannian gradients by acting on sections of vector bundles. The ROF denoising model based of this new gradient operator generalizes the Euclidean approach of [19] and its multidimensional extension [4], as well as the Riemannian ROF denoising model in [18]. The key idea is to treat the term ∇I as a vector-valued differential 1-form $\nabla^E I$, that we call **connection gradient** of I , where the operator ∇^E is a covariant derivative (also called connection). Given Riemannian metrics on the base manifold and vector bundle, a metric on the space of vector-valued differential 1-forms might be constructed, and consequently the norm of the connection gradient $\nabla^E I$ might be considered. Then, for particular choices of metrics and covariant derivative, the norm of $\nabla^E I$ corresponds to the norm of the Euclidean or Riemannian gradient.

In this paper, we focus on connection gradients where the covariant derivative is compatible with the metric of the vector bundle. In this context, the covariant derivative ∇^E has an adjoint operator ∇^{E*} and we show that both L1 and L2 norms minimization problems extend the Euclidean and Riemannian approaches in a natural way. Indeed, we show that the gradient descent flow for reaching the sections minimizing the L2 norm of connection gradient is the heat equation of a generalized Laplacian. Moreover, we show that the critical points of the L1 norm of connection gradient satisfy

$$\nabla^{E*} \left(\frac{\nabla^E I}{\|\nabla^E I\|} \right) = 0$$

It is a fact that nonlocal denoising methods, like Non Local Means [5] or BM3D [6] provide better results than local methods in terms of PSNR and Q-index [23] measures. The denoising model we propose in this paper is local, however we expect that it extends to a nonlocal model. Indeed, inspired by [9] where the Euclidean ROF model [19] is extended to a nonlocal model by the construction of a nonlocal gradient operator, we expect that our vector bundle ROF model extends to a nonlocal model by the construction of a nonlocal connection gradient operator.

The outline of the paper is the following. Sect. 2 is mainly theoretical. We first introduce the notion of connection gradient and its norm, and remind the definition of connection Laplacian on vector bundle. We relate them with well-known concepts of Euclidean and Riemannian geometry. Then, we restrict to the case where the covariant derivative is compatible with the metric of the vector bundle. We establish a link with the Beltrami framework of Sochen et al. (see e.g. [20]) by showing that sections minimizing the L2 norm of connection gradient are harmonic with respect to some Riemannian metric on the vector bundle. In Sect. 3, we present an application to color image denoising by considering the L1 norm of a suitable connection gradient as the regularizing term of a ROF denoising model. In particular, we discuss the discretization of the operator ∇^E

and its adjoint ∇^{E*} . We test our denoising method on the Kodak database [12] and compute both PSNR and Q-index measures. Results show that our method provides better results than the Split Bregman method [10] applied to ROF functional.

2 Generalized gradient on vector bundle

2.1 Definitions and examples

We refer to [21] for an introduction to differential geometry of fiber bundles. Given a vector bundle E , we denote by $\Gamma(E)$ the set of smooth sections of E .

Connection gradient

Definition 1. Let E be a vector bundle of rank m over a Riemannian manifold (M, g) of dimension n . Let ∇^E be a covariant derivative and h be a definite positive metric on E . Given $\varphi \in \Gamma(E)$, we call the term $\nabla^E \varphi \in \Gamma(T^*M \otimes E)$ the **connection gradient** of φ .

The metrics g on TM and h on E induce a definite positive metric $\langle \cdot, \cdot \rangle$ on $T^*M \otimes E$. Then, we define the norm of the connection gradient of φ as

$$\|\nabla^E \varphi\| := \sqrt{\langle \nabla^E \varphi, \nabla^E \varphi \rangle} = \sqrt{\sum_{i,j=1}^n g^{ij} h(\nabla_{\partial/\partial x_i}^E \varphi, \nabla_{\partial/\partial x_j}^E \varphi)} \quad (2)$$

where $(\partial/\partial x_1, \dots, \partial/\partial x_n)$ is the frame of TM induced by a coordinates system (x_1, \dots, x_n) of M .

Example 1. Let $E = C^\infty(M)$ be the vector bundle of rank 1 of smooth functions on a Riemannian manifold (M, g) . Let ∇^E be the trivial covariant derivative on E and h be the definite positive metric on E given by the scalar multiplication. From Definition 1, the connection gradient of a function f is its differential $df \in \Gamma(T^*M)$.

The musical isomorphism $\sharp: T^*M \longrightarrow TM$ maps df onto $\nabla_g f$, the Riemannian gradient of f . Moreover the norm of df is

$$\|df\| := \sqrt{\langle df, df \rangle} = \sqrt{\sum_{i,j=1}^n g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j}}, \quad (3)$$

and coincides with the norm of $\nabla_g f$.

Connection compatible with the metric

Definition 2. Let E be a vector bundle over a Riemannian manifold (M, g) , equipped with a definite positive metric h . A covariant derivative ∇^E on E is compatible with the metric h if it satisfies

$$dh(\varphi, \psi) = h(\nabla^E \varphi, \psi) + h(\varphi, \nabla^E \psi) \quad (4)$$

for any $\varphi, \psi \in \Gamma(E)$.

Example 2. On the vector bundle of smooth functions on a Riemannian manifold, the trivial covariant derivative is compatible with the metric given by the scalar multiplication on the fibers.

Example 3. The Levi-Civita connection on the tangent bundle TM of a Riemannian manifold (M, g) is compatible with the metric g . Actually, it is defined as the unique connection compatible with the metric and having a zero torsion tensor.

Assuming that E is associated with the principal bundle $PSO(E)$ of orthonormal frame fields of E , we have the following result

Proposition 1 (see e.g. Lawson et al. [14] (Prop. 4.4 p.103)). *There is a one-one correspondance between connection 1-forms on $PSO(E)$ and covariant derivatives on E that are compatible with the metric.*

Under the choice of a local trivializing section of $PSO(E)$, i.e. a local orthonormal frame with respect to h of the vector bundle E , a connection 1-form is a $\mathfrak{so}(n)$ -valued 1-form on M , i.e. $\omega \in \Gamma(T^*M \otimes \mathfrak{so}(n))$. More precisely, we have the formula

$$\nabla_X^E \varphi = d_X \varphi + \omega(X)(\varphi) \quad (5)$$

where $X \in \Gamma(TM)$.

Connection Laplacian Let $\nabla^{T^*M \otimes E}$ be the covariant derivative on $T^*M \otimes E$ defined as

$$\nabla^{T^*M \otimes E}(\eta \otimes \varphi) = \nabla^{T^*M} \eta \otimes \varphi + \eta \otimes \nabla^E \varphi$$

where ∇^{T^*M} is the covariant derivative on T^*M induced by the Levi-Civita covariant derivative on (TM, g) and ∇^E is a covariant derivative on E compatible with a definite positive metric h . The **adjoint** $\nabla^{E*} : \Gamma(T^*M \otimes E) \longrightarrow \Gamma(E)$ of the operator $\nabla^E : \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E)$ is the operator

$$\nabla^{E*} = -Tr \nabla^{T^*M \otimes E}$$

where Tr denotes the contraction with respect to the metric g . In others words, the following equality is satisfied

$$\int_M h(\nabla^{E*} \eta, \varphi) dM = \int_M \langle \eta, \nabla^E \varphi \rangle dM \quad (6)$$

assuming that φ has compact support.

Example 4. On the vector bundle of smooth functions on a Riemannian manifold (M, g) , the adjoint $d^*: \Gamma(T^*M) \longrightarrow C^\infty(M)$ of the trivial covariant derivative $d: C^\infty(M) \longrightarrow \Gamma(T^*M)$ is the operator

$$d^*\eta = - \sum_{i,j} \left(g^{ij} \partial_{x_i} \eta (\partial/\partial x_j) - \sum_k \Gamma_{ij}^k \eta (\partial/\partial x_k) \right)$$

where the symbols Γ_{ij}^k are the symbols of the Levi-Civita connection of (M, g) with respect to the frame $(\partial/\partial x_1, \dots, \partial/\partial x_n)$.

Definition 3. The **connection Laplacian** Δ^E is the second order differential operator on $\Gamma(E)$ defined as $\Delta^E = \nabla^{E*} \nabla^E$.

In the frame $(\partial/\partial x_1, \dots, \partial/\partial x_n)$ of (TM, g) associated to a local coordinates system (x_1, \dots, x_n) of M , we have

$$\Delta^E = - \sum_{ij} g^{ij} \left(\nabla_{\partial/\partial x_i}^E \nabla_{\partial/\partial x_j}^E - \sum_k \Gamma_{ij}^k \nabla_{\partial/\partial x_k}^E \right)$$

Note that the operator $\nabla^{E*} \nabla^E$ is also called **Bochner Laplacian**.

Example 5. The **Laplace-Beltrami operator** is the connection Laplacian (up to a sign) associated to the trivial covariant derivative d on the vector bundle of smooth functions on a Riemannian manifold (M, g) . In other words, it might be written

$$\Delta^E = - \sum_{ij} g^{ij} \left(\partial_{x_i} \partial_{x_j} - \sum_k \Gamma_{ij}^k \partial_{x_k} \right)$$

2.2 L2 minimization of connection gradient and Dirichlet energy

Let E be a vector bundle over a Riemannian manifold (M, g) equipped with a definite positive metric h and a covariant derivative ∇^E compatible with h . We have the following result

Proposition 2 (Lawson et al. [14] Prop. 8.1 p.154). *The operator Δ^E is non-negative and essentially self-adjoint. Furthermore,*

$$\int_M h(\Delta^E \varphi, \psi) dM = \int_M \langle \nabla^E \varphi, \nabla^E \psi \rangle dM \quad (7)$$

for all $\varphi, \psi \in \Gamma(E)$ provided that one of φ or ψ has compact support.

If M is compact, then $\Delta^E \varphi = 0$ if and only if $\nabla^E \varphi = 0$.

We observe that the right term of equality (7) corresponds to the Gâteaux derivative in the direction ψ of the energy

$$E(\varphi) = \int_M \|\nabla^E \varphi\|^2 dM \quad (8)$$

Hence the critical points of the energy (8) satisfy $\Delta^E \varphi = 0$. Moreover, they correspond to the minimum of the energy.

Heat equation and heat kernel of connection Laplacian The gradient descent method for reaching sections minimizing the energy (8) corresponds to the heat equation of the connection Laplacian Δ^E

$$\frac{\partial \varphi}{\partial t} + \Delta^E \varphi = 0 \quad (9)$$

Results about heat equation and heat kernel of connection Laplacian are well-established (see e.g. [2]).

Given a connection Laplacian Δ^E and $\varphi_0 \in \Gamma(E)$, there exist a smooth map called **heat kernel** of Δ^E and denoted by K such that the operator $e^{-t\Delta^E}$ defined by

$$(e^{-t\Delta^E} \varphi_0)(x) = \int_M K_t(x, y) \varphi_0(y) dM$$

satisfies the heat equation (9).

The heat kernel of a connection Laplacian has a series expansion of the form

$$\left(\frac{1}{4\pi t} \right)^{\frac{n}{2}} e^{-d(x,y)^2/4t} \Psi(d(x,y)^2) \sum_{i=0}^{+\infty} t^i \Phi_i(x, y, \Delta^E) J(x, y)^{-\frac{1}{2}} \quad (10)$$

where $\Phi_i(x, y, \Delta^E) \in \text{End}(E_y, E_x)$, n is the dimension of the base manifold M , and d stands for the geodesic distance on (M, g) . The function Ψ is such that the term $\Psi(d(x, y)^2)$ equals 1 if y is inside a normal neighborhood of x and 0 otherwise. At last, J are the Jacobians of the coordinates changes from usual coordinates systems to normal coordinates systems.

The leading term of the series (10) is

$$\left(\frac{1}{4\pi t} \right)^{n/2} e^{-d(x,y)^2/4t} \Psi(d(x,y)^2) \tau(x, y) J(x, y)^{-1/2} \quad (11)$$

where $\tau(x, y)$ is the parallel transport map on E associated to ∇^E along the unique geodesic joining x and y .

Example 6. In [22], convolution with the leading term (11) associated to the Laplace-Beltrami operator were applied to anisotropic diffusion of color images. It was extended in [1] to connection Laplacians with no trivial connections.

Parallel section and Harmonic map Harmonic maps between two Riemannian manifolds $\sigma: (M, g) \longrightarrow (N, Q)$ are defined as critical points of the **Dirichlet energy**

$$E(\sigma) = \int_M \text{trace}_g(\sigma^* h) dM = \int_M \|d\sigma\|^2 dM \quad (12)$$

The Euler-Lagrange equations of the functional (12) are

$$\tau(\sigma): = \text{trace}_g \nabla^{T^*M \otimes \sigma^{-1}(TN)} d\sigma = 0$$

Hence the harmonic maps σ satisfy $\tau(\sigma) = 0$. The quantity τ is called the **tension field**. We refer to [7] for a presentation of harmonic maps.

Note that the Dirichlet energy (12) is at the core of the Beltrami framework of Sochen et al. (see e.g. [20]).

Theorem 1 (Konderak [13]). *Let E be a vector bundle over a compact Riemannian manifold (M, g) equipped with a metric h and a covariant derivative ∇^E compatible with h . Let \tilde{h} be the Sasaki metric on E associated to (h, ∇^E, g) . Then $\sigma \in \Gamma(E)$ is a harmonic map $\sigma: (M, g) \rightarrow (E, \tilde{h})$ if and only if it is parallel, i.e. $\nabla^E \sigma = 0$.*

Hence, the sections minimizing the energy (8) are harmonic maps with respect to the Sasaki metric on the vector bundle.

2.3 Total Variation on vector bundle

Definition 4. *Let E be a vector bundle over a compact Riemannian manifold (M, g) equipped with a Riemannian metric h , and a covariant derivative ∇^E compatible with h . We define the total variation TV of $\varphi \in \Gamma(E)$ as*

$$TV(\varphi) = \int_M \|\nabla^E \varphi\| dM \quad (13)$$

Proposition 3. *The critical points of (13) are the sections φ satisfying*

$$-Tr \nabla^{T^*M \otimes E} \left(\frac{\nabla^E \varphi}{\|\nabla^E \varphi\|} \right) = 0 \quad (14)$$

Proof. Let ψ be a section with compact support. We have

$$\begin{aligned} \frac{d}{dt} (TV(\varphi + t\psi)) \Big|_{t=0} &= \int_M \left\langle \frac{\nabla^E \varphi}{\|\nabla^E \varphi\|}, \nabla^E \psi \right\rangle dM \\ &= - \int_M \left\langle Tr \nabla^{T^*M \otimes E} \left(\frac{\nabla^E \varphi}{\|\nabla^E \varphi\|} \right), \psi \right\rangle dM \end{aligned}$$

since $-Tr \nabla^{T^*M \otimes E}$ is the adjoint of ∇^E . Then, as ψ has compact support, we have

$$\frac{d}{dt} (TV(\varphi + t\psi)) \Big|_{t=0} = 0 \implies -Tr \nabla^{T^*M \otimes E} \left(\frac{\nabla^E \varphi}{\|\nabla^E \varphi\|} \right) = 0$$

□

Example 7. Let E be the vector bundle of smooth \mathbb{R}^m -valued functions defined on a compact domain of \mathbb{R}^n . In others words, E is a vector bundle of rank m equipped with the trivial connection and Euclidean scalar product over a

compact Euclidean manifold of dimension n . Then the total variation of a section φ is

$$TV(\varphi) = \int_M \sqrt{\sum_{i=1}^n \sum_{j=1}^m \left(\frac{\partial \varphi^j}{\partial x_i} \right)^2} dM \quad (15)$$

Formula (15) corresponds to the total variation defined by Blomgren et al. [4]. In particular, for $n = 2$ and $m = 1$, this is the total variation of Rudin et al. [19]. Hence, these two approaches may be viewed as the Euclidean restrictions of the total variation we construct in (13).

3 Application to color image denoising

3.1 ROF denoising model on vector bundle: the general case

Continuous setting Let E be a vector bundle of rank 3 equipped with a definite positive metric h and covariant derivative ∇^E compatible with h over a Riemannian manifold (M, g) of dimension 2. Let $I_0 \in BV(E)$ be a color image corrupted by additive Gaussian noise of deviation σ . We propose the denoising model

$$\arg \min_{I \in BV(E)} \int_M \frac{1}{2} \lambda \|I - I_0\|^2 d\Omega + \|\nabla^E I\| dM \quad (16)$$

where $d\Omega$ denotes the Euclidean measure on M and λ is a Lagrange multiplier associated with the noise level.

The gradient descent for reaching solutions of (16) is

$$\frac{\partial I}{\partial t} = -\lambda(I - I_0) + Tr \nabla^{T^*M \otimes E} \left(\frac{\nabla^E I}{\|\nabla^E I\|} \right), \quad I|_{t=0} = I_0 \quad (17)$$

Discrete setting We follow the approach in [19] where forward and backward finite difference operators are used for discretizing the trivial covariant derivative d and its adjoint $-div$. The key idea is to use the discrete version of the adjoint operator definition, which is written as follows in the context of connection gradient

$$\int_M -div \eta \varphi d\Omega = \int_M \langle \eta, d\varphi \rangle d\Omega$$

for $\varphi \in C^\infty(M)$ and $\eta \in \Gamma(T^*M)$. Then, using forward differences for discretizing $d\varphi$ implies that $div \eta$ must be discretized using backward differences.

We extend this approach by using the general definition of adjoint operator

$$\int_M h(\nabla^{E*} \eta, \varphi) dM = \int_M \langle \eta, \nabla^E \varphi \rangle dM \quad (18)$$

for $\eta \in \Gamma(T^*M \otimes E)$ and $\varphi \in \Gamma(E)$. Let us give the explicit expressions in the case of base manifold of dimension 2 and vector bundle of rank 3. Let $\varphi = \sum_{j=1}^3 \varphi^j e_j \in \Gamma(E)$ where (e_1, e_2, e_3) is orthonormal with respect to h . Under the forward finite difference operators for approximating $d\varphi^j$, we have

$$\nabla^E \varphi_{m,n} = \sum_{i=1}^2 \sum_{j=1}^3 \left[(\varphi_{m+\delta_{i1}, n+\delta_{i2}}^j - \varphi_{m,n}^j) + \sum_{k=1}^3 \varphi_{m,n}^k \Upsilon_{ik\ m,n}^j \right] dx_i \otimes e_j \quad (19)$$

where δ is the Kronecker symbol. Then, using the discrete form of (18) given by

$$\sum_{m,n} h_{m,n} (\nabla^{E*} \eta_{m,n}, \varphi_{m,n}) = \sum_{m,n} \langle \eta_{m,n}, \nabla^E \varphi_{m,n} \rangle_{m,n}$$

we obtain, for $\eta = \sum_{i=1}^2 \sum_{j=1}^3 \eta^{ij} dx_i \otimes e_j$, the expression

$$\nabla^{E*} \eta_{m,n} = \sum_{j=1}^3 \left[\sum_{i,k=1}^2 (g_{m-\delta_{k1}, n-\delta_{k2}}^{ik} \eta_{m-\delta_{k1}, n-\delta_{k2}}^{ij} - g_{m,n}^{ik} \eta_{m,n}^{ij}) + \sum_{r,s=1}^2 \sum_{p=1}^3 \eta_{m,n}^{rp} g_{m,n}^{rs} \Upsilon_{sj\ m,n}^p \right] e_j \quad (20)$$

As in the Euclidean case [19], forward finite difference operators on ∇^E imply backward finite difference operators on ∇^{E*} .

3.2 ROF denoising model on vector bundle: an example

Connection gradient suitable for color image processing Let $I = I^1 e_1 + I^2 e_2 + I^3 e_3$ be a color image defined on a domain Ω of \mathbb{R}^2 .

We construct a surface S embedded in $(\mathbb{R}^5, \|\cdot\|_2)$ parametrized by

$$\varphi: (x_1, x_2) \mapsto (x_1, x_2, \mu I^1(x_1, x_2), \mu I^2(x_1, x_2), \mu I^3(x_1, x_2))$$

where μ is a strictly positive constant. Let $(Z_1, Z_2, N_1, N_2, N_3)$ be an orthonormal frame field of $(\mathbb{R}^5, \|\cdot\|_2)$ where $Z_1, Z_2 \in \Gamma(TS)$.

Let E be the vector bundle of \mathbb{R}^5 -valued functions over the Euclidean manifold $(\Omega, \|\cdot\|_2)$. Let ∇^E be the covariant derivative on E given by the connection 1-form $\omega \equiv 0$ in the frame $(Z_1, Z_2, N_1, N_2, N_3)$. Denoting by P the change frame field from $(e_1, e_2, e_3, e_4, e_5)$ to $(Z_1, Z_2, N_1, N_2, N_3)$, the connection 1-form is given in the frame $(e_1, e_2, e_3, e_4, e_5)$ by

$$P dP^{-1} \quad (21)$$

As the connection 1-form ω is $\mathfrak{so}(\mathfrak{n})$ -valued, the covariant derivative ∇^E is compatible with the Euclidean metric on \mathbb{R}^5 .

Algorithm We test the denoising model (16) with this connection gradient.

The algorithm is the following:

1. Consider the (discrete) surface S parametrized by

$$\varphi: (m, n) \mapsto (m, n, \mu I_0^1(m, n), \mu I_0^2(m, n), \mu I_0^3(m, n))$$

2. Construct an orthonormal moving frame $(Z_1, Z_2, N_1, N_2, N_3)$ using Gram-Schmidt process where $Z_1, Z_2 \in \Gamma(TS)$, and denote by P the change frame field from $(e_1, e_2, e_3, e_4, e_5)$ to $(Z_1, Z_2, N_1, N_2, N_3)$.

3. Embed I_0 into the frame $(e_1, e_2, e_3, e_4, e_5)$: $(I_0^1, I_0^2, I_0^3) \rightarrow (0, 0, I_0^1, I_0^2, I_0^3)$.

4. Compute the components of I_0 in the frame $(Z_1, Z_2, N_1, N_2, N_3)$:

$$(J_0^1, J_0^2, J_0^3, J_0^4, J_0^5)^T := P^{-1}(0, 0, I_0^1, I_0^2, I_0^3)^T.$$

5. Perform the Euclidean ROF denoising algorithm on $(J_0^1, J_0^2, J_0^3, J_0^4, J_0^5)^T$ with stopping criteria

$$\frac{1}{|\Omega| \times 3} \sum_{x \in \Omega} \|J_t(x) - J_0(x)\|^2 \geq \sigma^2$$

or

$$\left| \frac{1}{|\Omega| \times 3} \sum_{x \in \Omega} \|J_{t+dt}(x) - J_0(x)\|^2 - \frac{1}{|\Omega| \times 3} \sum_{x \in \Omega} \|J_t(x) - J_0(x)\|^2 \right| \leq 0.0005$$

whichever happens first.

6. Compute the components of the result in the frame $(e_1, e_2, e_3, e_4, e_5)$:

$$(I_t^1, I_t^2, I_t^3, I_t^4, I_t^5)^T := P(J_t^1, J_t^2, J_t^3, J_t^4, J_t^5)^T, \text{ and return the function } (I_t^3, I_t^4, I_t^5).$$

Experiments We run the algorithm on the Kodak database [12], for $\sigma = 5, 10, 15, 20, 25$. We take $\mu = 0.0075$ for $\sigma = 5$, $\mu = 0.005$ for $\sigma = 10$, $\mu = 0.0045$ for $\sigma = 15$, $\mu = 0.004$ for $\sigma = 20$ and $\mu = 0.0035$ for $\sigma = 25$. The time step dt is 0.1. We compare the results with the split Bregman denoising method [10] tested online [8]. On Fig. 1, we compute the average increases of PSNR as well as the average percent increases of Q-index over the Kodak database (we define the Q-index of a color image as the mean of the Q-index on each component). Results show that our method improves the Split Bregman method. On Fig. 2, we compare the denoising methods on an image of the database. We observe that our method preserves more the details of the image. It comes from the choice of the moving frame $(Z_1, Z_2, N_1, N_2, N_3)$ which takes into account the local geometry of the image.

4 Conclusion

In this paper, we introduced a generalization of the Euclidean and Riemannian gradient operators in the context of vector bundle. We presented an application to image denoising by replacing the Euclidean gradient in the regularizing term of the Rudin-Osher-Fatemi denoising model by a generalized gradient on

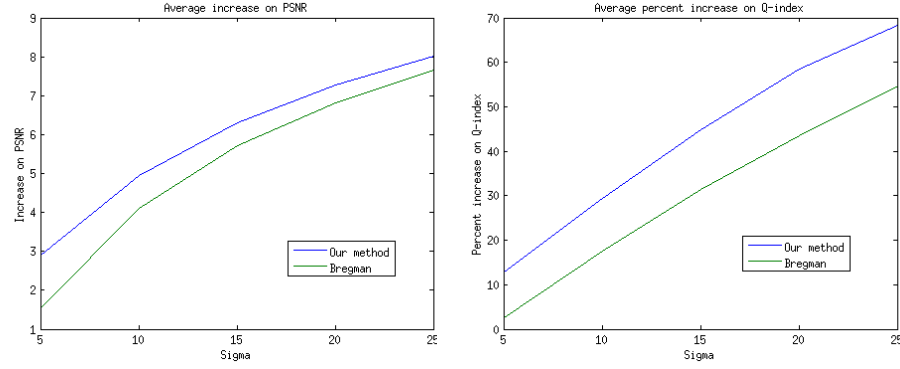


Fig. 1. Comparison of our method with Split Bregman. Left: PSNR increase for each method. Right: Percent increase on Q-index for each method. Values averaged over Kodak database.

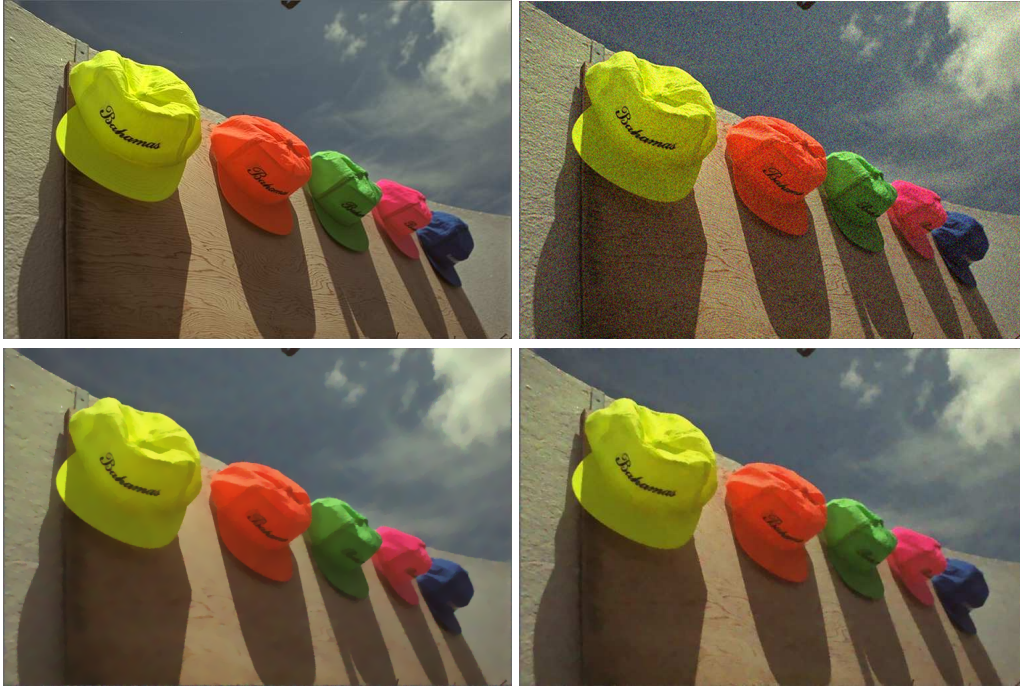


Fig. 2. Top: Original image (left) corrupted by additive white noise with $\sigma = 25$ (right). Bottom-left: Split Bregman denoising method. Bottom-right: Our denoising method.

vector bundle. By the gradient operator we considered, the denoising method is decomposed into 2 steps: first, a projection of the image on the tangent and non-

mal parts of a surface describing the image; then, an Euclidean ROF denoising method of the image in this moving frame. The results we got in terms of PSNR and Q-index makes the method be suitable. In some sense, the denoising process preserves the first order local geometry of the image. Inspired by [3] where it is shown that curvature information is less corrupted than first order local geometry under additive Gaussian noise, we are currently investigating projections taking into the curvature information of Riemannian manifold. Finally, we would like to point out that the step 2 of our denoising method might be extended to any denoising method. In particular, we expect that nonlocal denoising methods would increase significantly both PSNR and Q-index measures.

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