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CONFORMAL DIMENSION AND CANONICAL SPLITTINGS OF HYPERBOLIC GROUPS

MATIAS CARRASCO PIAGGIO

ABSTRACT. We prove a general criterion for a metric space to have conformal dimension one. The conditions are stated in terms of the existence of enough local cut points in the space. We then apply this criterion to the boundaries of hyperbolic groups and show an interesting relationship between conformal dimension and some canonical splittings of the group.

KEYWORDS: conformal dimension, local cut points, Gromov-hyperbolic groups, canonical splittings.

2010 SUBJECT CLASSIFICATION: 30L10, 51F99, 20F67, 28A78.

1. INTRODUCTION

In this article we give sufficient conditions for a compact metric space (X, d) to have conformal dimension one. The conformal dimension is a fundamental quasisymmetry invariant, introduced by Pansu in [Pan89]. Its original motivation is in the study of the quasiconformal structure of the boundary at infinity of a negatively curved space. For instance, any quasisymmetry invariant of the boundary of a hyperbolic group is a quasi-isometry invariant of the group. The understanding of conformal dimension has already given many applications in geometric group theory, in particular to the boundary characterization of Kleinian groups and to Cannon's conjecture [BonK05b]. See also [B06, Haïss08, Haïss12, Kle06, LP04, MT10] for other applications.

There are different related versions of this invariant; in this article we are concerned with the *Ahlfors regular conformal dimension*, a variant introduced by Bourdon and Pajot in [BP03]. It is defined by

$$\dim_{AR}(X, d) := \inf \{ \dim_H(X, \theta) : \theta \text{ is AR and } \theta \sim_{qs} d \},$$

where AR means Ahlfors regular, \dim_H denotes Hausdorff dimension, and $\theta \sim_{qs} d$ means that θ is a distance on X quasisymmetrically equivalent to d . That is, there exists an increasing homeomorphism $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\frac{\theta(x, a)}{\theta(y, a)} \leq \eta \left(\frac{d(x, a)}{d(y, a)} \right),$$

for all distinct points $x, y, a \in X$. We recall that a distance θ on X is Ahlfors regular of dimension $\alpha > 0$ if there exists a Radon measure μ on X and a constant $K \geq 1$ such that:

$$r^\alpha K^{-1} \leq \mu(B_r) \leq K r^\alpha,$$

for any ball B_r of radius $0 \leq r \leq \text{diam}_\theta X$. In that case, μ is comparable to the α -dimensional Hausdorff measure and $\alpha = \dim_H(X, \theta)$ is the Hausdorff dimension of (X, θ) . We write $\dim_{AR} X$ when there is no ambiguity on the metric d .

This kind of deformations can distort the Hausdorff dimension, and one can always quasisymmetrically deform the distance d to obtain arbitrarily large Hausdorff dimension. The conformal dimension measures the best shape of X , and it is in general very difficult

to compute. It is always bounded from below by the topological dimension $\dim_T X$. In particular, when the space is connected, $\dim_{AR} X \geq 1$ holds.

This article deals with the problem of under which conditions we can quasimetrically deform the distance d to obtain AR distances with Hausdorff dimension arbitrarily close to the topological dimension of the space. This question is of particular interest for the boundaries of hyperbolic groups. For instance, $\dim_{AR} \partial G = \dim_T \partial G = n \geq 1$ and is attained by a distance in the gauge if and only if ∂G is quasimetrically equivalent to the Euclidean sphere \mathbb{S}^n [BonK02b]. For $n = 1$, the same is true under the weaker hypothesis of ∂G being homeomorphic to \mathbb{S}^1 [CJ94, Gab92, Tu88]. But the problem is far from being understood in the general case, even for low topological dimension. In this paper we address the case when $\dim_T X = 1$.

Previously known non-trivial examples of spaces of conformal dimension one were very few, due to Bishop and Tyson [BT01], Pansu (see [BonK05b, Bu05, CMT99] for comments and generalizations), and Laakso (see Tyson and Wu [TW06] for generalizations). Although the techniques of proof are specific to each particular example, they are all related to the existence of local cut points. A point $x \in X$ is a *local cut point* if there is a connected open set $x \in U \subset X$ such that $U \setminus \{x\}$ is not connected. We also recall that X is a *doubling space* if there exists a constant N such that every ball can be covered using N balls of half its radius; and, that X is *linearly connected* if there exists a constant C such that for all $x, y \in X$, there exists a continuum J containing x and y of diameter less than or equal to $Cd(x, y)$. This is also known as the *bounded turning* condition. The following condition gives a scale invariant bound on the amount of local cut points needed to disconnect the space into small pieces.

Definition 1.1 (The UWS condition). *We say that a connected and compact metric space X has uniformly well spread local cut points —UWS for short— if there exists a constant $C \geq 1$ such that for any point $x \in X$ and $r > 0$, there is a finite set $P \subset B(x, r)$ verifying:*

- (1) $\#P \leq C$, and
- (2) no connected component of $X \setminus P$ can intersect both $B(x, \frac{r}{2})$ and $X \setminus \overline{B}(x, r)$.

We remark that one can always assume that the points of the subset P are local cut points of X ; this justifies the terminology. We obtain the following general criterion for conformal dimension one.

Theorem 1.2 (General criterion for conformal dimension one). *Let X be a doubling and compact metric space. If X is linearly connected and satisfies the UWS condition, then the (AR) conformal dimension of X is equal to one.*

The main ingredient of the proof is an unpublished result of S. Keith and B. Kleiner stating an equality between the conformal dimension and the critical exponent associated to the combinatorial modulus (a proof of this result is given in [Ca12]). A similar unpublished result to Theorem 1.2 was known by S. Keith and B. Kleiner; one of the motivations of the present article is to provide an accessible proof of this criterion and some of its consequences in the context of hyperbolic groups.

Note that Theorem 1.2 provides a large class of examples, including the examples mentioned before. A result of J. Mackay ensures that when the space X verifies a quantitative analogue of the topological conditions of being locally connected and without local cut points, then the conformal dimension is greater than one [Mac10]. Both criteria provide a clear conceptual picture of the relationship between conformal dimension and local cut points.

The rest of the paper is devoted to the boundaries of hyperbolic groups. They are part of a larger class consisting of *quasiselfsimilar* spaces (see Section 2.2). For this class, the hypotheses of Theorem 1.2 are equivalent to the following topological conditions: the space X is compact, connected, and verifies the *well spread* local cut points condition —WS for short: there exists a sequence of finite sets $P_n \subset X$ such that

$$\delta_n = \sup \{ \text{diam} A : A \in \mathcal{C}_n \} \rightarrow 0, \text{ when } n \rightarrow +\infty, \tag{WS}$$

where \mathcal{C}_n denotes the set of connected components of $X \setminus P_n$.

It is a remarkable fact that the topology of the boundary is reflected in the splittings of the group [Bow98a]. For example, if G is a one-ended hyperbolic group which is not virtually Fuchsian, then ∂G has a local cut point if and only if G splits over a virtually cyclic subgroup [Bow98a]. This motivates the problem of characterizing the hyperbolic groups whose boundary at infinity has conformal dimension one in terms of the properties of its canonical splittings. Recall that a splitting of G is given by an action of G on a simplicial tree \mathbb{T} , without edge inversions and of finite quotient [DD89].

The candidate groups for $\dim_{AR} \partial G = \dim_{\mathbb{T}} \partial G = 1$ are essentially obtained as repetitive amalgamated products or HNN-extensions of virtually Fuchsian and finite groups over elementary subgroups. To see this, note that if G is a hyperbolic group, we can decompose G repeatedly over finite and virtually cyclic subgroups, until —at least if there is no 2-torsion— all subgroups are finite, virtually Fuchsian, or one-ended without local cut points on the boundary [DP01, Va08]. If ∂G has no local cut points, then $\dim_{AR} \partial G > 1$ [Mac10].¹ Therefore, the question can be formulated as follows: is it true that if ∂H has a local cut point for every one-ended quasiconvex subgroup H of G , then $\dim_{AR} \partial G = 1$? The answer to this question seems to highly depend on how are embedded (in G) the one-ended quasiconvex subgroups of G .

We apply Bowditch’s work [Bow98a] on the structure of local cut points of the boundaries of one-ended hyperbolic groups, by relating the JSJ splitting of G with the WS property, to deduce the following partial answer:

Theorem 1.3 (Characterization of the WS property for hyperbolic groups). *Let G be a one-ended hyperbolic group. Then ∂G satisfies the WS property if and only if either*

- (i) ∂G is homeomorphic to the circle \mathbb{S}^1 , or
- (ii) all the rigid type vertices in the JSJ splitting of G are virtually free.

In particular, in this case we have $\dim_{AR} \partial G = 1$.

We refer to Section 3 for precise definitions. In the case (i) above the WS property is trivially verified, the interesting case is (ii), but we need to split the statement in two cases because the JSJ splitting is not well defined for virtually Fuchsian groups. Note that if G satisfies the hypothesis (ii) of Theorem 1.3, then the AR conformal dimension is never attained by a distance in the gauge. In terms of quasiconvex subgroups of G , the condition (ii) above is equivalent to the following: the limit set Λ_H of every one-ended quasiconvex subgroup H of G can be separated in ∂G by removing a pair of points (see Section 3).

In order to clarify the meaning of condition (ii) in Theorem 1.3, we give in Section 2 a local version of the WS property. Let \mathcal{E} be the union of all limit sets of the stabilizers of the edges of the JSJ splitting of G . For each $x \in \partial G$, we define the set $\mathcal{E}(x)$ as the set of points $y \in \partial G$ such that x and y are in the same connected component of $\partial G \setminus P$, for any finite subset P of \mathcal{E} , not containing x nor y . We call $\mathcal{E}(x)$ the *fiber* of x . In Section 2 we

¹We remark that the boundary of a one-ended hyperbolic group is either homeomorphic to the Sierpiński carpet or the Menger sponge if it has topological dimension one and no local cut points [KK00].

prove that the WS condition is equivalent to the triviality of all the fibers, i.e. $\mathcal{E}(x) = \{x\}$ for all $x \in \partial G$. Moreover, the proof of Theorem 1.3 tells us what are the fibers: let $x \in \partial G$, then the fiber $\mathcal{E}(x)$ is either equal to the singleton $\{x\}$ or to the set

$$\mathcal{E}(x) = \bigcup \{\Lambda_v(x) : x \in \Lambda_v, v \in \mathbb{T}_r\},$$

if there exists $v \in \mathbb{T}_r$ with $x \in \Lambda_v$, where \mathbb{T}_r denotes the set of rigid type vertices of the JSJ decomposition of G , Λ_v the limit set of the stabilizer of a vertex v , and if $x \in \Lambda_v$, $\Lambda_v(x)$ denotes the connected component of Λ_v containing x . This explains why the existence of a non-virtually free rigid type vertex in JSJ splitting of G prevents the boundary ∂G to satisfy the WS condition: the points in a non-trivial component of the limit set of a rigid vertex cannot be separated by a finite number of points.

REMARK. In order to rigorously justify our approach to the characterization of hyperbolic groups of conformal dimension one, we need to show that conformal dimension is stable under maximal splittings over finite subgroups, i.e. an action of G on a simplicial tree \mathbb{T} such that the stabilizers of the edges are finite and the stabilizers of the vertices have at most one end [Dun85, Sta68]. The precise statement is the following: denote by $\{v_1, \dots, v_n\}$ a set of representatives of the orbits of the vertices of \mathbb{T} , and by G_{v_i} their respective stabilizers. Then

- (1) $\dim_{AR} \partial G = 0$ if all the G_{v_i} are finite, or
- (2) $\dim_{AR} \partial G = \max \{\dim_{AR} \partial G_{v_i} : G_{v_i} \text{ is infinite}\}$ otherwise.

This fact is well known to specialists in the field but we were unable to find it in the literature. We mention that the techniques used here allows one to prove it and a proof can be found in [Ca11].

We end this introduction by showing how Theorem 1.2 and 1.3 can be used to provide new interesting computations of the conformal dimension. Indeed, one remarkable consequence of Theorem 1.3 is the existence of convex cocompact Kleinian groups for which the conformal dimension is different from a well known geometric invariant in the context of hyperbolic 3-manifolds. More precisely, let G be a one-ended convex cocompact Kleinian group (whose limit set is not the entire sphere) and let M be the compact hyperbolizable 3-manifold with boundary defined by $(\mathbb{H}^3 \cup \Omega) / G$, where \mathbb{H}^3 denotes the real hyperbolic 3-space and Ω is the discontinuity set of the action of G on the Riemann sphere, so that $G \simeq \pi_1(M)$. Consider the collection of all complete hyperbolic 3-manifolds N which are homeomorphic to the interior of M , see [CMT99]. Each N is uniformized by a Kleinian group G_N so that N is isometric to the quotient \mathbb{H}^3 / G_N . Define $d(N)$ to be the Hausdorff dimension of the limit set of G_N equipped with the Euclidean metric (note that this dimension only depends on N), and let

$$D(M) := \inf_N d(N).$$

Note that one always has $\dim_{AR} \partial \pi_1(M) \leq D(M)$. In [CMT99] the authors completely characterize the compact hyperbolizable 3-manifolds for which $D(M) = 1$. This is the case if and only if M is a *generalized book of I-bundles*, see also [CMT99] for the definition. When M is a generalized book of I-bundles, there is no rigid vertex in the JSJ decomposition of $\pi_1(M)$. And therefore we obtain the following:

Corollary 1.4 ($\dim_{AR} \partial \pi_1(M) < D(M)$). *Let M be a compact hyperbolizable 3-manifold as above. If all the rigid vertices of the JSJ decomposition of $\pi_1(M)$ are virtually free, and there exists at least one rigid vertex, then $\dim_{AR} \partial \pi_1(M) < D(M)$.*

That is, for such a group there is a definite gap between the infimum of Hausdorff dimensions of Kleinian deformations and the conformal dimension. See Section 3.6 for a concrete example. The inequality $D(M) > 1$ is a (not direct) consequence of Thurston’s relative compactness theorem. It would be interesting to know if the conformal dimension of such a group can be obtained as the infimum of Hausdorff dimensions of the limit sets of convex cocompact actions of G on $\text{CAT}(-1)$ spaces. See [Bou97] Theorem 1.2 for a similar type of result for hyperbolic buildings.

REMARK. Many of the ideas of this paper can also be applied to the study of conformal dimension for the repellers of dynamical systems induced by a certain class of expanding branched coverings, i.e. the topologically cxc maps, see [HP09] for a precise definition. A dynamical sufficient condition for conformal dimension one also holds in this context. Nevertheless, the tools involved are quite different, so it will be explained elsewhere [Ca13].

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Outline of the paper. The paper is divided into two parts. In the first part, Section 2, we treat the problem in the general setting and we prove Theorem 1.2. Next we give a local version of the WS condition in terms of fibers, Lemma 2.2. Finally, we show that the WS property is equivalent to the hypotheses of Theorem 1.2 in the quasiselfsimilar case, this corresponds to Lemma 2.3.

The rest of the paper is devoted to hyperbolic groups. We start Section 3 by improving some well known facts about quasiconvex splittings of hyperbolic groups, see Section 3.2. We then prove some general lemmas about virtually free hyperbolic groups in Section 3.3. We apply Bowditch’s work on the structure of local cut points of the boundaries of one-ended hyperbolic groups to deduce Theorem 1.3. We end by showing how to construct explicit examples of groups verifying the hypotheses of Corollary 1.4 in Section 3.6.

2. A GENERAL CRITERION FOR CONFORMAL DIMENSION ONE

2.1. Proof of Theorem 1.2. See the Remark at the end of this Section for an idea of the proof. We start by remarking that the UWS property is equivalent to the following: there exists a function $C : (0, 1) \rightarrow \mathbb{R}_+$ such that for all $x \in X$ and $0 < s < r \leq \text{diam}X$, there is a finite set $P \subset B(x, r)$ of cardinality bounded from above by $C(s/r)$, and such that no connected component of $X \setminus P$ can intersect both $B(x, s)$ and $X \setminus \overline{B}(x, r)$. In fact, suppose the UWS condition is satisfied, and take $0 < s < r \leq \text{diam}X$. Let

$$n := \left\lceil \frac{2}{r-s} \right\rceil + 1, \text{ and } \epsilon = \frac{1}{4n}.$$

Consider the compact set $K = \overline{B}(x, r - n^{-1}) \setminus B(x, s + n^{-1})$. For every $y \in K$, the ball $B(y, 2\epsilon)$ is contained in $\overline{B}(x, r) \setminus B(x, s)$. By the doubling condition, we can cover K by less than M balls $B(y_i, \epsilon)$, with $y_i \in K$. Since $\text{diam}K \leq 2r$, the constant M can be chosen depending only on

$$\frac{\epsilon}{r} \asymp \frac{1}{nr} \asymp \frac{1-s/r}{2}.$$

For each center $y_i \in K$, consider a set $P_i \subset B(y_i, 2\epsilon)$, with $\#P_i \leq C$, given by the UWS condition. Then it suffices to take $P = \bigcup_i P_i$, which is contained in $B(x, r)$ and of cardinal number less than or equal to $M \cdot C$.

We must show that if X is a doubling, compact metric space satisfying LC and UWS, then $\dim_{AR} X = 1$. The main ingredient of the proof is the combinatorial modulus. Using an appropriate sequence of finite coverings of X whose mesh tends to zero, we can define combinatorial versions of conformal moduli, from which we are able to compute the conformal dimension of the space.

Consider $\{X_i\}$ a sequence of maximal a^{-i} -separated sets, with $a > 1$ a big enough constant. We associate the coverings \mathcal{S}_i by the balls $B(x, a^{-i})$ where $x \in X_i$. We write $\mathcal{S} := \bigcup_k \mathcal{S}_k$. The combinatorial modulus is defined as follows. Let $n, k \geq 1$, for every ball $B \in \mathcal{S}_k$, we consider the family of curves $\Gamma(B)$ in X that “join” the ball B with the complement of the ball $2B$, i.e. $\gamma \cap B \neq \emptyset$ and $\gamma \cap X \setminus 2B \neq \emptyset$. Here $2B$ denotes the ball with the same center as B and twice its radius. Given $p > 0$, we define the p -combinatorial modulus of the “annulus” $(B, 2B)$ by

$$M_{p,n}(B) := \inf_{\rho} \text{Vol}_p(\rho), \text{ where } \text{Vol}_p(\rho) := \sum_{A \in \mathcal{S}_{k+n}} \rho(A)^p,$$

and where the infimum is taken over all weight functions $\rho : \mathcal{S}_{k+n} \rightarrow \mathbb{R}_+$ which are $\Gamma(B)$ -admissible, i.e. for any curve $\gamma \in \Gamma(B)$, we have

$$\ell_{\rho}(\gamma) = \sum_{A \cap \gamma \neq \emptyset} \rho(A) \geq 1.$$

Therefore, for each $p > 0$, we obtain a sequence $\{M_{p,n}\}_n$, where $M_{p,n}$ is the p -combinatorial modulus of X at scale n :

$$M_{p,n} := \sup_{B \in \mathcal{S}} M_{p,n}(B).$$

In other words, the modulus $M_{p,n}$ takes into account all the “annuli” of X with a fixed radius ratio equal to 2.

We look at the asymptotic behavior of the sequence $\{M_{p,n}\}_n$, and its dependence on p . For example, for fixed p , $\{M_{p,n}\}_n$ verifies a sub-multiplicative inequality, see [Ca12]. It is therefore natural to consider the *critical exponent* Q defined by

$$Q := \inf \left\{ p > 0 : \liminf_n M_{p,n} = 0 \right\}.$$

We derive Theorem 1.2 from the following result.

Theorem 2.1 ([Ca12] Theorem 3.12 and Corollary 3.15). *Let X be a doubling, linearly connected, compact metric space. Then Q is equal to the AR conformal dimension of X .*

Therefore, we must prove that $Q = 1$. For this, we will prove that $M_{p,n} \lesssim \eta_n^{p-1}$ for all $p > 1$, where η_n is a sequence of positive real numbers that tends to zero as $n \rightarrow \infty$. Indeed, let $x \in X_k$ and write $r = a^{-k}$. Recall that $\Gamma(B)$ is the family of curves of X joining $B(x, r)$ and $X \setminus B(x, 2r)$. We denote by $P = P(B) \subset 3/2 \cdot B$ the finite set given by the UWS property, which verifies that $\gamma \cap P \neq \emptyset$ for any $\gamma \in \Gamma(B)$, and that $\#P \leq C$, where C is a uniform constant.

We have $\Gamma(B) \subset \bigcup_{z \in P} \Gamma_z$, where Γ_z is the family of curves γ in X such that $z \in \gamma$ and $\gamma \cap X \setminus B(z, s)$, where $s = r/3$. In particular, we obtain

$$M_{p,n}(B) \leq \sum_{z \in P} \text{Mod}_p(\Gamma_z, \mathcal{S}_{k+n}) \leq C \max_{z \in P} \{\text{Mod}_p(\Gamma_z, \mathcal{S}_{k+n})\}.$$

We must bound from above the combinatorial modulus of Γ_z . We take $m \geq 1$, and for $i = 0, \dots, m-1$, we set

$$A_i = \overline{B(z, 2^{-i}s)} \setminus B(z, 2^{-(i+1)}s).$$

By the UWS property, for each $z \in P$ and each $i = 0, \dots, m-1$, there exists a finite set $R_{z,i} \subset A_i$, with cardinal number bounded from above by a universal constant K , such that any curve γ of X verifying $\gamma \cap B(z, 2^{-(i+1)}s) \neq \emptyset$ and $\gamma \cap B(z, 2^{-i}s) \neq \emptyset$, must pass by $R_{z,i}$. For $n \geq 1$ such that $a^{-(k+n)} \leq 2^{-m}s$, consider the set

$$U = \{A \in \mathcal{S}_{k+n} : A \cap R \neq \emptyset\},$$

where

$$R := \bigcup_{i=0}^{m-1} R_{z,i}.$$

We define $\rho : \mathcal{S}_{k+n} \rightarrow \mathbb{R}_+$ by

$$\rho(A) := \begin{cases} \frac{1}{m} & \text{if } A \in U \\ 0 & \text{otherwise.} \end{cases} \quad (2.1)$$

On the one hand, since any curve $\gamma \in \Gamma_z$ must cross each A_i for $i = 0, \dots, m-1$, we have that ρ is Γ_z -admissible. On the other hand, there exists a constant M , which depends only on the doubling constant of X and a , such that

$$\#U \leq M \cdot \# \left\{ \bigcup_{i=0}^{m-1} R_{z,i} \right\} \leq (MK) \cdot m := K'm.$$

Therefore,

$$\text{Vol}_p(\rho) = \frac{\#U}{m^p} \leq \frac{K'}{m^{p-1}}.$$

This shows that $\text{Mod}_p(\Gamma_z, \mathcal{S}_{k+n}) \leq K'm^{1-p}$. Thus, it suffices to take

$$\eta_n := \left[\frac{n \log a - \log 3}{\log 2} \right]^{-1}.$$

This ends the proof of Theorem 1.3.

REMARK. We give an heuristic idea of what is the effect of the quasisymmetric deformations of the original distance d on X induced by the combinatorial modulus. Intuitively, the dimension of a distance θ on X is $\leq p$ if the cardinal of a maximal ϵr -separated set in a ball of radius r is bounded from above by $C\epsilon^{-p}$. Using the same notations as in the proof, the weights defined by (2.1) produce a distance θ for which the balls of the covering \mathcal{S}_{k+n} which do not intersect R have a very small diameter, and the balls in U have diameter almost equal to $1/m$ times the diameter s' of $B(z, s)$. This implies that the set of centers of the balls in U , i.e. intersecting the the set R , becomes very close to be a s'/m -separated set in $B(z, s)$ for θ , and its cardinal number is bounded from above by $K'm$. This shows why the dimension of θ is close to one, and tends to one when $m \rightarrow +\infty$. See Figure 2.1.

REMARK. The UWS condition fails when there is a infinite number of definite diameter pairwise disjoint curves in the space. Therefore, it is not difficult to find examples of spaces of conformal dimension equal to 1, but not verifying the UWS property. However, the LC and UWS assumptions are in some sense optimal. One can construct an example of a compact connected space which verifies WS and LC, but not UWS, and has AR conformal dimension strictly bigger than 1; and, also a space which verifies UWS, but not LC, and of Ahlfors regular conformal dimension strictly bigger than 1.

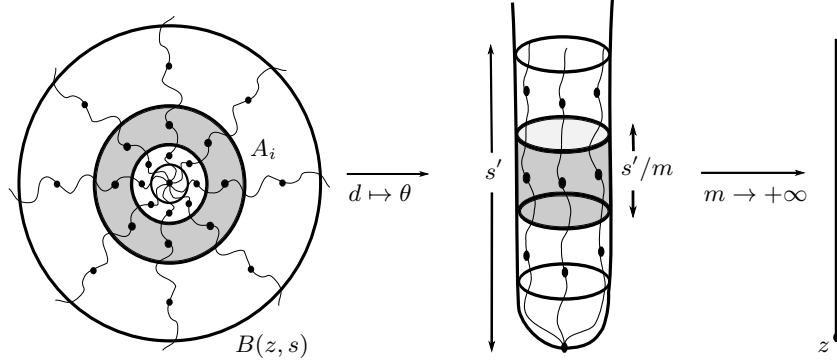


FIGURE 2.1. Idea of the proof of Theorem 1.2. The points represent the set R given by the UWS condition applied to each annulus A_i . Some curves passing through the annuli are drawn.

2.2. Equivalence between the different conditions. The aim of this Section is to relate the three different conditions on the existence of local cut points mentioned in the Introduction; i.e. the UWS, the WS, and the triviality of the fibers conditions.

Let X be a connected and compact metric space, and let \mathcal{E} be a countable subset of local cut points of X . Recall that for each $x \in X$, we define $\mathcal{E}(x)$ (the *fiber* of x) as the set of points $y \in X$ such that x and y are in the same connected component of $X \setminus P$ for any finite subset P of \mathcal{E} not containing x and y . If X verifies the WS condition, we can consider \mathcal{E} to be the union of the sets of local cut points P_n given by the definition. In that case, all fibers are trivial. When X is locally connected, the converse also holds.

Lemma 2.2 (Trivial fibers and WS). *Let X be a compact connected metric space. Then the following statements are equivalent:*

- X satisfies the WS property.
- X is locally connected and there is a countable subset \mathcal{E} of local cut points of X such that $\mathcal{E}(x) = \{x\}$ for all $x \in X$.

Note that since X is compact, local connectivity implies local arcwise connectivity. It is clear that if X satisfies UWS, then it verifies also WS. The converse is true in the self-similar case. Recall that X is *quasiselfsimilar* if there exist a distortion function $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and constants $c_0 > 0$ and $r_0 > 0$, such that for any $0 < r \leq \text{diam}X$ and any $x \in X$, there exists an open η -quasisymmetric embedding $\phi : B(x, r) \rightarrow U$ where $U \subset X$, with $U \supset B(\phi(x), r_0)$, and with $\text{diam}B(\phi(x), r_0) \geq c_0$. This definition implies that X is doubling and uniformly perfect, and that if X is connected and locally connected, then X is LC (see [Ca11] Chapter 2, Prop. 2.9). The boundaries of hyperbolic groups, when endowed with visual metrics, provide an important class of quasiselfsimilar spaces (with linear distortion function η), see Section 3.

Lemma 2.3 (UWS and WS for self-similar spaces). *Let X be a compact connected quasiselfsimilar space. Then the following statements are equivalent:*

- X satisfies the WS property.
- X satisfies the LC and UWS properties.

We obtain the following corollary of Theorem 1.2: if X is a compact connected quasiselfsimilar space, and satisfies the WS property, then $\dim_{AR} X = 1$. It is important to

note that the WS condition can be stated without using a metric, only with the uniform structure of the compact set X . It is therefore, a topological property.

2.2.1. *Proof of Lemma 2.2.* The arguments involved in the proof are rather elementary, we include them for clarity and completeness of presentation. See also [Ca11] Chapter 5, Prop. 5.2 for more details.

WS implies local connectivity. We can assume that the sequence $\{P_n\}_n$ in the definition of WS is increasing. Denote by \mathcal{C}_n the set of connected components of $X \setminus P_n$, and for $\delta > 0$,

$$\mathcal{C}_n(\delta) = \{C \in \mathcal{C}_n : \text{diam}C \geq \delta\}.$$

The set $\mathcal{C}_n(\delta)$ is always finite. Indeed, we have $\#\mathcal{C}_n(\delta) \leq \#P_m$ for m large enough so that $\delta_m < \delta$. This implies that if $\{C_m\}_m$ is a sequence of different connected components in \mathcal{C}_n , then

$$\text{diam}C_m \rightarrow 0, \text{ when } m \rightarrow +\infty. \tag{2.2}$$

Note that $X \setminus P_n$ is an open set and its topological boundary coincides with P_n . Then, the Boundary Bumping Theorem (see Chapter 5 of [Na92]) implies that $\overline{C} \cap P_n \neq \emptyset$ for all $C \in \mathcal{C}_n$. Let $x \in X$ and $\varepsilon > 0$. We take $n \geq 1$ such that $\delta_n < \varepsilon/2$, and set

$$U := \bigcup \{C : C \in \mathcal{C}_n, x \in \overline{C}\} \cup \{x\}.$$

Then U is a connected neighborhood of x by (2.2). Since $\text{diam}U \leq 2\delta_n < \varepsilon$, we obtain the local connectivity at x .

Trivial fibers implies WS. Let x be a point in X , and suppose $\mathcal{E}(x) = \{x\}$. We claim that for any radius $r \in (0, \text{diam}X)$, there exists a connected open set U containing x , which is contained in the ball $B(x, r)$, and whose frontier is a finite subset of \mathcal{E} .

Indeed, write $\mathcal{E} \setminus \{x\} = \{p_n : n \geq 1\}$. Let $\mathcal{E}_n = \{p_i : 1 \leq i \leq n\}$ for each $n \geq 1$, and define U_n to be the connected component of $X \setminus \mathcal{E}_n$ containing x . Then $\{U_n\}$ is a decreasing sequence of open connected sets containing x . For each n , the frontier of U_n is finite and is contained in \mathcal{E}_n . Since $\mathcal{E}(x) = \{x\}$, one checks that

$$\bigcap_n \overline{U}_n = \bigcap_n U_n = \{x\}.$$

This proves the claim. In particular, when $\mathcal{E}(x) = \{x\}$ for all $x \in X$, we can find for all $r \in (0, \text{diam}X)$, a finite open covering \mathcal{U} of X such that for all $U \in \mathcal{U}$: the frontier of U is a finite subset of \mathcal{E} and $\text{diam}U \leq 2r$. Finally, X satisfies the WS property: consider P_r the union of the frontiers ∂U , with $U \in \mathcal{U}$. Then P_r is a finite set, and all connected components of $X \setminus P_r$ have diameter less than or equal to $2r$.

2.2.2. *Proof of Lemma 2.3.* We prove now that for quasiselfsimilar spaces, the WS condition is equivalent to LC and UWS. Since when X verifies WS, it is locally connected, by self-similarity it is also LC. We now prove UWS.

For each $x \in X$ and $0 < r \leq \text{diam}X$, let $\phi : B(x, 2r) \rightarrow U \subset X$ be a η -quasisymmetric homeomorphism given by the definition of self-similarity. Let $\{P_n\}$ be the sequence of finite sets given by the definition of WS. We claim that there exists $n \geq 1$ such that for all $x \in X$ and $0 < r \leq \text{diam}X$, a connected component of $X \setminus R_n$ cannot intersect both $B(x, \frac{r}{2})$ and $X \setminus \overline{B}(x, r)$, where $R_n := \phi^{-1}(P_n \cap U)$. Since the cardinal number of R_n is bounded from above by that of P_n , this shows that X verifies UWS.

Suppose by contradiction that the claim is false. Then for each n , there exists $x_n \in X$, $0 < r_n \leq \text{diam}X$ and a connected component Y_n of $X \setminus R_n$ intersecting both $B(x_n, r_n/2)$

and $X \setminus B(x_n, r_n)$. Since X is locally connected and every R_n is finite, the component Y_n is arcwise connected. Let γ_n be a curve in Y_n satisfying $\gamma_n \cap B(x_n, r_n/2) \neq \emptyset$ and $\gamma_n \cap X \setminus B(x_n, r_n) \neq \emptyset$. Then $\text{diam}\gamma \geq r_n/2$, and therefore (see [Hei01], Prop. 10.8)

$$\frac{\text{diam}\phi_n(\gamma_n)}{\text{diam}\phi(B(x_n, 2r_n))} \geq \frac{1}{2} \cdot \eta \left(\frac{\text{diam}B(x_n, 2r_n)}{\text{diam}\gamma_n} \right)^{-1} \geq \frac{1}{2\eta(8)} := \delta. \quad (2.3)$$

The connected components of $X \setminus P_n$ have their diameter bounded from above by δ_n , which tends to zero as n tends to infinity. In particular, for n large enough, $\phi_n(\gamma_n) \cap P_n \neq \emptyset$. But this is a contradiction, since $\gamma_n \cap R_n = \emptyset$. This ends the proof.

3. THE WS PROPERTY FOR HYPERBOLIC GROUPS

In this section we give the proof of Theorem 1.3. Section 3.2.2 contains some estimates of the diameter of the limit sets of edge and vertex stabilizers of a quasiconvex splitting of G . Section 3.2.4 contains the specific estimates for the further study of virtually free groups in Section 3.3.

The main result of Section 3.3 is Lemma 3.15. This is the key lemma for the proof of Theorem 1.3. The lemma says that if v is a virtually free vertex of the JSJ splitting of G , then for a given scale $\delta > 0$, we can find a finite set F_v of edges incident to v , such that:

- removing their limit sets Λ_e from Λ_v produces a partition of Λ_v into small pieces ($\text{diam} \leq \delta$), and
- with the property that if e is an edge incident to v not in F_v , then Λ_e is contained in exactly one piece of this partition; i.e. Λ_e does not “connect” different pieces.

This consist in an inductive step of the proof. Section 3.5 contains the proof of the theorem, which is summarized in Proposition 3.18. Basically, the proof consists in constructing a big enough finite subtree T_δ of T in such a way that removing the limit sets of the vertices of T_δ lives only pieces of diameter less than δ . Then we can apply the previous lemma to decompose each one of the vertices of T_δ in a coherent fashion, producing the desired partition of the boundary of G .

In Section 3.6 we construct a simple example of a convex cocompact Kleinian group verifying the hypotheses of Corollary 1.4. We invite the reader to look at this example, and also Example 3.3.1, before going into the details of the proof.

3.1. Preliminaries. We begin by recalling some definitions and establishing some notations and properties which will be useful in the sequel. Let G be a Gromov-hyperbolic group and let \mathcal{C} be a Cayley graph of G with respect to some finite symmetric generating set S . Equip \mathcal{C} with the length distance which makes each edge of \mathcal{C} isometric to the interval $[0, 1]$, so that the action of G by left multiplication on \mathcal{C} is a geometric action. See [CDP90, GH90] for an introduction to the subject. We can identify the boundary of G with $\partial\mathcal{C}$. Consider the canonical conformal gauge on the boundary of \mathcal{C} induced by the visual metrics. In the following, we fix such a distance θ on $\partial\mathcal{C}$, and we recall that it satisfies

$$\theta(x, y) \asymp \exp(-\epsilon(x|y)), \quad \forall x, y \in \partial\mathcal{C},$$

where $\epsilon > 0$ and $(x|y)$ denotes the Gromov product of x and y viewed from the identity element of G . This metric is Ahlfors regular of dimension h/ϵ whenever G is non-elementary; here h is the volume entropy of the action of G on \mathcal{C} [Coo93]. In particular, $(\partial\mathcal{C}, \theta)$ is a doubling uniformly perfect compact metric space.

Recall also that if $x, x', y, y' \in \partial\mathbb{C}$ are four distinct points, their cross-ratio is defined by

$$[x, x', y, y'] := \frac{\theta(x, y)}{\theta(x, y')} \cdot \frac{\theta(x', y')}{\theta(x', y)}.$$

A homeomorphism $f : \partial\mathbb{C} \rightarrow \partial\mathbb{C}$ is η -quasi-Möbius if for all distinct $x, x', y, y' \in \partial\mathbb{C}$, we have

$$[f(x), f(x'), f(y), f(y')] \leq \eta([x, x', y, y'])$$

see [Väis84]. When G is a hyperbolic group, there exists a constant $C = C(\epsilon)$ such that the group G acts on $\partial\mathbb{C}$ by η -quasi-Möbius homeomorphisms with $\eta(t) = C \cdot t$ (see for example [Haïss08] Theorem 3.1). By the work of P. Tukia and B. Bowditch, G acts properly discontinuously and cocompactly on $\partial^3\mathbb{C}$, the space of triples of distinct points of $\partial\mathbb{C}$. Cocompactness on triples means that there is $\delta > 0$ such that for any triple $\{x_1, x_2, x_3\}$ of distinct points of $\partial\mathbb{C}$, there exists $g \in G$ such that $\theta(gx_i, gx_j) \geq \delta$, $i \neq j$. This implies that G is a uniform convergence group.

Quasi-Möbius and quasiasymmetric maps are closely related: if $f : B(x, r) \rightarrow U \subset \partial\mathbb{C}$ is a η -quasi-Möbius homeomorphism, and if there is a constant $\lambda > 0$ and three points $x_1, x_2, x_3 \in B(x, r)$, such that

$$\theta(x_i, x_j) \geq \frac{\text{diam}B(x, r)}{\lambda} \text{ and } \theta(f(x_i), f(x_j)) \geq \frac{\text{diam}U}{\lambda}, \quad (3.1)$$

then f is η_λ -quasiasymmetric, with $\eta_\lambda(t) = 2\lambda\eta(2\lambda t)$. Therefore, $\partial\mathbb{C}$ is a quasiselfsimilar space (see also [Haïss08] Prop. 4.6): if G is non-elementary, there exist $r_0 > 0$, $c_0 > 0$ and a linear distortion function $\eta(t) = C't$ such that for any $x \in \partial\mathbb{C}$ and any radius $r > 0$, there is $g \in G$ such that $g(B(x, r)) \supset B(g(x), r_0)$, $\text{diam}B(g(x), r_0) \geq c_0$ and $g|_{B(x, r)}$ is η -quasiasymmetric. This is also known as the Sullivan *conformal elevator principle* [Sul82].

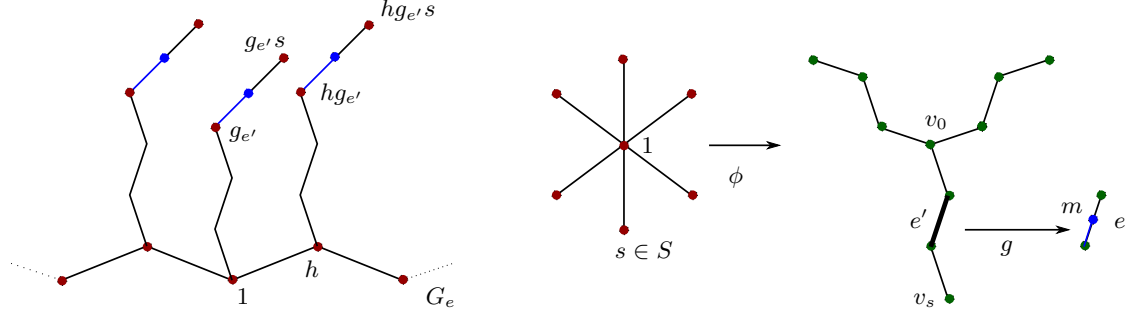
3.2. Quasiconvex splittings and the boundary. Let \mathbb{T} be a simplicial tree, we denote by \mathbb{T}_0 and by \mathbb{T}_1 its vertices and its edges. We equip the geometric realization of \mathbb{T} with the length distance that makes each edge isometric to the interval $[0, 1]$. Consider a simplicial action of G on \mathbb{T} , without edge inversions and such that the quotient \mathbb{T}/G is finite, so that G is isomorphic to the fundamental group of a graph groups. Assume also that the action is minimal, i.e. there is no proper G -invariant sub-tree of \mathbb{T} . We will use the letters e for an edge of \mathbb{T} and l for an edge of \mathbb{C} .

Given $v \in \mathbb{T}_0$ and $e \in \mathbb{T}_1$, we denote by G_v and by G_e the stabilizers of v and e respectively. We also let Λ_v and Λ_e be the limit set of G_v and G_e respectively. Thus, if $e \in \mathbb{T}_1$ and $v, w \in \mathbb{T}_0$ are the endpoints of e , then $G_e = G_v \cap G_w$.

Recall that a subset Y of \mathbb{C} is *quasiconvex* if there exists a constant $K \geq 0$, such that any point on any geodesic of \mathbb{C} joining two points of Y is at distance less than or equal to K from Y . We say that a subgroup H of G is quasiconvex if H is a quasiconvex subset of \mathbb{C} . In particular, H is also a hyperbolic group, the inclusion $H \hookrightarrow \mathbb{C}$ is a quasi-isometric embedding, and ∂H is canonically identified with the limit set $\Lambda_H \subset \partial\mathbb{C}$. If H_1 and H_2 are two quasiconvex subgroups of G , then $H_1 \cap H_2$ is also quasiconvex and $\Lambda_{H_1 \cap H_2} = \Lambda_{H_1} \cap \Lambda_{H_2}$ [Sho90, BH99]. In the sequel we suppose that G_e is quasiconvex for every edge $e \in \mathbb{T}_1$.

3.2.1. Some useful definitions. The following definition will be important in relating the action of G on \mathbb{T} and the boundary of \mathbb{C} . Fix an arbitrary vertex $v_0 \in \mathbb{T}_0$.

Definition 3.1. *We define an equivariant continuous function $\phi : \mathbb{C} \rightarrow \mathbb{T}$ by setting $\phi(1) = v_0$ and $\phi(g) = g \cdot v_0$ for all $g \in G$. This determines ϕ on G , the vertices of \mathbb{C} . If $l = (g, h)$ is an edge of \mathbb{C} , we extend ϕ so that it sends l linearly to the unique arc of \mathbb{T} that joins $\phi(g)$ and $\phi(h)$.*

FIGURE 3.1. The set C_e .

The image of an edge of \mathcal{C} is either a vertex or an arc of \mathbb{T} . In particular, the image of ϕ is a G -invariant subtree of \mathbb{T} , and since the action of G is minimal, ϕ is surjective. For each $s \in S$, we let $l_s = (1, s)$ be the edge between 1 and s , and $v_s = s \cdot v_0$. Then $\phi(l_s) = [v_0, v_s]$ is the unique arc of \mathbb{T} joining v_0 and v_s . Therefore, if $l = (g, gs)$ is an edge of \mathcal{C} , then $\phi(l) = g \cdot [v_0, v_s]$. Since S is a finite set,

$$L := \max \{ \text{length}([v_0, v_s]) : s \in S \} < \infty,$$

and ϕ is Lipschitz continuous with constant L .

Let us introduce some notation:

- Denote by dist_H the Hausdorff distance between closed subsets of \mathcal{C} . If A is a closed subset of \mathcal{C} , we denote by Λ_A the set $\overline{A} \setminus A$, where the closure is taken in the space $\mathcal{C} \cup \partial\mathcal{C}$.
- Let $e \in \mathbb{T}_1$, we denote by m_e its mid-point and $M_e := \phi^{-1}(m_e)$.
- For $v \in \mathbb{T}_0$, let \mathbb{T}_v^e be the closure of the connected component of $\mathbb{T} \setminus \{m_e\}$ which does not contain v , and let $C_e^v = \phi^{-1}(\mathbb{T}_v^e)$. For $v = v_0$, we simply write \mathbb{T}_e and C_e ; we also denote by \mathbb{T}_e^- the other component of $\mathbb{T} \setminus \{m_e\}$ and $C_e^- = \phi^{-1}(\mathbb{T}_e^-)$.
- We denote by E_v the set of edges of \mathbb{T} which are incident to v .
- For a vertex $v \in \mathbb{T}_0$, let S_v be the star-shaped connected subset of \mathbb{T} formed by v and all the segments joining v to m_e with $e \in E_v$. Let $M_v := \phi^{-1}(S_v)$. Note that M_e and M_v are respectively G_e and G_v invariant, and for every edge $e \in E_v$, we have $M_e \subset M_v$.
- For every vertex $v \neq v_0$ of \mathbb{T} , there exists exactly one edge $e^* \in E_v$ such that $C_{e^*}^v = C_{e^*}^-$, i.e. such that $C_{e^*}^v$ contains v_0 . Let $E_v^* = E_v \setminus \{e^*\}$.

Finally, we say that a function $f : A \rightarrow [0, \infty)$ defined on a set A , tends to zero, $f(a) \rightarrow 0$ for $a \in A$, if the set $\{a \in A : f(a) \geq \delta\}$ is finite for all $\delta > 0$. We define similarly $f(a) \rightarrow \infty$ for $a \in A$: for all $N > 0$, the set $\{a \in A : f(a) \leq N\}$ is finite.

3.2.2. Estimating the diameter of Λ_e and Λ_v . The estimates that follow are improvements of some results proved in [Bow98a, Bow98b]. Our aim is to deduce some topological properties of $\partial\mathcal{C}$ from the action of G on \mathbb{T} .

Lemma 3.2. *For each edge $e \in \mathbb{T}_1$, there exists a constant $K = K(e) > 0$ such that*

$$\text{dist}_H(M_e, G_e) \leq K. \quad (3.2)$$

In particular, $\Lambda_{M_e} = \Lambda_e$.

Proof. Let $e \in \mathbb{T}_1$. We first show that every point of M_e is at distance less than or equal to K from G_e , where K is a constant that depends on e . Indeed, if $l = (g, gs)$ is an edge

of \mathbb{C} with $g \in G$ and $s \in S$, then $M_e \cap l$ has at most one point. For fixed $s \in S$, consider the set

$$H_{s,e} = \{g \in G : \exists e' \in [v_0, v_s] \text{ s.t. } g \cdot e' = e\}.$$

Then $M_e \cap l \neq \emptyset$ if and only if $g \in H_{s,e}$. If for $e' \in [v_0, v_s]$ we choose arbitrarily —if any— $g_{e'} \in H_{s,e}$ such that $g_{e'} \cdot e' = e$, then all the others $g \in H_{s,e}$ sending e' to e are of the form $g = hg_{e'}$, with $h \in G_e$. So for any $g \in H_{s,e}$, there exists $e' \in [v_0, v_s]$ such that

$$\text{dist}(G_e, g) \leq d(h, g) = d(h, hg_{e'}) = d(1, g_{e'}),$$

see Figure 3.2.1. Since the cardinal number of S and that of the edges in each $[v_0, v_s]$ is finite, it suffices to take

$$K := 1 + \max\{d(1, g_{e'}) : e' \in [v_0, v_s], s \in S\} < \infty.$$

By surjectivity of ϕ , there exists at least one $s \in S$ such that $H_{s,e} \neq \emptyset$ (i.e. there exists an edge $e' \in [v_0, v_s]$ such that one can choose $g_{e'}$ as above). Since for all $h \in G_e$, the element $g_h = hg_{e'} \in G$ satisfies $g_h \cdot e' = e$, we have $M_e \cap (g_h, g_h s) \neq \emptyset$. That is, for all $h \in G_e$, we also have $\text{dist}(h, M_e) \leq K$, and therefore, $\text{dist}_H(M_e, G_e) \leq K$. \square

The following lemma is proved in Section 1 of [Bow98a]. We include the proof because we need a uniform control on the constants involved.

Lemma 3.3. *The sets $\{M_e : e \in \mathbb{T}_1\}$ are uniformly quasiconvex. In addition,*

$$\text{diam } \Lambda_e = \text{diam } \Lambda_{M_e} \lesssim \exp(-\epsilon \text{dist}(1, M_e)). \quad (3.3)$$

Proof. Let $\{e_1, \dots, e_n\}$ be a set of representatives of edge orbits. First, note that M_{e_i} is quasiconvex because it is at bounded Hausdorff distance K_i from G_{e_i} .

Moreover, the quasiconvexity constant $K_{qc}(i)$ of M_{e_i} depends only on K_i and the hyperbolicity constant of \mathbb{C} . But since $M_{g \cdot e} = g \cdot M_e$ for any $g \in G$, the sets M_e are in fact uniformly quasiconvex, with constant

$$K_{qc} = \max\{K_{qc}(i) : i = 1, \dots, n\}.$$

Finally, this implies that for any geodesic γ joining two points of M_e , we have

$$\text{dist}(\gamma, 1) \geq \text{dist}(M_e, 1) - K_{qc}.$$

This shows (3.3). \square

Lemma 3.4. *The sets $\{C_e : e \in \mathbb{T}_1\}$ are uniformly quasiconvex of constant K_{qc} . In particular,*

$$\text{diam } \Lambda_{C_e} \lesssim \exp(-\epsilon \text{dist}(1, C_e)). \quad (3.4)$$

In addition,

$$\text{dist}(1, C_e) = \text{dist}(1, M_e) \rightarrow \infty \text{ for } e \in \mathbb{T}_1. \quad (3.5)$$

Proof. If γ is a geodesic joining two points of C_e , then, since \mathbb{T} is a tree, the connected components of $\gamma \setminus C_e$ join two points of M_e . But $M_e \subset C_e$, so the sets C_e , with $e \in \mathbb{T}_1$, are K_{qc} quasiconvex. Inequality (3.4) is proved in a similar way as (3.3).

Note that mid-points of different edges of \mathbb{T} are at distance at least one. So if $x \in M_e$ and $y \in M_{e'}$ for e and e' two different edges of \mathbb{T} , then

$$d(x, y) \geq L^{-1} \cdot d(m_e, m_{e'}) \geq L^{-1}.$$

Since in each edge l of \mathbb{C} there is at most one element of M_e , the number of edges e such that $M_e \cap l \neq \emptyset$ is bounded from above by L . Therefore, there is at most a finite number of sets M_e which intersects the ball $B(1, n)$ of \mathbb{C} . That is, $\text{dist}(1, M_e) \rightarrow \infty$ for $e \in \mathbb{T}_1$.

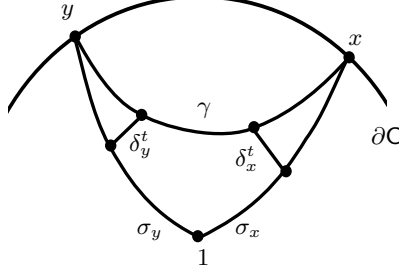


FIGURE 3.2. Proof of Lemma 3.5.

Finally, since every geodesic γ which joins 1 to a point of C_e has to pass through M_e , we obtain $\text{dist}(1, C_e) \geq \text{dist}(1, M_e)$. The other inequality follows from $M_e \subset C_e$. \square

REMARK. Lemma 3.3 still holds, and the proof is analogous, if we replace in its statement the set M_e by M_v . Also $\{M_v : v \in T_0\}$ is a locally finite cover of C by K_{qc} -quasiconvex sets, and $\Lambda_{M_v} = \Lambda_v$.

If γ is a geodesic of C , we let $\gamma_{\pm} = \gamma(\pm\infty)$. We say that γ is asymptotic to γ_+ and γ_- . Denote by $C_0 \geq 1$ a uniform constant, that depends only on the hyperbolicity of C , such that for any pair of points x and y of ∂C and any geodesic γ asymptotic to x and y , we have

$$C_0^{-1} \exp(-\epsilon \text{dist}(1, \gamma)) \leq \theta(x, y) \leq C_0 \exp(-\epsilon \text{dist}(1, \gamma)). \quad (3.6)$$

In the following lemma, we show that any geodesic asymptotic to a point of Λ_{C_e} and to a point of $\Lambda_{C_e^-}$ must pass uniformly close from M_e .

Lemma 3.5. *There is a uniform constant K , which depends only on the hyperbolicity constant of C , such that*

$$\text{dist}(\gamma, M_e) \leq K, \quad (3.7)$$

for each edge $e \in T_1$, any two points $x, y \in \partial C$ with $x \in \Lambda_{C_e}$ and $y \in \Lambda_{C_e^-}$, and any geodesic γ of C with $\gamma_+ = x$ and $\gamma_- = y$.

Proof. We consider $\sigma_x : [0, +\infty) \rightarrow C$ and $\sigma_y : [0, +\infty) \rightarrow C$ two geodesic rays from $1 \in C$ such that $\sigma_x(+\infty) = x$ and $\sigma_y(+\infty) = y$. By hyperbolicity, there exists a uniform constant K , and a constant $T < +\infty$ which depends on x and y , such that if $t \in [T, +\infty)$, then

$$\max \{ \text{dist}(\gamma(t), \sigma_x), \text{dist}(\gamma(-t), \sigma_y) \} \leq K.$$

For $t \geq T$, we consider the curve

$$\gamma_t = \sigma_y^t * \delta_y^t * \gamma|_{[-t, t]} * \delta_x^t * \sigma_x^t,$$

where δ_x^t (resp. δ_y^t) is a geodesic segment of length less than or equal to K , joining $\gamma(t)$ (resp. $\gamma(-t)$) with a point $\sigma_x(u_t)$ (resp. $\sigma_y(v_t)$); the curve σ_x^t (resp. σ_y^t) is the restriction of σ_x (resp. σ_y) to the interval $[u_t, \infty)$ (resp. $(-\infty, v_t]$). See Figure 3.2.

Note that $\phi(\gamma_t)$ joins a point in T_e^- to a point in T_e . In particular, m_e belongs to $\phi(\gamma_t)$, so $\gamma_t \cap M_e \neq \emptyset$. Let p be a point in the intersection, take t large enough so that p does not belong to $\sigma_x^t \cup \sigma_y^t$. In this case, either p belongs to γ or to $\delta_x^t \cup \delta_y^t$. Therefore, we obtain a point in γ which is at distance smaller than or equal to K from $p \in M_e$. \square

3.2.3. *Description of $\partial\mathbb{C}$ from the action on \mathbb{T} .* One can associate to \mathbb{T} an ideal boundary $\partial\mathbb{T}$, formed by the geodesic rays from v_0 . We are interested in $\partial\mathbb{T}$ only as a set. See [Bow98a] Proposition 1.3, for a proof of the following proposition.

Proposition 3.6. *Assume that the stabilizers G_e of the edges $e \in \mathbb{T}_1$ are quasiconvex. Then there exists a natural G -invariant partition $\partial\mathbb{C} = \partial_\infty\mathbb{C} \cup \partial_0\mathbb{C}$ with*

$$\partial_0\mathbb{C} = \bigcup_{v \in \mathbb{T}_0} \Lambda_v, \quad (3.8)$$

and $\partial_\infty\mathbb{C} = \{\text{rays } \gamma : \phi(\gamma) \text{ is unbounded}\}$. Moreover, there exists a bijection

$$j : \partial_\infty\mathbb{C} \rightarrow \partial\mathbb{T}.$$

We merely describe the function j . Let $\gamma : [0, \infty) \rightarrow \mathbb{C}$ be a geodesic ray from 1, and denote by $l_n = (g_n, g_{n+1})$ the edges of γ , where $g_n = \gamma(n)$ for $n \geq 0$. Let $v \in \mathbb{T}_0$, and suppose that $n \leq m$ are such that $\phi(l_n) \cap \mathbb{S}_v \neq \emptyset$ and $\phi(l_m) \cap \mathbb{S}_v \neq \emptyset$. Since \mathbb{M}_v is quasiconvex, $\gamma([n, m])$ is at uniformly bounded distance from \mathbb{M}_v . Therefore, if there is an infinite number of n such that $\phi(l_n) \cap \mathbb{S}_v \neq \emptyset$, then $\lim_n \gamma(n) \in \Lambda_v$.

Suppose now that for every $v \in \mathbb{T}_0$, the set of edges l_n of γ such that $\phi(l_n) \cap \mathbb{S}_v \neq \emptyset$ is finite. We have $\phi(l_s) \cap \mathbb{S}_{v_0} \neq \emptyset$, because $\gamma(0) = 1 \in G$ and $\phi(1) = v_0$. Then there exists a sequence of edges $\{e_i\}$ of \mathbb{T} , and an increasing sequence of positive integers $\{n_i\}$, such that for all $i \geq 0$, we have $\phi(\gamma([n_i, \infty))) \subset \mathbb{T}_{e_i}$, where $e_i = (v_i, v_{i+1})$ and $\mathbb{T}_{e_{i+1}} \subset \mathbb{T}_{e_i}$. The sequence of edges $\{e_i\}$ defines a geodesic ray r from v_0 in \mathbb{T} , and therefore, a point of $\partial\mathbb{T}$. This ray depends only on the class $[\gamma]$, and we can define $j([\gamma]) = r$.

Let $r : [0, \infty) \rightarrow \mathbb{T}$ be a ray from v_0 , and let $\{e_i = (v_i, v_{i+1})\}_{i \geq 0}$ be the edges of r . Lemma 3.4 implies that

$$\bigcap_{i \geq 0} \overline{\mathbb{C}_{e_i}} = \{x\} \subset \partial\mathbb{C},$$

and one checks that $j^{-1}(r) = x$.

3.2.4. *The case of finite edge stabilizers.* We suppose in this section that the stabilizers of the edges of \mathbb{T} are finite, in particular, quasiconvex, and the stabilizers of the vertices have at most one end. We call DS such a splitting [Dun85, Sta68]. For every vertex $v \in \mathbb{T}_0$, either the stabilizer G_v is finite and its limit set Λ_v is empty, or G_v is infinite and Λ_v is connected. The DS splitting describes the structure of connected components of $\partial\mathbb{C}$.

Let $\partial\mathbb{T}_e^\pm$ be the set of rays of \mathbb{T} which eventually are contained in \mathbb{T}_e^\pm . Note that in this case $\Lambda_e = \emptyset$ for any edge $e \in \mathbb{T}_1$, so

$$\Lambda_{\mathbb{C}_e^-} \cap \Lambda_{\mathbb{C}_e} = \Lambda_e = \emptyset,$$

and the sets $\Lambda_{\mathbb{C}_e}$ are open and closed in $\partial\mathbb{C}$.

For any element $g \in G$ we set $|g| = d(1, g)$. Let $\mathbf{E}_0 = \{e_1, \dots, e_r\}$ be a set of representatives of edge orbits. For each edge $e \in \mathbb{T}_1$ we choose, and fix for the sequel, $g_e \in G$ such that $e = g_e \cdot e_i$. If g and g' are two elements of G such that $e = g \cdot e_i = g' \cdot e_i$, then $d(g, g') \leq \eta$, where

$$\eta = \max \{|h| : h \in G(e_i), i = 1, \dots, r\}.$$

Then the set of elements $g \in G$ such that $e = g \cdot e_i$ is contained in $B_{\mathbb{C}}(g_e, \eta)$. This implies that for each edge $e \in \mathbb{T}_1$, the number $|g_e|$ does not depend, up to an additive constant, on the choice of g_e .

Recall that for each $g \in G$ and $e_i \in \mathbf{E}_0$ with $e = g \cdot e_i$, we have $\mathbb{M}_e = g \cdot \mathbb{M}_{e_i}$. Since from Lemma 3.2, the Hausdorff distance between \mathbb{M}_{e_i} and G_{e_i} is finite, each \mathbb{M}_{e_i} is finite and there exists a radius R_i such that $\mathbb{M}_{e_i} \subset B_{\mathbb{C}}(x_i, R_i)$, with $x_i \in \mathbb{M}_{e_i}$. If $R := \max\{R_i\}$,

then for any $e \in \mathbb{T}_1$, there exists $x_e \in M_e$ such that $M_e \subset B_{\mathbb{C}}(x_e, R)$. In addition, we have the following estimate:

Lemma 3.7. *For each edge $e \in \mathbb{T}_1$, we have $\text{dist}(1, M_e) = |g_e| + O(1)$.*

Proof. Let $d_i = \text{dist}(1, M_{e_i})$ for $i = 1, \dots, r$. We set $d = \max\{d_i\}$. Then for each edge $e \in \mathbb{T}_1$, we have $\text{dist}(g_e, M_e) = \text{dist}(1, M_{e_i}) = d_i$ for some $i \in \{1, \dots, r\}$. So M_e is contained in the ball of radius $R + d$ centered at g_e . This ends the proof. \square

We obtain by Lemma 3.4 the following corollary:

Corollary 3.8. *For each edge $e \in \mathbb{T}_1$, we have $\text{diam } \Lambda_{C_e} \lesssim \exp(-\epsilon|g_e|)$.*

REMARK. Similar arguments show that for every vertex $v \in \mathbb{T}_0$ of finite stabilizer, M_v is contained in a ball of definite radius. For each edge $e \in E_v$, we have $M_e \subset M_v$, and therefore, $\text{dist}(1, M_v) = |g_e| + O(1)$ for any edge $e \in M_v$. Furthermore, we see that E_v is finite.

Corollary 3.9. *For each edge $e \in \mathbb{T}_1$, we have $\text{dist}(\Lambda_{C_e}, \Lambda_{C_e^-}) \gtrsim \exp(-\epsilon|g_e|)$.*

Proof. Consider a geodesic γ asymptotic to $x \in \Lambda_{C_e}$ and $y \in \Lambda_{C_e^-}$. By inequality (3.7), $\text{dist}(w, \gamma) \leq K$ for some $w \in M_e$. So we have

$$\text{dist}(1, \gamma) \leq \text{dist}(\gamma, w) + d(w, 1) \leq K + R + d + |g_e| = |g_e| + O(1).$$

Since $(x|y) = \text{dist}(1, \gamma) + O(1)$, we obtain

$$\theta(x, y) \asymp \exp(-\epsilon \text{dist}(1, \gamma)) \gtrsim \exp(-\epsilon|g_e|).$$

This proves the lemma. \square

A general property of uniformly perfect spaces allows us to improve the previous estimate. We denote by C_P the constant of uniform perfectness.

Lemma 3.10. *Let X be a uniformly perfect metric space. For any nonempty subset W of X , we have*

$$\text{dist}(W, X \setminus W) \leq C_P \min\{\text{diam}W, \text{diam}(X \setminus W)\}. \quad (3.9)$$

Proof. There is nothing to prove if $\text{diam}W \geq \text{diam}X/C_P$, so suppose that W is a subset of X of diameter smaller than $\text{diam}X/C_P$. We let $r = \text{diam}W$, and consider $x \in W$. Since $W \subset B(x, r)$ and $B(x, C_P r) \setminus B(x, r) \neq \emptyset$, there exists $y \in B(x, C_P r) \cap X \setminus W$. Therefore,

$$\text{dist}(W, X \setminus W) \leq d(x, y) \leq C_P r = C_P \text{diam}W.$$

The lemma follows by symmetry. \square

Corollary 3.11. *For each edge $e \in \mathbb{T}_1$, we have*

$$\text{dist}(\Lambda_{C_e}, \Lambda_{C_e^-}) \asymp \exp(-\epsilon|g_e|).$$

Proof. By Corollary 3.9 it suffices to show the upper bound. But since Λ_{C_e} is the complement of $\Lambda_{C_e^-}$, and since we have $\text{diam } \Lambda_{C_e} \lesssim \exp(-\epsilon|g_e|)$, we can apply (3.9). \square

Lemma 3.12. *Let v be a vertex of \mathbb{T}_0 of finite stabilizer. Then for every pair of edges $e, e' \in E_v^*$, we have*

$$\text{dist}(\Lambda_{C_e^v}, \Lambda_{C_{e'}^v}) \asymp \exp(-\epsilon|g_e|). \quad (3.10)$$

Proof. We know from the remark following Corollary 3.8, that $|g_e| = |g_{e'}| + O(1)$. Moreover, since $\Lambda_{\mathbb{C}_{e'}}$ is contained in the complement of $\Lambda_{\mathbb{C}_e}$, by Corollary 3.11 we have

$$\text{dist}\left(\Lambda_{\mathbb{C}_e}, \Lambda_{\mathbb{C}_{e'}}\right) \gtrsim \exp(-\epsilon|g_e|).$$

Let us show the upper bound. Let x be a point of $\Lambda_{\mathbb{C}_e}$ and y a point of $\Lambda_{\mathbb{C}_{e'}}$. The geodesic rays from 1, σ_x and σ_y , representing respectively x and y , must pass through $x_v \in M_e$ and $y_v \in M_{e'}$ respectively. Denote by $\tilde{\sigma}_x$ (resp. $\tilde{\sigma}_y$) the restriction of σ_x (resp. σ_y) starting at x_v (resp. y_v). Since $M_e \cup M_{e'} \subset M_v$, which is contained in a ball of definite radius R , there exists a geodesic σ of length less than $2R$ that joins x_v and y_v . Let $\gamma_{x,y} := \tilde{\sigma}_y * \tilde{\sigma} * \tilde{\sigma}_x$ be the concatenation of these geodesics. By hyperbolicity of \mathbb{C} , there exists a constant K and a geodesic γ of \mathbb{C} , asymptotic to x and y , such that $\text{dist}_H(\gamma, \gamma_{x,y}) \leq K$. Then

$$\text{dist}(1, \gamma) \geq \text{dist}(1, M_v) - (2R + K) = |g_e| + O(1).$$

Therefore, $\theta(x, y) \lesssim \exp(-\epsilon|g_e|)$. This proves the lemma. \square

3.3. Some general lemmas on virtually free groups. The purpose of this section is to prove a key lemma on virtually free groups, Lemma 3.15, for the proof of Theorem 1.3. Let us begin by the following general remark. Let x and y be two different points on $\partial\mathbb{C}$, and suppose that the stabilizer $\text{Stab}(\{x, y\})$ is infinite —it is then a two-ended quasiconvex subgroup of G . We claim that for fixed $\delta > 0$, there exists a finite set $\{g_1, \dots, g_N\} \subset G$ such that if $g \in G$ verifies $\theta(g \cdot x, g \cdot y) \geq \delta$, then it can be written as

$$g = g_i h, \text{ for some } i \in \{1, \dots, N\} \text{ and } h \in \text{Stab}(\{x, y\}). \quad (3.11)$$

Indeed, let γ be a geodesic asymptotic to x and y , and fix any point $w_0 \in \gamma$. By assumption, $\text{dist}(1, g \cdot \gamma) \leq K$ for some constant which depends only on ϵ , the hyperbolicity constant of \mathbb{C} and δ . Suppose $g \cdot w$ is the nearest point of $g \cdot \gamma$ to 1. Since the stabilizer of $\{x, y\}$ is quasiconvex, there exists a uniform constant C , and $h_w \in \text{Stab}(\{x, y\})$, such that $d(h_w \cdot w_0, w) \leq C$. This implies that

$$gh_w \cdot w_0 \in B_{\mathbb{C}}(1, K + C),$$

and therefore, that gh_w belongs to a finite subset of G .

We suppose now that G is a virtually free group. Let \mathbb{T} be the simplicial tree associated to the DS splitting of G , so in particular edge and vertex stabilizers are all finite subgroups of G . In this case, the map j defined in Proposition 3.6 provides a bijection between $\partial\mathbb{C}$ and $\partial\mathbb{T}$; i.e. for every geodesic ray γ in \mathbb{C} , the image $\phi(\gamma)$ is unbounded in \mathbb{T} .

Let $\mathcal{H} := \{H_1, \dots, H_k\}$ be a finite collection of two-ended subgroups of G . For each $i \in \{1, \dots, k\}$ the limit set Λ_i of H_i consists of exactly two points of $\partial\mathbb{C}$, which we denote by h_i^+ and h_i^- . Define the *decomposition space* associated to \mathcal{H} by $D := \partial\mathbb{C} / \sim$, where two points x and y of $\partial\mathbb{C}$ are identified if and only if, either $x = y$, or there exists $g \in G$ such that $\{x, y\} = g \cdot \Lambda_i$. We denote by $p : \partial\mathbb{C} \rightarrow D$ the projection defined by \sim and we equip D with the quotient topology. Note that any point of D has at most two pre-images under p . By (3.11), D is a compact Hausdorff topological space. In the sequel, we will always suppose that D is connected. We recall that a saturated set A in $\partial\mathbb{C}$ is by definition a subset of $\partial\mathbb{C}$ such that if $A \cap p^{-1}(y) \neq \emptyset$ for some $y \in D$, then $p^{-1}(y) \subset A$.

Lemma 3.13. *If e is an edge of \mathbb{T} , then there exist $i \in \{1, \dots, k\}$ and $g \in G$ such that $g \cdot h_i^\pm \in \Lambda_{\mathbb{C}_e}$ and $g \cdot h_i^\mp \in \Lambda_{\mathbb{C}_e^-}$.*

Proof. If not, the sets $\Lambda_{\mathbb{C}_e}$ and $\Lambda_{\mathbb{C}_e^-}$ are open, closed and saturated sets in $\partial\mathbb{C}$, and therefore, the set $U := p(\Lambda_{\mathbb{C}_e})$ is a proper open and closed subset of D . \square

Let e be an edge of \mathbb{T} . We set

$$\mathcal{P}_e := \left\{ g \in G : \exists i \in \{1, \dots, k\} \text{ s.t. } g \cdot h_i^\pm \in \Lambda_{C_e}, g \cdot h_i^\mp \in \Lambda_{C_e^-} \right\}.$$

By Lemma 3.13, for any edge e of \mathbb{T} the set \mathcal{P}_e is not empty. Let

$$P_e := \bigcup_{g \in \mathcal{P}_e} g \cdot \Lambda_i,$$

and let $D_e = p(P_e)$ be its projection onto D .

Lemma 3.14. *For any edge e of \mathbb{T} , the set P_e is finite and $D \setminus D_e$ is disconnected.*

Proof. Fix e an edge of \mathbb{T} . Let us first prove that P_e is finite. Indeed, if $g \in G$ and $i \in \{1, \dots, k\}$ are such that $g \cdot h_i^\pm \in \Lambda_{C_e}$ and $g \cdot h_i^\mp \in \Lambda_{C_e^-}$, then

$$\theta(g \cdot h_i^\pm, g \cdot h_i^\mp) \geq \text{dist}(\Lambda_{C_e}, \Lambda_{C_e^-}) := c_e > 0,$$

where c_e only depends on e . By (3.11), there exists a finite subset F_e of G such that $g = g'h$, with $g' \in F_e$ and $h \in \text{Stab}(\Lambda_i)$. Therefore, $g \cdot \Lambda_i = g' \cdot \Lambda_i$, which implies that P_e is finite.

Let us show that D_e is a cut-set in D . Since P_e is finite, it is closed in $\partial\mathcal{C}$. Since $p(\Lambda_{C_e}) \cap p(\Lambda_{C_e^-}) = D_e$, we see that $\Lambda_{C_e} \setminus P_e$ and $\Lambda_{C_e^-} \setminus P_e$ are open and closed saturated sets in $\partial\mathcal{C}$. Taking the projection of these sets by p gives a non-trivial partition of $D \setminus D_e$. \square

REMARK. It is important to note that P_e is finite because $\Lambda_e = \emptyset$. The finiteness conclusion of Lemma 3.14 does not hold in general if the edge stabilizers are not finite. This is a key point for the proof of Theorem 1.3.

We prove now the main lemma of this section.

Lemma 3.15. *Let $\delta > 0$. There exists a finite set E_δ of edges of \mathbb{T} such that if we denote by $D_\delta := \bigcup_{e \in E_\delta} D_e$ and by $P_\delta := p^{-1}(D_\delta)$, then $\partial\mathcal{C} \setminus P_\delta$ can be partitioned into a finite union of open and closed saturated sets A with $\text{diam} A \leq \delta$. In particular, if U is a connected component of $D \setminus D_\delta$, then*

$$\text{diam } p^{-1}(U) \leq \delta.$$

Moreover, this implies that if for some $g \in G$ and $i \in \{1, \dots, k\}$ we have $g \cdot \Lambda_i \subset \partial\mathcal{C} \setminus P_\delta$, then $g \cdot \Lambda_i \subset A$ for some subset A of the partition.

Proof. Let $\eta > 0$ and let $E_{\eta,0}$ be the finite subset of edges of \mathbb{T} such that $\text{diam} \Lambda_{C_e} \geq \eta$, and let \mathbb{T}_η be its convex hull in \mathbb{T} , so that \mathbb{T}_η is a finite subtree of \mathbb{T} . Let E_η be the set of edges of \mathbb{T}_η . For each vertex $v \in \mathbb{T}_\eta$ we denote by $E_v^*(\eta) := E_v \setminus E_\eta$. We can assume without loss of generality that $v_0 \in \mathbb{T}_\eta$, so that for each vertex v of \mathbb{T}_η , and each edge $e \in E_v^*(\eta)$, we have $\Lambda_{C_e^v} = \Lambda_{C_e}$, and in particular, that $\text{diam } \Lambda_{C_e^v} \leq \eta$. We set

$$D_\eta := \bigcup_{e \in E_\eta} D_e.$$

If v_1 and v_2 are two different vertices of \mathbb{T}_η , and if $e_j \in E_{v_j}^*(\eta)$, $j = 1, 2$, then

$$p\left(\Lambda_{C_{e_1}^{v_1}}\right) \cap p\left(\Lambda_{C_{e_2}^{v_2}}\right) \subset D_\eta. \quad (3.12)$$

Let $P_\eta := p^{-1}(D_\eta)$. For each vertex v of \mathbb{T}_η , let A_v be the union of the sets $\Lambda_{C_e^v} \setminus P_\eta$ with $e \in E_v^*(\eta)$. Then A_v is open and closed in $\partial\mathcal{C} \setminus P_\eta$, and is a saturated set. This provides a partition of $\partial\mathcal{C} \setminus P_\eta$ by open and closed saturated sets. Therefore, if U is a connected

component of $D \setminus D_\eta$, there exists a vertex v of \mathbb{T}_η such that $U \subset p(A_v)$. Since A_v is saturated, this implies that $p^{-1}(U) \subset A_v$.

We end by estimating the diameter of the sets A_v . Indeed, by Corollary 3.8 and Lemma 3.12 we have

$$\begin{aligned} \text{diam} A_v &\leq 2 \max_{e \in E_v^*(\eta)} \{\text{diam} \Lambda_{C_e^v}\} + \max_{e, e' \in E_v^*(\eta)} \text{dist}(\Lambda_{C_e^v}, \Lambda_{C_{e'}^v}) \\ &\lesssim \max_{e \in E_v^*(\eta)} \{\text{diam} \Lambda_{C_e^v}\} \leq K\eta, \end{aligned}$$

where K is a uniform constant. So it is enough to take η small enough so that $K\eta \leq \delta$. \square

REMARK. We use the fact that G is virtually free in the following key point of the proof of Lemma 3.15. The sets A_v introduced in the proof do not contain the limit sets Λ_v for a vertex $v \in \mathbb{T}_\eta$. When G is virtually free this is not a problem since $\Lambda_v = \emptyset$ for every vertex $v \in \mathbb{T}_1$. Therefore the sets A_v form a covering of $\partial C \setminus P_\eta$ (note also that ∂C can be identified with $\partial \mathbb{T}$ when G is virtually free). Equation (3.12) is always true by construction, so in a general DS splitting one needs to add $\Lambda_v \setminus P_\eta$ to A_v in order to obtain a partition of $\partial C \setminus P_\eta$. This prevents the diameter of A_v to be small. Nevertheless, since the diameter of $p^{-1}(x)$ tends to zero for $x \in D$, there is only a finite number of classes “connecting” two given limit sets Λ_v and Λ_w . Therefore, adding a finite number of points to P_η if necessary, one can separate $p(\Lambda_v)$ and $p(\Lambda_w)$ by removing a finite number of points of D .

REMARK. The set P_δ in the statement of Lemma 3.15 can be assumed to contain any given finite set of equivalence classes. In fact, if $\{x, y\} = g \cdot \Lambda_i$ is a pair of equivalent points, they belong to some piece A of the partition of $\partial C \setminus P_\delta$. Then it suffices to replace P_δ by $P_\delta \cup \{x, y\}$ and A by $A \setminus \{x, y\}$; the statement remains unchanged.

3.4. The JSJ splitting and local cut points. The aim of this section is to recall the basic properties of Bowditch’s JSJ splitting for hyperbolic groups. By a *virtually Fuchsian* group we mean a non-elementary hyperbolic group that acts properly discontinuously and by isometries on the real hyperbolic plane \mathbb{H}^2 . We say that the group is *cocompact* virtually Fuchsian if the action is. The action is not necessarily faithful, but its kernel is finite. If H is a virtually Fuchsian group, we say that H is *convex cocompact* if the action is cocompact on its convex core. Recall that its convex core is the minimal H -invariant closed convex subset of \mathbb{H}^2 . In this case, the *peripheral* subgroups of H are the stabilizers of the boundary connected components of its convex core.

Let G be a one-ended hyperbolic group which is not a cocompact virtually Fuchsian group. Bowditch’s JSJ decomposition theorem (see [Bow98a] Theorem 0.1 and Theorem 5.28) asserts that there exists a minimal simplicial action of G on a simplicial tree \mathbb{T} , without edge inversions and with finite quotient \mathbb{T}/G , whose edge stabilizers are virtually cyclic, and the vertices of \mathbb{T} are of three types:

- (\mathbb{T}_C) *Virtually cyclic*: the stabilizer is a maximal virtually cyclic subgroup of G . Its valence in \mathbb{T} is finite and at least two. The limit set Λ_v corresponds to a pair of local cut points of ∂C .
- (\mathbb{T}_S) *Surface or MHF*: the stabilizer is a quasiconvex, non-elementary virtually free convex cocompact Fuchsian subgroup of G . Its peripheral groups are precisely the stabilizers of its incident edges. They are maximal in the following sense: if H occurs as the vertex group of a finite splitting of G over virtually cyclic subgroups, in such a way that H admits a convex cocompact Fuchsian action on \mathbb{H}^2 whose

peripheral groups are the stabilizers of its incident edges, then H is contained in a subgroup of surface type.

- (T_R) *Rigid*: the stabilizer is quasiconvex, non-elementary and not of Surface type, and admits no decomposition over virtually cyclic subgroups relative to the stabilizers of its incident edges. The limit set of a rigid type vertex is characterized by the following property: Λ_v is a maximal closed subsets of $\partial\mathcal{C}$ with the property that it cannot be separated by two points of $\partial\mathcal{C}$.

Partition the vertices of the JSJ tree T as $T_C \cup T_S \cup T_R$, which are respectively the vertices of virtually cyclic, surface and rigid type. These types are mutually exclusive, are preserved by the action of G , and two adjacent vertices are never of the same type. The splitting is quasi-isometry invariant and the maximality of the splitting can be formulated as follows: if x is any local cut point of $\partial\mathcal{C}$, then there exists a vertex $v \in T_C \cup T_S$ such that $x \in \Lambda_v$.

To construct the simplicial tree T , Bowditch uses the structure of local cut points of $\partial\mathcal{C}$. Recall that $\partial\mathcal{C}$ is a locally connected metric space with no *global* cut points. The limit set Λ_e of $e \in T_1$ consists of exactly two points which are fixed points of any element of infinite order in G_e . Since $\Lambda_{C_e^-} \cap \Lambda_{C_e} = \Lambda_e$, the limit set Λ_e separates the boundary of \mathcal{C} , and the two points of Λ_e are local cut points. There are only a finite number of connected components of $\partial\mathcal{C} \setminus \Lambda_e$, and every component has Λ_e as frontier. In particular, the sets $\Lambda_{C_e^\pm}$ are connected.

The vertices of surface type T_S are virtually free (with a peripheral structure) and the limit set Λ_v is a Cantor set of local cut points. The stabilizer of Λ_v , for the action of G on $\partial\mathcal{C}$, is the same as the stabilizer of v for the action of G on T , i.e. G_v . Bowditch named them *maximal hanging Fuchsian*, or MHF for short. Let us explain in more detail the natural bijection between the edges of T incident to v , and the peripheral subgroups of G_v . The limit set of the peripheral subgroups are the ‘‘jumps’’ of the Cantor set Λ_v , which can be defined using the embedding of Λ_v into S^1 given by the action of G_v on \mathbb{H}^2 . Denote by J_v the set of jumps. For $\tau = \{x, y\} \in J_v$, denote by \mathcal{C}_τ the set of connected components of $\partial\mathcal{C} \setminus \tau$. Then the set of connected components \mathcal{C}_v of $\partial\mathcal{C} \setminus \Lambda_v$ is given by

$$\mathcal{C}_v = \bigcup_{\tau \in J_v} \mathcal{C}_\tau. \quad (3.13)$$

This means that two points are separated by Λ_v if and only if they are separated by a jump. Therefore, in order to study the WS property (or equivalently, the fibers) it is enough to consider the local cut points which are in the limit sets of the edges of T :

$$\mathcal{E} := \bigcup_{e \in T_1} \Lambda_e.$$

EXAMPLE 3.3.1 Let us finish our discussion of the JSJ splitting by giving a simple example of a rigid type vertex. Consider two isometric copies of a closed hyperbolic surface S and S' . Let γ be a closed geodesic in S , and suppose that γ is filling; i.e. the connected components of $S \setminus \gamma$ are simply connected. Denote by γ' the copy of γ in S' . Consider the complex

$$X = S \sqcup S' / \gamma \sim \gamma',$$

and let $G = \pi_1(X)$. Then the quotient graph T/G of the JSJ splitting of G has three vertices: $\langle \gamma \rangle \in T_C$, $\pi_1(S) \in T_R$ and $\pi_1(S') \in T_R$. To see this, note that since γ is a filling geodesic, given any two points $x, y \in \Lambda_{\pi_1(S)}$ there exists a component of $\partial\mathcal{C} \setminus \{x, y\}$ which

contains $\Lambda_{\pi_1(S)} \setminus \{x, y\}$. That is, $\Lambda_{\pi_1(S)}$ cannot be separated in $\partial\mathcal{C}$ by a pair of points. Note that $\partial\mathcal{C}$ does not satisfy the WS property.

3.5. Proof of Theorem 1.3.

3.5.1. *All rigid type vertices are virtually free implies WS.* Assume now that in the JSJ splitting of G all rigid type vertices are virtually free. In this case, all the vertices of \mathbb{T} are either non-elementary virtually free or virtually cyclic.

Let v be a vertex in $\mathbb{T}_S \cup \mathbb{T}_R$, so G_v is non-elementary virtually free. Recall that E_v is the set of edges of \mathbb{T} which are incident to v . Since the action of G on \mathbb{T} preserves the type of a vertex and two adjacent vertices are of different type, the stabilizer of E_v is also G_v . This implies that the number of G -orbits of edges incident to v is the same as the number of G_v -orbits, and therefore E_v/G_v is finite. Let $\{f_1, \dots, f_{k_v}\}$ be a finite set of representatives of E_v/G_v , and write $H_j(v)$ the two-ended subgroup $G\langle f_j \rangle$ of G_v . We consider the decomposition space D_v associated to v and we denote by $p_v : \Lambda_v \rightarrow D_v$ the quotient map like in Section 3.3. Another equivalent way to construct D_v is by identifying the points of $\partial\mathcal{C}$ which are in the same $\Lambda_{C_e^v}$ for $e \in E_v$. Since $\partial\mathcal{C}$ is connected, the same is true for D_v . Moreover, the space D_v is either homeomorphic to the circle if $v \in \mathbb{T}_S$, or it is locally connected without cut-points and cut-pairs if $v \in \mathbb{T}_R$ [CaMa11, Haïss12]. Therefore, we can apply the results of Section 3.3.

The subsets used in Section 3.3 to separate the decomposition space D_v were constructed by projecting a finite union of sets of the form $g \cdot \Lambda_{f_j}$ with $g \in G_v$ and $j \in \{1, \dots, k_v\}$. Therefore, in this case, they are the projection of a finite union of Λ_e with $e \in E_v$. We can state Lemma 3.15 in the following way: for all $\delta > 0$, there exists a finite set of edges $\{e_1, \dots, e_n\}$, where n depends on v and δ , of E_v such that if we set $P_v := \bigcup_{i=1}^n \Lambda_{e_i}$, then

$$\Lambda_v \setminus P_v = \bigcup_{i=1}^m A_i,$$

where the sets A_i are disjoint, open and closed in $\Lambda_v \setminus P_v$, and saturated sets. Here m also depends on v and δ .

Moreover, for any other edge $e \in E_v$, there exists $i \in \{1, \dots, m\}$ such that $\Lambda_e \subset A_i$. Given such a partition, we denote by $E_v(i)$, for $i = 1, \dots, m$, the edges e incident to v such that $\Lambda_e \subset A_i$. Naturally,

$$E_v = \{e_1, \dots, e_n\} \cup \bigcup_{i=1}^m E_v(i),$$

is a partition of all edges incident to v . Let

$$K_i(v) = A_i \cup \bigcup_{e \in E_v(i)} \Lambda_{C_e^v} \text{ for } i = 1, \dots, m.$$

Lemma 3.16. *The sets $\{K_i(v)\}_{i=1}^m$ are pairwise disjoint and closed in $\partial\mathcal{C} \setminus P_v$.*

Proof. They are pairwise disjoint: the A_i are pairwise disjoint, and by definition, the intersection of two sets $\Lambda_{C_e^v}$ and $\Lambda_{C_{e'}^v}$, with $e \neq e'$ in E_v , is contained in $\Lambda_e \cap \Lambda_{e'}$. But if $e \in E_v(i)$ and $e' \in E_v(j)$ with $j \neq i$, then $\Lambda_e \cap \Lambda_{e'} = \emptyset$.

They are closed: we adapt an argument taken from [Bow98b], see Lemma 4.6 therein. If $x \notin K_i(v) \cup P_v$, then $\text{dist}(x, A_i) = \eta > 0$, because A_i is closed in $\partial\mathcal{C} \setminus P_v$ and $x \notin A_i$. If $e \in E_v(i)$ is an edge such that $\Lambda_{C_e^v} \cap B(x, \eta/2) \neq \emptyset$, then $\text{diam } \Lambda_{C_e^v} \geq \eta/2$ because $\Lambda_{C_e^v} \cap A_i \neq \emptyset$. So, there is only a finite number of edges $e \in E_v(i)$ such that $\Lambda_{C_e^v} \cap B(x, \delta/2) \neq \emptyset$. But each $\Lambda_{C_e^v}$ is closed, so we have $\text{dist}(x, K_i(v)) > 0$. Then $K_i(v)$ is closed in $\partial\mathcal{C} \setminus P_v$. \square

We can partition the boundary of \mathbb{C} as

$$\partial\mathbb{C} = \bigcup_{i=1}^n \left(\Lambda_{\mathbb{C}_{e_i}^v} \setminus P_v \right) \cup \bigcup_{i=1}^m K_i(v) \cup P_v, \quad (3.14)$$

and the sets of the first and second union are closed in $\partial\mathbb{C} \setminus P_v$. Furthermore, we have

$$\text{diam } K_i(v) \leq \text{diam } A_i + 2 \cdot \max_{e \in \mathbf{E}_v(i)} \{ \text{diam } \Lambda_{\mathbb{C}_e^v} \}, \quad (3.15)$$

Summing up we can obtain the following lemma.

Lemma 3.17. *Let $v \in \mathbb{T}_S \cup \mathbb{T}_R$ be a vertex of \mathbb{T} . Then for any $\delta > 0$, there exists a finite set of edges $e_1, \dots, e_n \in \mathbf{E}_v$ such that $\text{diam } K_i(v) \leq \delta$ for all $i = 1, \dots, m$.*

Proof. Let $\delta > 0$. There is a finite subset $\mathbf{E}_1 \subset \mathbf{E}_v$, such that if $e \in \mathbf{E}_v \setminus \mathbf{E}_1$, the diameter of $\Lambda_{\mathbb{C}_e^v}$ is bounded from above by $\delta/4$. Also by the previous paragraph, there exists a finite subset $\mathbf{E}_2 = \{e_1, \dots, e_n\} \subset \mathbf{E}_v$, which we can assume containing \mathbf{E}_1 , such that $\text{diam } A_i \leq \delta/2$. Therefore, from (3.15), we have $\text{diam } K_i(v) \leq \delta$. \square

We summarize the proof of the WS property in the following proposition.

Proposition 3.18. *Let G be a one-ended hyperbolic group which is not a cocompact virtually Fuchsian group. Suppose that in the JSJ splitting of G all rigid type vertices are virtually free. Then for any $\delta > 0$, there exists a finite set $\{e_1, \dots, e_N\}$ of edges of \mathbb{T} , such that*

$$\max \{ \text{diam } U : U \in \mathcal{U} \} \leq \delta,$$

where \mathcal{U} is the set of connected components of $\partial\mathbb{C} \setminus P$, and $P = \bigcup_{i=1}^N \Lambda_{e_i}$. In particular, $\partial\mathbb{C}$ satisfies the WS property.

Proof. We chose v_0 in the Definition 3.1 to be any vertex in $\mathbb{T}_S \cup \mathbb{T}_R$. Let $\delta > 0$, and consider the set \mathbf{E}_δ of edges $e \in \mathbb{T}_1$ such that $\text{diam } \Lambda_{\mathbb{C}_e} \geq \delta$. By Lemma 3.4, \mathbf{E}_δ is a finite set. Let \mathbf{V}_δ be the union of $\{v_0\}$ and the vertices of the edges in \mathbf{E}_δ . Let \mathbb{T}_δ be the convex hull of \mathbf{V}_δ . Then \mathbb{T}_δ is a finite subtree of \mathbb{T} . We can suppose that for a vertex $w \in \mathbb{T}_C \cap \mathbb{T}_\delta$, all the edges incident to w are in \mathbb{T}_δ , because the valence of w in \mathbb{T} is finite. In this way, all the terminal vertices of \mathbb{T}_δ are in $\mathbb{T}_S \cup \mathbb{T}_R$.

We will prove that it is enough to take P to be the union of the limit sets of a finite collection of edges belonging to a slightly bigger set than \mathbb{T}_δ . Let v be a vertex in \mathbb{T}_δ . We need to consider four cases:

- *Case 1:* Suppose $v = v_0$. By Lemma 3.17, there is

$$\mathbf{F}_{v_0} := \{e_1(v_0), \dots, e_{n_{v_0}}(v_0)\} \subset \mathbf{E}_{v_0},$$

such that

$$\text{diam } K_i(v_0) \leq \delta, \text{ for all } i = 1, \dots, m_{v_0}. \quad (3.16)$$

We can assume that the edges of \mathbb{T}_δ incident to v_0 belong to \mathbf{F}_{v_0} . Denote by H_{v_0} the union of $\Lambda_{e_i(v_0)}$ for $i = 1, \dots, n_{v_0}$. Therefore, $\partial\mathbb{C}$ can be partitioned as in (3.14).

- *Case 2:* Suppose $v \neq v_0$ is any non-terminal vertex in $(\mathbb{T}_S \cup \mathbb{T}_R) \cap \mathbb{T}_\delta$. There is exactly one edge e^* in \mathbf{E}_v such that $\mathbb{T}_{e^*}^v = \mathbb{T}_{e^*}^-$. It necessarily belongs to \mathbb{T}_δ because $v_0 \in \mathbb{T}_\delta$. For the others, we have $\mathbb{T}_e^v = \mathbb{T}_e$. By Lemma 3.17, there are edges

$$\mathbf{F}_v := \{e_1(v), \dots, e_{n_v}(v)\} \subset \mathbf{E}_v,$$

such that

$$\text{diam } K_i(v) \leq \delta, \text{ for all } i = 1, \dots, m_v. \quad (3.17)$$

We can assume that the edges of \mathbb{T}_δ incident to v belong to F_v . Denote by P_v the union of $\Lambda_{e_i(v)}$ for $i = 1, \dots, n_v$. Therefore, $\Lambda_{C_{e^*}}$ is partitioned as

$$\Lambda_{C_{e^*}} = \bigcup_{F_v \setminus \{e^*\}} \left(\Lambda_{C_{e_j(v)}} \setminus P_v \right) \cup \bigcup_{i=1}^{m_v} K_i(v) \cup P_v. \quad (3.18)$$

- *Case 3:* Suppose that $v \in \mathbb{T}_C \cap \mathbb{T}_\delta$. Then all the edges incident to v are in \mathbb{T}_δ . Let e^* be the edge incident to v such that $\mathbb{T}_{e^*}^v = \mathbb{T}_{e^*}^-$. If we write the other edges incident to v as $E_v^* = \{e_i = (v, v_i) : i = 1, \dots, n_v\}$, then

$$\Lambda_{C_{e^*}} = \bigcup_{i=1}^{n_v} \Lambda_{C_{e_i}} = \bigcup_{i=1}^{n_v} \Lambda_{C_{e_i}^v}. \quad (3.19)$$

The intersection of two sets of this decomposition is equal to $\Lambda_v = \Lambda_{e^*} = \Lambda_{e_i}$, for $i = 1, \dots, n_v$. This allows us to replace $\Lambda_{C_{e^*}}$ by the union of $\Lambda_{C_{e_i}}$, where the v_i all belong to $(\mathbb{T}_S \cup \mathbb{T}_R) \cap \mathbb{T}_\delta$.

- *Case 4:* Suppose that v is a terminal vertex of \mathbb{T}_δ . Let as before e^* be the edge incident to v such that $\mathbb{T}_{e^*}^v = \mathbb{T}_{e^*}^-$, which in this case is the unique edge of \mathbb{T}_δ incident to v . Then $E_v^* \subset \mathbb{T}_1 \setminus \mathbb{T}_\delta$, and by construction for any $e \in E_v^*$, we have

$$\text{diam } \Lambda_{C_e^v} = \text{diam } \Lambda_{C_e} \leq \delta. \quad (3.20)$$

Then the set $\Lambda_{C_{e^*}}$ can be decomposed as in (3.18), and all the sets in the first and second union are of diameter less than or equal to δ .

Starting from v_0 , a repetitive combination of the partitions obtained in Cases 1 to 4 give us the set P . Indeed, we obtain a partition $\partial\mathcal{C} = A \cup B \cup P$, where

$$A := \bigcup_{v \in (\mathbb{T}_S \cup \mathbb{T}_R) \cap \mathbb{T}_\delta} \bigcup_{i=1}^{m_v} K_i(v), \quad B := \bigcup_{v \in \text{ter}(\mathbb{T}_\delta)} \bigcup_{e \in E_v^*} (\Lambda_{C_e} \setminus P_v), \quad (3.21)$$

where $\text{ter}(\mathbb{T}_\delta)$ denote the set of terminal vertices of \mathbb{T}_δ , and

$$P := \bigcup_{v \in (\mathbb{T}_S \cup \mathbb{T}_R) \cap \mathbb{T}_\delta} P_v \quad (3.22)$$

The sets in the decomposition (3.21) defining A and B are pairwise disjoint and closed in $\partial\mathcal{C} \setminus P$, and of diameter less than or equal to δ . This implies that any connected component of $\partial\mathcal{C} \setminus P$ is contained in one of these sets. This ends the proof. \square

3.5.2. *WS is not satisfied when a non-virtually free rigid type vertex exists.* Suppose there exists $v \in \mathbb{T}_R$ such that G_v is not virtually free. First, decompose G_v over finite groups, i.e. the DS splitting of G_v . Note that points in different connected components of Λ_v can be separated by removing a finite number of local cut points of $\partial\mathcal{C}$, see the remark after Lemma 3.15. Consider a vertex w of the DS splitting of G_v whose stabilizer G_w is one-ended. We will show that Λ_w cannot be separated by a finite set of local cut points of $\partial\mathcal{C}$.

Suppose that g is a loxodromic element of G_w , and denote by $\Lambda_g = \{g^+, g^-\}$ its fixed points. The limit set Λ_w is locally connected and without *global* cut points, so $\Lambda_w \setminus \Lambda_g$ has a finite number of connected components; which we denote by Ω_i , $i = 1, \dots, n$, $n \geq 1$. Note that since the infinite cyclic group $\langle g \rangle$ permutes the components $\{\Omega_i\}$, up to taking a big enough power of g , we can suppose that each Ω_i is fixed by g . The set $\Lambda_w \setminus \Lambda_g$ is contained in a connected component Ω of $\partial\mathcal{C} \setminus \Lambda_g$, because v is a rigid vertex of G . Take

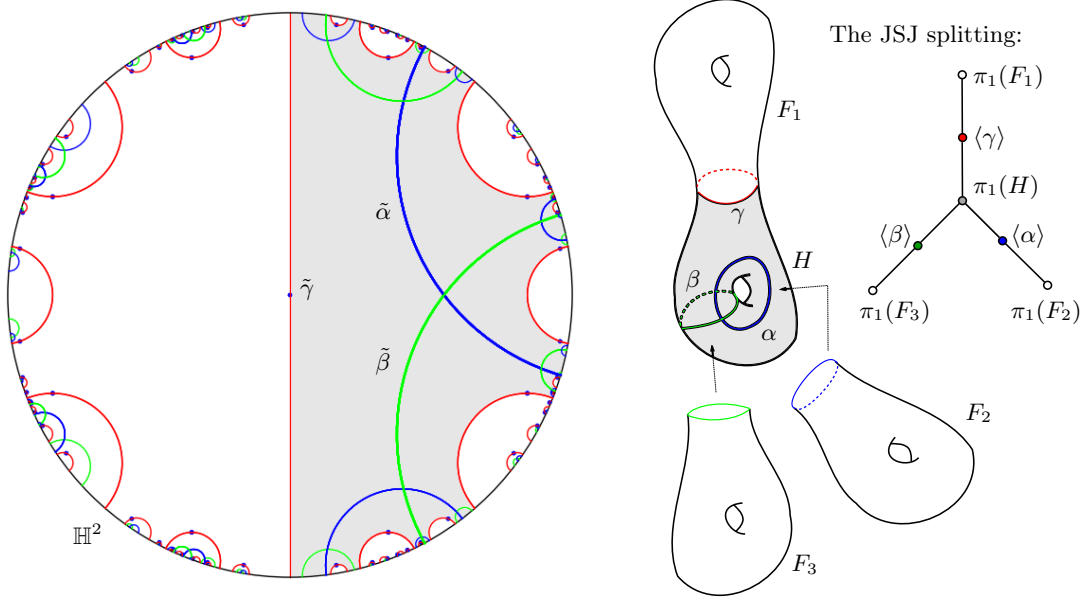


FIGURE 3.3. This figure shows an example of a convex cocompact Kleinian group $\pi_1(M)$ for which $D(M) > 1 = \dim_{AR} \partial\pi_1(M)$. Here H is a surface homeomorphic to the torus minus an open disk (with boundary γ). The curves α and β generate $\pi_1(H)$, and we have $\gamma = [a, b] = aba^{-1}b^{-1}$. The lifts of these curves to the hyperbolic plane are drawn on the left. The group is the fundamental group of the complex obtained by gluing to H three copies, F_i , $i = 1, 2, 3$, of H along the multicurve $\{\alpha, \beta, \gamma\}$. All the vertices of the JSJ decomposition of $\pi_1(M)$ are virtually free, and the conjugates of $\pi_1(H)$ are rigid vertices. This also implies that M is not a generalized book of I -bundles.

$i \neq j$ and let $\alpha : [0, 1] \rightarrow \Omega$ be a curve with $\alpha(0) \in \Omega_i$ and $\alpha(1) \in \Omega_j$. Given any $\delta > 0$, there exists $k \geq 0$ such that the curves $\alpha_k := g^k \cdot \alpha$ and $\alpha_{-k} := g^{-k} \cdot \alpha$ are respectively contained in the balls $B(g^+, \delta)$ and $B(g^-, \delta)$. This means that the components of $\Lambda_w \setminus \Lambda_g$ can be connected by a curve of ∂C contained in an arbitrarily small ball centered at the fix points of g .

We need to consider two cases:

- *Case 1:* Suppose first that Λ_w is not homeomorphic to the circle. Consider the JSJ decomposition of G_w with associated tree \mathbb{T}_w . If it is trivial, we are done because there are no local cut points in Λ_w . If not, it is enough to consider the local cut points of Λ_w which are in the limit set of an edge e of \mathbb{T}_w , see (3.13). Note that $\Lambda_e = \{g^+, g^-\}$ is the set of fixed points of a loxodromic g .

Let $\{e_1, \dots, e_m\}$ be any finite subset of edges of \mathbb{T}_w . Consider a $\delta > 0$ small enough so that the balls centered at $P := \bigcup_{i=1}^m \Lambda_{e_i}$ and of radius δ are disjoint. Write $\Lambda_{e_i} := \{g_i^+, g_i^-\}$ and for each i consider the curves α_i^\pm contained in the ball $B(g_i^\pm, \delta)$ as before. Then the set

$$(\Lambda_w \setminus P) \cup \bigcup_{i=1}^m (\alpha_i^+ \cup \alpha_i^-)$$

is a connected subset of $\partial\mathcal{C} \setminus P$ containing $\Lambda_w \setminus P$. Since $\text{diam}(\Lambda_w \setminus P) = \text{diam}\Lambda_w$, $\partial\mathcal{C}$ does not satisfy the WS condition.

- *Case 2:* Suppose now that Λ_w is homeomorphic to the circle. Then any pair of distinct points $\{x, y\}$ of Λ_w is a cut-pair, and

$$\Lambda_w \setminus \{x, y\} = \Omega_1 \cup \Omega_2.$$

Take a curve α connecting Ω_1 and Ω_2 as before. Given $\delta > 0$, there exists a loxodromic element of G_w such that $g^+ \in B(x, \delta)$ and $g^- \in B(y, \delta)$. Therefore, applying exactly the same argument as in Case 1 we show that $\Lambda_w \setminus P$ is contained in a connected component of $\partial\mathcal{C} \setminus P$ for any finite subset P of Λ_w .

This ends the proof.

3.6. A simple example with $\dim_{AR} \partial\pi_1(M) < D(M)$. In this section we give a simple example of a one-ended convex cocompact Kleinian group $G = \pi_1(M)$ for which the conclusion of Corollary 1.4 holds; namely, that $\dim_{AR} \partial\pi_1(M) < D(M)$ where M is the hyperbolizable 3-manifold with boundary whose interior is isometric to \mathbb{H}^3/G .

Let H be a handlebody with boundary S a closed surface of genus $g \geq 2$, and let $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ be a multicurve on S ; i.e. the elements of Γ are disjoint simple closed curves on S . Suppose that Γ is chosen so that the boundary of any properly embedded disk or annulus in H intersects at least one of the curves in Γ . Consider a small smooth neighborhood A of Γ in S so that $A = \{A_1, \dots, A_n\}$ is a collection of disjoint annuli in S .

Let T_i for $i = 1, \dots, n$ be n copies of the solid torus $D^2 \times S^1$. On the boundary S_i of T_i let $A_{T,i}$ be a smooth annulus whose core curve generates the $\pi_1(T_i)$. We glue together each solid torus T_i to H identifying A_i with $A_{T,i}$. Denote by R the 3-manifold obtained.

For each i , let B_i be a collection of $m_i \geq 1$ disjoint smooth annuli in S_i for which each core curve generates $\pi_1(T_i)$. Suppose that B_i is also disjoint from $A_{T,i}$. Denote by $m = m_1 + \dots + m_n$. Let F be an I-bundle (over a surface which is not necessarily connected) with vertical boundary consisting of exactly m smooth annuli. Glue F to R along the annuli B_i , $i = 1, \dots, n$, and call M the 3-manifold with boundary obtained. Let $G = \pi_1(M)$.

By Thurston hyperbolization theorem, G is isomorphic to a discrete, one-ended convex cocompact subgroup of $\text{Iso}(\mathbb{H}^3)$. See [KK00] for a similar construction and for more details. The JSJ decomposition of G has exactly one conjugacy class of rigid vertices which are all free groups; i.e. the conjugates of $\pi_1(H)$. Thus we are in the hypotheses of Theorem 1.3 and $\dim_{AR} \partial G = 1$. On the other hand, M is not a generalized book of I-bundles, and therefore, by Theorem 2.9 in [CMT99] we have $D(M) > 1$. See also Figure 3.3.

REFERENCES

- [BT01] C. Bishop, J. Tyson, Conformal dimension of the antenna set, Proc. Amer. Math. Soc. 129 :3631-3636, 2001.
- [B06] M. Bonk, *Quasiconformal geometry of fractals*, Proceedings Internat. Congress Math. (Madrid, 2006), Europ. Math. Soc., Zürich, 2006, 1349-1373.
- [BonK02a] M. Bonk, B. Kleiner, *Quasisymmetric parametrizations of two-dimensional metric spheres*, Invent. Math. 150 (2002), 127-183.
- [BonK02b] M. Bonk, B. Kleiner, *Rigidity for quasi-Möbius group actions*, J. Diff. Geom. 61 (2002), 81-106.
- [BonK05a] M. Bonk, B. Kleiner, *Quasi-hyperbolic planes in hyperbolic groups*, Proc. Amer. Math. Soc. 133 (2005), no. 9, 2491-2494.
- [BonK05b] M. Bonk, B. Kleiner, *Conformal dimension and Gromov hyperbolic groups with 2-sphere boundary*, Geom. Topol. 9 219-246, 2005.

- [Bou97] M. Bourdon, *Immeubles hyperboliques, dimension conforme et rigidité de Mostow*, Geom. Funct. Anal. Vol. 7 (1997) 245-268.
- [BP03] M. Bourdon, H. Pajot, *Cohomologie ℓ_p et espaces de Besov*, Journal für die Reine und Angewandte Mathematik Volume 558 (2003), 85-108.
- [Bow98a] B. Bowditch, *Cut points and canonical splittings of hyperbolic groups*, Acta Math. 180 (1998), no. 2, 145-186.
- [Bow98b] B. Bowditch, *Boundaries of strongly accessible hyperbolic groups*, The Epstein birthday schrift, 51-97, Geom. Topol. Monogr., 1, Geom. Topol. Publ., Coventry, 1998.
- [BH99] M. Bridson, A. Haefliger, *Metric spaces of non-positive curvature*, vol. 319 of Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, 1999.
- [Bu05] S. Buyalo, *Volume entropy of hyperbolic graph surfaces*, Erg. Th. Dyn. Syst. Vol. 25 Iss. 02 (2005), 403-417.
- [CMT99] R. Canary, Y. Minsky, E. Taylor, *Spectral theory, Hausdorff dimension and the topology of hyperbolic 3-manifolds*, J. Geom. Analysis 9 (1999), 18-40.
- [Ca11] M. Carrasco Piaggio, *Gauge conforme des espaces métriques compacts*, PhD Thesis, Aix-Marseille Université, 2011. Available online at <http://tel.archives-ouvertes.fr/tel-00645284>
- [Ca12] M. Carrasco Piaggio, *On the conformal gauge of a compact metric space*, Ann. Sci. École Norm. Sup. to appear.
- [Ca13] M. Carrasco Piaggio, *Conformal dimension of Julia sets*, in preparation.
- [CaMa11] C. Cashen, N. Macura, *Line patterns in free groups*, Geom. Topol. 15 (2011), no. 3, 1419-1475.
- [CJ94] A. Casson, D. Jungreis, *Convergence groups and Seifert fibered 3-manifolds*, Invent. Math. 118, 3 (1994), 441-456.
- [CDP90] M. Coornaert, T. Delzant, A. Papadopoulos, *Géométrie et théorie des groupes*. No. 1441 Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1990.
- [Coo93] M. Coornaert, *Mesures de Patterson-Sullivan sur le bord d'un espace hyperbolique au sens de Gromov*, Pacific J. Math. 159(1993), 241-270.
- [DP01] T. Delzant, L. Potyagailo, *Accessibilité hiérarchique des groupes de présentation finie*, Topology 40 (2001), no. 3, 617-629.
- [DD89] W. Dicks, M. J. Dunwoody, *Groups acting on graphs*, Cambridge Studies in Advanced Mathematics, 17. Cambridge University Press, Cambridge, 1989. xvi+283 pp.
- [Dun85] M. J. Dunwoody, *The accessibility of finitely presented groups*, Invent. Math. 81, 3 (1985), 449-457.
- [Gab92] D. Gabai, *Convergence groups are Fuchsian groups*, Ann. of Math. (2) 136, 3 (1992), 447-510.
- [GH90] E. Ghys, P. de la Harpe, *Sur les Groupes Hyperboliques d'après Mikhael Gromov*. Progress in Mathematics. Birkhäuser, Boston-Basel-Berlin, 1990.
- [Hei01] J. Heinonen, *Lectures on analysis on metric spaces*, Universitext. Springer-Verlag, New York, 2001.
- [Haïss08] P. Haïssinsky, *Géométrie quasiconforme, analyse au bord des espaces métriques hyperboliques et rigidités, d'après Mostow, Pansu, Bourdon, Pajot, Bonk, Kleiner...* Séminaire Bourbaki 60, 2007/08, n 993
- [Haïss12] P. Haïssinsky, *Hyperbolic groups with planar boundary*, preprint 2012.
- [HP09] P. Haïssinsky, K. Pilgrim, *Coarse Expanding Conformal Dynamics*. Astérisque 325 2009.
- [KK00] M. Kapovich, B. Kleiner, *Hyperbolic groups with low-dimensional boundary*, Ann. Sci. de ENS Paris, t. 33 (2000) p. 647-669.
- [Kle06] B. Kleiner, *The asymptotic geometry of negatively curved spaces: uniformization, geometrization, and rigidity*, International Congress of Mathematicians. Vol. II, pp. 743-768, Eur. Math. Soc., Zürich, 2006.
- [LP04] G. Lupo-Krebs, H. Pajot, *Dimensions conformes, espaces Gromov-hyperboliques et ensembles autosimilaires*, Séminaire de Théorie spectrale et géométrie, tome 22 (2003-2004), 153-182.
- [Mac10] J. Mackay, *Spaces with conformal dimension greater than one*, Duke Math. J. Volume 153, Number 2 (2010), 211-227.
- [MT10] J. Mackay, J. Tyson, *Conformal dimension: theory and application*, University Lecture Series, AMS 2010.
- [Na92] S. B. Nadler, *Continuum Theory. An introduction*, Marcel Dekker, (1992).
- [Pan89] P. Pansu, *Dimension conforme et sphère à l'infini des variétés à courbure négative*. Ann. Acad. Sci. Fenn. Ser. A I Math. 14 (1989), 177-212.
- [Sho90] H. Short, *Quasiconvexity and a theorem of Howson's*, dans Group Theory from a Geometrical Viewpoint (Trieste, 1990), pp. 168-176. World Sci. Publishing, River Edge, NJ, 1991.

- [Sta68] J. R. Stallings, *On torsion-free groups with infinitely many ends*, Ann. of Math. (2) 88 (1968), 312-334.
- [Su83] D. Sullivan, *Conformal dynamical systems*, Geometric dynamics (Rio de Janeiro, 1981), Lecture Notes in Math., 1007, Springer, Berlin (1983), 725-752.
- [Sul82] D. Sullivan, *Seminar on hyperbolic geometry and conformal dynamical systems*, Preprint IHES, 1982.
- [Tu88] P. Tukia, *Homeomorphic conjugates of Fuchsian groups*, J. Reine Angew. Math. 391 (1988), 1-54.
- [TW06] J. Tyson, J. Wu, *Quasiconformal dimensions of self-similar fractals*. Rev. Mat. Iberoamericana 22 (2006), 205-258.
- [Väis84] J. Väisälä, *Quasi-Möbius maps*, J. Analyse Math. 44(1984/85), 218-234.
- [Va08] D. Vavricheck, *Strong accessibility for hyperbolic groups*, Algebraic & Geometric Topology, 8 (2008), 1459-1479.

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