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Matias Carrasco Piaggio

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CONFORMAL DIMENSION AND CANONICAL SPLITTINGS OF HYPERBOLIC GROUPS

MATIAS CARRASCO PIAGGIO

ABSTRACT. We prove a general criterion for a metric space to have conformal dimension one. The conditions are stated in terms of the existence of enough local cut points in the space. We then apply this criterion to the boundaries of hyperbolic groups and show an interesting relationship between conformal dimension and some canonical splittings of the group.

KEYWORDS: conformal dimension, local cut points, Gromov-hyperbolic groups, canonical splittings.

2010 SUBJECT CLASSIFICATION: 30L10, 51F99, 20F67, 28A78.

1. INTRODUCTION

In this article we give sufficient conditions on a compact metric space (X, d) to have conformal dimension one. The conformal dimension is a fundamental quasimetry invariant, introduced by Pansu in [Pan89]. Its original motivation is in the study of the quasiconformal structure of the boundary at infinity of a negatively curved space. For instance, any quasimetry invariant of the boundary of a hyperbolic group is a quasimetry invariant of the group. The understanding of the conformal dimension is essential and has already given many applications in geometric group theory, in particular to the boundary characterization of Kleinian groups and to Cannon's conjecture [BonK05b]. See also [B06, Haïss08, Haïss12, Kle06, LP04, MT10] for other applications.

There are different related versions of this invariant; in this article we are concerned with the *Ahlfors regular conformal dimension*, a variant introduced by Bourdon and Pajot in [BP03]. It is defined by

$$\dim_{AR}(X, d) := \inf \{ \dim_H(X, \theta) : \theta \text{ is AR and } \theta \sim_{qs} d \},$$

where AR means Ahlfors regular, \dim_H denotes Hausdorff dimension, and $\theta \sim_{qs} d$ means that θ is a distance on X quasimetrically equivalent to d . That is, there exists an increasing homeomorphism $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\frac{\theta(x, a)}{\theta(y, a)} \leq \eta \left(\frac{d(x, a)}{d(y, a)} \right),$$

for all distinct points $x, y, a \in X$. We recall that a distance θ on X is Ahlfors regular of dimension $\alpha > 0$ if there exists a Radon measure μ on X and a constant $K \geq 1$ such that:

$$K^{-1} \leq \frac{\mu(B_r)}{r^\alpha} \leq K,$$

for any ball B_r of radius $0 < r \leq \text{diam}_\theta X$. In that case, μ is comparable to the α -dimensional Hausdorff measure and $\alpha = \dim_H(X, \theta)$ is the Hausdorff dimension of (X, θ) . We write $\dim_{AR} X$ when there is no ambiguity on the metric d .

This kind of deformations can distort the Hausdorff dimension, and one can always quasimetrically deform the distance d to obtain a distance of arbitrarily large Hausdorff

dimension. The conformal dimension measures the best shape of X , and it is in general difficult to compute. It is always bounded from below by the topological dimension $\dim_T X$. In particular, when the space is connected, $\dim_{AR} X \geq 1$ holds.

This article deals with the problem of under which conditions we can quasimetrically deform the distance d to obtain AR distances with Hausdorff dimension arbitrarily close to the topological dimension of the space. This question is of particular interest for the boundaries of hyperbolic groups. For instance, $\dim_{AR} \partial G = \dim_T \partial G = n \geq 1$ and is attained by a distance in the gauge if and only if ∂G is quasimetrically equivalent to the Euclidean sphere \mathbb{S}^n [BonK02b]. For $n = 1$, the same is true under the weaker hypothesis of ∂G being homeomorphic to \mathbb{S}^1 [CJ94, Gab92, Tu88]. But the problem is far from being understood in the general case, even for low topological dimension. In this paper we address the case when $\dim_T X = 1$.

Previously known non-trivial examples of spaces of conformal dimension one were very few, due to Bishop and Tyson [BT01], Pansu (see [BonK05b, Bu05, CMT99] for comments and generalizations), and Laakso (see Tyson and Wu [TW06] for generalizations). Although the techniques of proof are specific to each particular example, they are all related to the existence of local cut points. A point $x \in X$ is a *local cut point* if there is a connected open set $x \in U \subset X$ such that $U \setminus \{x\}$ is not connected. We also recall that X is a *doubling space* if there exists a constant N such that every ball can be covered using N balls of half its radius; and, that X is *linearly connected* if there exists a constant C such that for all $x, y \in X$, there exists a continuum J containing x and y of diameter less than or equal to $Cd(x, y)$. This is also known as the *bounded turning* condition. The following condition gives a scale invariant bound on the amount of local cut points needed to disconnect the space into small pieces.

Definition 1.1 (The UWS condition). *We say that a connected and compact metric space X has uniformly well spread local cut points —UWS for short— if there exists a constant $C \geq 1$ such that for any point $x \in X$ and $r > 0$, there is a finite set $P \subset B(x, r)$ verifying:*

- (1) $\#P \leq C$, and
- (2) no connected component of $X \setminus P$ can intersect both $B(x, \frac{r}{2})$ and $X \setminus \overline{B}(x, r)$.

We remark that one can always assume that the points of the subset P are local cut points of X ; this justifies the terminology. We obtain the following general criterion for conformal dimension one.

Theorem 1.2 (General criterion for conformal dimension one). *Let X be a doubling and compact metric space. If X is linearly connected and satisfies the UWS condition, then the (AR) conformal dimension of X is equal to one.*

The main ingredient of the proof is a result of S. Keith and B. Kleiner stating an equality between the conformal dimension and the critical exponent associated to the combinatorial modulus (a proof of this result is given in [Ca12]). A similar unpublished result to Theorem 1.2 was known by S. Keith and B. Kleiner; one of the motivations of the present article is to provide an accessible proof of this criterion and some of its consequences in the context of hyperbolic groups.

Note that Theorem 1.2 provides a large class of examples. On the opposite, a result of J. Mackay ensures that when the space X verifies a quantitative analogue of the topological conditions of being locally connected and without local cut points, then the conformal dimension is greater than one [Mac10]. Both criteria provide a clear conceptual picture of the relationship between conformal dimension and local cut points.

The rest of the paper is devoted to the boundaries of hyperbolic groups. They are part of a larger class consisting of *quasiselfsimilar* spaces (see Section 2.2). For this class, the hypotheses of Theorem 1.2 are equivalent to the following topological conditions: the space X is compact, connected, and verifies the *well spread* local cut points condition —WS for short: there exists a sequence of finite sets $P_n \subset X$ such that

$$\delta_n = \sup \{ \text{diam} A : A \in \mathcal{C}_n \} \rightarrow 0, \text{ when } n \rightarrow +\infty, \tag{WS}$$

where \mathcal{C}_n denotes the set of connected components of $X \setminus P_n$.

It is a remarkable fact that the topology of the boundary is reflected in the splittings of the group [Bow98a]. This motivates the problem of characterizing the hyperbolic groups whose boundary at infinity has conformal dimension one in terms of the properties of its canonical splittings.

If G is a hyperbolic group, we can decompose G repeatedly over finite and virtually cyclic subgroups, until (at least if there is no 2-torsion) all subgroups are finite, virtually Fuchsian, or one-ended without local cut points on the boundary [DP01, Va08]. Therefore, the candidate groups G for $\dim_{AR} \partial G = \dim_T \partial G = 1$ are essentially obtained as repetitive amalgamated products or HNN-extensions of virtually Fuchsian and finite groups over elementary subgroups.

A maximal decomposition of G over finite subgroups is given by an action of G on a simplicial tree T , without edge inversions and of finite quotient, such that the stabilizers of the edges are finite, and the stabilizers of the vertices have at most one end [Dun85, Sta68]. We call this action a DS splitting of G . It describes the connected components of ∂G . Our first result regarding the boundary of a hyperbolic group is the following.

Theorem 1.3 (Stability under splittings over finite groups). *Let G be a hyperbolic group, and let T be the tree associated to the DS splitting. We denote by $\{v_1, \dots, v_M\}$ a set of representatives of the orbits of the vertices of T , and by $G(v_i)$ their respective stabilizers. Then*

- (1) $\dim_{AR} \partial G = 0$ if all the $G(v_i)$ are finite, or
- (2) $\dim_{AR} \partial G = \max \{ \dim_{AR} \partial G(v_i) : G(v_i) \text{ is infinite} \}$ otherwise.

We remark that Theorem 1.3 is the analogue, in the broader context of hyperbolic groups, of a result proved in [CMT99] for hyperbolic 3-manifolds (see Theorem 2.7 therein).

Assume now that G is one-ended (∂G is connected). If G is not virtually Fuchsian, then ∂G has a local cut point if and only if G splits over a virtually cyclic subgroup [Bow98a], and if G has no local cut points, then $\dim_{AR} \partial G > 1$ [Mac10]. In the latter case, if in addition $\dim_T G = 1$, then ∂G is homeomorphic to the Sierpiński carpet or the Menger sponge [KK00]. Therefore, the question can be formulated as follows: is $\dim_{AR} \partial G = 1$ if ∂G does not contains a Sierpiński carpet or a Menger sponge? Or even: is it true that if ∂H has a local cut point for every one-ended quasiconvex subgroup H of G , then $\dim_{AR} \partial G = 1$? Whether or not under this hypothesis the ∂G verifies the WS condition highly depends on how are embedded (in G) the one-ended quasiconvex subgroups of G .

We apply Bowditch’s work [Bow98a] on the structure of local cut points of the boundaries of one-ended hyperbolic groups, by relating the JSJ splitting of G with the WS property, to deduce the following partial answer:

Theorem 1.4 (All rigid are virtually free is equivalent to WS). *Let G be a one-ended hyperbolic group and suppose that G is not a cocompact virtually Fuchsian group. Then ∂G satisfies the WS property if and only if all the rigid type vertices in the JSJ splitting of G are virtually free. In particular, in this case we have $\dim_{AR} \partial G = 1$.*

We refer to Section 3 for precise definitions. Note that if G satisfies the hypotheses of Theorem 1.4, then the AR conformal dimension is never attained by a distance in the gauge. An interesting reformulation of Theorem 1.4 is the following: let \mathcal{E} be the union of all limit sets of the stabilizers of the edges of the JSJ splitting of G . For each $x \in \partial G$, we define the set $\mathcal{E}(x)$ as the set of points $y \in \partial G$ such that x and y are in the same connected component of $\partial G \setminus P$, for any finite subset P of \mathcal{E} , not containing x nor y . We call $\mathcal{E}(x)$ the *fiber* of x . Denote by T_R the set of rigid vertices of the JSJ decomposition of G , and by Λ_v the limit set of the stabilizer of a vertex v . If $x \in \Lambda_v$, with $v \in T_R$, denote by $C_v(x)$ the connected component of Λ_v containing x . Define

$$C(x) := \bigcup \{C_v(x) : x \in \Lambda_v, v \in T_R\}.$$

In Section 2 we prove that the WS condition is equivalent to the triviality of all the fibers, i.e. $\mathcal{E}(x) = \{x\}$ for all $x \in \partial G$. Moreover, the proof of Theorem 1.4 implies:

Corollary 1.5 (Fibers). *Let G be a one-ended hyperbolic group which is not a cocompact virtually Fuchsian group, and let $x \in \partial G$. Then the fiber $\mathcal{E}(x)$ is equal to either $C(x)$ if there exists $v \in T_R$ with $x \in \Lambda_v$, or the singleton $\{x\}$ otherwise.*

One remarkable consequence of Theorem 1.4 is the existence of convex cocompact Kleinian groups for which the conformal dimension is not equal to a well known geometric invariant in the context of hyperbolic 3-manifolds. The analogy of Theorem 1.3 with the results in [CMT99] does not hold for Theorem 1.4. More precisely, let G be a one-ended convex cocompact Kleinian group (whose limit set is not the entire sphere) and let M be the hyperbolic 3-manifold with boundary whose interior is \mathbb{H}^3/G ; here \mathbb{H}^3 is the real hyperbolic space. Consider the space $TT(M)$ of all complete hyperbolic 3-manifolds N which are homeomorphic to the interior of M [CMT99]. Each N is uniformized by a Kleinian group G_N so that N is isometric to the quotient \mathbb{H}^3/G_N . Define $d(N)$ to be the Hausdorff dimension of the limit set of G_N (note that this dimension only depends on N), and let

$$D(M) = \inf\{d(N) : N \in TT(M)\}.$$

In [CMT99] the authors completely characterize the hyperbolic 3-manifolds for which $D(M) = 1$. This is the case if and only if M is a *generalized book of I-bundles*. When M is a generalized book of I-bundles, there is no rigid vertex in the JSJ decomposition of $\pi_1(M)$. And therefore we obtain the following:

Corollary 1.6. *Let M be a hyperbolizable 3-manifold as before. If all the rigid vertices of the JSJ decomposition of $\pi_1(M)$ are virtually free, and there exists at least one rigid vertex, then $\dim_{AR} \partial\pi_1(M) < D(M)$.*

See Section 3.6 for a concrete example. For such a group, we have $D(M) > 1 = \dim_{AR} \partial\pi_1(M)$; in other words, restricted to Kleinian deformations, the infimum of Hausdorff dimensions is greater than the conformal dimension of the group. The inequality $D(M) > 1$ is a consequence of Thurston's relative compactness theorem. See [Bou97] Theorem 1.2 for a similar type of result for hyperbolic buildings.

REMARK. Many of the ideas of this paper can also be applied to the study of conformal dimension for the repellers of dynamical systems induced by a certain class of expanding branched coverings, namely, the topologically exc maps (see [HP09] for their definition and basic properties). A dynamical sufficient condition for conformal dimension one also holds in this context [Ca13].

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Outline of the paper. The paper is divided into two parts. In the first part, Section 2, we treat the problem in the general setting and we prove Theorem 1.2. Next we give a local version of the WS condition in terms of fibers, Lemma 2.2. Finally, we show that the WS property is equivalent to the hypotheses of Theorem 1.2 in the quasiselfsimilar case, this corresponds to Lemma 2.3.

The rest of the paper is devoted to the second part. We start Section 3 by improving some well known facts about quasiconvex splittings of hyperbolic groups, see Section 3.1. This results are then used to prove that the AR conformal dimension is stable under splittings over finite groups, Theorem 1.3. We then prove some general lemmas about virtually free hyperbolic groups in Section 3.3. We apply Bowditch’s work on the structure of local cut points of the boundaries of one-ended hyperbolic groups to deduce Theorem 1.4. We end by showing how to construct explicit examples of groups verifying the hypotheses of Corollary 1.6 in Section 3.6.

2. A GENERAL CRITERION FOR CONFORMAL DIMENSION ONE

2.1. Proof of Theorem 1.2. We start by remarking that the UWS property is equivalent to the following: there exists a function $C : (0, 1) \rightarrow \mathbb{R}_+$ such that for all $x \in X$ and $0 < s < r \leq \text{diam}X$, there is a finite set $P \subset B(x, r)$ of cardinality bounded from above by $C(s/r)$, and such that no connected component of $X \setminus P$ can intersect both $B(x, s)$ and $X \setminus \overline{B}(x, r)$. In fact, suppose the UWS condition is satisfied, and take $0 < s < r \leq \text{diam}X$. Let

$$n := \left\lceil \frac{2}{r - s} \right\rceil + 1, \text{ and } \epsilon = \frac{1}{4n}.$$

Consider the compact set $K = \overline{B}(x, r - n^{-1}) \setminus B(x, s + n^{-1})$. For every $y \in K$, the ball $B(y, 2\epsilon)$ is contained in $\overline{B}(x, r) \setminus B(x, s)$. By the doubling condition, we can cover K by less than M balls $B(y_i, \epsilon)$, with $y_i \in K$. Since $\text{diam}K \leq 2r$, the constant M can be chosen depending only on

$$\frac{\epsilon}{r} \asymp \frac{1}{nr} \asymp \frac{1 - s/r}{2}.$$

For each of the centers $y_i \in K$, consider a set $P_y \subset B(y, 2\epsilon)$, with $\#P_y \leq C$, given by the UWS condition. Then it suffices to take $P = \bigcup_i P_i$, which is contained in $B(x, r)$ and of cardinal number less than or equal to $M \cdot C$.

We must show that if X is a doubling, compact metric space satisfying LC and UWS, then $\text{dim}_{AR} X = 1$. The main ingredient of the proof is the combinatorial modulus. Using an appropriate sequence of finite coverings of X whose mesh tends to zero, we can define combinatorial versions of conformal moduli, from which we are able to compute the conformal dimension of the space.

Consider $\{X_i\}$ a sequence of maximal a^{-i} -separated sets, with $a > 1$ a big enough constant. We associate the coverings \mathcal{S}_i by the balls $B(x, a^{-i})$ where $x \in X_i$. We write $\mathcal{S} := \bigcup_k \mathcal{S}_k$. The combinatorial modulus is defined as follows. Let $n, k \geq 1$, for every ball $B \in \mathcal{S}_k$, we consider the family of curves $\Gamma(B)$ in X that “join” the ball B with the complement of the ball $2B$, i.e. $\gamma \cap B \neq \emptyset$ and $\gamma \cap X \setminus 2B \neq \emptyset$. Here $2B$ denotes the ball with the same center as B and twice its radius. Given $p > 0$, we define the p -combinatorial

modulus of the “annulus” $(B, 2B)$ by

$$M_{p,n}(B) := \inf_{\rho} \text{Vol}_p(\rho), \text{ where } \text{Vol}_p(\rho) := \sum_{A \in \mathcal{S}_{k+n}} \rho(A)^p,$$

and where the infimum is taken over all weight functions $\rho : \mathcal{S}_{k+n} \rightarrow \mathbb{R}_+$ which are $\Gamma(B)$ -admissible, i.e. for any curve $\gamma \in \Gamma(B)$, we have

$$\ell_{\rho}(\gamma) = \sum_{A \cap \gamma \neq \emptyset} \rho(A) \geq 1.$$

Therefore, for each $p > 0$, we obtain a sequence $\{M_{p,n}\}_n$, where $M_{p,n}$ is the p -combinatorial modulus of X at scale n :

$$M_{p,n} := \sup_{B \in \mathcal{S}} M_{p,n}(B).$$

In other words, the modulus $M_{p,n}$ takes into account all the “annuli” of X with a fixed radius ratio equal to 2.

We look at the asymptotic behavior of the sequence $\{M_{p,n}\}_n$, and its dependence on p . For example, for fixed p , $\{M_{p,n}\}_n$ verifies a sub-multiplicative inequality, see [Ca12]. It is therefore natural to consider the *critical exponent* Q defined by

$$Q := \inf \left\{ p > 0 : \liminf_n M_{p,n} = 0 \right\}.$$

Recall that X is said to be *uniformly perfect* if there exists a constant $c > 0$ such that $\text{diam} B_r \geq cr$ for all ball B_r of radius $r > 0$ in X . For instance, X is uniformly perfect if it is connected. If $A \subset X$ and $r > 0$, then $V_r(A)$ denotes the r -neighborhood of A . We derive Theorem 1.2 from the following result.

Theorem 2.1 ([Ca12] Theorem 3.12 and Corollary 3.15). *Let X be a doubling, uniformly perfect, compact metric space. Suppose that X also verifies the following two conditions:*

- (1) *(Uniform linear connectivity of components) There exists a constant $K_{\ell} \geq 1$ such that any connected component of X is K_{ℓ} -linearly connected.*
- (2) *(Uniform separation of components) There exists a constant $K_s \geq 1$ such that: for all $0 < r \leq \text{diam} X$, there exists a covering \mathcal{W}_r of X , by open and closed sets, such that for all $W \in \mathcal{W}_r$, we have $\text{dist}(W, X \setminus W) \geq r/K_s$ and there exists a connected component Y of X with $Y \subset W \subset V_r(Y)$.*

Then $Q = \dim_{AR} X$. This is the case, in particular, when X is linearly connected.

If in addition, X is quasiregular (with linear distortion function) and verifies

- (3) *(Diameter of the connected components tends to zero) For all $\delta > 0$, there are only finitely many connected components of X which have diameter greater than or equal to δ ,*

then $\dim_{AR} X = \sup \{\dim_{AR} Y : Y \text{ is a connected component of } X\}$. We remark that, by convention, we set $\dim_{AR} Y = 0$ if Y is a point.

Therefore, we must prove that $Q = 1$. For this, we will prove that $M_{p,n} \lesssim \eta_n^{p-1}$ for all $p > 1$, where η_n is a sequence of positive real numbers that tends to zero as $n \rightarrow \infty$. Indeed, let $x \in X_k$ and write $r = a^{-k}$. Recall that $\Gamma(B)$ is the family of curves of X joining $B(x, r)$ and $X \setminus B(x, 2r)$. We denote by $P = P(B) \subset 3/2 \cdot B$ the finite set given by the UWS property, which verifies that $\gamma \cap P \neq \emptyset$ for any $\gamma \in \Gamma(B)$, and that $\#P \leq C$, where C is a uniform constant.

We have $\Gamma(B) \subset \bigcup_{z \in P} \Gamma_z$, where Γ_z is the family of curves γ in X such that $z \in \gamma$ and $\gamma \cap X \setminus B(z, s)$, where $s = r/3$. In particular, we obtain

$$M_{p,n}(B) \leq \sum_{z \in P} \text{Mod}_p(\Gamma_z, \mathcal{S}_{k+n}) \leq C \max_{z \in P} \{\text{Mod}_p(\Gamma_z, \mathcal{S}_{k+n})\}.$$

We must bound from above the combinatorial modulus of Γ_z . We take $m \geq 1$, and for $i = 0, \dots, m-1$, we set

$$A_i = \overline{B(z, 2^{-i}s)} \setminus B(z, 2^{-(i+1)}s).$$

By the UWS property, for each $z \in P$ and each $i = 0, \dots, m-1$, there exists a finite set $R_{z,i} \subset A_i$, with cardinal number bounded from above by a universal constant K , such that any curve γ of X verifying $\gamma \cap B(z, 2^{-(i+1)}s) \neq \emptyset$ and $\gamma \cap B(z, 2^{-i}s) \neq \emptyset$, must pass by $R_{z,i}$. For $n \geq 1$ such that $a^{-(k+n)} \leq 2^{-m}s$, consider the set

$$U = \{A \in \mathcal{S}_{k+n} : A \cap R \neq \emptyset\},$$

where

$$R := \bigcup_{i=0}^{m-1} R_{z,i}.$$

We define $\rho : \mathcal{S}_{k+n} \rightarrow \mathbb{R}_+$ by

$$\rho(A) := \begin{cases} \frac{1}{m} & \text{if } A \in U \\ 0 & \text{otherwise.} \end{cases}$$

On the one hand, since any curve $\gamma \in \Gamma_z$ must cross each A_i for $i = 0, \dots, m-1$, we have that ρ is Γ_z -admissible. On the other hand, there exists a constant M , which depends only on the doubling constant of X and a , such that

$$\#U \leq M \cdot \# \left\{ \bigcup_{i=0}^{m-1} R_{z,i} \right\} \leq (MK) \cdot m := K'm.$$

Therefore,

$$\text{Vol}_p(\rho) = \frac{\#U}{m^p} \leq \frac{K'}{m^{p-1}}.$$

This shows that $\text{Mod}_p(\Gamma_z, \mathcal{S}_{k+n}) \leq K'm^{1-p}$. Thus, it suffices to take

$$\eta_m := \left[\frac{n \log a - \log 3}{\log 2} \right]^{-1}.$$

This ends the proof of Theorem 1.4.

REMARK. The UWS condition fails when there is a infinite number of definite diameter pairwise disjoint curves in the space. Therefore, it is not difficult to find examples of spaces of conformal dimension equal to 1, but not verifying the UWS property. However, the LC and UWS assumptions are in some sense optimal. One can construct an example of a compact connected space which verifies WS and LC, but not UWS, and has AR conformal dimension strictly bigger than 1; and, also a space which verifies UWS, but not LC, and of Ahlfors regular conformal dimension strictly bigger than 1.

2.2. Equivalence between the different conditions. The goal of the next subsections is to relate the three different conditions on the existence of local cut points mentioned in the Introduction; i.e. the UWS, the WS, and the triviality of the fibers conditions.

Let X be a connected and compact metric space, and let \mathcal{E} be a countable subset of local cut points of X . Recall that for each $x \in X$, we define $\mathcal{E}(x)$ (the *fiber* of x) as the set of points $y \in X$ such that x and y are in the same connected component of $X \setminus P$ for any finite subset P of \mathcal{E} not containing x and y . If X verifies the WS condition, we can consider \mathcal{E} to be the union of the sets of local cut points P_n given by the definition. In that case, all fibers are trivial. When X is locally connected, the converse also holds.

Lemma 2.2 (Trivial fibers and WS). *Let X be a compact connected metric space. Then the following statements are equivalent:*

- X satisfies the WS property.
- X is locally connected and there is a countable subset \mathcal{E} of local cut points of X such that $\mathcal{E}(x) = \{x\}$ for all $x \in X$.

Note that since X is compact, local connectivity implies local arcwise connectivity. It is clear that if X satisfies UWS, then it verifies also WS. The converse is true in the self-similar case. Recall that X is *quasiselfsimilar* if there exist a distortion function $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and constants $c_0 > 0$ and $r_0 > 0$, such that for any $0 < r \leq \text{diam}X$ and any $x \in X$, there exists an open η -quasisymmetric embedding $\phi : B(x, r) \rightarrow U$ where $U \subset X$, with $U \supset B(\phi(x), r_0)$, and with $\text{diam}B(\phi(x), r_0) \geq c_0$. This definition implies that X is doubling and uniformly perfect, and that if X is connected and locally connected, then X is LC (see [Ca11] Chapter 2, Prop. 2.9). The boundaries of hyperbolic groups, when endowed with visual metrics, provide an important class of quasiselfsimilar spaces (with linear distortion function η), see Section 3.

Lemma 2.3 (UWS and WS for self-similar spaces). *Let X be a compact connected quasiselfsimilar space. Then the following statements are equivalent:*

- X satisfies the WS property.
- X satisfies the LC and UWS properties.

We obtain the following corollary of Theorem 1.2: if X is a compact connected quasiselfsimilar space, and satisfies the WS property, then $\dim_{AR} X = 1$. It is important to note that the WS condition can be stated without using a metric, only with the uniform structure of the compact set X . It is therefore, a topological property.

2.2.1. Proof of Lemma 2.2. The arguments involved in the proof are rather elementary, we include them for clarity and completeness of presentation. See also [Ca11] Chapter 5, Prop. 5.2 for more details.

WS implies local connectivity. We can assume that the sequence $\{P_n\}_n$ in the definition of WS is increasing. We denote by \mathcal{C}_n the set of connected components of $X \setminus P_n$, and for $\delta > 0$,

$$\mathcal{C}_n(\delta) = \{C \in \mathcal{C}_n : \text{diam}C \geq \delta\}.$$

Note that the set $\mathcal{C}_n(\delta)$ is always finite. Indeed, we have $\#\mathcal{C}_n(\delta) \leq \#P_n$ for n large enough so that $\delta_n < \delta$. This implies that if $\{C_m\}_m$ is a sequence of different connected components in \mathcal{C}_n , then

$$\text{diam}C_m \rightarrow 0, \text{ when } m \rightarrow +\infty. \tag{2.1}$$

Note that $X \setminus P_n$ is an open set and its topological boundary coincides with P_n . Then, the Boundary Bumping Theorem (see Chapter 5 of [Na92]) implies that $\overline{C} \cap P_n \neq \emptyset$ for all $C \in \mathcal{C}_n$.

Let $x \in X$ and $\varepsilon > 0$. We take $n \geq 1$ such that $\delta_n < \varepsilon/2$. We set

$$U := \bigcup \{C : C \in \mathcal{C}_n, x \in \overline{C}\} \cup \{x\}.$$

Then U is connected. Moreover, U contains x in its interior by (2.1). Since $\text{diam}U \leq 2\delta_n < \varepsilon$, we obtain the local connectivity at x .

Trivial fibers implies WS. Let x be a point in X , and suppose $\mathcal{E}(x) = \{x\}$. We claim that for any radius $r \in (0, \text{diam}X)$, there exists a connected open set U containing x , which is contained in the ball $B(x, r)$, and whose frontier is a finite subset of \mathcal{E} .

Indeed, write $\mathcal{E} \setminus \{x\} = \{p_n : n \geq 1\}$. Let $\mathcal{E}_n = \{p_i : 1 \leq i \leq n\}$ for each $n \geq 1$, and define U_n to be the connected component of $X \setminus \mathcal{E}_n$ containing x . Then $\{U_n\}$ is a decreasing sequence of open connected sets containing x . For each n , the frontier of U_n is finite and is contained in \mathcal{E}_n . Since $\mathcal{E}(x) = \{x\}$, one checks that

$$\bigcap_n \overline{U}_n = \bigcap_n U_n = \{x\}.$$

This proves the claim. In particular, when $\mathcal{E}(x) = \{x\}$ for all $x \in X$, we can find for all $r \in (0, \text{diam}X)$, a finite open covering \mathcal{U} of X such that for all $U \in \mathcal{U}$: the frontier of U is a finite subset of \mathcal{E} and $\text{diam}U \leq 2r$. Finally, X satisfies the WS property: consider P_r the union of the frontiers ∂U , with $U \in \mathcal{U}$. Then P_r is a finite set, and all connected components of $X \setminus P_r$ have diameter less than or equal to $2r$.

2.2.2. Proof of Lemma 2.3. We prove now that for quasiselfsimilar spaces, the WS condition is equivalent to LC and UWS. Since when X verifies WS, it is locally connected, by self-similarity it is also LC. We now prove UWS.

For each $x \in X$ and $0 < r \leq \text{diam}X$, let $\phi : B(x, 2r) \rightarrow U \subset X$ be a η -quasisymmetric homeomorphism given by the definition of self-similarity. Let $\{P_n\}$ be the sequence of finite sets given by the definition of WS. We claim that there exists $n \geq 1$ such that for all $x \in X$ and $0 < r \leq \text{diam}X$, a connected component of $X \setminus R_n$ cannot intersect both $B(x, \frac{r}{2})$ and $X \setminus \overline{B}(x, r)$, where $R_n := \phi^{-1}(P_n \cap U)$. Since the cardinal number of R_n is bounded from above by that of P_n , this shows that X verifies UWS.

Suppose by contradiction that the claim is false. Then for each n , there exists $x_n \in X$, $0 < r_n \leq \text{diam}X$ and a connected component Y_n of $X \setminus R_n$ intersecting both $B(x_n, r_n/2)$ and $X \setminus B(x_n, r_n)$. Since X is locally connected and every R_n is finite, the component Y_n is arcwise connected. Let γ_n be a curve in Y_n satisfying $\gamma_n \cap B(x_n, r_n/2) \neq \emptyset$ and $\gamma_n \cap X \setminus B(x_n, r_n) \neq \emptyset$. Then $\text{diam}\gamma_n \geq r_n/2$, and therefore (see [Hei01], Prop. 10.8)

$$\frac{\text{diam}\phi_n(\gamma_n)}{\text{diam}\phi(B(x_n, 2r_n))} \geq \frac{1}{2} \cdot \eta \left(\frac{\text{diam}B(x_n, 2r_n)}{\text{diam}\gamma_n} \right)^{-1} \geq \frac{1}{2\eta(8)} := \delta. \quad (2.2)$$

The connected components of $X \setminus P_n$ have their diameter bounded from above by δ_n , which tends to zero as n tends to infinity. In particular, for n large enough, $\phi_n(\gamma_n) \cap P_n \neq \emptyset$. But this is a contradiction, since $\gamma_n \cap R_n = \emptyset$. This ends the proof.

3. THE WS PROPERTY FOR HYPERBOLIC GROUPS

In this section we give the proofs of Theorem 1.3 and Theorem 1.4. We begin by recalling some definitions and establishing some notations and properties which will be useful in the

sequel. Let G be a Gromov-hyperbolic group and $Z = \mathcal{C}(G, S)$ be a Cayley graph with respect to some finite symmetric generating set S . Equip Z with the length distance which makes each edge of Z isometric to the interval $[0, 1]$, so that the action of G by left multiplication on Z is a geometric action. See [CDP90, GH90] for an introduction to the subject. We can identify the boundary of G with ∂Z . Consider the canonical conformal gauge associated with the boundary of Z , induced by the visual metrics. In the following, we fix such a distance $\theta = \theta$ on ∂Z , with $\epsilon > 0$ small enough, and we recall that it satisfies

$$\theta(x, y) \asymp \exp(-\epsilon(x|y)), \quad \forall x, y \in \partial Z,$$

where $(x|y)$ denotes the Gromov product of x and y viewed from the identity element of G . This metric is Ahlfors regular of dimension h/ϵ whenever G is non-elementary; here h is the volume entropy of the action of G on Z [Coo93]. In particular, $(\partial Z, \theta)$ is a doubling uniformly perfect compact metric space.

Recall also that if $a, b, c, d \in \partial Z$ are four distinct points, their cross-ratio is defined by

$$[a, b, c, d] := \frac{\theta(a, b)}{\theta(a, c)} \cdot \frac{\theta(c, d)}{\theta(b, d)}.$$

A homeomorphism $f : \partial Z \rightarrow \partial Z$ is η -quasi-Möbius if for all distinct $a, b, c, d \in \partial Z$, we have

$$[f(a), f(b), f(c), f(d)] \leq \eta([a, b, c, d])$$

(see [Väis84]). When G is a hyperbolic group, there exists a constant $C = C(\epsilon)$ such that the group G acts on ∂Z by η -quasi-Möbius homeomorphisms with $\eta(t) = C \cdot t$ (see for example [Haïss08] Theorem 3.1).

By the work of P. Tukia and B. Bowditch, G acts properly discontinuously and cocompactly on $\partial^3 Z$, the space of triples of distinct points of ∂Z . Cocompactness on triples means that there is $\delta > 0$ such that for any triple $\{x_1, x_2, x_3\}$ of distinct points of ∂Z , there exists $g \in G$ such that $\theta(gx_i, gx_j) \geq \delta$, $i \neq j$. This property implies that G is a uniform convergence and uniformly quasi-Möbius group.

Quasi-Möbius and quasimetric maps are closely related: if $f : B(x, r) \rightarrow U \subset \partial Z$ is a η -quasi-Möbius homeomorphism, and if there is a constant $\lambda > 0$ and three points $x_1, x_2, x_3 \in B(x, r)$, such that

$$\theta(x_i, x_j) \geq \frac{\text{diam}B(x, r)}{\lambda} \quad \text{and} \quad \theta(f(x_i), f(x_j)) \geq \frac{\text{diam}U}{\lambda}, \quad (3.1)$$

then f is η_λ -quasimetric, with $\eta_\lambda(t) = 2\lambda\eta(2\lambda t)$.

Finally, note that this properties imply the quasi-selfsimilarity of ∂Z (see also [Haïss08] Prop. 4.6): suppose G is a non-elementary hyperbolic group, then there exists $r_0 > 0$, $c_0 > 0$ and a distortion function $\eta(t) = C' \cdot t$ such that for any $x \in \partial Z$ and any radius $r > 0$, there exists $g \in G$ such that $g(B(x, r)) \supset B(g(x), r_0)$, $\text{diam}B(g(x), r_0) \geq c_0$ and $g|_{B(x, r)}$ is η -quasimetric. This is also known as the Sullivan *conformal elevator principle* [Sul82].

3.1. Quasiconvex splittings and the boundary. We recall the definition of quasiconvex subgroup. A subset Y of a geodesic metric space Z is *quasiconvex* if there exists a constant $K \geq 0$, such that any point on any geodesic of Z joining two points of Y is at distance less than or equal to K from Y . We say that a subgroup H of G is quasiconvex if H is a quasiconvex subset of Z . This definition is independent of the generators considered to construct the distance on Z , and is indeed, equivalent to: the inclusion $H \hookrightarrow Z$ is a quasi-isometric embedding. So H is also a hyperbolic group and the boundary ∂H is canonically identified with the limit set $\Lambda_H \subset \partial Z$. The limit set is by definition the set

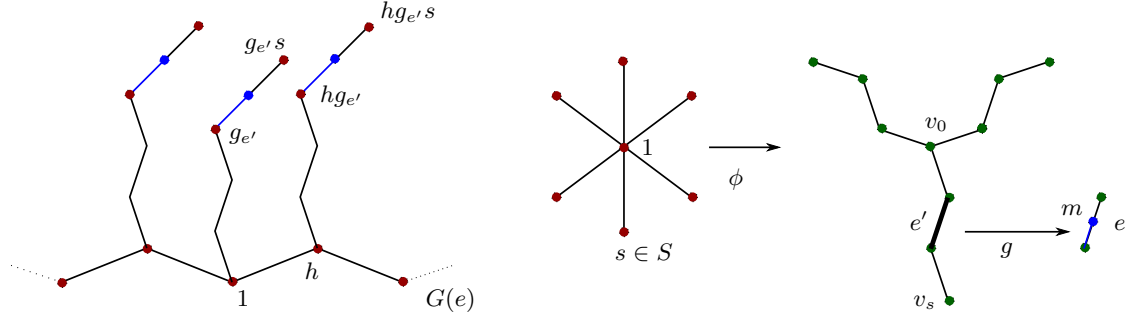


FIGURE 3.1. The set $Q(e)$.

all of accumulation points of H in ∂Z . If H_1 and H_2 are two quasiconvex subgroups of G , then $H_1 \cap H_2$ is also quasiconvex and $\Lambda_{H_1 \cap H_2} = \Lambda_{H_1} \cap \Lambda_{H_2}$ (see [Sho90], [BH99]).

Let T be a simplicial tree, we denote by T_0 and by T_1 the vertices and the edges of T . We equip the geometric realization of T with the length distance that makes each edge isometric to the interval $[0, 1]$. Consider a simplicial action of G on T , without edge inversions and such that the quotient T/G is finite, so that G is isomorphic to the fundamental group of a graph groups. Assume also that the action is minimal, i.e. there is no proper G -invariant sub-tree of T . Given $v \in T_0$ and $e \in T_1$, we denote by $G(v)$ and by $G(e)$ the stabilizers of v and e respectively. We also let Λ_v and Λ_e be the limit set of $G(v)$ and $G(e)$ respectively. Thus, if $e \in T_1$ and $v, w \in T_0$ are the endpoints of e , then $G(e) = G(v) \cap G(w)$.

3.1.1. *Estimating the diameter of Λ_e and Λ_v .* The estimates that follow are improvements of some results proved in [Bow98a, Bow98b]. Our aim is to deduce some topological properties of ∂Z from the action of G on T . Suppose now that $G(e)$ is quasiconvex for every edge $e \in T_1$.

Definition 3.1. *We define an equivariant continuous function $\phi : Z \rightarrow T$ by setting $\phi(1) = v_0$, where $v_0 \in T_0$ is any vertex, and $\phi(g) = g \cdot v_0$ for all $g \in G$. This determines ϕ on G , the vertices of Z . If $e = (g, h)$ is an edge of Z , we extend ϕ so that ϕ sends e linearly to the unique arc of T that joins $\phi(g)$ and $\phi(h)$.*

The image of an edge of Z is either a vertex or an arc of T . In particular, the image of ϕ is a G -invariant subtree of T , and since the action of G is minimal, ϕ is surjective. For each $s \in S$, we let $I_s = (1, s)$ be the edge between 1 and s , and $v_s = s \cdot v_0$, so $\phi(I_s) = [v_0, v_s]$ is the unique arc of T joining v_0 and v_s . Therefore, if $I = (g, gs)$ is an edge of Z , with $g \in G$ and $s \in S$, then $\phi(I) = g \cdot [v_0, v_s]$. Since S is a finite set,

$$L := \max \{ \text{length}([v_0, v_s]) : s \in S \} < \infty,$$

so ϕ is indeed Lipschitz continuous with constant L .

NOTATION. In the sequel, dist_H denotes the Hausdorff distance between closed subsets of Z . If A is a closed subset of Z , we denote by ∂A the set $\overline{A} \setminus A$, where the closure is taken in the space $Z \cup \partial Z$. Let $e \in T_1$, we denote by $m(e)$ its mid-point and $Q(e) := \phi^{-1}(m(e))$. For $v \in T_0$, let $T_v(e)$ be the connected component of $T \setminus \{m(e)\}$ which does not contain v , and $Z_v(e) := \phi^{-1}(T_v(e))$. For $v = v_0$, we simply write $T(e)$ and $Z(e)$, and we also denote by $-T(e)$ the the other component of $T \setminus \{m(e)\}$ and $-Z(e) := \phi^{-1}(-T(e))$.

We denote by $N(v)$ the set of edges of T which are incident to v . Let $M(v)$ be the connected set formed by v and all the segments joining v to $m(e)$, with $e \in N(v)$, and consider $Q(v) := \phi^{-1}(M(v))$. Note that $Q(e)$ and $Q(v)$ are respectively $G(e)$ and $G(v)$

invariant, and for every edge $e \in N(v)$, we have $Q(e) \subset Q(v)$. For every vertex $v \neq v_0$ of T , there exists exactly one edge $e_v \in N(v)$ such that $\partial Z_v(e_v) = -\partial Z(e_v)$, i.e. such that $\partial Z_v(e_v)$ contains v_0 . Let $N^*(v) = N(v) \setminus \{e_v\}$.

Lemma 3.2. *For each edge $e \in T_1$, there exists a constant $K = K(e) > 0$ such that*

$$\text{dist}_H(Q(e), G(e)) \leq K. \quad (3.2)$$

In particular, $\partial Q(e) = \overline{Q(e)} \setminus Q(e) = \Lambda_e$.

Proof. Let $e \in T_1$. We first show that every point of $Q(e)$ is at distance less than or equal to K_e from $G(e)$, where K_e is a constant that depends on e . Indeed, if $I = (g, gs)$ is an edge of Z with $g \in G$ and $s \in S$, then $Q(e) \cap I$ has at most one point. For fixed $s \in S$, consider the set

$$H_{s,e} = \{g \in G : \exists e' \in [v_0, v_s] \text{ s.t. } g \cdot e' = e\}.$$

Then $Q(e) \cap I \neq \emptyset$ if and only if $g \in H_{s,e}$. If for $e' \in [v_0, v_s]$ we choose arbitrarily —if any— $g_{e'} \in H_{s,e}$ such that $g_{e'} \cdot e' = e$, then all the others $g \in H_{s,e}$ sending e' to e are of the form $g = hg_{e'}$, with $h \in G(e)$. So for any $g \in H_{s,e}$ there exists $e' \in [v_0, v_s]$ such that

$$\text{dist}(G(e), g) \leq d(h, g) = d(h, hg_{e'}) = d(1, g_{e'})$$

(see Figure 3.1.1). Since the cardinal number of S and that of the edges in each $[v_0, v_s]$ is finite, it suffices to take

$$K_e := 1 + \max\{d(1, g_{e'}) : e' \in [v_0, v_s], s \in S\} < \infty.$$

By surjectivity of ϕ , there exists at least one $s \in S$ such that $H_{s,e} \neq \emptyset$ (i.e. there exists an edge $e' \in [v_0, v_s]$ such that one can choose $g_{e'}$ as above). Since for all $h \in G(e)$, the element $g_h = hg_{e'} \in G$ satisfies $g_h \cdot e' = e$, we have $Q(e) \cap (g_h, g_h s) \neq \emptyset$. That is, for all $h \in G(e)$, we also have $\text{dist}(h, Q(e)) \leq K_e$, and therefore, $\text{dist}_H(Q(e), G(e)) \leq K_e$. \square

The following lemma is proved in Section 1 of [Bow98a]. We include the proof because we need a uniform control on the constants involved.

Lemma 3.3. *The sets $\{Q(e) : e \in T_1\}$ are uniformly quasiconvex. In addition,*

$$\text{diam } \Lambda_e = \text{diam } \partial Q(e) \lesssim \exp(-\epsilon \text{dist}(1, Q(e))). \quad (3.3)$$

Proof. Let $\{e_1, \dots, e_n\}$ be a set of representatives of edge orbits. First, note that $Q(e_i)$ is quasiconvex because it is at bounded Hausdorff distance K_i from $G(e_i)$. Moreover, the quasiconvexity constant $K_{qc}(i)$ of $Q(e_i)$ depends only on K_i and the hyperbolicity constant of Z . But since $Q(g \cdot e) = g \cdot Q(e)$ for any $g \in G$, the sets $Q(e)$ are in fact uniformly quasiconvex, with constant

$$K_{qc} = \max\{K_{qc}(i) : i = 1, \dots, n\}.$$

Finally, this implies that for any geodesic γ joining two points of $Q(e)$, we have

$$\text{dist}(\gamma, 1) \geq \text{dist}(Q(e), 1) - K_{qc}.$$

This shows (3.3). \square

NOTATION. If $f : A \rightarrow [0, \infty)$ is a real function defined on a set A , we say that $f(a) \rightarrow 0$ for $a \in A$, if the set $\{a \in A : f(a) \geq \epsilon\}$ is finite for all $\epsilon > 0$. We define similarly $f(a) \rightarrow \infty$ for $a \in A$: for all $N > 0$, the set $\{a \in A : f(a) \leq N\}$ is finite.

Lemma 3.4. *The sets $\{Z(e) : e \in T_1\}$ are uniformly quasiconvex of constant K_{qc} . In particular,*

$$\text{diam } Z(e) \lesssim \exp(-\epsilon \text{dist}(1, Z(e))). \quad (3.4)$$

In addition,

$$\text{dist}(1, Z(e)) = \text{dist}(1, Q(e)) \rightarrow \infty \text{ for } e \in T_1. \quad (3.5)$$

Proof. If γ is a geodesic joining two points of $Z(e)$, then, since T is a tree, the connected components of $\gamma \setminus Z(e)$ joins two points of $Q(e)$. But $Q(e) \subset Z(e)$, so the sets $Z(e)$, with $e \in T_1$, are K_{qc} quasiconvex. Inequality (3.4) is proved in a similar way as (3.3).

Note that mid-points of different edges are at distance at least one. So if $x \in Q(e)$ and $y \in Q(e')$ for e and e' two different edges of T , then

$$d(x, y) \geq L^{-1} \cdot d(m(e), m(e')) \geq L^{-1}.$$

Since in each edge I of Z there is at most one element of $Q(e)$, the number of edges e such that $Q(e) \cap I \neq \emptyset$ is bounded from above by L . Therefore, there is at most a finite number of sets $Q(e)$ which intersects the ball $B(1, n)$ of Z . That is, $\text{dist}(1, Q(e)) \rightarrow \infty$ for $e \in T_1$.

Finally, since every geodesic γ which joins 1 to a point of $Z(e)$ has to pass through $Q(e)$, we obtain $\text{dist}(1, Z(e)) \geq \text{dist}(1, Q(e))$. The other inequality follows from $Q(e) \subset Z(e)$. \square

REMARK. Lemma 3.3 still holds, and the proof is analogous, if we replace in its statement the set $Q(e)$ by $Q(v)$. Also $\{Q(v) : v \in T_0\}$ is a locally finite cover of Z by K_{qc} -quasiconvex sets, and $\partial Q(v) = \Lambda_v$.

If γ is a geodesic of Z , we let $\gamma_+ = \gamma(+\infty)$ and $\gamma_- = \gamma(-\infty)$. We say that γ is asymptotic to γ_+ and γ_- . Denote by $C_0 \geq 1$ a uniform constant, that depends only on the hyperbolicity of Z , such that for any pair of points x and y of ∂Z and any geodesic γ asymptotic to x and y , we have

$$C_0^{-1} \exp(-\epsilon \text{dist}(1, \gamma)) \leq \theta(x, y) \leq C_0 \exp(-\epsilon \text{dist}(1, \gamma)). \quad (3.6)$$

In the following lemma, we show that any geodesic asymptotic to a point of $\partial Z(e)$ and to a point of $-\partial Z(e)$ must pass uniformly close to $Q(e)$.

Lemma 3.5. *There is a uniform constant K , which depends only on the hyperbolicity of G , such that*

$$\text{dist}(\gamma, Q(e)) \leq K, \quad (3.7)$$

for each edge $e \in T_1$, any two points $x, y \in \partial Z$ with $x \in \partial Z(e)$ and $y \in -\partial Z(e)$, and any geodesic γ of Z with $\gamma_+ = x$ and $\gamma_- = y$.

Proof. We consider $\sigma_x : [0, +\infty) \rightarrow Z$ and $\sigma_y : [0, +\infty) \rightarrow Z$ two geodesic rays from $1 \in Z$ such that $[\sigma_x] = x$ and $[\sigma_y] = y$. We denote by $\{I_k\}_k$ and $\{J_k\}_k$ the sequence of edges belonging to σ_x and σ_y respectively. By hyperbolicity, there exists a uniform constant K , and a constant $T < +\infty$ which depends on x and y , such that if $t \in [T, +\infty)$, then

$$\max \{ \text{dist}(\gamma(t), \sigma_x), \text{dist}(\gamma(-t), \sigma_y) \} \leq K.$$

For $t \geq T$, we consider the curve

$$\gamma_t = \sigma_y^t * \delta_y^t * \gamma|_{[-t, t]} * \delta_x^t * \sigma_x^t,$$

where δ_x^t (resp. δ_y^t) is a geodesic segment of length less than or equal to K , joining $\gamma(t)$ (resp. $\gamma(-t)$) with a point $\sigma_x(u_t)$ (resp. $\sigma_y(v_t)$); the curve σ_x^t (resp. σ_y^t) is the restriction of σ_x (resp. σ_y) to the interval $[u_t, \infty)$ (resp. $(-\infty, v_t]$). See Figure 3.2.

Note that $\phi(\gamma_t)$ joins a point in $-T(e)$ to a point in $T(e)$. In particular, $m(e)$ belongs to $\phi(\gamma_t)$, so $\gamma_t \cap Q(e) \neq \emptyset$. Let p be a point of intersection of $Q(e)$ with γ_t . We take t

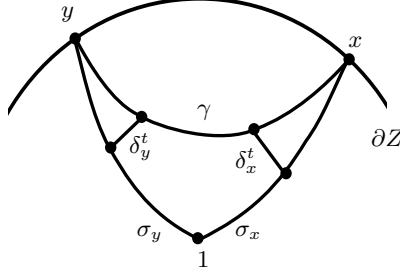


FIGURE 3.2. Proof of Lemma 3.5.

large enough so that p does not belong to $\sigma_x^t \cup \sigma_y^t$. In this case, either p belongs to γ or to $\delta_x^t \cup \delta_y^t$. Therefore, we get a point in γ which is at distance smaller than or equal to K from $p \in Q(e)$. \square

3.1.2. *Description of ∂Z from the action on T .* One can associate to T an ideal boundary ∂T , formed by the geodesic rays from v_0 . We are interested in ∂T only as a set. See [Bow98a] Proposition 1.3, for a proof of the following proposition.

Proposition 3.6. *Assume that the stabilizers $G(e)$ of the edges $e \in T_1$ are quasiconvex. Then there exists a natural G -invariant partition $\partial G = \partial_\infty G \cup \partial_0 G$ with*

$$\partial_0 G = \bigcup_{v \in T_0} \Lambda_v, \quad (3.8)$$

and $\partial_\infty G = \{\text{rays } \gamma : \phi(\gamma) \text{ is unbounded}\}$. Moreover, there exists a bijection

$$j : \partial_\infty G \rightarrow \partial T.$$

We merely describe the function j . Let $\gamma : [0, \infty) \rightarrow Z$ be a geodesic ray from 1. We denote $I_n = (g_n, g_{n+1})$ the edges of γ , where $g_n = \gamma(n)$ for $n \geq 0$. Let $v \in T_0$, and suppose that $n \leq m$ are such that $\phi(I_n) \cap M(v) \neq \emptyset$ and $\phi(I_m) \cap M(v) \neq \emptyset$. Since $Q(v)$ is quasiconvex, $\gamma([n, m])$ is at uniformly bounded distance from $Q(v)$. Therefore, if there is an infinite number of n such that $\phi(I_n) \cap M(v) \neq \emptyset$, then $\lim_n \gamma(n) \in \Lambda_v$.

Suppose now that for every $v \in T_0$, the set of edges $I_n \in \gamma$ such that $\phi(I_n) \cap M(v) \neq \emptyset$ is finite. We have $\phi(I_s) \cap M(v_0) \neq \emptyset$, because $\gamma(0) = 1 \in G$ and $\phi(1) = v_0$. Then there exists a sequence of edges $\{e_i\}$ of T , and an increasing sequence of positive integers $\{n_i\}$, such that for all $i \geq 0$, we have $\phi(\gamma([n_i, \infty))) \subset T(e_i)$, where $e_i = (v_i, v_{i+1})$ and $T(e_{i+1}) \subset T(e_i)$. The sequence of edges $\{e_i\}$ defines a geodesic ray r from v_0 in T , and therefore, a point of ∂T . This ray depends only on the class $[\gamma]$, and we can define $j([\gamma]) = r$.

Let $r : [0, \infty) \rightarrow T$ be a ray from v_0 , and let $\{e_i = (v_i, v_{i+1})\}_{i \geq 0}$ be the edges of r . Lemma 3.4 implies that

$$\bigcap_{i \geq 0} \overline{Z(e_i)} = \{x\} \subset \partial Z,$$

and one checks that $j^{-1}(r) = x$.

3.2. Proof of Theorem 1.3. Recall that the DS splitting of G is given by a simplicial action of G on a simplicial tree T , without edge inversions and with finite quotient, such that the stabilizers of the edges of T are finite, in particular quasiconvex, and the stabilizers of the vertices have at most one end [Dun85, Sta68]. Since G is a hyperbolic group, this means that for every vertex $v \in T_0$, either the stabilizer $G(v)$ is finite and its limit set Λ_v is empty or $G(v)$ is infinite and Λ_v is connected.

Let $\phi : Z \rightarrow T$ and v_0 be as before. Let $\pm\partial T(e)$ be the set of rays of T which eventually are contained in $\pm T(e)$. Note that in this case $\Lambda_e = \emptyset$ for any edge $e \in T_1$, so

$$-\partial Z(e) \cap \partial Z(e) = \Lambda_e = \emptyset,$$

and the sets $\partial Z(e)$ are open and closed in ∂Z . The proof of Theorem 1.3 is divided into several lemmas.

We note that the DS splitting describes the structure of connected components of ∂Z . More precisely, the components of ∂Z are either points, or of the form $g(\Lambda_{v_i})$ for some $g \in G$ and some $i \in \{1, \dots, M\}$ with $G(v_i)$ infinite. From Theorem 2.1, Theorem 1.3 follows from the following three properties:

- (Uniform linear connectivity of components) There exists a constant $K_\ell \geq 1$ such that any connected component of ∂Z is K_ℓ -linearly connected.
- (Uniform separation of components) There exists a constant $K_s \geq 1$ such that: for all $0 < r \leq \text{diam } \partial Z$, there exists a covering \mathcal{W}_r of ∂Z , by open and closed sets, such that for all $W \in \mathcal{W}_r$, we have $\text{dist}(W, \partial Z \setminus W) \geq r/K_s$ and there exists a connected component Y of ∂Z with $Y \subset W \subset V_r(Y)$.
- (Diameter of the connected components tends to zero) For all $\delta > 0$, there are only finitely many connected components of ∂Z which have diameter greater than or equal to δ .

Since G is a uniform quasi-Möbius group, it follows that each connected component of ∂Z is linearly connected with uniform constant. Also since G is a uniform convergence group we also have that the diameter of connected components tends to zero, see also Corollary 3.8. Therefore, we must show that ∂Z satisfies the uniform separation condition on the components.

NOTATION. For any element $g \in G$ we set $|g| = d(1, g)$. Let $E_0 = \{e_1, \dots, e_N\}$ be a set of representatives of edges orbits. For each edge $e \in T_1$ we choose, and fix for the sequel, $g_e \in G$ such that $e = g_e \cdot e_i$. If g and g' are two elements of G such that $e = g \cdot e_i = g' \cdot e_i$, then $d(g, g') \leq \eta$, where

$$\eta = \max \{|h| : h \in G(e_i), i = 1, \dots, N\}.$$

Then the set of elements $g \in G$ such that $e = g \cdot e_i$ is contained in $B(g_e, \eta)$. This implies that for each edge $e \in T_1$, the number $|g_e|$ does not depend, up to an additive constant, on the choice of g_e .

Recall that for each $g \in G$ and $e_i \in E_0$ with $e = g \cdot e_i$, we have $Q(e) = g \cdot Q(e_i)$. Since from Lemma 3.2, the Hausdorff distance between $Q(e_i)$ and $G(e_i)$ is finite, each $Q(e_i)$ is finite and there exists a radius R_i such that $Q(e_i) \subset B(x_i, R_i)$, with $x_i \in Q(e_i)$. If $R := \max\{R_i\}$, then for any $e \in T_1$, there exists $x_e \in Q(e)$ such that $Q(e) \subset B(x_e, R)$. In addition, we have the following estimate:

Lemma 3.7. *For each edge $e \in T_1$, we have $\text{dist}(1, Q(e)) = |g_e| + O(1)$.*

Proof. Let $d_i = \text{dist}(1, Q(e_i))$ for $i = 1, \dots, N$. We set $d = \max\{d_i\}$. Then for each edge $e \in T_1$, we have $\text{dist}(g_e, Q(e)) = \text{dist}(1, Q(e_i)) = d_i$ for some $i \in \{1, \dots, N\}$. So $Q(e)$ is contained in the ball of radius $R + d$ centered at g_e . This ends the proof. \square

We obtain by Lemma 3.4 the following corollary:

Corollary 3.8. *For each edge $e \in T_1$, we have $\text{diam } \partial Z(e) \lesssim \exp(-\epsilon|g_e|)$.*

REMARK. Similar arguments show that we can suppose the constant R to be large enough, so that for every vertex $v \in T_0$ of finite stabilizer, $Q(v)$ is contained in a ball of radius R . For each edge $e \in N(v)$, we have $Q(e) \subset Q(v)$. And therefore $\text{dist}(1, Q(v)) = |g_e| + O(1)$ for any edge $e \in N(v)$. Furthermore, we see that $N(v)$ is finite.

We use Lemma 3.6 to bound from below the distance between $\partial Z(e)$ and $-\partial Z(e)$.

Corollary 3.9. *For each edge $e \in T_1$, we have $\text{dist}(\partial Z(e), -\partial Z(e)) \gtrsim \exp(-\epsilon|g_e|)$.*

Proof. Consider a geodesic γ asymptotic to $x \in \partial Z(e)$ and $y \in -\partial Z(e)$. By inequality (3.7), $\text{dist}(w, \gamma) \leq K$ for some $w \in Q(e)$. So we have

$$\text{dist}(1, \gamma) \leq \text{dist}(\gamma, w) + d(w, 1) \leq K + R + d + |g_e| := |g_e| + C.$$

Since $(x|y) = \text{dist}(1, \gamma) + O(1)$, we obtain

$$\theta(x, y) \asymp \exp(-\epsilon \text{dist}(1, \gamma)) \gtrsim \exp(-\epsilon|g_e|).$$

This proves the lemma. \square

A general property of uniformly perfect spaces allows us to improve the previous estimate. We denote by C_P the constant of uniform perfectness.

Lemma 3.10. *Let X be a uniformly perfect metric space. For any nonempty subset W of X , we have*

$$\text{dist}(W, X \setminus W) \leq C_P \min\{\text{diam}W, \text{diam}(X \setminus W)\}. \quad (3.9)$$

Proof. There is nothing to prove if $\text{diam}W \geq \text{diam}X/C_P$, so suppose that W is a subset of X of diameter smaller than $\text{diam}X/C_P$. We let $r = \text{diam}W$, and consider $x \in W$. Since $W \subset B(x, r)$ and $B(x, C_P r) \setminus B(x, r) \neq \emptyset$, there exists $y \in B(x, C_P r) \cap X \setminus W$. Therefore,

$$\text{dist}(W, X \setminus W) \leq d(x, y) \leq C_P r = C_P \text{diam}W.$$

The lemma follows by symmetry. \square

Corollary 3.11. *For each edge $e \in T_1$, we have*

$$\text{dist}(\partial Z(e), -\partial Z(e)) \asymp \exp(-\epsilon|g_e|).$$

Proof. By Corollary 3.9 it suffices to show the upper bound. But since $\partial Z(e)$ is the complement of $-\partial Z(e)$, and since we have $\text{diam}\partial Z(e) \lesssim \exp(-\epsilon|g_e|)$, we get the result from (3.9). \square

Lemma 3.12. *Let v be a vertex of T_0 of finite stabilizer. Then for every pair of edges $e, e' \in N^*(v)$, we have*

$$\text{dist}(\partial Z_v(e), \partial Z_v(e')) \asymp \exp(-\epsilon|g_e|). \quad (3.10)$$

Proof. We know from the remark following Corollary 3.8, that $|g_e| = |g_{e'}| + O(1)$. Moreover, since $\partial Z_v(e') \subset -\partial Z_v(e)$, by Corollary 3.11 we have

$$\text{dist}(\partial Z_v(e), \partial Z_v(e')) \geq \text{dist}(\partial Z_v(e), -\partial Z_v(e)) \asymp \exp(-\epsilon|g_e|).$$

Let us show the upper bound. Let x be a point of $\partial Z_v(e)$ and y a point $\partial Z_v(e')$. The geodesic rays from 1, σ_x and σ_y , representing respectively x and y , must pass through $x_v \in Q(e)$ and $y_v \in Q(e')$ respectively. Since these two sets are contained in $Q(v)$, which is contained in a ball of radius R , there exists a geodesic σ_v of length less than $2R$ that joins x_v and y_v . Let $\gamma_{x,y} := \sigma_y * \sigma_v * \sigma_x$ be the concatenation of these geodesics. By

hyperbolicity of Z , there exists a constant K and a geodesic γ of Z , asymptotic to x and y , such that $\text{dist}_H(\gamma, \gamma_{x,y}) \leq K$. Then

$$\text{dist}(1, \gamma) \geq \text{dist}(1, Q(v)) - (2R + K) = |g_e| + O(1).$$

Therefore, $\theta(x, y) \lesssim \exp(-\epsilon|g_e|)$. This proves the lemma. \square

We will need the following fact (see [BonK02a] Lemma 3.2): if $\phi : \partial Z \rightarrow \partial Z$ is a η -quasi-Möbius homeomorphism, then there exists an increasing homeomorphism $\hat{\eta} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, which depends only on η , such that for all compact disjoint subsets E and F of ∂Z , we have

$$\Delta(\phi(E), \phi(F)) \leq \hat{\eta}(\Delta(E, F)), \quad (3.11)$$

where

$$\Delta(E, F) = \frac{\text{dist}(E, F)}{\min\{\text{diam}E, \text{diam}F\}}$$

is the relative distance between E and F . In particular, since G acts by η -quasi-Möbius homeomorphisms on ∂Z , the inequality (3.11) is true if we replace ϕ by any $g \in G$. Finally, we can prove the main estimate of this section.

Lemma 3.13. *For every vertex $v \in T_0$ with nonempty limit set Λ_v , and every $e \in N(v)$, we have*

$$\text{dist}(\Lambda_v, \partial Z_v(e)) \lesssim \min\{\text{diam}\Lambda_v, \text{diam}\partial Z_v(e)\}. \quad (3.12)$$

Proof. Recall that $E_0 = \{e_1, \dots, e_N\}$ is a set of representatives of T_1/G . Since for all i , the compact sets Λ_{v_i} and $\partial Z_{v_i}(e_i)$ are disjoint, where v_i is an endpoint of e_i , if Λ_{v_i} is nonempty, we obtain

$$\delta(e_i, v_i) := \Delta(\Lambda_{v_i}, \partial Z_{v_i}(e_i)) \in (0, +\infty).$$

We set $\delta = \max\{\Delta(e_i, v_i)\}$, where the maximum is taken over all edges $e_i \in E_0$ and all endpoints v_i which have nonempty limit set. Let e be an edge of T_1 and v be an endpoint of e . There exists $e_i \in E_0$ and an endpoint v_i of e_i , such that $g_e \cdot e_i = e$ and $g_e \cdot v_i = v$. Note that $g_e \cdot \Lambda_{v_i} = \Lambda_v$ and $g_e \cdot \partial Z_{v_i}(e_i) = \partial Z_v(e)$. Then, from inequality (3.11), we have

$$\Delta(\Lambda_v, \partial Z_v(e)) = \Delta(g_e \cdot \Lambda_{v_i}, g_e \cdot \partial Z_{v_i}(e_i)) \leq \hat{\eta}(\Delta(\Lambda_{v_i}, \partial Z_{v_i}(e_i))) \leq \hat{\eta}(\delta).$$

Finally, this implies

$$\text{dist}(\Lambda_v, \partial Z_v(e)) \leq \hat{\eta}(\delta) \cdot \min\{\text{diam}\Lambda_v, \text{diam}\partial Z_v(e)\}.$$

This ends the proof. \square

We can now finish the proof of Theorem 1.3.

Lemma 3.14. *We let $r_n := \exp(-\epsilon n)$ for each n . There exists a constant $C_1 \geq 1$, such that for all $n \geq 1$, there exists a covering of ∂Z*

$$\mathcal{U}_n := \{W_1, \dots, W_N\},$$

such that for all $j \in \{1, \dots, N\}$, we have:

- $\text{dist}(W_j, \partial Z \setminus W_j) \geq C_1^{-1} r_n$, and
- either there exists a vertex $v_j \in T_0$ with $\Lambda_{v_j} \subset W_j \subset V_{C_1 r_n}(\Lambda_{v_j})$, or there exists a point $x_j \in W_j$ such that $W_j \subset B(x_j, C_1 r_n)$

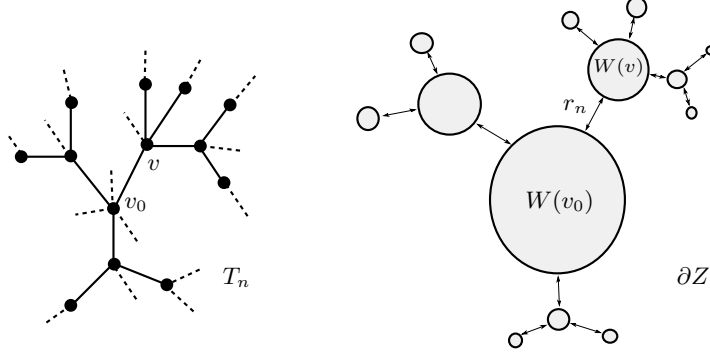


FIGURE 3.3. Construction of the covering in the proof of Lemma 3.14.

Proof. From Corollary 3.8, and Lemmas 3.11 and 3.13, for all $v \in T_0$ with nonempty limit set, and every edge $e \in N^*(v)$, we have

$$\begin{aligned} \text{diam} \partial Z_v(e) &\lesssim \exp(-\epsilon |g_e|) \lesssim \text{dist}(\partial Z_v(e), -\partial Z_v(e)) \\ &\leq \text{dist}(\partial Z_v(e), \Lambda_v) \lesssim \text{diam} \partial Z_v(e), \end{aligned}$$

i.e. all these quantities are comparable. Similarly, by Lemma 3.12, if v is a vertex with finite stabilizer, then for any pair of edges $e, e' \in N^*(v)$, we have

$$\text{dist}(\partial Z_v(e), \partial Z_v(e')) \asymp \exp(-\epsilon |g_e|).$$

We denote by C a uniform constant that bounds from above all the comparison constants which appear in the above inequalities. For $n \geq 1$, we write $B_n := B_Z(1, n)$ the ball in Z centered at 1 and of radius n . We consider the set $T_1(n)$ of edges $e \in T_1$ such that $g_e \in B_n$, and let T_n be the convex hull of $T_1(n)$. Then T_n is a finite subtree of T .

Suppose first that $v_0 \notin T_n$; i.e. that for every edge $e \in N(v_0)$, we have $|g_e| > n$. If Λ_{v_0} is nonempty, we obtain

$$\max \{ \text{diam} \partial Z(e), \text{dist}(\partial Z(e), \Lambda_{v_0}) \} \leq C \exp(-\epsilon |g_e|) \leq Cr_n. \quad (3.13)$$

But this implies that for every edge $e \in N(v_0)$, we have $\partial Z(e) \subset V_{2Cr_n}(\Lambda_{v_0})$, so $\partial Z = V_{2Cr_n}(\Lambda_{v_0})$. If $G(v_0)$ is finite, then $N(v_0)$ is finite, and for every edge $e \in N(v_0)$, we have $\text{diam} \partial Z(e) \leq Cr_n$. In this case it suffices to take $W(e) = \partial Z(e)$ for each $e \in N(v_0)$ to obtain the desired covering.

Assume now that $v_0 \in T_n$, so for every vertex v of T , the edge e_v belongs to T_n , and $N(v) \setminus (T_n)_1 \subset N^*(v)$. In particular, for any edge $e \in N(v) \setminus (T_n)_1$, we have $\partial Z_v(e) = \partial Z(e)$. We claim that for every edge e of T_n , we have $|g_e| \leq n + O(1)$; or equivalently, that $\text{diam} \partial Z(e) \gtrsim r_n$. Indeed, if e is an edge of T_n which does not belong to $T_1(n)$, then there exists an edge $f \in T_1(n) \cap T(e)$. Since $\partial Z(f) \subset \partial Z(e)$, we obtain

$$C^{-1}r_n \leq C^{-1} \exp(-\epsilon |g_f|) \leq \text{diam} \partial Z(f) \leq \text{diam} \partial Z(e) \leq C \exp(-\epsilon |g_e|),$$

and therefore, $|g_e| \leq n + 2\epsilon^{-1} \log(C)$.

For each vertex v of T_n , denote by $(T_n)_1(v)$ the set of edges of T_n which are incident to v . We set

$$W(v) := \Lambda_v \cup \bigcup_{e \in N(v) \setminus (T_n)_1(v)} \partial Z_v(e) = \partial Z \setminus \bigcup_{e \in (T_n)_1(v)} \partial Z_v(e).$$

First case: Suppose $\Lambda_v \neq \emptyset$. For each edge $e \in N(v) \setminus (T_n)_1(v)$, as in (3.13), we have

$$\max \{ \text{diam} \partial Z_v(e), \text{dist}(\partial Z_v(e), \Lambda_v) \} \leq C \exp(-\epsilon |g_e|) \leq Cr_n,$$

because $g_e \notin B_n$. Then, $\partial Z_v(e) \subset V_{2Cr_n}(\Lambda_v)$, and since $e \in N(v) \setminus (T_n)_1(v)$ is arbitrary, we have $W(v) \subset V_{2Cr_n}(\Lambda_v)$.

Second case: Suppose $\Lambda_v = \emptyset$. Let $e \in N(v) \setminus (T_n)_1(v)$ be any edge. For any other edge $e' \in N(v) \setminus (T_n)_1(v)$, we have

$$\text{dist}(\partial Z_v(e), \partial Z_v(e')) \asymp \text{diam} \partial Z_v(e) \asymp \text{diam} \partial Z_v(e') \lesssim r_n.$$

So if we choose any point $x_v \in \partial Z_v(e)$, we obtain $W(v) \subset B_\epsilon(x_v, 2Cr_n)$. Also, since for each edge $e \in (T_n)_1(v)$, we have

$$\text{dist}_\epsilon(\partial Z_v(e), -\partial Z_v(e)) \geq C^{-1} \exp(-\epsilon |g_e|) \geq C^{-3} r_n,$$

and $W(v) \subset -\partial Z_v(e)$, it is true that $\text{dist}(W(v), \partial Z \setminus W(v)) \geq C^{-3} r_n$.

Therefore, we construct a finite covering

$$\partial Z = \bigcup_{v \in T_n} W(v),$$

where for each $v \in T_n$, if the limit set Λ_v is nonempty, we have $\Lambda_v \subset W(v) \subset V_{2Cr_n}(\Lambda_v)$, or, that there exists a point $x_v \in W(v)$ such that $W(v) \subset B_\epsilon(x_v, 2Cr_n)$ otherwise. Moreover, the distance between each element of this collection and its complement, is bounded from below by $C^{-3} r_n$. This ends the proof. \square

3.3. Some general lemmas on virtually free groups. We prove in this section some lemmas on virtually free groups that will be useful in the proof of Theorem 1.4. Nevertheless, we start by the following general remark. Let x and y be two different points on ∂Z , and suppose that the stabilizer $\text{Stab}(\{x, y\})$ is infinite—it is then a two-ended quasiconvex subgroup of G . We claim that for fixed $\delta > 0$, there exists a finite set $\{g_1, \dots, g_N\} \subset G$ such that if $g \in G$ verifies $\theta(g \cdot x, g \cdot y) \geq \delta$, then it can be written as

$$g = g_i h, \text{ for some } i \in \{1, \dots, N\} \text{ and } h \in \text{Stab}(\{x, y\}). \quad (3.14)$$

Indeed, let γ be a geodesic asymptotic to x and y , and fix any point $w_0 \in \gamma$. By assumption, $\text{dist}(1, g \cdot \gamma) \leq K$ for some constant which depends only on ϵ , the hyperbolicity constant of Z and δ . Suppose $g \cdot w$ is the nearest point of $g \cdot \gamma$ to 1. Since the stabilizer of $\{x, y\}$ is quasiconvex, there exists a uniform constant C , and $h_w \in \text{Stab}(\{x, y\})$, such that $d(h_w \cdot w_0, w) \leq C$. This implies that

$$gh_w \cdot w_0 \in B_Z(1, K + C),$$

and therefore, that gh_w belongs to a finite subset of G .

We suppose now that G is a virtually free group. Let T be the simplicial tree associated to the DS splitting of G , so in particular edge and vertex stabilizers are all finite subgroups of G . We take as before a continuous equivariant map $\phi : Z \rightarrow T$, where Z is any locally finite Cayley graph of G . In this case, the map j defined in Proposition 3.6 provides a bijection between ∂Z and ∂T ; i.e. for every geodesic ray γ in Z , the image $\phi(\gamma)$ is unbounded in T .

Let $\mathcal{H} := \{H_1, \dots, H_k\}$ be a finite collection of two-ended subgroups of G . For each $i \in \{1, \dots, k\}$ the limit set Λ_i of H_i consists of exactly two points of ∂Z , which we denote by h_i^+ and h_i^- . Define the *decomposition space* associated to \mathcal{H} by $D := \partial Z / \sim$, where two points x and y of ∂Z are identified if and only if, either $x = y$, or there exists $g \in G$ such that $\{x, y\} = g \cdot \Lambda_i$. We denote by $p : \partial Z \rightarrow D$ the projection defined by \sim and we equip D with the quotient topology. Note that any point of D has at most two pre-images under

p. By (3.14), D is a compact Hausdorff topological space. In the sequel, we will always suppose that D is connected. We recall that a saturated set A in ∂Z is by definition a subset of ∂Z such that if $A \cap p^{-1}(y) \neq \emptyset$ for some $y \in D$, then $p^{-1}(y) \subset A$.

Lemma 3.15. *If e is an edge of T , then there exist $i \in \{1, \dots, k\}$ and $g \in G$ such that $g \cdot h_i^\pm \in \partial Z(e)$ and $g \cdot h_i^\mp \in -\partial Z(e)$.*

Proof. If not, the sets $\partial Z(e)$ and $-\partial Z(e)$ are open, closed and saturated sets in ∂Z , and therefore, the set $U := p(\partial Z(e))$ is a proper open and closed subset of D . \square

NOTATION. Let e be an edge of T . We set

$$\mathcal{H}_e := \{g \in G : \exists i \in \{1, \dots, k\} \text{ s.t. } g \cdot h_i^\pm \in \partial Z(e), g \cdot h_i^\mp \in -\partial Z(e)\}.$$

By Lemma 3.15, for any edge e of T the set \mathcal{H}_e is not empty. Let

$$H_e := \bigcup_{g \in \mathcal{H}_e} g \cdot \Lambda_i,$$

and let $P_e = p(H_e)$ be its projection onto D .

Lemma 3.16. *For any edge e of T , the set H_e is finite and $D \setminus P_e$ is disconnected.*

Proof. Fix e an edge of T . Let us first prove that T_e is finite. Indeed, if $g \in G$ and $i \in \{1, \dots, k\}$ are such that $g \cdot h_i^\pm \in \partial Z(e)$ and $g \cdot h_i^\mp \in -\partial Z(e)$, then

$$\theta(g \cdot h_i^\pm, g \cdot h_i^\mp) \geq \text{dist}(\partial Z(e), -\partial Z(e)) := c_e > 0,$$

where c_e only depends on e . By (3.14), there exists a finite subset G_e of G such that $g = g'h$, with $g' \in G_e$ and $h \in \text{Stab}(\Lambda_i)$. Therefore, $g \cdot \Lambda_i = g' \cdot \Lambda_i$, which implies that H_e is finite.

Let us show that P_e is a cut-set in D . Since H_e is finite, it is closed in ∂Z . Since $p(\partial Z(e)) \cap p(-\partial Z(e)) = P_e$, we see that $\partial Z(e) \setminus H_e$ and $-\partial Z(e) \setminus H_e$ are open and closed saturated sets in ∂Z . Taking the projection of these sets by p gives a non-trivial partition of $D \setminus P_e$. \square

We prove now the main lemma of this section.

Lemma 3.17. *Let $\delta > 0$. There exists a finite set E_δ of edges of T such that if we denote by $P_\delta := \bigcup_{e \in E_\delta} P_e$ and by $H_\delta := p^{-1}(P_\delta)$, then $\partial Z \setminus H_\delta$ can be partitioned into a finite union of open and closed saturated sets A with $\text{diam} A \leq \delta$. In particular, if U is a connected component of $D \setminus P_\delta$, then*

$$\text{diam} p^{-1}(U) \leq \delta.$$

Moreover, this implies that if for some $g \in G$ and $i \in \{1, \dots, k\}$ we have $g \cdot \Lambda_i \subset \partial Z \setminus H_\delta$, then $g \cdot \Lambda_i \subset A$ for some subset A of the partition.

Proof. Let $\eta > 0$ and let E_0 be the finite subset of edges of T such that $\text{diam} \partial Z(e) \geq \eta$, and let T_η be its convex hull in T , so that T_η is a finite subtree of T . Let E_η be the set of edges of T_η . For each vertex $v \in T_\eta$ we denote by $N_0(v) := N(v) \setminus E_\eta$. We can assume without loss of generality that $v_0 \in T_\eta$, so that for each vertex v of T_η , and each edge $e \in N_0(v)$, we have $\partial Z_v(e) = \partial Z(e)$, and in particular, that $\text{diam}(\partial Z_v(e)) \leq \eta$. We set

$$P_\eta := \bigcup_{e \in E_\eta} P_e.$$

If v_1 and v_2 are two different vertices of T_η , and if $e_j \in N_0(v_j)$, $j = 1, 2$, then

$$p(\partial_{v_1} Z(e_1)) \cap p(\partial_{v_2} Z(e_2)) \subset P_\eta.$$

Let $H_\eta := p^{-1}(P_\eta)$. For each vertex v of T_η , let A_v be the union of the sets $\partial Z_v(e) \setminus T_\eta$ with $e \in N_0(v)$. Then A_v is an open and closed (in $\partial Z \setminus H_\eta$) saturated set. This provides a partition of $\partial Z \setminus H_\eta$ by open and closed saturated sets. Therefore, if U is a connected component of $D \setminus P_\eta$, there exists a vertex v of T_η such that $U \subset p(A_v)$. Since A_v is saturated, this implies that $p^{-1}(U) \subset A_v$.

We end by estimating the diameter of the sets A_v . Indeed, by Corollary 3.8 and Lemma 3.12 we have

$$\begin{aligned} \text{diam} A_v &\leq 2 \max_{e \in N_0(v)} \{\text{diam} \partial Z_v(e)\} + \max_{e, e' \in N_0(v)} \text{dist}(\partial Z_v(e), \partial Z_v(e')) \\ &\lesssim \max_{e \in N_0(v)} \{\text{diam} \partial Z_v(e)\} \leq K\eta, \end{aligned}$$

where K is a uniform constant. So it is enough to take η small enough so that $K\eta \leq \delta$. \square

3.4. The JSJ splitting and local cut points. The goal of this section is to recall the basic properties of Bowditch’s JSJ splitting for hyperbolic groups. A non-elementary hyperbolic group —infinite and not virtually cyclic— G is said *virtually Fuchsian* if it acts properly discontinuously and by isometries on the real hyperbolic plane \mathbb{H}^2 . We say that G is cocompact virtually Fuchsian if the action is.

The action is not necessarily faithful, but its kernel is finite. By [CJ94, Gab92, Tu88], a non-elementary hyperbolic group is cocompact virtually Fuchsian if and only if its boundary is homeomorphic to the circle \mathbb{S}^1 . More generally, if G is a virtually Fuchsian group, we say that G is *convex cocompact* if the action is cocompact on its convex core, but not cocompact on \mathbb{H}^2 . Recall that its convex core is the minimal G -invariant closed convex subset of \mathbb{H}^2 . In this case, the *peripheral* subgroups of G are the stabilizers of the boundary connected components of its convex core.

We recall the following theorem due to Bowditch (see [Bow98a] Theorem 0.1 and Theorem 5.28).

Theorem 3.18 (JSJ splitting). *Let G be a one-ended hyperbolic group which is not a cocompact virtually Fuchsian group. Then there exists a minimal simplicial action of G on a simplicial tree T , without edge inversions and with finite quotient T/G , whose edge stabilizers are virtually cyclic. The vertices of T are of three types:*

- *Virtually cyclic: the stabilizer is a maximal virtually cyclic subgroup of G . Its valence in T is finite and at least two.*
- *Surface or MHF: the stabilizer is a quasiconvex, non-elementary convex cocompact virtually Fuchsian subgroup of G . Its peripheral groups are precisely the stabilizers of its incident edges.*
- *Rigid: the stabilizer is quasiconvex, non-elementary and not of Surface type.*

These types are mutually exclusive and are preserved by the action of G . Two adjacent vertices are never of the same type. The splitting is quasi-isometry invariant and it is maximal in the following sense: the rigid type vertices admit no decomposition over virtually cyclic subgroups relative to the stabilizers of its incident edges.

To construct the simplicial tree T , Bowditch uses the structure of local cut points of ∂G . Indeed, any local cut point of ∂G belongs to the limit set Λ_v of a vertex $v \in T$ of virtually cyclic or surface type. The limit set Λ_e of $e \in T_1$ consists of exactly two points which are fixed points of any element of infinite order in $G(e)$.

Recall that ∂Z is a locally connected metric space with no global cut points. Since $-\partial Z(e) \cap \partial Z(e) = \Lambda_e$, the limit set Λ_e separates the boundary of Z , and the two points of

Λ_e are local cut points. There are only a finite number of connected components of $\partial Z \setminus \Lambda_e$, and every component has Λ_e as frontier. In particular, the sets $\pm \partial Z(e)$ are connected.

We partition the vertices of the JSJ tree T as $T_C \cup T_S \cup T_R$, which are respectively the vertices of virtually cyclic, surface and rigid type. The vertices of surface type T_S are virtually free (with peripheral subgroups) and the limit set Λ_v a Cantor set of local cut points. The stabilizer of Λ_v , for the action of G on ∂Z , is the same as the stabilizer of v for the action of G on T , i.e. $G(v)$. The subgroup $G(v)$ of G is said maximal hanging Fuchsian, or MHF for short. There is a natural bijection between the edges of T incident to v , and the peripheral subgroups of $G(v)$. The limit set of the peripheral subgroups are the "jumps" of the Cantor set Λ_v ; denote by J_v the set of jumps. For $\tau = \{x, y\} \in J_v$, denote by \mathcal{C}_τ the set of connected components of $\partial Z \setminus \tau$. Then the set of connected components \mathcal{C}_v of $\partial Z \setminus \Lambda_v$ is given by

$$\mathcal{C}_v = \bigcup_{\tau \in J_v} \mathcal{C}_\tau.$$

This means that in order to study the WS property (or equivalently, the fibers) it is enough to consider the local cut points which are in the limit sets of the edges of T :

$$\mathcal{E} := \bigcup_{e \in T_1} \Lambda_e.$$

The vertices of rigid type T_R correspond to non-elementary subgroups (and not T_S) and are characterized by the following: the limit set cannot be separated by two points of ∂Z , and Λ_v is maximal for this property.

3.5. Proof of Theorem 1.4.

3.5.1. *All rigid vertices are virtually free implies WS.* Assume now that in the JSJ splitting of G all rigid type vertices are virtually free. In this case, all vertices of T are either non-elementary virtually free or virtually cyclic. Let v be a vertex in $T_S \cup T_R$, so $G(v)$ is non-elementary virtually free. Recall that $N(v)$ is the set of edges of T which are incident to v . Since the action of G on T preserves the type of a vertex and two adjacent vertices are of different type, the stabilizer of $N(v)$ is also $G(v)$. This implies that the number of G -orbits of edges incident to v is the same as the number of $G(v)$ -orbits, and therefore $N(v)/G(v)$ is finite. Let $\{f_1, \dots, f_{k_v}\}$ be a finite set of representatives of $N(v)/G(v)$, and re-denote by $H_j(v)$ the two-ended subgroup $G(f_j)$ of $G(v)$. We consider the decomposition space D_v associated to v and we denote by $p_v : \Lambda_v \rightarrow D_v$ the quotient map like in Section 3.3. Another equivalent way to construct D_v is by identifying the points of ∂Z which are in the same $\partial Z(e)$ for $e \in N(v)$. Since ∂Z is connected, the same is true for D_v . Moreover, the space D_v is either homeomorphic to the circle if $v \in T_S$, or it is locally connected without cut-points and cut-pairs if $v \in T_R$ [CaMa11, Haïss12]. Therefore, we can apply the results of Section 3.3.

The subsets used in Section 3.3 to separate the decomposition space D_v were constructed by projecting a finite union of sets of the form $g \cdot \Lambda_{f_j}$ with $g \in G(v)$ and $j \in \{1, \dots, k_v\}$. Therefore, in this case, they are the projection of a finite union of Λ_e with $e \in N(v)$. We can state Lemma 3.17 in the following way: for all $\delta > 0$, there exists a finite set of edges $\{e_1, \dots, e_n\}$ (here $n = n(v, \delta)$) of $N(v)$ such that if we set $H_\delta(v) := \bigcup_{i=1}^n \Lambda_{e_i}$, then

$$\Lambda_v \setminus H_\delta(v) = \bigcup_{i=1}^{n+1} A_i,$$

where the sets A_i are disjoint open and closed (in $\Lambda_v \setminus H_\delta(v)$) saturated sets ($n+1$ is the number of vertices of the finite tree associated to the edges, see the proof of Lemma 3.17). Moreover, for any other edge $e \in N(v)$, there exists a $i \in \{1, \dots, n+1\}$ such that $\Lambda_e \subset A_i$. Given such a partition, we denote by $N_i(v)$ for $i = 1, \dots, n+1$ the edges e incident to v such that $\Lambda_e \subset A_i$. Naturally,

$$N(v) = \{e_1, \dots, e_n\} \cup \bigcup_{i=1}^{n+1} N_i(v),$$

is a partition of all edges incident to v . Let

$$K_i(v) = A_i \cup \bigcup_{e \in N_i(v)} \partial Z_v(e) \text{ for } i = 1, \dots, n+1.$$

Lemma 3.19. *The sets $\{K_i(v)\}_{i=1}^n$ are pairwise disjoint and closed in $\partial Z \setminus H_\delta(v)$.*

Proof. They are pairwise disjoint: the A_i are pairwise disjoint, and by definition, the intersection of two sets $\partial Z_v(e)$ and $\partial Z_v(e')$, with $e \neq e'$ in $N(v)$, is contained in $\Lambda_e \cap \Lambda_{e'}$. But if $e \in N_i(v)$ and $e' \in N_j(v)$ with $j \neq i$, then $\Lambda_e \cap \Lambda_{e'} = \emptyset$.

They are closed: we adapt an argument taken from [Bow98b] (see Lemma 4.6). If $x \notin K_i(v) \cup H_\delta(v)$, then $\text{dist}(x, A_i) = \eta > 0$ ($x \notin A_i$ which is closed in $\partial Z \setminus H_\delta(v)$). If $e \in N_i(v)$ is such that $\partial Z_v(e) \cap B(x, \eta/2) \neq \emptyset$, then since $\partial Z_v(e) \cap A_i \neq \emptyset$, we obtain $\text{diam } \partial Z_v(e) \geq \eta/2$. So, there is only a finite number of edges $e \in N_i(v)$ such that $\partial Z_v(e) \cap B(x, \delta/2) \neq \emptyset$. Since each $\partial Z_v(e)$ is closed, we have $\text{dist}(x, K_i(v)) > 0$, so $K_i(v)$ is closed in $\partial Z \setminus H_\delta(v)$. \square

We can partition the boundary of G as

$$\partial Z = \bigcup_{i=1}^n (\partial Z_v(e_i) \setminus H_\delta(v)) \cup \bigcup_{i=1}^{n+1} K_i(v) \cup H_\delta(v), \quad (3.15)$$

and the sets of the first and second union are closed in $\partial Z \setminus H_\delta(v)$. Furthermore, we have

$$\text{diam } K_i(v) \leq \text{diam } A_i + 2 \cdot \max_{e \in N_i(v)} \{\text{diam } \partial Z_v(e)\}, \quad (3.16)$$

so we can prove the following lemma.

Lemma 3.20. *Let $v \in T_S \cup T_R$ be a vertex of T . Then for any $\delta > 0$, there exists a finite set of edges $e_1, \dots, e_n \in N(v)$ such that $\text{diam } K_i(v) \leq \delta$ for all $i = 1, \dots, n+1$.*

Proof. Let $\delta > 0$. There is a finite subset $N_1 \subset N(v)$, such that if $e \in N(v) \setminus N_1$, then $\text{diam } \partial Z_v(e) \leq \delta/4$. Also by the previous paragraph, there exists a finite subset $N_2 = \{e_1, \dots, e_n\} \subset N(v)$, which we can assume containing N_1 , such that $\text{diam } A_i \leq \delta/2$. Therefore, from (3.16), we have $\text{diam } K_i(v) \leq \text{diam } A_i + 2 \cdot \max \{\text{diam } \partial Z_v(e)\} \leq \delta/2 + 2 \cdot \delta/4 = \delta$. \square

Proposition 3.21. *Let G be a one-ended hyperbolic group which is not a cocompact virtually Fuchsian group. Suppose that in the JSJ splitting of G all rigid type vertices are virtually free. Then for any $\delta > 0$, there exists a finite set $\{e_1, \dots, e_N\}$ of edges of H , such that*

$$\max \{\text{diam } U : U \in \mathcal{U}\} \leq \delta,$$

where \mathcal{U} is the set of connected components of $\partial Z \setminus H$, and $H = \bigcup_{i=1}^N \Lambda_{e_i}$. In particular, ∂Z satisfies the WS property.

Proof. Let $\delta > 0$. We fix v_0 any vertex in $T_S \cup T_R$. Consider the set E_δ of edges $e \in T_1$ such that $\text{diam } \partial Z(e) \geq \delta$. By Lemma 3.4, E_δ is a finite set.

Let V_δ be the union of $\{v_0\}$ and the vertices of the edges in E_δ . Let T_δ be the convex hull of V_δ . Then T_δ is a finite subtree of T .

If $w \in (T_\delta)_1 \cap T_C$, since the valence of w in T is finite, we can add to T_δ all the edges incident to w , so that T_δ is still finite. In this way, we can suppose that all the terminal vertices of T_δ are in $T_S \cup T_R$.

Let v be any vertex in $(T_S \cup T_R) \cap (T_\delta)_0$. There is exactly one edge e in $N(v)$ such that $T_v(e) = -T(e)$. It necessarily belongs to $(T_\delta)_1$ because $v_0 \in T_\delta$. For the others, we have $T_v(e) = T(e)$. By Lemma 3.20, there are edges $E(v) := \{e_1(v), \dots, e_{n_v}(v)\} \in N(v)$, such that

$$\text{diam } K_i(v) \leq \delta, \text{ for all } i = 1, \dots, n_v + 1. \quad (3.17)$$

Denote by H_v the union of $\Lambda_{e_i(v)}$ for $i = 1, \dots, n_v$. Therefore, $\partial Z \setminus H_v$ is partitioned as in (3.15). We can assume that the edges of T_δ incident to v belong to $\{e_1(v), \dots, e_{n_v}(v)\}$.

Take now $e = (v, w) \in (T_\delta)_1$ an edge such that $d(w, v_0) \geq d(v, v_0)$, and that $w \in (T_\delta)_0 \cap T_C$ (thus, $v \in T_S \cup T_R$). If we let $\{e_j = (v_j, w) : j = 1, \dots, m_w\} := N(w) \setminus \{e\}$, then

$$\partial Z(e) = \bigcup_{j=1}^{m_w} \partial Z(e_j) = \bigcup_{j=1}^{m_w} \partial Z_w(e_j). \quad (3.18)$$

The intersection of two sets of this decomposition is equal to Λ_w (which is equal to Λ_{e_j} , for $j = 1, \dots, m_w$, and also to Λ_e). This allows us to replace $\partial Z(e)$ by the union of $\partial Z(e_j)$, where the v_j all belong to $(T_S \cup T_R) \cap (T_\delta)_0$. Note that all the vertices which are an endpoint of an edge incident to v are in T_C . For $i = 1, \dots, n_v$, denote by w_i the other endpoint of $e_i(v)$, and by f_j for $j = 1, \dots, m_{w_i}$ the other edges incident to w_i . Then, we can decompose $\partial Z \setminus H_v$ as

$$\partial Z = \bigcup_{i=1}^{n_v} \bigcup_{j=1}^{m_{w_i}} (\partial Z_w(f_j) \setminus H_v) \cup \bigcup_{i=1}^{n_v+1} K_i(v) \cup H_v, \quad (3.19)$$

The argument can be repeated inductively until we reach all the terminal vertices \mathcal{T} of T_δ . For $u \in \mathcal{T}$, there is exactly one edge e_u of $N(u)$ which is in $(T_\delta)_1$ (i.e. that which verifies $T_u(e) = -T(e)$). Then $N(u) \setminus \{e_u\} \subset T_1 \setminus (T_\delta)_1$, and therefore, for all $u \in \mathcal{T}$ and any $e \in E(u) \setminus \{e_u\}$, we have

$$\text{diam } \partial Z_u(e) = \text{diam } \partial Z(e) \leq \delta. \quad (3.20)$$

Then the set $\partial Z(e_u)$ can be decomposed as

$$\partial Z(e_u) = \bigcup_{e \in N(u) \setminus \{e_u\}} (\partial Z_u(e) \setminus H_u) \cup \bigcup_{i=1}^{n_u+1} K_i(u) \cup H_u, \quad (3.21)$$

and all the sets in the first and second union are of diameter less than or equal to δ . We obtain then a decomposition $\partial Z = A \cup B \cup H$, where

$$A := \bigcup_{v \in (T_S \cup T_R) \cap (T_\delta)_1} \bigcup_{i=1}^{n_v+1} K_i(v), \quad B := \bigcup_{u \in \mathcal{T}} \bigcup_{e \in N(u) \setminus \{e_u\}} (\partial Z_u(e) \setminus H_u), \quad (3.22)$$

and

$$H := \bigcup_{v \in (T_S \cup T_R) \cap (T_\delta)_1} H_v \quad (3.23)$$

The sets in the decomposition (3.22) defining A and B are pairwise disjoint and closed in $\partial Z \setminus H$, and of diameter less than or equal to δ . This implies that any connected component of $\partial Z \setminus H$ is contained in one of these sets. This ends the proof. \square

3.5.2. *WS is not satisfied when a non-virtually free rigid type vertex exists.* Suppose there exists $v \in T_R$ such that $G(v)$ is not virtually free. First, decompose $G(v)$ over finite groups and consider a vertex w of this decomposition such that its stabilizer in $G(v)$, $G(w)$, is one-ended. We will show that Λ_w cannot be separated by a finite set of local cut points of ∂Z . Note that Λ_w can be separated (with a finite number of local cut points) from the other connected components of Λ_v : the proof follows the same arguments as in 3.5.1. In particular, points on different connected components of Λ_w are not in the same fiber.

Suppose first that Λ_w is not homeomorphic to the circle. Consider the JSJ decomposition of $G(w)$ with associated tree T_w . If it is trivial, we are done because there are no local cut points in Λ_w . If not, it is enough to consider the local cut points of Λ_w which are in the limit set of an edge e of T_w . Denote by C_i , $i = 1, \dots, n$, the connected components of $\Lambda_w \setminus \Lambda_e$. Since v is a rigid vertex of G , the set $\Lambda_w \setminus \Lambda_e$ is contained in a connected set C of ∂Z . If $x \in C_i$ and $y \in C_j$, with $i \neq j$, then there is a curve in ∂Z avoiding Λ_e and connecting x and y . Therefore, we can suppose that $C = \bigcup_{i=1}^m C_i \cup \alpha$, where α is a curve in $\partial Z \setminus \Lambda_e$ intersecting all the C_i 's. Note that $\Lambda_e = \{x^+, x^-\}$ is the set of fixed points of a loxodromic g , so given any $\delta > 0$, there exists $n \geq 1$ such that $\alpha_n := g^n \cdot \alpha$ and $\alpha_{-n} := g^{-n} \cdot \alpha$ are respectively contained in the balls $B(x^+, \delta)$ and $B(x^-, \delta)$. Note that $\alpha_{\pm n}$ intersects all the C_i 's. Therefore, given any two points of $\Lambda_w \cap B(x^\pm, \delta)$, there exists a curve in the ball connecting them and avoiding x^\pm .

Let $\{e_1, \dots, e_m\}$ be any finite subset of edges of T_w . Consider a $\delta > 0$ small enough so that the balls centered at $P := \bigcup_{i=1}^m \Lambda_{e_i}$ and of radius δ are disjoint. Let $\Lambda_{e_i} := \{x_i^+, x_i^-\}$ and for each i consider the curves α_i^+ and α_i^- contained in the i 's balls as before. Then the set

$$(\Lambda_w \setminus P) \cup \bigcup_{i=1}^m (\alpha_i^+ \cup \alpha_i^-)$$

is a connected set containing $\Lambda_w \setminus P$. Since $\text{diam}(\Lambda_w \setminus P) = \text{diam} \Lambda_w$, ∂Z does not satisfies the WS condition.

If Λ_w is homeomorphic to the circle, the same argument as before works. In fact, the fixed points $\text{Fix}(g)$ of any loxodromic $g \in G(w)$ is a cut-pair of Λ_w .

3.6. **A simple example with $\dim_{AR} \partial\pi_1(M) < D(M)$.** In this section we give a simple example of a one-ended convex cocompact Kleinian group $G = \pi_1(M)$ for which the conclusion of Corollary 1.6 holds; namely, that $\dim_{AR} \partial\pi_1(M) < D(M)$ where M is the hyperbolizable 3-manifold with boundary whose interior is isometric to \mathbb{H}^3/G .

Let H be a handlebody with boundary S a closed surface of genus $g \geq 2$, and let $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ be a multicurve on S ; i.e. the elements of Γ are disjoint simple closed curves on S . Suppose that Γ is chosen so that the boundary of any properly embedded disk or annulus in H intersects at least one of the curves in Γ . Consider a small smooth neighborhood A of Γ in S so that $A = \{A_1, \dots, A_n\}$ is a collection of disjoint annuli in S .

Let T_i for $i = 1, \dots, n$ be n copies of the solid torus $D^2 \times S_1$. On the boundary S_i of T_i let $A_{T,i}$ be a smooth annulus whose core curve generates the $\pi_1(T_i)$. We glue together each solid torus T_i to H identifying A_i with $A_{T,i}$. Denote by R the 3-manifold obtained.

For each i , let B_i be a collection of $m_i \geq 1$ disjoint smooth annuli in S_i for which each core curve generates $\pi_1(T_i)$. Suppose that B_i is also disjoint from $A_{T,i}$. Denote by $m = m_1 + \dots + m_n$. Let F be an I-bundle (over a surface which is not necessarily

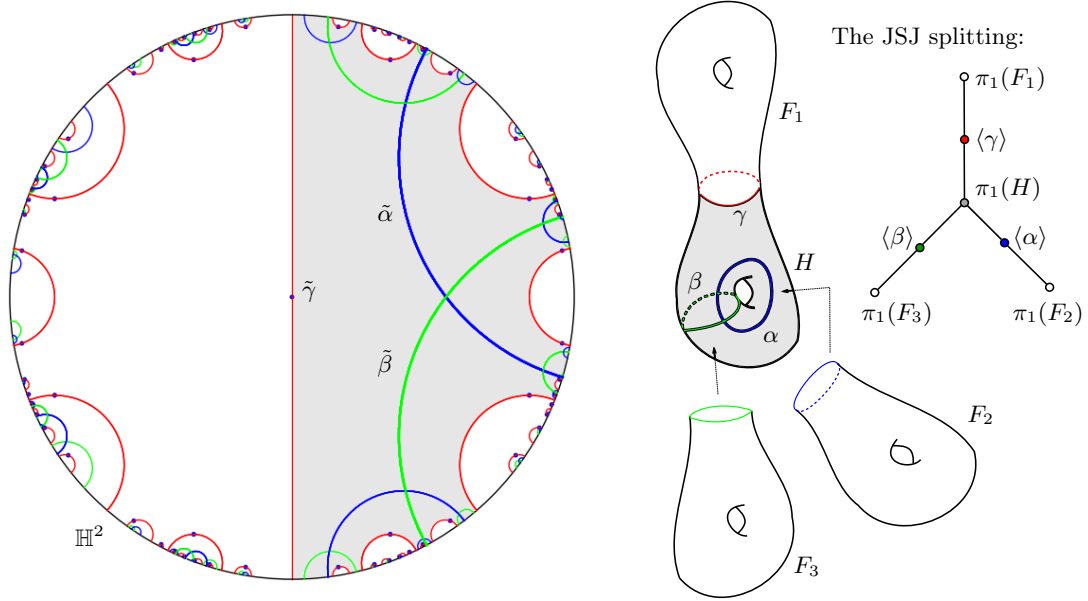


FIGURE 3.4. This figure shows an example of a convex cocompact Kleinian group $\pi_1(M)$ for which $D(M) > 1 = \dim_{AR} \partial\pi_1(M)$. Here H is a surface homeomorphic to the torus minus an open disk (with boundary γ). The curves α and β generate $\pi_1(H)$, and we have $\gamma = [a, b] = aba^{-1}b^{-1}$. The lifts of these curves to the hyperbolic plane are drawn on the left. The group is the fundamental group of the complex obtained by gluing to H three copies, F_i , $i = 1, 2, 3$, of H along the multicurve $\{\alpha, \beta, \gamma\}$. All the vertices of the JSJ decomposition of $\pi_1(M)$ are virtually free, and the conjugates of $\pi_1(H)$ are rigid vertices. This also implies that M is not a generalized book of I -bundles.

connected) with vertical boundary consisting of exactly m smooth annuli. Glue F to R along the annuli B_i , $i = 1, \dots, n$, and call M the 3-manifold with boundary obtained. Let $G = \pi_1(M)$.

By Thurston hyperbolization theorem, G is isomorphic to a discrete, one-ended convex cocompact subgroup of $\text{Iso}(\mathbb{H}^3)$. See [KK00] for a similar construction and for more details. The JSJ decomposition of G has exactly one conjugacy class of rigid vertices which all free groups; i.e. the conjugates of $\pi_1(H)$. Thus we are in the hypotheses of Theorem 1.4 and $\dim_{AR} \partial G = 1$. On the other hand, M is not a generalized book of I -bundles, and therefore, by Theorem 2.9 in [CMT99] we have $D(M) > 1$. See also Figure 3.4.

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E-mail address: `matias@cmi.univ-mrs.fr`

LABORATOIRE D'ANALYSE, TOPOLOGIE ET PROBABILITÉS, AIX-MARSEILLE UNIVERSITÉ.