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# Improving Monte Carlo simulations by Dirichlet forms

Nicolas Bouleau

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## Abstract

Equipping the probability space with a local Dirichlet form with square field operator  $\Gamma$  and generator  $A$  allows to improve Monte Carlo simulations of expectations and densities as soon as we are able to simulate a random variable  $X$  together with  $\Gamma[X]$  and  $A[X]$ . We give examples on the Wiener space, on the Poisson space and on the Monte Carlo space. When  $X$  is real-valued we give an explicit formula yielding the density at the speed of the law of large numbers.

## Résumé

Nous montrons que, dans les situations où l'espace de probabilité est équipé d'une forme de Dirichlet locale avec carré du champ  $\Gamma$  et générateur  $A$ , la possibilité de simuler une variable aléatoire  $X$  ainsi que  $\Gamma[X]$  et  $A[X]$  permet d'accélérer le calcul de l'espérance de  $X$  et de sa densité. Nous donnons des exemples dans les cas de l'espace de Wiener, de l'espace de Poisson et de l'espace de Monte Carlo. Lorsque  $X$  est à valeurs réelles nous donnons une formule explicite permettant d'obtenir la densité à la vitesse de la loi des grands nombres. *Pour citer cet article : N. Bouleau, C. R. Acad. Sci. Paris, Ser. I ... (2005).*

## 1 Introduction

The efficiency of Dirichlet forms is known in order to obtain existence of densities under weak hypotheses (cf [3]). We show here that they are still useful for the computation of such densities. Our framework is an error structure  $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$ , i.e. a probability space equipped with a local Dirichlet form  $(\mathcal{E}, \mathbb{D})$  admitting a square field operator  $\Gamma$  (cf [2],[3]). The associated  $L^2$ -generator is denoted  $(A, \mathcal{D}A)$ .

We consider a random variable  $X \in \mathcal{D}A$  such that  $X$ ,  $\Gamma[X]$  and  $A[X]$  are simulatable.

Example 1. Wiener space.

Let us consider a stochastic differential equation (sde) defined on the Wiener space equipped with the Ornstein-Uhlenbeck error structure (cf [2],[3])

$$X_t = x_0 + \int_0^t \sigma(X_s, s) dB_s + \int_0^t r(X_s, s) ds$$

By the functional calculus for the operators  $\Gamma$  and  $A$ , if the coefficients are smooth, the

triplet  $(X_t, \Gamma[X_t], A[X_t])$  is a diffusion, solution to the equation

$$\begin{aligned} \begin{pmatrix} X_t \\ \Gamma[X_t] \\ A[X_t] \end{pmatrix} &= \begin{pmatrix} x_0 \\ 0 \\ 0 \end{pmatrix} + \int_0^t \begin{bmatrix} \sigma(X_s, s) & 0 & 0 \\ 0 & 2\sigma'_x(X_s, s) & 0 \\ -\frac{1}{2}\sigma(X_s, s) & \frac{1}{2}\sigma''_{x^2}(X_s, s) & \sigma'_x(X_s, s) \end{bmatrix} \begin{pmatrix} 1 \\ \Gamma[X_s] \\ A[X_s] \end{pmatrix} dB_s \\ &+ \int_0^t \begin{bmatrix} r(X_s, s) & 0 & 0 \\ \sigma^2(X_s, s) & 2r'_x(X_s, s) + \sigma_x'^2(X_s, s) & 0 \\ 0 & \frac{1}{2}r''_{x^2}(X_s, s) & r'_x(X_s, s) \end{bmatrix} \begin{pmatrix} 1 \\ \Gamma[X_s] \\ A[X_s] \end{pmatrix} ds \end{aligned}$$

Denoting  $Y_t$  the column vector  $(X_t, \Gamma[X_t], A[X_t])$  this equation writes  $Y_t = Y_0 + \int_0^t a(Y_s, s)dB_s + \int_0^t b(Y_s, s)ds$  and applying the Euler scheme with mesh  $\frac{1}{n}$  on  $[0, T] : Y_t^n = Y_0 + \int_0^t a(Y_{\frac{[ns]}{n}}^n, \frac{[ns]}{n})dB_s + \int_0^t b(Y_{\frac{[ns]}{n}}^n, \frac{[ns]}{n})ds$ . yields a process  $Y_t^n = (X_t^n, (\Gamma[X])_t^n, (A[X])_t^n)^t$  for which it is easy to verify that  $\Gamma[X_t^n] = (\Gamma[X])_t^n$  and  $A[X_t^n] = (A[X])_t^n$ .

By known results (cf [1] [4] [5]) in order to compute the density of  $X_T$ , we may approximate it by the solution  $X_T^n$  of the Euler scheme. Thus, we have then to simulate  $X_T^n$  in a situation where we are also able to simulate  $\Gamma[X_T^n]$  and  $A[X_T^n]$ .

Example 2. Poisson space.

Let  $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu, \mathbf{d}, \gamma)$  be an error structure on  $\mathbb{R}^d$ ,  $(a, \mathcal{D}a)$  its generator. Let  $N$  be a Poisson point process defined on  $(\Omega, \mathcal{A}, \mathbb{P})$  with state space  $\mathbb{R}^d$  and intensity measure  $\mu$ .  $(\Omega, \mathcal{A}, \mathbb{P})$  may be equipped with a so-called "white" error structure  $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$  (cf [2]) with the following properties : if  $h \in \mathcal{D}a$  then  $N(h) \in \mathcal{D}A$ ,  $\Gamma[N(h)] = N(\gamma[h])$  and  $A[N(h)] = N(a[h])$ .

In order to simulate  $N(\xi)$  we have only to draw a finite (poissonian) number of i.i.d. random variables with law  $\mu$  so that we are indeed in a situation where  $N(h)$ ,  $\Gamma[N(h)]$ , and  $A[N(h)]$  are simulatable.

Example 3. Monte Carlo space.

Let  $X = F(U_0, U_1, \dots, U_m, \dots; V_0, V_1, \dots, V_n, \dots)$  be a random variable defined on the space  $([0, 1]^{\mathbb{N}}, \mathcal{B}([0, 1]^{\mathbb{N}}), dx^{\mathbb{N}}) \times ([0, 1]^{\mathbb{N}}, \mathcal{B}([0, 1]^{\mathbb{N}}), dx^{\mathbb{N}})$  where the  $U_i$  are the coordinates of the first factor with respect to which  $X$  is supposed to be regular,  $V_j$  the ones of the second factor with respect to which  $X$  is supposed to be irregular or discontinuous (rejection method, etc.).

Let us put on the  $U_i$  the following error structure

$$([0, 1]^{\mathbb{N}}, \mathcal{B}([0, 1]^{\mathbb{N}}), dx^{\mathbb{N}}, \mathbb{D}, \Gamma) = ([0, 1], \mathcal{B}([0, 1]), dx, \mathbf{d}, \gamma)^{\mathbb{N}}$$

where  $(\mathbf{d}, \gamma)$  is the closure of the operator  $\gamma[u](x) = x^2(1-x)^2u'^2(x)$  for  $u \in \mathcal{C}^1([0, 1])$ .

Then under natural regularity assumptions, we have  $\Gamma[X] = \sum_{i=0}^{\infty} F_i'^2 U_i^2 (1-U_i)^2$ . and

$$A[X] = \sum_{i=0}^{\infty} \left( \frac{1}{2} F_{ii}'' U_i^2 (1-U_i)^2 + F_i' U_i (1-U_i) (1-2U_i) \right)$$

so that  $X$ ,  $\Gamma[X]$  and  $A[X]$  are simulatable.

## 2 Diminishing the bias

Let  $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$  be an error structure. For  $X \in (\mathcal{DA})^d$ ,  $\underline{\text{var}}[X]$  denotes the covariance matrix of  $X$ ,  $A[X]$  the column vector with components  $(A[X_1], \dots, A[X_d])$ ,  $\underline{\Gamma}[X]$  is the matrix  $\Gamma[X_i, X_j]$  and  $\sqrt{\underline{\Gamma}[X]}$  denotes the positive symmetric square root of  $\underline{\Gamma}[X]$ .

We follow the idea that the random variable  $X + \varepsilon A[X] + \sqrt{\varepsilon} \sqrt{\underline{\Gamma}[X]} G$  where  $G$  is an exogeneous independent reduced Gaussian variable, has almost the same law as  $X$ . Starting from the fundamental relation of the functional calculus on  $A$ , an integration by parts argument gives the following lemma.

**Lemma 2.1** *Let  $X \in (\mathcal{DA})^d$ . we suppose that  $X$  possesses a conditional density  $\eta(x, \gamma, a)$  given  $\underline{\Gamma}[X] = \gamma$  et  $A[X] = a$  such that  $x \mapsto \eta(x, \gamma, a)$  be  $\mathcal{C}^2$  with bounded derivatives. Then  $\forall x \in \mathbb{R}^d$*

$$\mathbb{E}[-(A[X])^t \nabla_x \eta(x, \underline{\Gamma}[X], A[X]) + \frac{1}{2} \text{trace}(\underline{\Gamma}[X].\text{Hess}_x \eta)(x, \underline{\Gamma}[X], A[X])] = 0.$$

**Theorem 2.2** *Let  $X$  be as in the preceding lemma, the conditional density  $\eta(x, \gamma, a)$  being  $\mathcal{C}^3$  bounded with bounded derivatives. When  $\varepsilon \rightarrow 0$ , the quantity*

$$\frac{1}{\varepsilon^2} (\mathbb{E}[g(x - X - \varepsilon A[X], \varepsilon \underline{\Gamma}[X])] - f(x))$$

has a finite limit equal to

$$\frac{1}{2} \mathbb{E}[(A[X])^t (\text{Hess}_x \eta)(x, \underline{\Gamma}[X], A[X]) A[X] - \sum_{i,j,k=1}^d A[X_i] \Gamma[X_j, X_k] \eta'''_{x_i x_j x_k}(x, \underline{\Gamma}[X], A[X])].$$

*Proof.* If we write  $\mathbb{E}[g(x - X - \varepsilon A[X], \varepsilon \underline{\Gamma}[X])] = \int \mu(d\gamma, da) \int g(x - y - \varepsilon a, \varepsilon \gamma) \eta(y, \gamma, a) dy = \int \mu(d\gamma, da) \mathbb{E} \eta(x - \varepsilon a - \sqrt{\varepsilon} \sqrt{\gamma} G, \gamma, a)$  where  $G$  is an  $\mathbb{R}^d$ -valued reduced Gaussian variable, and if we expand with respect to  $\sqrt{\varepsilon}$  and take the expectation, terms in  $\sqrt{\varepsilon}$  and  $\varepsilon \sqrt{\varepsilon}$  vanish because  $G$  and  $G^3$  are centered and the term in  $\varepsilon$  vanishes also thanks to the lemma. This gives the result.

About the variance, we obtain

**Proposition 2.3** *Let  $X$  satisfying the assumptions of the lemma and such that  $(\det \underline{\Gamma}[X])^{-\frac{1}{2}} \in L^1$ , then*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{d/2} \mathbb{E} g^2(x - X - \varepsilon A[X], \varepsilon \underline{\Gamma}[X]) = \lim_{\varepsilon \rightarrow 0} \varepsilon^{d/2} \text{var} g(x - X - \varepsilon A[X], \varepsilon \underline{\Gamma}[X]) = \mathbb{E} \left[ \frac{\eta(x, \underline{\Gamma}[X], A[X])}{(4\pi)^{d/2} \sqrt{\det \underline{\Gamma}[X]}} \right].$$

The quantity  $\mathbb{E} g(x - X - \varepsilon A[X], \varepsilon \underline{\Gamma}[X])$  is obtained by simulation with the law of large numbers, so that the approximation  $\hat{f}$  of the density  $f$  of  $X$  is

$$\hat{f}(x) = \frac{1}{N} \sum_{n=1}^N g(x - X_n - \varepsilon(A[X])_n, \varepsilon(\underline{\Gamma}[X])_n)$$

where the indices  $n$  denote independent drawings. The preceding results show that, with respect to the usual kernel method, the speed, in the sense of the  $L^2$ -norm, is the same as if the dimension was divided by 2.

### 3 Direct formulae

In the case where  $X$  is real-valued, if in addition to  $X, A[X], \Gamma[X]$  we are able to simulate  $\Gamma[X, \frac{1}{X}]$ , it is possible to obtain the density of  $X$  at the speed of the law of large numbers thanks to the following formulae :

**Theorem 3.1** *a) If  $X \in \mathcal{DA}$  with  $\Gamma[X] \in \mathbb{ID}$  and  $\Gamma[X] > 0$  a.s. then  $X$  has a density  $f$  which possesses an l.s.c. version  $\tilde{f}$  given by*

$$\tilde{f}(x) = \lim_{\varepsilon \downarrow 0} \uparrow \frac{1}{2} \mathbb{E} \left( \text{sign}(x - X) \left( \Gamma[X, \frac{1}{\varepsilon + \Gamma[X]}] + \frac{2A[X]}{\varepsilon + \Gamma[X]} \right) \right).$$

*b) If in addition  $\frac{1}{\Gamma[X]} \in \mathbb{ID}$ , then  $X$  has a density  $f$  which is absolutely continuous and given by*

$$f(x) = \frac{1}{2} \mathbb{E} \left( \text{sign}(x - X) \left( \Gamma[X, \frac{1}{\Gamma[X]}] + \frac{2A[X]}{\Gamma[X]} \right) \right).$$

The proof is based on the relation

$$\mathbb{E}[\varphi''(X) \frac{\Gamma[X]}{\varepsilon + \Gamma[X]}] = -\mathbb{E}[\varphi'(X) \left( \Gamma[X, \frac{1}{\varepsilon + \Gamma[X]}] + \frac{2A[X]}{\varepsilon + \Gamma[X]} \right)].$$

valid for any  $\mathcal{C}^2$ -function  $\varphi$  with bounded derivatives which comes from the functional calculus using the general relation  $\mathcal{E}[u, v] = -\langle A[u], v \rangle \forall u \in \mathcal{DA} \forall v \in \mathbb{ID}$ , and then applying it with  $\varphi = \sqrt{\lambda^2 + (y - x)^2}$  in order to get the monotone convergence result.

Under the hypotheses of theorem 3.1, as soon as  $G \in \mathbb{ID} \cap L^\infty$ , there are similar formulae for conditional expectations  $\mathbb{E}[G|X = x]$  :

$$f(x) \mathbb{E}[G|X = x] = \frac{1}{2} \mathbb{E} \left( \text{sign}(x - X) \left( \Gamma[X, \frac{G}{\Gamma[X]}] + \frac{2GA[X]}{\Gamma[X]} \right) \right).$$

Let us finally remark that in these formulae, the factor on the right of  $\text{sign}(x - X)$  is centered and a variance optimisation may be performed thanks to an arbitrary deterministic function as done in [4] where direct formulae similar to those of section 3 are given in the case of the Wiener space involving Skorokhod integrals instead of Dirichlet forms.

### References

- [1] BALLY V., TALAY D. "The law of the Euler scheme for stochastic differential equations : II. Convergence rate of the density", *Monte Carlo Methods and Appl.* vol 104, No1, 43-80 (1996)
- [2] BOULEAU N. *Error Calculus for Finance and Physics, the Language of Dirichlet Forms*, De Gruyter, 2003.
- [3] BOULEAU N., HIRSCH F. *Dirichlet Forms and Analysis on Wiener Space*, De Gruyter, 1991.

- [4] KOHATSU-HIGA A., PETERSSON R. "Variance reduction methods for simulation of densities on Wiener space", *SIAM J. Numer. Anal.* Vol 40, No2, 431-450, (2002)
- [5] MALLIAVIN P., THALMAIER A. "Numerical error for SDE: Asymptotic expansion and hyperdistributions", *C. R. Acad. Sci. Paris ser. I* 336 (2003) 851-856