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# ON DISCRETE SPECTRUM OF COMPLEX PERTURBATIONS OF FINITE BAND SCHRÖDINGER OPERATORS

L. GOLINSKII AND S. KUPIN

*Dedicated to Nikolai Nikolski  
on occasion of his 70-th birthday*

ABSTRACT. Let  $H_0 = -\Delta + V_0$  be a finite band Schrödinger operator with a real-valued potential. We study its complex perturbation  $H = H_0 + V$  defined in the form sense, and obtain a Lieb–Thirring type inequalities for the discrete spectrum of  $H$  in the case when  $V_0 \in L^\infty(\mathbb{R}^d)$  and  $V \in L^p(\mathbb{R}^d)$ ,  $p > \max(d/2, 2)$ .

## INTRODUCTION

Different characteristics of the distribution of the discrete spectrum for a complex perturbation of a model differential self-adjoint operator (*e.g.*, a Laplacian on  $\mathbb{R}^d$ , a discrete Laplacian on  $\mathbb{Z}^d$ , etc.) were studied, for instance, in Frank-Laptev-Lieb-Seiringer [4], Borichev-Golinskii-Kupin [1], and Demuth-Hansmann-Katriel [3]. This paper focuses on the case when the model self-adjoint operator contains a periodic smooth background.

So, let  $V_0$  be a real-valued measurable function on  $\mathbb{R}^d$ ,  $d \geq 2$ , such that  $\text{ess inf } V_0 > -\infty$ , and the Schrödinger operator

$$(0.1) \quad H_0 = -\Delta + V_0$$

is self-adjoint,  $H_0^* = H_0$ . The standing assumption is that the spectrum  $\sigma(H_0)$  is finite band, i.e.,

$$(0.2) \quad \sigma(H_0) = \sigma_{\text{ess}}(H_0) = J = \bigcup_{k=1}^{n+1} [a_k, b_k], \quad b_{n+1} = +\infty.$$

For instance, an impressive result of Parnovsky [7], confirming the Bethe–Sommerfeld conjecture, says that this is the case when  $V_0$  is a smooth periodic function on  $\mathbb{R}^d$ .

Furthermore, consider

$$H = H_0 + V,$$

where  $V$  is a complex-valued potential, defined as the form sum. If  $V$  is relatively compact perturbation of  $H_0$ , that is,  $\text{dom}(V) \supset \text{dom}(H_0)$ , and

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$V(H_0 - z)^{-1}$  is a compact operator for  $z \in \rho(H_0)$ , then by the celebrated theorem of Weyl (see, e.g., [6, Section IV.5.6])  $\sigma_{ess}(H) = \sigma_{ess}(H_0)$  and

$$\sigma(H) = J \dot{\cup} \sigma_d(H)$$

(disjoint union), where the discrete spectrum  $\sigma_d(H)$  of  $H$ , i.e., the set of isolated eigenvalues of finite algebraic multiplicity, can accumulate only on  $J$ . The main goal of the paper is to obtain certain quantitative bounds for the rate of this accumulation.

The assumption on the potentials looks as follows

$$(0.3) \quad V_0 \in L^\infty(\mathbb{R}^d), \quad V \in L^p(\mathbb{R}^d), \quad p > \max(d/2, 2).$$

Under assumption (0.3)  $H$  is a well-defined, closed and  $m$ -sectorial operator, and there is  $\omega_1 < 0$  such that

$$(0.4) \quad \sigma(H) \subset \overline{N(H)} \subset \{z : \operatorname{Re} z \geq \omega_1\},$$

where  $N(H) = \{(Hf, f) : f \in \operatorname{dom}(H), \|f\|_2 \leq 1\}$  is the numerical range of  $H$  (see, e.g., [6, Chapter VI]). Moreover,  $H$  appears to be a relatively compact (even  $\mathcal{S}_p$ ) perturbation of  $H_0$ ,  $\mathcal{S}_p$  being the Schatten–von Neumann class of compact operators.

**Theorem 0.1.** *Let  $H_0$  be a finite band Schrödinger operator in  $\mathbb{R}^d$ ,  $d \geq 2$ ,  $V_0, V$  satisfy (0.3), and*

$$(0.5) \quad \omega := \min(\omega_1, a_1) - 1,$$

$a_1$  in (0.2) is the leftmost edge of  $\sigma(H_0)$ . Then

$$(0.6) \quad \sum_{z \in \sigma_d(H)} \frac{d^p(z, J)}{(1 + |z|)^{2p}} \leq C(J, p, d) |\omega|^{2p} (1 + \|V_0\|_\infty)^p \|V\|_p^p,$$

where a positive constant  $C(J, p, d)$  depends on  $J, p, d$ .

There is an elementary way to specify  $\omega$  and eliminate it from the final expression. The price we pay is an additional factor in the right hand side.

**Theorem 0.2.** *Under assumptions (0.3)*

$$(0.7) \quad \sum_{z \in \sigma_d(H)} \frac{d^p(z, J)}{(1 + |z|)^{2p}} \leq C(J, p, d) (1 + \|V_0\|_\infty)^p (1 + \|V\|_p)^{p \frac{2p+d}{2p-d}} \|V\|_p^p,$$

where a positive constant  $C(J, p, d)$  depends on  $J, p, d$ .

## 1. PROOF OF THE MAIN RESULTS

The first key ingredient of the proof is the following result of Hansmann [5, Theorem 1]. Let  $A_0 = A_0^*$  be a bounded self-adjoint operator on the Hilbert space,  $A$  a bounded operator with  $A - A_0 \in \mathcal{S}_p$ ,  $p > 1$ . Then

$$(1.1) \quad \sum_{\lambda \in \sigma_d(A)} d^p(\lambda, \sigma(A_0)) \leq K \|A - A_0\|_{\mathcal{S}_p}^p,$$

$K$  is an explicit (in a sense) constant, which depends only on  $p$ . We set

$$A_0(\omega) = R(\omega, H_0) = (H_0 - \omega)^{-1}, \quad A(\omega) = R(\omega, H) = (H - \omega)^{-1},$$

$\omega$  is defined in (0.5). By (0.2), (0.4)  $\omega \in \rho(H_0) \cap \rho(H)$ .

Let  $\lambda = \lambda_\omega(z) = (z - \omega)^{-1}$ . The Spectral Mapping Theorem [2, Lemma 8.1.9, Theorem 11.2.2] implies that

$$\lambda \in \sigma_d(A(\omega)) \quad (\lambda \in \sigma(A_0(\omega))) \quad \iff \quad z \in \sigma_d(H) \quad (z \in \sigma(H_0)).$$

The second ingredient of the proof of Theorem 0.1 is the following distortion lemma for linear fractional transformations. Of course, one can give several proofs of the lemma. We have chosen to give the most elementary (though, rather long) one. One of its advantages is that the quantity  $M$  in (1.2) is optimal.

**Lemma 1.1.** *For  $J$  (0.2) and any  $\omega \leq a_1 - 1$  the bound*

$$(1.2) \quad \frac{d(\lambda_\omega(z), \lambda_\omega(J))}{d(z, J)} \geq \frac{M}{|z - \omega|(|z - \omega| + a_{n+1} - \omega)}, \quad z \in \mathbb{C} \setminus J,$$

holds with  $M = M(J)$  depending on  $J$ , but not on  $\omega$ . The value of  $M(J)$  can be explicitly computed, see (1.11), (1.12).

*Proof.* Set

$$J = \bigcup_1^{n+1} J_k, \quad I = \lambda_\omega(J) = \bigcup_{k=1}^{n+1} I_k,$$

$I_k = \lambda_\omega(J_k) = [\beta_k, \alpha_k]$ ,  $k = 1, \dots, n$ , and  $I_{n+1} = [\beta_{n+1}, \alpha_{n+1}]$ ,  $\beta_{n+1} = 0$ .

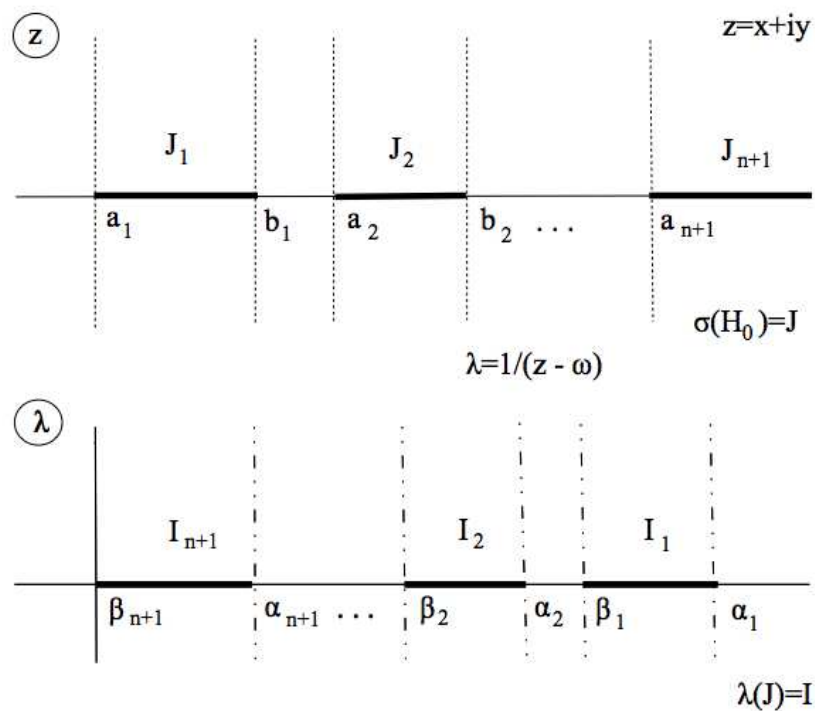


FIGURE 1. Sets  $\sigma(H_0) = J$  and  $\lambda(J) = I$  with map  $\lambda = \lambda_\omega(z) = \frac{1}{z-\omega}$ .

Let us begin with the case  $\omega = 0$ , so  $a_1 > 1$ , and put  $\lambda = \lambda_0$ . If  $z = x + iy$  and  $x = \operatorname{Re} z \leq 0$ , then  $\operatorname{Re} \lambda = x|z|^{-2} \leq 0$  and so

$$(1.3) \quad \frac{d(\lambda, I)}{d(z, J)} = \frac{|\lambda|}{|z - a_1|} = \frac{1}{|z||z - a_1|} \geq \frac{1}{|z|(|z| + a_1)}.$$

Similarly, if  $x \in J$ , then  $x \geq a_1$ ,  $0 < \operatorname{Re} \lambda = x|z|^{-2} \leq a_1^{-1} = \alpha_1$ , and

$$d(\lambda, [0, \alpha_1]) = |\operatorname{Im} \lambda| = \frac{|y|}{|z|^2}, \quad d(z, J) = |y|,$$

and so

$$(1.4) \quad \frac{d(\lambda, I)}{d(z, J)} \geq \frac{d(\lambda, [0, \alpha_1])}{d(z, J)} = \frac{1}{|z|^2} > \frac{1}{|z|(|z| + a_1)}.$$

Next, fix  $x$  in  $k$ 's gap,  $b_k < x < a_{k+1}$ ,  $k = k(x) = 0, 1, \dots, n$  (we put  $b_0 = 0$  and consider  $(0, a_1)$  as a gap). Then

$$d(z, J) = \min(|z - b_k|, |z - a_{k+1}|), \quad k = 1, 2, \dots, n; \quad d(z, J) = |z - a_1|, \quad k = 0.$$

Define two sets of positive numbers  $\{y_j(\alpha)\}$ ,  $\{y_j(\beta)\}$ ,  $j = k + 1, \dots, n + 1$  by equalities

$$\operatorname{Re}(\lambda(x + iy_j(\alpha))) = \alpha_j, \quad \operatorname{Re}(\lambda(x + iy_j(\beta))) = \beta_j,$$

or, equivalently,  $y_j(\alpha) = x(a_j - x)$ ,  $y_j(\beta) = x(b_j - x)$ . We also put  $y_k(\beta) = 0$ , so

$$0 = y_k(\beta) < y_{k+1}(\alpha) < y_{k+1}(\beta) < \dots < y_{n+1}(\alpha) < y_{n+1}(\beta) = +\infty.$$

Assume first that  $y_j(\beta) < |y| < y_{j+1}(\alpha)$ ,  $j = k, \dots, n$ ,  $k = 0, 1, \dots, n$ , which means that  $\alpha_{j+1} < \operatorname{Re} \lambda < \beta_j$ , so we have "gaps for  $\lambda$ ". Then

$$\begin{aligned} d(\lambda, I) &= \min(|\lambda - \alpha_{j+1}|, |\lambda - \beta_j|) = \frac{1}{|z|} \min\left(\frac{|z - a_{j+1}|}{a_{j+1}}, \frac{|z - b_j|}{b_j}\right) \\ &\geq \frac{d(z, J)}{|z|} \min\left(\frac{1}{a_{j+1}}, \frac{1}{b_j}\right) = \frac{d(z, J)}{|z|a_{j+1}}, \end{aligned}$$

and so

$$(1.5) \quad \frac{d(\lambda, I)}{d(z, J)} \geq \frac{1}{|z|a_{j+1}} \geq \frac{1}{|z|(|z| + a_{j+1})}.$$

Assume next that  $y_{j+1}(\alpha) \leq |y| \leq y_{j+1}(\beta)$ ,  $j = k, \dots, n$ ,  $k = 0, 1, \dots, n$ , which means that  $\beta_{j+1} \leq \operatorname{Re} \lambda \leq \alpha_{j+1}$ , so we have "bands for  $\lambda$ ". Now  $d(\lambda, I) = |\operatorname{Im} \lambda| = |y||z|^{-2}$  and

$$\begin{aligned} d(z, J) &\leq |z - a_{k+1}| \leq |y| + a_{k+1} - x = |y| + \frac{y_{k+1}^2(\alpha)}{x} \leq |y| + \frac{y_{j+1}^2(\alpha)}{x} \\ &\leq |y| \left(1 + \frac{y_{j+1}^2(\alpha)}{x}\right) = |y| \left(1 + \sqrt{\frac{a_{j+1} - x}{x}}\right), \end{aligned}$$

so

$$(1.6) \quad \frac{d(\lambda, I)}{d(z, J)} \geq \left(1 + \sqrt{\frac{a_{j+1} - x}{x}}\right)^{-1} \frac{1}{|z|^2} \geq \left(1 + \sqrt{\frac{a_{n+1}}{x}}\right)^{-1} \frac{1}{|z|(|z| + a_{n+1})}.$$

If  $k \geq 1$  then  $x > b_1$ , and it follows from (1.6) that

$$(1.7) \quad \frac{d(\lambda, I)}{d(z, J)} \geq \left(1 + \sqrt{\frac{a_{n+1}}{b_1}}\right)^{-1} \frac{1}{|z|(|z| + a_{n+1})},$$

and it remains to handle the case  $k = 0$ , that is,  $0 < x < a_1$ . By the above assumption  $|y| \geq y_1(\alpha) = \sqrt{x(a_1 - x)}$ . If  $|y| \leq 2x$  then

$$2\sqrt{x} > \frac{|y|}{\sqrt{x}} \geq \sqrt{a_1 - x}, \quad x > \frac{a_1}{5},$$

and by (1.6) and  $d(z, J) = |z - a_1|$

$$(1.8) \quad \frac{d(\lambda, I)}{d(z, J)} \geq \left(1 + \sqrt{\frac{5a_{n+1}}{a_1}}\right)^{-1} \frac{1}{|z|(|z| + a_{n+1})}.$$

For  $|y| \geq 2x$  we have directly

$$y^2 \geq 4x^2, \quad |y| \geq \frac{2|z|}{\sqrt{5}}, \quad d(\lambda, I) = \frac{|y|}{|z|^2} \geq \frac{2}{\sqrt{5}|z|},$$

and so

$$(1.9) \quad \frac{d(\lambda, I)}{d(z, J)} \geq \frac{2}{\sqrt{5}} \frac{1}{|z|(|z| + a_1)}.$$

The combination of (1.3)–(1.9) gives

$$(1.10) \quad \frac{d(\lambda, I)}{d(z, J)} \geq \left(1 + \sqrt{\frac{5a_{n+1}}{a_1}}\right)^{-1} \frac{1}{|z|(|z| + a_{n+1})},$$

which is (1.2) for  $\omega = 0$ .

To work out the general case, note that the shift of variable leads to

$$\begin{aligned} \frac{d(\lambda_\omega(z), \lambda_\omega(J))}{d(z, J)} &\geq \frac{c(J, \omega)}{|z - \omega|(|z - \omega| + a_{n+1} - \omega)}, \\ c(J, \omega) &= \left(1 + \sqrt{\frac{5a_{n+1} - \omega}{a_1 - \omega}}\right)^{-1}, \end{aligned}$$

so we have to find a uniform bound from below for  $c(J, \omega)$ . If  $a_1 > 0$  then

$$(1.11) \quad \begin{aligned} \frac{a_{n+1} - \omega}{a_1 - \omega} &< \frac{a_{n+1}}{a_1}, \\ c(J, \omega) &> M(J) = \left(1 + \sqrt{\frac{5a_{n+1}}{a_1}}\right)^{-1}. \end{aligned}$$

If  $a_1 \leq 0$ , then by the hypothesis  $\omega < a_1 - 1$  we see that

$$(1.12) \quad \begin{aligned} \frac{a_{n+1} - \omega}{a_1 - \omega} &= \frac{a_{n+1} - a_1}{a_1 - \omega} + 1 < 1 + a_{n+1} - a_1, \\ c(J, \omega) &> M(J) = \left(1 + \sqrt{5(1 + a_{n+1} - a_1)}\right)^{-1}. \end{aligned}$$

The proof of the lemma is complete.  $\square$

*Proof of Theorem 0.1.* For  $\omega$  (0.5) consider an operator

$$\begin{aligned}
 (1.13) \quad W &= W(\omega) = VR(\omega, H_0) \\
 &= V(-\Delta - \omega)^{-1}(-\Delta - \omega)(H_0 - \omega)^{-1} \\
 &= V(-\Delta - \omega)^{-1}(1 - V_0(H_0 - \omega)^{-1}),
 \end{aligned}$$

so

$$\begin{aligned}
 R(\omega, H) - R(\omega, H_0) &= -R(\omega, H)(H - H_0)R(\omega, H_0) = -R(\omega, H)VR(\omega, H_0) \\
 &= -R(\omega, H)W.
 \end{aligned}$$

The choice of  $\omega$  and (0.4) imply  $d(\omega, \overline{N(H)}) \geq 1$ , and by [6, Theorem V.3.2]

$$(1.14) \quad \|R(\omega, H)\| \leq \frac{1}{d(\omega, \overline{N(H)})} \leq 1.$$

The next step is to show that  $W \in \mathcal{S}_p$  and to obtain the bound for  $\|W\|_{\mathcal{S}_p}$ . We write  $V(-\Delta - \omega)^{-1} = V(x)g_\omega(-i\nabla)$  with  $g_\omega(x) = (|x|^2 - \omega)^{-1}$ ,  $x \in \mathbb{R}^d$ . By [8, Theorem 4.1]

$$\|V(-\Delta - \omega)^{-1}\|_{\mathcal{S}_p} \leq (2\pi)^{-d/p} \|V\|_p \|g_\omega\|_p, \quad p \geq 2.$$

Furthermore, [3, Lemma 3.6] estimates the value

$$\|g_\omega\|_p^p \leq C_1 \frac{|\omega|^{d/2-1}}{d(\omega, \mathbb{R}_+)^{p-1}}, \quad p > \max(d/2, 2).$$

Here and in what follows  $C_k = C_k(p, d)$ ,  $k = 1, 2, \dots$  denote positive constants, which depend only on  $p$  and  $d$ . So,

$$\|V(-\Delta - \omega)^{-1}\|_{\mathcal{S}_p} \leq C_2 \frac{|\omega|^{\frac{d}{2p} - \frac{1}{p}}}{d^{1-\frac{1}{p}}(\omega, \mathbb{R}_+)} \cdot \|V\|_p.$$

Looking back at (1.13), we see that

$$(1.15) \quad \|W\|_{\mathcal{S}_p} \leq C_2 \left\{ 1 + \frac{\|V_0\|_\infty}{d(\omega, J)} \right\} \frac{|\omega|^{\frac{d}{2p} - \frac{1}{p}}}{d^{1-\frac{1}{p}}(\omega, \mathbb{R}_+)} \|V\|_p.$$

Recalling (0.2) and the choice of  $\omega$ , we have

$$d(\omega, J) = |\omega - a_1| \geq 1, \quad d(\omega, \mathbb{R}_+) = |\omega| \geq 1,$$

and so with  $p > \max(d/2, 2)$  and  $q := 1 - d/2p > 0$

$$\begin{aligned}
 (1.16) \quad \|W\|_{\mathcal{S}_p} &\leq C_2 \left( 1 + \frac{\|V_0\|_\infty}{|\omega - a_1|} \right) \frac{\|V\|_p}{|\omega|^q} \\
 &\leq C_2 (1 + \|V_0\|_\infty) \|V\|_p.
 \end{aligned}$$

Finally, by (1.14)

$$\|R(\omega, H) - R(\omega, H_0)\|_{\mathcal{S}_p} \leq \|R(\omega, H)\| \|W\|_{\mathcal{S}_p} \leq \|W\|_{\mathcal{S}_p} \leq C_2 (1 + \|V_0\|_\infty) \|V\|_p.$$

We go back and apply (1.1) with

$$A_0 = A_0(\omega) = R(\omega, H_0), \quad A = A(\omega) = R(\omega, H),$$

so

$$(1.17) \quad \begin{aligned} \sum_{\lambda \in \sigma_d(A(\omega))} d^p(\lambda, I) &= \sum_{\lambda \in \sigma_d(A(\omega))} d^p(\lambda, \sigma(A_0(\omega))) \\ &\leq K \|R(\omega, H) - R(\omega, H_0)\|_{\mathcal{S}_p}^p \\ &\leq C_3 (1 + \|V_0\|_\infty)^p \|V\|_p^p, \end{aligned}$$

$C_3 = KC_2^p$ . Lemma 1.1 yields

$$\sum_{z_k \in \sigma_d(H)} \frac{d^p(z_k, J)}{|z_k - \omega|^p (|z_k - \omega| + a_{n+1} - \omega)^p} \leq \frac{C_3}{M^p} (1 + \|V_0\|_\infty)^p \|V\|_p^p,$$

and it remains only to estimate the denominator on the left hand side. It follows from (0.5) that

$$(1.18) \quad \begin{aligned} |z_k - \omega| &\leq (1 + |\omega|)(1 + |z_k|) \leq 2|\omega|(1 + |z_k|), \\ |a_{n+1} - \omega| &\leq C(J)(1 + |\omega|), \end{aligned}$$

and we come to

$$\sum_{z_k \in \sigma_d(H)} \frac{d^p(z_k, J)}{(1 + |z_k|)^{2p}} \leq C_4(J, p, d) |\omega|^{2p} (1 + \|V_0\|_\infty)^p \|V\|_p^p,$$

as claimed. Theorem 0.1 is proved.  $\square$

The proof of Theorem 0.1 shows that the bound (0.6) essentially depends on the parameter  $\omega$ . Roughly speaking, it comes from a bound from below of  $\inf \operatorname{Re} \sigma(H)$ , and so it seems to be rather important to estimate this quantity in terms of  $V_0$  and  $V$  only.

*Proof of Theorem 0.2.* Let  $\operatorname{Re} z < a_1^- = \min(a_1, 0)$ . Then  $z \in \rho(H_0)$ ,

$$d(z, \sigma(H_0)) = |z - a_1|, \quad d(z, \mathbb{R}_+) = |z|,$$

and as in (1.16)

$$\|W(z)\|_{\mathcal{S}_p} \leq C_2 \left(1 + \frac{\|V_0\|_\infty}{|z - a_1|}\right) \frac{\|V\|_p}{|z|^q}, \quad p > \max(d/2, 2),$$

with  $q = 1 - d/2p > 0$ . Put

$$\Omega = \{\operatorname{Re} z < a_1^-\} \cap \{|z - a_1| > \|V_0\|_\infty\} \cap \{|z|^q > 4C_2(1 + \|V\|_p)\}.$$

We have

$$(1.19) \quad \|W(z)\|_\infty \leq \|W(z)\|_{\mathcal{S}_p} \leq 2C_2 \cdot \frac{\|V\|_p}{4C_2(1 + \|V\|_p)} < \frac{1}{2}, \quad z \in \Omega,$$

so  $I + W$  is invertible and  $\|(I + W)^{-1}\| < 2$ .

An identity  $H - z = (1 + W(z))(H_0 - z)$  and (1.19) show that  $\Omega \subset \rho(H_0) \cap \rho(H)$ . We write the difference of the resolvents in another way

$$R(z, H) - R(z, H_0) = -R(z, H_0)(1 + W(z))^{-1}W(z)$$

to obtain for  $z \in \Omega$

$$\begin{aligned} \|R(z, H) - R(z, H_0)\|_{\mathcal{S}_p} &\leq \|R(z, H_0)\| \|(1 + W(z))^{-1}\| \|W(z)\|_{\mathcal{S}_p} \\ &\leq \frac{2}{|z - a_1|} \frac{\|V\|_p}{2(1 + \|V\|_p)} = \frac{\|V\|_p}{|z - a_1|(1 + \|V\|_p)}. \end{aligned}$$

Let us now choose  $z = \omega' < 0$  as

$$(1.20) \quad \frac{|\omega'|}{4} = |a_1| + |a_{n+1}| + 1 + \|V_0\|_\infty + (4C_2(1 + \|V\|_p))^{1/q}.$$

It is easy to check that  $\{\lambda : \operatorname{Re} \lambda < \frac{\omega'}{2}\} \subset \Omega$ , so in particular,  $\omega' \in \Omega$ ,  $|\omega' - a_1| > 1$ , and hence

$$\|R(\omega', H) - R(\omega', H_0)\|_{\mathcal{S}_p} \leq \frac{\|V\|_p}{1 + \|V\|_p}.$$

Once again, (1.1) says

$$(1.21) \quad \sum_{\lambda \in \sigma_d(A(\omega'))} d^p(\lambda, \sigma(A_0(\omega'))) \leq K \left( \frac{\|V\|_p}{1 + \|V\|_p} \right)^p, \quad p > \max\left(\frac{d}{2}, 2\right),$$

and, using Lemma 1.1, we come to

$$\sum_{z_k \in \sigma_d(H)} \frac{d^p(z_k, J)}{|z_k - \omega'|^p (|z_k - \omega'| + a_{n+1} - \omega')^p} \leq KM^{-p} \left( \frac{\|V\|_p}{1 + \|V\|_p} \right)^p.$$

By the choice of  $\omega'$  (1.20),  $\operatorname{Re} z_k \geq \omega'/2$ , so

$$|z_k - \omega'| \geq \frac{|\omega'|}{2} > \frac{|a_{n+1}| + |\omega'|}{4},$$

and

$$\begin{aligned} |z_k - \omega'| + a_{n+1} + |\omega'| &< 5|z_k - \omega'|, \\ |z_k - \omega'| (|z_k - \omega'| + a_{n+1} + |\omega'|) &< 5|z_k - \omega'|^2. \end{aligned}$$

Next,  $|z_k - \omega'| \leq 2|\omega'|(1 + |z_k|)$ , as in (1.18), and hence

$$\begin{aligned} \sum_{z_k \in \sigma_d(H)} \frac{d^p(z_k, J)}{(1 + |z_k|)^{2p}} &\leq C(J, p, d) |\omega'|^{2p} \left( \frac{\|V\|_p}{1 + \|V\|_p} \right)^p \\ &\leq C(J, p, d) (1 + \|V_0\|_\infty)^{2p} (1 + \|V\|_p)^{p \frac{2p+d}{2p-d}} \|V\|_p^p. \end{aligned}$$

The proof is complete.  $\square$

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