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A BLASCHKE-TYPE CONDITION FOR ANALYTIC FUNCTIONS ON FINITELY CONNECTED DOMAINS. APPLICATIONS TO COMPLEX PERTURBATIONS OF A FINITE-BAND SELFADJOINT OPERATOR

L. GOLINSKII AND S. KUPIN

Abstract. This is a sequel of the article by Borichev–Golinskii–Kupin [2] where the authors obtain Blaschke-type conditions for special classes of analytic functions in the unit disk which satisfy certain growth hypotheses. These results were applied to get Lieb–Thirring inequalities for complex compact perturbations of a selfadjoint operator with a simply connected resolvent set.

The first result of the present paper is an appropriate local version of the Blaschke-type condition from [2]. We apply it to obtain a similar condition for an analytic function in a finitely connected domain of a special type. Such condition is by and large the same as a Lieb–Thirring type inequality for complex compact perturbations of a selfadjoint operator with a finite-band spectrum. A particular case of this result is the Lieb–Thirring inequality for a selfadjoint perturbation of the Schatten class of a periodic (or a finite-band) Jacobi matrix. The latter result seems to be new in such generality even in this framework.

Introduction

Let $e = \{\alpha_j, \beta_j\}_{j=1,\dots,n+1} \subset \mathbb{R}$ be a set of distinct points. We suppose that

$$-\infty < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \dots < \alpha_{n+1} < \beta_{n+1} < +\infty. \tag{0.1}$$

Let also

$$\mathfrak{e} = \bigcup_{j=1}^{n+1} \mathfrak{e}_j, \quad \mathfrak{e}_j = [\alpha_j, \beta_j], \tag{0.2}$$

and $\Omega := \bar{\mathbb{C}} \backslash \mathfrak{e}$. For a function f analytic in Ω, $f \in \mathcal{A}(\Omega)$, Z_f stands for the set of the zeros counting the multiplicities. By $\mathrm{d}(\lambda, M)$ we denote the distance between a point λ and a set M.

Our main functional theoretic result looks as follows.

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Theorem 0.1. *Let $f \in \mathcal{A}(\Omega)$, $|f(\infty)| = 1$, and, for $p, q \geq 0$*

$$\log|f(\lambda)| \leq \frac{K_1}{\mathrm{d}^p(\lambda, \mathfrak{e})\, \mathrm{d}^q(\lambda, e)}. \tag{0.3}$$

Then for any $0 < \varepsilon < 1$

$$\sum_{\lambda \in Z_f} \mathrm{d}^{p+1+\varepsilon}(\lambda, \mathfrak{e})\ \mathrm{d}^{a(p,q,\varepsilon)}(\lambda, e)\, (1 + |\lambda|)^{b(p,q,\varepsilon)} \leq C \cdot K_1, \tag{0.4}$$

where

$$a(p, q, \varepsilon) = \frac{(p + 2q - 1 + \varepsilon)_+ - (p + 1 + \varepsilon)}{2},$$

$$b(p, q, \varepsilon) = (p + q - 1 + \varepsilon)_+ - \frac{(p + 2q - 1 + \varepsilon)_+ + p + 1 + \varepsilon}{2}.$$

As usual, $x_+ = \max\{x, 0\}$. Here and in the sequel $C = C(\mathfrak{e}, p, q, \varepsilon)$ stands for a generic positive constant which depends on indicated parameters. Of course, inequality (0.4) looks somewhat cumbersome, and it can be simplified in specific situations. Here are two examples.

Corollary 0.2. *Let $f \in \mathcal{A}(\Omega)$, $|f(\infty)| = 1$, and, for $p, q \geq 0$, $p+q \geq 1$*

$$\log|f(\lambda)| \leq \frac{K_1}{\mathrm{d}^p(\lambda, \mathfrak{e})\, \mathrm{d}^q(\lambda, e)}.$$

Then for any $0 < \varepsilon < 1$

$$\sum_{\lambda \in Z_f} \frac{\mathrm{d}^{p+1+\varepsilon}(\lambda, \mathfrak{e})\ \mathrm{d}^{q-1}(\lambda, e)}{1 + |\lambda|} \leq C \cdot K_1. \tag{0.5}$$

The case $q = 0$ is important for applications.

Corollary 0.3. *Let $f \in \mathcal{A}(\Omega)$, $|f(\infty)| = 1$, and*

$$\log|f(\lambda)| \leq \frac{K_1}{\mathrm{d}^p(\lambda, \mathfrak{e})}, \quad p \geq 0. \tag{0.6}$$

Then for any $0 < \varepsilon < 1$

$$\sum_{\lambda \in Z_f} \frac{\mathrm{d}^{p+1+\varepsilon}(\lambda, \mathfrak{e})}{\mathrm{d}(\lambda, e)(1 + |\lambda|)} \leq C \cdot K_1, \tag{0.7}$$

as long as $p \geq 1$, and

$$\sum_{\lambda \in Z_f} \frac{\mathrm{d}^{p+1+\varepsilon}(\lambda, \mathfrak{e})}{(\mathrm{d}(\lambda, e)\, (1 + |\lambda|))^{(p+1+\varepsilon)/2}} \leq C \cdot K_1 \tag{0.8}$$

for $p < 1$.

All operators appearing in the present paper act on a separable Hilbert space H. Consider a (bounded) selfadjoint operator A_0 defined on H. We suppose it to be finite-band, i.e., for its spectrum

$$\sigma(A_0) = \sigma_{ess}(A_0) = \mathfrak{e},$$

where $\mathfrak{e}$ looks like in (0.2). A typical example here is a double infinite periodic Jacobi matrix. Let $B \in \mathcal{S}_p$, the Schatten class of operators, $p \geq 1$. We do not suppose B to be selfadjoint. By the Weyl theorem, see, e.g., [8], the essential spectrum $\sigma_{ess}(A)$, $A = A_0 + B$, coincides with $\sigma_{ess}(A_0)$.

We want to have some information on the distribution of the discrete spectrum $\sigma_d(A) := \sigma(A) \backslash \sigma_{ess}(A)$, which consists of eigenvalues of finite algebraic multiplicity. It is clear that the points from $\sigma_d(A)$ can only accumulate to $\mathfrak{e}$. Here is the quantitative version of this intuition.

Theorem 0.4. *Let A_0 be as described above, $B \in \mathcal{S}_p$ and $A = A_0 + B$. Then, for $0 < \varepsilon < 1$ and $p \geq 1$*

$$\sum_{\lambda \in \sigma_d(A)} \frac{\mathrm{d}^{p+1+\varepsilon}(\lambda, \mathfrak{e})}{\mathrm{d}(\lambda, e)\,(1 + |\lambda|)} \leq C \cdot \|B\|_{\mathcal{S}_p}, \tag{0.9}$$

and for $0 \leq p < 1$

$$\sum_{\lambda \in Z_f} \frac{\mathrm{d}^{p+1+\varepsilon}(\lambda, \mathfrak{e})}{(\mathrm{d}(\lambda, e)\,(1 + |\lambda|))^{(p+1+\varepsilon)/2}} \leq C \cdot \|B\|_{\mathcal{S}_p}. \tag{0.10}$$

In such generality the above inequality seems to be new even for the case when A is a selfadjoint perturbation of a periodic selfadjoint Jacobi matrix A_0.

Remark 0.5. The case $n = 0$, i.e., $\sigma(A_0) = [\alpha, \beta]$, is not exceptional. The point is that for $e = \{\alpha, \beta\}$

$$C_1|(\lambda - \alpha)(\lambda - \beta)| \leq \mathrm{d}(\lambda, e)(1 + |\lambda|) \leq C_2|(\lambda - \alpha)(\lambda - \beta)|, \quad \lambda \in \mathbb{C},$$

with absolute constants $C_{1,2}$, so we come to Theorem 2.3 from [2].

For the Lieb–Thirring inequalities for nonselfadjoint compact perturbations of the discrete Laplacian see also Golinskii–Kupin [7], Hansmann–Katriel [9]. A few interesting results of the same flavor on Lieb–Thirring inequalities for selfadjoint Jacobi matrices and Schrödinger operators are in Hundertmark–Simon [10], Damanik–Killip–Simon [3] and Frank–Simon [6].

As usual, we write $\mathbb{D} = \{z : |z| < 1\}$ for the unit disk, $\mathbb{T} = \{z : |z| = 1\}$ for the unit circle, and $B(w_0, r) = \{w : |w - w_0| < r\}$ for balls in the complex plane. Sometimes, we label the balls by the variable of the corresponding complex plane, i.e. $B_w(z_0, r)$ ($B_\lambda(z_0, r)$) stays for a ball in the w-plane (the λ-plane), respectively.

1. Local version of Borichev–Golinskii–Kupin Theorem

We begin with the result of Borichev–Golinskii–Kupin [2, Theorem 0.2] and its version in [9, Theorem 4].

Theorem 1.1. *Let $I = \{\zeta_j\}_{j=1,\dots,k}$ be a finite subset of $\mathbb{T}$, $f \in \mathcal{A}(\mathbb{D})$, $|f(0)| = 1$, and for $p', q', s \geq 0$*

$$\log|f(z)| \leq \frac{K|z|^s}{\mathrm{d}^{p'}(z,\mathbb{T})\ \mathrm{d}^{q'}(z,I)}\,, \quad z \in \mathbb{D}.$$

Then for any $0 < \varepsilon < 1$

$$\sum_{z\in Z_f} \frac{\mathrm{d}^{p'+1+\varepsilon}(z,\mathbb{T})}{|z|^{(s-1+\varepsilon)_+}}\ \mathrm{d}^{(q'-1+\varepsilon)_+}(z,I) \leq C(I,p',q',\varepsilon)\cdot K.$$

Our goal here is to prove a local version of the above result (cf. [4, Theorem 7]).

Let $G \subset \bar{\mathbb{D}}$ be an open circular polygon, $0 \in G$, with vertices $\{u_i\} \in \mathbb{T}$, and sides (arcs) $\tau_i = [u_i, u_{i+1}]$, $i = 1, 2, \dots, 2N$, $u_{2N+1} = u_1$ (see Figure 1). The arcs τ_{2j} lie on $\mathbb{T}$, and τ_{2j-1}, which we call the inner sides of G, lie on some orthocircles, that is, circles orthogonal to $\mathbb{T}$. Put

$$\Delta_1 = \partial G \cap \mathbb{D} = \left\{\bigcup_{j=1}^{N}(u_{2j-1}, u_{2j})\right\}, \quad \Delta_2 = \partial G \cap \mathbb{T} = \left\{\bigcup_{j=1}^{N}[u_{2j}, u_{2j+1}]\right\},$$

so

$$\partial G = \Delta_1 \cup \Delta_2.$$

Let $E = \{\zeta_j\}_{j=1,\dots,k} \subset \Delta_2$ be a selected finite subset of the unit circle. We take $\tilde{G} \subset G$ to be a properly "shrunk" circular polygon, in such a way that $E \subset \tilde{\Delta}_2$, see again Figure 1. The notation for $\tilde{G}$ is the same as for G up to "waves" referring to the first set. So, for instance, the vertices of $\tilde{G}$ are $\tilde{u}_i$,

$$\partial\tilde{G} = \tilde{\Delta}_1 \cup \tilde{\Delta}_2, \quad \tilde{\Delta}_1 = \partial\tilde{G} \cap \mathbb{D}, \quad \tilde{\Delta}_2 = \partial\tilde{G} \cap \mathbb{T}.$$

It is important that $\min_j \mathrm{d}(\tau_{2j-1}, \tilde{\tau}_{2j-1}) = \mathrm{d}' > 0$, $j = 1, 2, \dots, N$.

Consider a conformal map w, $w : \mathbb{D} \to G$, normalized by $w(0) = 0$, $w'(0) > 0$. Sometimes, to indicate explicitly the variables, we will write $w : \mathbb{D}_z \to G_w$.

Put $\tilde{D} = w^{-1}(\tilde{G}) \subset \mathbb{D}_z$ and introduce

- preimages of vertices $v_j = w^{-1}(u_j), \tilde{v}_j = w^{-1}(\tilde{u}_j)$, $\quad j = 1, \dots, 2N$,
- preimages of sides $\tilde{\tau}_j = w^{-1}(\tau_j) \subset \mathbb{T}_z$, $\quad j = 1, \dots, 2N$.
- preimages of selected points $I = \{\xi_j = w^{-1}(\zeta_j)\}$, $j = 1, \dots, k$.

Clearly, I is contained in the closure of $\tilde{D}$.

For short, we write $w = w(z)$. Here is a couple of elementary properties of w:

- $\mathrm{d}(z, \mathbb{T}_z) = 1 - |z| \leq 1 - |w| = \mathrm{d}(w, \mathbb{T}_w)$ by the Schwarz lemma.
- By [11, Corollary 1.4], $\mathrm{d}(w, \partial G) \asymp |w'(z)|\,(1-|z|) = |w'(z)|\ \mathrm{d}(z, \mathbb{T})$. Since $z \in \tilde{D}$ if and only if $w \in \tilde{G}$, and $|w'(z)| \asymp 1$ for $z \in \tilde{D}$, then

$$\mathrm{d}(w, \partial G) \asymp \mathrm{d}(z, \mathbb{T}), \quad z \in \tilde{D}. \tag{1.1}$$

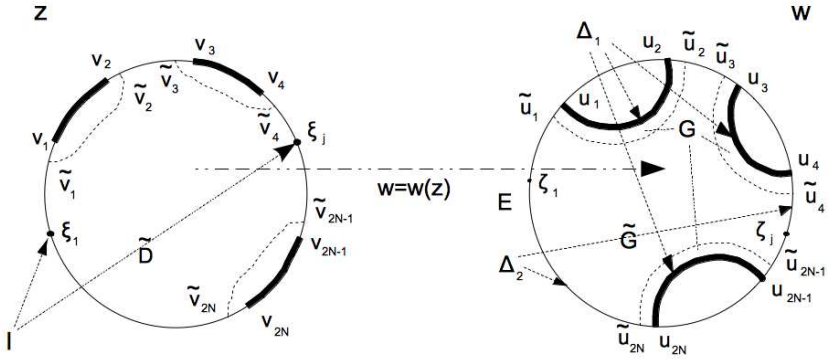

FIGURE 1. The domains $G, \tilde{G}$ and the map w.

Here and in what follows the equivalence relation $A \asymp B$ means that $c_1 \le A/B \le c_2$ for generic positive constants c_i which depend only on G and E. Similarly,

$$\mathrm{d}(w, E) \asymp \mathrm{d}(z, I), \quad z \in \mathbb{D}. \tag{1.2}$$

Indeed, for $z \in \tilde{D}$, $w \in \tilde{G}$, we have $|w'(z)|, |z'(w)| \asymp 1$. For $z \in \mathbb{D} \backslash \tilde{D}$ both sides in (1.2) are equivalent to 1.

Let now $f \in \mathcal{A}(G)$, $|f(0)| = 1$, and assume that for some $p', q', s \ge 0$

$$\log |f(w)| \le \frac{K|w|^s}{\mathrm{d}^{p'}(w, \mathbb{T})\ \mathrm{d}^{q'}(w, E)}, \quad w \in G. \tag{1.3}$$

Consider a function $F(z) = f(w(z)) \in \mathcal{A}(\mathbb{D}_z)$. By using the first property of w, equivalence $|w| \asymp |z|$, and (1.2), we obtain

$$\log |F(z)| \le \frac{K|z|^s}{\mathrm{d}^{p'}(z, \mathbb{T})\ \mathrm{d}^{q'}(z, I)}, \quad z \in \mathbb{D}.$$

Theorem 1.1 now implies

$$\sum_{z \in Z_F} \frac{\mathrm{d}^{p'+1+\varepsilon}(z, \mathbb{T})}{|z|^{(s-1+\varepsilon)_+}}\ \mathrm{d}^{(q'-1+\varepsilon)_+}(z, I) \le C \cdot K$$

for any $0 < \varepsilon < 1$, and, by far,

$$\sum_{z \in \tilde{D} \cap Z_F} \frac{\mathrm{d}^{p'+1+\varepsilon}(z, \mathbb{T})}{|z|^{(s-1+\varepsilon)_+}}\ \mathrm{d}^{(q'-1+\varepsilon)_+}(z, I) \le C \cdot K.$$

Of course, $Z_f = w(Z_F)$, so by (1.1) and (1.2)

$$\sum_{w \in \tilde{G} \cap Z_F} \frac{\mathrm{d}^{p'+1+\varepsilon}(w, \partial G)}{|w|^{(s-1+\varepsilon)_+}}\ \mathrm{d}^{(q'-1+\varepsilon)_+}(w, E) \le C \cdot K.$$

Let us show that $\mathrm{d}(w,\partial G) \ge C\,\mathrm{d}(w,\mathbb{T}_w)$, as long as $w \in \tilde{G}$. Indeed, if $\mathrm{d}(w,\partial G) = \mathrm{d}(w,\Delta_2)$, then $\mathrm{d}(w,\partial G) \ge \mathrm{d}(w,\mathbb{T}_w)$. Otherwise, $\mathrm{d}(w,\partial G) = \mathrm{d}(w,\Delta_1)$, so $\mathrm{d}(w,\partial G) \ge \mathrm{d}'$ and

$$\mathrm{d}(w,\mathbb{T}_w) = 1 - |w| \le 1 \le \frac{\mathrm{d}(w,\partial G)}{\mathrm{d}'}\,,$$

as claimed. Hence

$$\sum_{w\in\tilde{G}\cap Z_F} \frac{\mathrm{d}^{p'+1+\varepsilon}(w,\mathbb{T}_w)}{|w|^{(s-1+\varepsilon)_+}}\ \mathrm{d}^{(q'-1+\varepsilon)_+}(w,E) \le C\cdot K. \tag{1.4}$$

That is, we have proven

Theorem 1.2. *Let $f \in \mathcal{A}(G), |f(0)| = 1$, and for $p', q', s \ge 0$*

$$\log|f(w)| \le \frac{K|w|^s}{\mathrm{d}^{p'}(w,\mathbb{T})\ \mathrm{d}^{q'}(w,E)}\,, \quad w \in G.$$

Then, (1.4) *holds for any $0 < \varepsilon < 1$.*

It goes without saying that the similar counterpart of Theorem 0.3 from [2] is also valid in the present setting.

2. Uniformization, Fuchsian groups, and all that

In this section we are aimed at proving Theorem 0.1 with the help of Theorem 1.2.

We start reminding the celebrated uniformization theorem of Klein–Koebe–Poincaré [1, Ch. III], which is one of the key ingredients of the proof. The result is valid for arbitrary Riemann surfaces, but we will formulate it for the so called planar domains, since this is enough for our purposes. Recall that a discrete group of Möbius transformations Γ (of $\mathbb{D}$ on itself) is called a Fuchsian group. The discreteness means that any orbit $\{\gamma(z)\}_{\gamma\in\Gamma}$ is a discrete set in the relative topology of $\mathbb{D}$.

Let $\Omega \subset \bar{\mathbb{C}}$ be a domain with the boundary containing more than two points, and $\lambda_0 \in \Omega$. The uniformization theorem says that there exists a covering map $\lambda : \mathbb{D} \to \Omega$, which is unique provided the normalization conditions $\lambda(0) = \lambda_0$, $\lambda'(0) > 0$ are set. Moreover, the map is automorphic with respect to a certain Fuchsian group Γ, i.e., $\lambda\circ\gamma = \lambda$ for any $\gamma \in \Gamma$. Symbolically, we write

$$\Omega \simeq \mathbb{D}/\Gamma,$$

where two points $z, w \in \mathbb{D}$ are equivalent with respect to Γ if and only if there is a $\gamma \in \Gamma$ such that $w = \gamma(z)$. For further terminology on the subject, we refer to [1, Ch. III], [5]; see also Simon [13] for a recent presentation.

We will focus upon the special case $\Omega = \bar{\mathbb{C}}\backslash\mathfrak{e}$, described in (0.1). The standard normalization now is

$$\lambda(0) = \infty, \qquad \lim_{w\to 0} w\lambda(w) = \kappa(\mathfrak{e}) > 0. \tag{2.1}$$

The properties of the Fuchsian group Γ in this situation are well-studied, see [13, Chapter 9.6]. In particular, Γ is a free nonabelian group with n generators $\{\gamma_j\}_{j=1}^n$. The fundamental domain $\mathcal{F}$ (more precisely, its interior $\mathcal{F}^{int}$) is a circular polygon in $\mathbb{D}$, its topological boundary in $\mathbb{D}$ consists of n orthocircles in $\mathbb{C}_+$ and their complex conjugates, and there is a finite distance in $\overline{\mathbb{D}}$ between the different orthocircles, see Figure 2. We label the vertices of $\mathcal{F}$ by $E = \lambda^{-1}(e) = \{w_j\}$.

The following relations for the covering map are crucial in the sequel.

Lemma 2.1. *Let $w \in \overline{\mathcal{F}}$, closure in $\overline{\mathbb{D}}$, and $\lambda = \lambda(w)$. Then*

$$\mathrm{d}(\lambda, e) \asymp \frac{\mathrm{d}^2(w, E)}{|w|} \tag{2.2}$$

and

$$\mathrm{d}(\lambda, \mathfrak{e}) \asymp \frac{\mathrm{d}(w, \mathbb{T}_w)\, \mathrm{d}(w, E)}{|w|}\,. \tag{2.3}$$

Proof. In the case $w \in B(0, r)$ both (2.2) and (2.3) are obvious, since

$$\mathrm{d}(\lambda, e) \asymp \mathrm{d}(\lambda, \mathfrak{e}) \asymp |\lambda| \asymp \frac{1}{|w|}\,, \quad \mathrm{d}(w, \mathbb{T}_w) \asymp \mathrm{d}(w, E) \asymp 1$$

by (2.1). So we assume $|w| \geq r$.

Put

$$B_j := B_w(w_j, r) \cap \mathcal{F}^{int}, \quad B := \bigcup B_j,$$

with small enough $r = r(\mathfrak{e})$, so B_j are disjoint. The argument is based on the properties of the covering map (cf., e.g., [13, Theorem 9.6.4]):

(1) λ can be extended analytically to a certain domain, which contains $\overline{\mathcal{F}^{int}}$;
(2) λ is one-one in $\mathcal{F}^{int}$, and $\lambda'(w) = 0$ if and only if $w = w_j$;
(3) for $w \in B_j$, we have

$$\lambda(w) = \lambda(w_j) + C_j(w - w_j)^2 + O((w - w_j)^3), \tag{2.4}$$

and $C_j \neq 0$.

By (2.4), we have for $w \in B_j$

$$\mathrm{d}(\lambda, e) = |\lambda(w) - \lambda(w_j)| \asymp |w - w_j|^2 = \mathrm{d}(w, E)^2 \asymp \frac{\mathrm{d}(w, E)^2}{|w|}\,,$$

so (2.2) is true on B. For $w \in \overline{\mathcal{F}^{int}} \backslash (B \cup B(0, r))$

$$\mathrm{d}(\lambda, e) \asymp \mathrm{d}(w, E) \asymp |w| \asymp 1,$$

and the proof of (2.2) is complete.

To prove (2.3) for $|w| \geq r$ we begin with its simple half

$$\mathrm{d}(\lambda, \mathfrak{e}) \leq C\, \mathrm{d}(w, E)\ \mathrm{d}(w, \mathbb{T}_w). \tag{2.5}$$

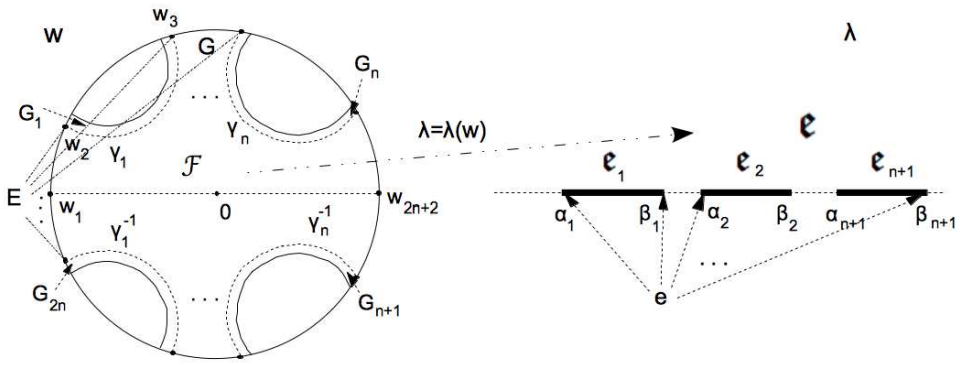

FIGURE 2. Uniformization of the domain Ω and the map λ.

For $w \in B_j$ take $\zeta \in \mathbb{T}_{\mathcal{F}} = \mathbb{T} \cap \overline{\mathcal{F}}$ so that $|w-\zeta| = \mathrm{d}(w, \mathbb{T}_{\mathcal{F}})$. By (2.4)

$$|\lambda(\zeta) - \lambda(w)| \le \max_{z\in[w,\zeta]} |\lambda'(z)|\,|\zeta - w| \le C|w - w_j|\,|\zeta - w|$$
$$= C\,\mathrm{d}(w, E)\ \mathrm{d}(w, \mathbb{T}_{\mathcal{F}}).$$

Since $|\lambda(\zeta) - \lambda(w)| \ge \mathrm{d}(\lambda, \mathfrak{e})$ and $\mathrm{d}(w, \mathbb{T}_w) \asymp \mathrm{d}(w, \mathbb{T}_{\mathcal{F}})$, (2.5) holds for $w \in B_j$. The similar argument applies in the case $w \in \overline{\mathcal{F}^{int}} \setminus (B \cup B(0,r))$, where $|\lambda'| \asymp 1$, so (2.5) is proved.

Suppose next, that $\mathrm{d}(\lambda, \mathfrak{e}) \ge C\,\mathrm{d}(\lambda, e)$. Then by (2.2) for $|w| \ge r$

$$\mathrm{d}(\lambda, \mathfrak{e}) \ge C\,\mathrm{d}^2(w, E) \ge C\,\mathrm{d}(w, E)\,\mathrm{d}(w, \mathbb{T}_w),$$

which is opposite to (2.5), so (2.3) is true. Hence it remains to consider the case

$$\mathrm{d}(\lambda, \mathfrak{e}) \le \delta\ \mathrm{d}(\lambda, e), \tag{2.6}$$

δ is small enough.

We apply a version of [11, Corollary 1.4], which reads

$$\mathrm{d}(g, \partial\Omega_2) \asymp |g'(w)|\ \mathrm{d}(w, \partial\Omega_1), \tag{2.7}$$

$g : \Omega_1 \to \Omega_2$ is a conformal map of bounded domains Ω_j. Let $\Omega_2 = B(0,R) \cap \mathbb{C}_-$ be a large semidisk, such that $\mathfrak{e} \subset \partial\Omega_2$, $g = \lambda$ restricted on the preimage of the later set (the part of $\mathcal{F}^{int}$ in the upper half plane away from the origin). The part of (2.6) in $\mathbb{C}_-$ is a union $T = \cup T_j$ of small isosceles triangles T_j with bases $\mathfrak{e}_j$. It is clear from the properties of the covering map that

$$\mathrm{d}(\lambda, \partial\Omega_2) = \mathrm{d}(\lambda, \mathfrak{e}), \qquad \lambda \in T,$$

$$\mathrm{d}(w, \partial\Omega_1) \asymp \mathrm{d}(w, \mathbb{T}_w), \quad |\lambda'(w)| \asymp \mathrm{d}(w, E), \qquad w \in \lambda^{(-1)}(T),$$

so by (2.7)

$$\mathrm{d}(\lambda, \mathfrak{e}) \asymp \mathrm{d}(w, E) \cdot \mathrm{d}(w, \mathbb{T}_w).$$

The proof is complete. □

Proof of Theorem 0.1. Let $\lambda = \lambda(w) : \mathbb{D}_w \to \Omega_\lambda$ be the covering map with normalization (2.1), Γ the corresponding Fuchsian group with generators $\{\gamma_j\}_{j=1}^n$, $E = \lambda^{-1}(e)$ the vertices of $\mathcal{F}$. Put $\gamma_{2n+1-k} := \gamma_k^{(-1)}$, $k = 1, \dots, n$.

Let $f \in \mathcal{A}(\Omega)$ satisfy (0.3). It is clear that $|f(\infty)| = 1$. We put $F(w) := f(\lambda(w))$. Then $F \in \mathcal{A}(\mathbb{D})$ and automorphic with respect to Γ. By Lemma 2.1

$$\log|F(w)| \le \frac{K_1|w|^{p+q}}{\mathrm{d}^p(w,\mathbb{T})\,\mathrm{d}^{p+2q}(w,E)}\,, \quad w \in \mathcal{F}. \tag{2.8}$$

The special structure of Γ and $\mathcal{F}$ enables one to "inflate" the domain $\mathcal{F}^{int}$ slightly to get another polygon G, so that

$$\mathcal{F} \subset G \subset \mathcal{F} \bigcup \left(\bigcup_{j=1}^{2n} \gamma_j(\mathcal{F}) \right), \quad \gamma_{n+k}(\mathcal{F}) = \overline{\gamma_k(\mathcal{F})}, \quad k = 1, \dots, n.$$

The distance between the corresponding inner sides of G and $\mathcal{F}^{int}$ is strictly positive.

It is not hard to see that bound (2.8) actually holds in the bigger polygon G. Indeed, let $G_j \subset G \backslash \mathcal{F}^{int}$ be an "annular segment" between the corresponding inner sides of G and $\mathcal{F}^{int}$, so $G \backslash \mathcal{F}^{int} = \cup_{j=1}^{2n} G_j$. We have to check (2.8) on each G_j. For $w \in G_j$ there is a unique $z \in \mathcal{F}^{int}$ so that $w = \gamma_j(z)$. Since

$$\begin{aligned} \mathrm{d}(w,\mathbb{T}) &= \mathrm{d}(\gamma_j(z), \gamma_j(\mathbb{T})) \asymp d(z,\mathbb{T}), \\ \mathrm{d}(z,E) &= \mathrm{d}(\gamma_j^{-1}(w), E) \asymp \mathrm{d}(w, \gamma_j(E)) \ge C\,\mathrm{d}(w,E), \end{aligned}$$

where we used in an essential way that the number of generators is finite, we see that for $w \in G_j$

$$\begin{aligned} \log|F(w)| &= \log|F(z)| \le \frac{K_1|z|^{p+q}}{\mathrm{d}^p(z,\mathbb{T})\ \mathrm{d}^{p+2q}(z,E)} \\ &\le \frac{CK_1|w|^{p+q}}{\mathrm{d}^p(w,\mathbb{T})\ \mathrm{d}^{p+2q}(w,E)}, \end{aligned}$$

the first equality being exactly the automorphic property of F. Theorem 1.2 with $s = p+q$ then yields

$$\sum_{w \in \tilde{G} \cap Z_F} \frac{\mathrm{d}^{p+1+\varepsilon}(w,\mathbb{T}_w)}{|w|^{(p+q-1+\varepsilon)_+}}\ \mathrm{d}^{(p+2q-1+\varepsilon)_+}(w,E) \le C \cdot K_1 \tag{2.9}$$

for $0 < \varepsilon < 1$, where $\tilde{G}$ is another polygon with $\mathcal{F}^{int} \subset \tilde{G} \subset G$. The more so, the same inequality holds for $w \in \overline{\mathcal{F}^{int}} \cap Z_F$.

It remains only to go back to $f \in \mathcal{A}(\Omega)$ and its zero set Z_f. Note that although each point from Z_f has infinitely many preimages in $\mathbb{D}$, we can restrict ourselves with those in $\overline{\mathcal{F}^{int}}$. It follows easily from

the properties of the covering map (see the proof of Lemma 2.1) that $1+|\lambda| \asymp \frac{1}{|w|}$. Hence, (2.2) yields

$$\mathrm{d}(w,E) \asymp \left(\frac{\mathrm{d}(\lambda,e)}{1+|\lambda|}\right)^{1/2},$$

and, with the help of (2.3)

$$\mathrm{d}(w,\mathbb{T}_w) \asymp \frac{\mathrm{d}(\lambda,\mathfrak{e})}{(\mathrm{d}(\lambda,e)\,(1+|\lambda|))^{1/2}}.$$

Substitution of the above relations in (2.9) gives (0.4), and the proof of Theorem 0.1 is complete. □

3. Applications to complex perturbations of a finite-band selfadjoint operator

Consider a bounded finite-band selfadjoint operator A_0, defined on H. Let $A = A_0 + B$, $B \in \mathcal{S}_p$, with $p \geq 1$, B is not supposed to be selfadjoint.

The Schatten classes $\mathcal{S}_p$ form a nested family of operator ideals, that is,

(1) if $p < q$, then $\mathcal{S}_p \subset \mathcal{S}_q$ and $\|\cdot\|_{\mathcal{S}_q} \leq \|\cdot\|_{\mathcal{S}_p}$;
(2) if P is a bounded operator, and $Q \in \mathcal{S}_p$, then $PQ, QP \in \mathcal{S}_p$ and $\|PQ\|_{\mathcal{S}_p}, \|QP\|_{\mathcal{S}_p} \leq \|P\|\|Q\|_{\mathcal{S}_p}$.

More information on the classes $\mathcal{S}_p$ can be found in monographs [8] and [12].

Given $p \geq 1$ put $\lceil p \rceil := \min\{j \in \mathbb{N} : \ j \geq p\}$. The following object known as a *regularized perturbation determinant*

$$g_p(\lambda) := \det{}_{\lceil p \rceil}(A-\lambda)(A_0-\lambda)^{-1}$$

is well defined, $g_p \in \mathcal{A}(\Omega)$, $\Omega = \overline{\mathbb{C}} \backslash \sigma(A_0)$. The basic property of g_p relates its zero set and the discrete spectrum of A:

$\lambda \in Z_{g_p}$ with order k if and only if $\lambda \in \sigma_d(A)$ with algebraic multiplicity k.

Furthermore, for $\lambda \in \Omega$ the bound

$$\log|g_p(\lambda)| \leq C_p\,\|(A_0-\lambda)^{-1}\|^p\,\|B\|^p_{\mathcal{S}_p}$$

holds, see, e.g., [12]. For the selfadjoint and finite-band operator A_0 the latter turns into

$$\log|g_p(\lambda)| \leq C_p\,\frac{\|B\|^p_{\mathcal{S}_p}}{\mathrm{d}^p(\lambda,\mathfrak{e})},$$

which is exactly (0.6).

Theorem 0.4 thus follows by a straightforward application of Corollary 0.3.

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