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ITÔ DIFFUSIONS, MODIFIED CAPACITY AND HARMONIC MEASURE. APPLICATIONS TO SCHRÖDINGER OPERATORS.

S. DENISOV, S. KUPIN

ABSTRACT. We observe that some special Itô diffusions are related to scattering properties of a Schrödinger operator on \mathbb{R}^d , $d \geq 2$. We introduce Feynman-Kac type formulae for these stochastic processes which lead us to results on the preservation of the a.c. spectrum of the Schrödinger operator. To better understand the analytic properties of the processes, we construct and study a special version of the potential theory. The modified capacity and harmonic measure play an important role in these considerations. Various applications to Schrödinger operators are also given. For example, we relate the presence of the absolutely continuous spectrum to the geometric properties of the support of the potential.

INTRODUCTION

The main motivation for this paper was an exciting open problem in the spectral theory of a multi-dimensional Schrödinger operator. Namely, we are interested in the spectral properties of the operator

$$(0.1) \quad H = H_V = -\Delta + V$$

acting on $L^2(\mathbb{R}^d)$, $d \geq 2$. One of the central problems of the topic is to understand under what conditions on V the absolutely continuous (a.c., to be brief) spectrum $\sigma_{ac}(H)$ is present and what its essential support is. Putting this a bit differently, we want to know what perturbations V preserve the a.c. spectrum as compared to $H_0 = -\Delta$. Of course, $\sigma(H_0) = \sigma_{ac}(H_0) = \mathbb{R}_+$. The related physical intuition is that if the potential decays sufficiently fast at infinity, there is some a.c. spectrum which means that the scattering properties of the medium modeled by the operator are not too bad.

For instance, Deift-Killip [13] proved in one-dimensional case that the condition $V \in L^p(\mathbb{R}_+)$ leads to $\sigma_{ac}(H) = \mathbb{R}_+$ if $p \leq 2$. If $p > 2$, there are some V such that the spectrum is singular [29]. For $d \geq 2$, a counterpart of the result by Deift and Killip was conjectured by Simon [43]. His question is: does

$$(0.2) \quad \int_{\mathbb{R}^d} \frac{V^2(x)}{|x|^{d-1} + 1} dx < \infty$$

imply that $\sigma_{ac}(H) = \mathbb{R}_+$? Here one might also need more regularity of the potential V for the operator H_V to be well-defined, but this is a minor issue.

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While being wide open for the case \mathbb{R}^d or \mathbb{Z}^d , $d \geq 2$, the problem has a neat solution on a Cayley tree (Bethe lattice), see Denisov [15], Denisov-Kiselev [14]. Since this construction serves as a starting point to the present investigation, it seems instructive to recall some of its details.

Assume that the Cayley tree \mathbb{B} is rooted with the root (the origin) denoted by O , O has two neighbors and other vertices have one ascendant and two descendants (the actual number of descendants is not important but it should be the same for all points $X \neq O$). The set of vertices of the tree is denoted by $\mathbb{V}(\mathbb{B})$. For an $f \in \ell^2(\mathbb{V}(\mathbb{B}))$, define the free Laplacian by

$$(H_0 f)_n = \sum_{\text{dist}(i,n)=1} f_i, \quad n \in \mathbb{V}(\mathbb{B})$$

One can show rather easily [15, Sect. 2] that the spectrum of H_0 is purely a.c. on $[-2\sqrt{2}, 2\sqrt{2}]$. Assume now that V is a bounded potential on $\mathbb{V}(\mathbb{B})$ so that

$$H = H_0 + V$$

is well-defined. Denote the spectral measure related to delta function at O by σ_O ; the density of its absolutely continuous part is σ'_O . Take $w(\lambda) = (4\pi)^{-1}(8 - \lambda^2)^{1/2}$ and let $\rho_O(\lambda) = \sigma'_O(\lambda)w^{-1}(\lambda)$.

Consider also the probability space on the set of nonintersecting paths in \mathbb{B} that go from the origin to infinity. This space is constructed by assigning the Bernoulli random variable to each vertex and the outcome of Bernoulli trial (0 or 1) then corresponds to whether the path (stemming from the origin) goes to the “left” or to the “right” descendant at the next step. Notice also that (discarding a set of Lebesgue measure zero) each path is in one-to-one correspondence with a point on the interval $[0, 1]$ by the binary decomposition of reals. In this way, the “infinity” for \mathbb{B} can be identified with $[0, 1]$. For any $t \in [0, 1]$, we can then define the function ϕ as

$$\phi(t) = \sum_{n=1}^{\infty} V^2(x_n)$$

where the path $\{x_n\} \subset \mathbb{V}(\mathbb{B})$ corresponds to t . This function does not have to be finite at any point t but it is well-defined and is Lebesgue measurable. See [14] for

Theorem 0.1. *For any bounded V ,*

$$\begin{aligned} \int_{-2\sqrt{2}}^{2\sqrt{2}} w(\lambda) \log \rho_O(\lambda) d\lambda &\geq \log \mathbb{E} \left\{ \exp \left[-\frac{1}{4} \sum_{n=1}^{\infty} V^2(x_n) \right] \right\} \\ &= \log \int_0^1 \exp \left(-\frac{1}{4} \phi(t) \right) dt \end{aligned}$$

where the expectation is taken with respect to all paths $\{x_n\}$ and the probability space defined above. In particular, if the right hand side is finite, then $[-2\sqrt{2}, 2\sqrt{2}] \subseteq \sigma_{ac}(H)$.

The proof of the theorem is based on the adjusted form of sum rules in the spirit of Killip-Simon [28]. Higher order sum rules are applied to different classes of potentials in Kupin [31].

Notice that ϕ is always nonnegative, therefore the right hand side is bounded away from $-\infty$ iff $V \in \ell^2$ with a positive probability. This is the true multi-dimensional L^2 -condition. The simple application of Jensen's inequality then immediately implies that the estimate

$$\int \phi(t) dt = \sum_{n=0}^{\infty} 2^{-n} \sum_{\text{dist}(X, O)=n} V^2(X) < \infty$$

guarantees $[-2\sqrt{2}, 2\sqrt{2}] \subseteq \sigma_{ac}(H)$. The last condition is precisely the analogue of (0.2) for the Cayley tree. Indeed, the factor 2^n is the “area” of the sphere of radius n in \mathbb{B} and is exactly the counterpart of $|x|^{d-1}$ in (0.2).

Remark. Assume that in the above model there is a set F of vertices on which V is uncontrollable and $V = 0$ on $\mathbb{V}(\mathbb{B}) \setminus F$. Then, the theorem says that the a.c. spectrum of H contains $[-2\sqrt{2}, 2\sqrt{2}]$ as long as there are “enough” paths that do not visit F .

The substantial part of this paper is devoted to the study of analogous phenomena in the case of \mathbb{R}^d (see, e.g., Theorem 4.1). In its first part, we obtain a result similar to Theorem 0.1 for a Schrödinger operator (0.1). Besides various technical difficulties, an immediate problem we run into is the question of how to introduce a probability space of paths in \mathbb{R}^d similar to the one appearing in (0.3).

It turns out rather naturally that the right probability space of paths is given by Itô's stochastic calculus or, more precisely, by a (stationary) Itô stochastic differential equation of the form

$$dX_t = p(X_t)dt + dB_t, \quad X_0 = x^0$$

where B_t is a \mathbb{R}^d -Brownian motion, $x^0 \in \mathbb{R}^d$, and p is defined by (1.1). The solution $\{X_t\}$ to this equation is called the Itô diffusion. The probability measure on the probability space of paths is denoted by $\mathbb{P} = \mathbb{P}_{X^0}$ and the corresponding expectation is \mathbb{E}_{x^0} . The coefficient p is termed a drift. We refer to the nice books by Øksendal [38] and Bass [7] in this connection; see also Section 1 for more details.

As a corollary of Feynman-Kac type formulae proved in Section 2, we get the following theorem.

Theorem (= Theorem 2.2). *Let $V \geq 0$ be continuous, $f \in L^2(\mathbb{R}^d)$, $f \geq 0$ and f have a compact support. Let σ_f be the spectral measure of f with respect to H_V . We have*

$$(0.3) \quad \exp \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\log \sigma'_f(k^2)}{1+k^2} dk \right] \geq C_f \int f(x^0) \mathbb{E}_{x^0} \left[\exp \left(-\frac{1}{2} \int_0^{\infty} V(X_\tau) d\tau \right) \right] dx^0$$

and the constant $C_f > 0$ does not depend on V .

Above, σ'_f stands for the density of the absolutely continuous component of σ_f . This theorem can be viewed as a counterpart of Theorem 0.1 although

the L^2 -summability over the path is replaced by the stronger L^1 -condition. Here is one of its corollaries.

Corollary (=Corollary 2.1). *Let V and f be as above. Then*

$$\int_{\mathbb{R}} \frac{\log \sigma'_f(k^2)}{1+k^2} dk \geq -C_{1,f} \int_{\mathbb{R}^d} \frac{V(x)}{|x|^{d-1}+1} dx - C_{2,f}$$

It is appropriate to compare this to results of Laptev-Naboko-Safronov [34, 35]. For a potential V , we write $V_{\pm} = \max(\pm V, 0)$ so that $V = V_+ - V_-$, $|V| = V_+ + V_-$. Let Δ_D be the Laplace operator on $\mathbb{R}^d \setminus B(0, 1)$ with the Dirichlet boundary condition on the unit sphere. Take $H = -\Delta_D + V$. We have

Theorem 0.2 ([34]). *Let $V \in \ell^\infty(\mathbb{Z}^d, L^q(\Pi_1))$, $q > d/2$, and*

$$\int_{\mathbb{R}^d} V_-^{(d+1)/2}(x) dx < \infty, \quad \int_{\mathbb{R}^d} \frac{V_+(x)}{|x|^{d-1}+1} dx < \infty$$

Then

$$\int_{\mathbb{R}_+} \frac{\log \sigma'_f(E)}{(1+E^{3/2})E^{1/2}} dE > -\infty$$

The function $f \in L^2(\mathbb{R}^d)$ is bounded, spherically symmetric and has compact support, $\Pi_1 = [0, 1]^d \subset \mathbb{R}^d$.

The discovery of deep and fruitful relations between the Brownian motion and Schrödinger operator probably goes back as far as to Wiener [45], Feynman [20] and Kac [25, 26]. Chung-Zhao [10] use these ideas to make connections with gauge theory, gaugeability, the properties of the discrete spectrum and eigenfunctions of the Schrödinger operator. Aizenman-Simon [4] apply this technique to study the properties of the evolution semigroup of H_V ; see Simon [42] for an overview of the topic. The new aspects of our approach are that, first, we work with an “appropriately modified Brownian motion” (= the Itô diffusion), and not its “classical” version. Second, this allows us to bypass several steps of computations and to get rather directly to the spectral measure of H_V and, especially, its a.c. component.

Further analysis requires a good understanding of stochastic integrals appearing in the right hand side of (0.3). To simplify the picture, we then assume that $V = 0$ on a domain $\Omega \subset \mathbb{R}^d$ and V is of arbitrary size on $E = \Omega^c$. The second part of the article deals with stochastic integrals (0.3) from the point of view of the potential theory induced by $\{X_t\}$. To give an idea of the results obtained in this direction, we give some definitions for $d = 3$; Section 3.1 contains more details. Consider operators

$$L^{\pm} = \frac{1}{2} \Delta \mp \partial_{x_1}$$

We say that a function u is L^{\pm} -harmonic on Ω , if $L^{\pm}u = 0$. Since L^{\pm} are second order elliptic operators, one can show that L^{\pm} -harmonic functions possess many usual properties of harmonic functions (i.e., the max/min principle, Harnack principle etc., see Landis [32]).

We want to build the potential theory for these operators. Modulo some technical aspects, the construction follows the lines of the classical case

related to $L = \Delta$; see, for instance, Landkof [33], Hayman-Kennedy [24] and Garnett-Marshall [21]. So, let $z = (z_1, z')$, $\xi = (\xi_1, \xi') \in \mathbb{R}^3$ with $z, \xi' \in \mathbb{R}^2$. We introduce the potentials

$$K^\pm(z, \xi) = 2G_0(z, \xi)e^{\pm(z_1 - \xi_1)}$$

where G_0 is Green's function for $(-\Delta + 1)^{-1}$ on \mathbb{R}^3 , that is

$$G_0(z, \xi) = G_0(z, \xi; i) = \frac{1}{4\pi} \frac{e^{-|z - \xi|}}{|z - \xi|}$$

Let E be a compact subset of \mathbb{R}^3 and $\mathcal{P}(E)$ be the set of probability Borel measures on E . For $\mu \in \mathcal{P}(E)$, set

$$U_\mu^\pm(z) = \int_E K^\pm(z, \xi) d\mu(\xi)$$

to be the corresponding potential. Consider

$$C^\pm(E) = \left(\inf_{\mu \in \mathcal{P}(E)} \sup_{z \in \mathbb{R}^3} U_\mu^\pm(z) \right)^{-1}$$

It turns out that, by Theorem 3.1, $C^\pm(E)$ are equal and their common value is denoted by $C(E)$. We call $C(E)$ the modified capacity of E . More involved results along with some developments of this theory are in Section 3.

In Section 4, we give applications of the introduced techniques to the spectral theory of Schrödinger operator. For example, we construct an obstacle E with the following properties:

- $\mathbb{P}(X_t \text{ does not hit } E) > 0$ and hence $\sigma_{ac}(H) = \mathbb{R}_+$.
- any ray, issued from the origin, intersects E infinitely many times.

It is interesting to compare this example to the result by Amrein-Pearson [5], where the authors show that $\sigma_{ac}(H) = \mathbb{R}_+$ if there is a sufficiently thick obstacle-free cone. Our results suggest that the phenomenon of the preservation of the a.c. spectrum is much finer and is of a capacity nature. In particular, we present an example where the above ‘‘cone condition’’ is not satisfied but the a.c. spectrum is nevertheless preserved. The Appendices A and B contain some bounds on modified harmonic measure with respect to the operators L^\pm introduced above.

We conclude the introduction with a few words on the notations. We write $B(x, r) = \{y : |y - x| < r\}$, $r > 0$, for an open ball in \mathbb{R}^d , and $\text{int}(A)$ is the interior of an $A \subset \mathbb{R}^d$. Σ_r is a sphere of radius $r > 0$ in \mathbb{R}^d . For a domain Ω in \mathbb{R}^d , $L^p(\Omega)$ is the usual Lebesgue space and

$$W^{1,2}(\Omega) = \left\{ f : \int_\Omega (|f|^2 + |\nabla f|^2) dx < \infty \right\}$$

the derivatives being understood in the distributional sense. We write

$$f(x) \approx g(x), \quad x \rightarrow x_0$$

if

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 1$$

and

$$f(x) \sim g(x), \quad \text{for } x \in I$$

if

$$C_1 < \frac{f(x)}{g(x)} < C_2, \quad x \in I$$

with some $C_1, C_2 > 0$. We also write $f(x) \lesssim g(x)$ on I if $f(x) < Cg(x)$ for $x \in I$ with some $C > 0$.

As for the probabilistic notation, let as usual $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, that is, Ω is the probability set, \mathcal{F} is a σ -algebra of measurable subsets from Ω , and \mathbb{P} is a probability measure, $\mathbb{P}(\Omega) = 1$, defined on sets from \mathcal{F} . We say that $X : \Omega \rightarrow \mathbb{R}$ is a (scalar) random variable, if it is \mathcal{F} -measurable. Of course, $X = (X_j)_{j=1, \dots, d} : \Omega \rightarrow \mathbb{R}^d$ is a vector-valued random variable, if all X_j are scalar random variables. Recall that for an $A \in \mathcal{F}$ and a random variable $X = (X_j)_{j=1, \dots, d}$ we write

$$\mathbb{P}(A) = \int_A d\mathbb{P}(\omega), \quad \mathbb{E}(X) = \int_{\Omega} X(\omega) d\mathbb{P}(\omega).$$

We refer to the books by Øksendal [38] and Karatzas-Shreve [27] for the rest of the probabilistic notation used in the article.

1. A STOCHASTIC DIFFERENTIAL EQUATION

The following stochastic differential equation will play an important role later on. The choice of its coefficients, for instance, is crucial for the proof of Theorem 2.1; see the discussion preceding the result for more details.

So, consider the Lipschitz vector field

$$(1.1) \quad p(x) = \left(\frac{I'_\nu(|x|)}{I_\nu(|x|)} - \nu|x|^{-1} \right) \cdot \frac{x}{|x|}, \quad \nu = (d-2)/2$$

where I_ν denotes the modified Bessel function [1, Sect. 9.6]. The asymptotics of I_ν, I'_ν at zero and at infinity are given by Abramowitz-Stegun [1], formulae (9.6.10), (9.7.1), and (9.7.3). They yield

$$(1.2) \quad p(x) = \left(1 - \left(\nu + \frac{1}{2} \right) |x|^{-1} + \underline{O}(|x|^{-2}) \right) \cdot x|x|^{-1}, \quad |x| \rightarrow \infty$$

$$(1.3) \quad p(x) = \left(\frac{|x|}{2(\nu+1)} + \underline{O}(|x|^3) \right) \cdot x|x|^{-1}, \quad |x| \rightarrow 0$$

In fact, it is the asymptotics of p at infinity that plays the key role in later analysis and other p can be considered. However, the choice we make (i.e., formula (1.1)) will greatly simplify the calculations later on (see, e.g., (1.6)).

Then, fix any point $x^0 \in \mathbb{R}^d$ and consider the following stochastic process

$$(1.4) \quad dX_t = p(X_t)dt + dB_t, \quad X_0 = x^0$$

with the drift given by p (1.1). The solution to this diffusion process exists and all trajectories are continuous almost surely. Its generating operator [38, Sect. 7.3] is given by

$$(1.5) \quad A = \frac{1}{2} \Delta + p \cdot \nabla$$

We need to understand better the properties of the trajectories X_t . Consider the radial component $Z_t = |X_t|$. By Ito's formula [38, Sect. 4.2], we have

$$dZ_t = d\tilde{B}_t + \left(\frac{1}{2Z_t} + \frac{I'_\nu(Z_t)}{I_\nu(Z_t)} \right) dt$$

where \tilde{B}_t is a one-dimensional Brownian motion [38, Sect. 8.4]. The generating operator for Z_t [38, Sect. 7.3] is given by

$$D = \frac{1}{2} \frac{d^2}{dx^2} + \alpha(x) \frac{d}{dx}$$

where

$$\alpha(x) = \frac{1}{2x} + \frac{I'_\nu(x)}{I_\nu(x)}$$

From (1.2), we get $\alpha(x) = 1 + \underline{O}(x^{-2})$ for $x \rightarrow \infty$ so all paths of Z_t go to infinity almost surely. Notice that

$$p(x) = \nabla \log (|x|^{-\nu} I_\nu(|x|))$$

and consider $Q = |p|^2 + \operatorname{div} p$. Recall that ([1], formula (9.6.1)),

$$r^2 I''_\nu(r) + r I'_\nu(r) = (r^2 + \nu^2) I_\nu(r)$$

Since

$$\Delta f(|x|) = \frac{1}{r^{d-1}} \partial_r (r^{d-1} \partial_r f)$$

with $r = |x|$, we have

$$\Delta(|x|^{-\nu} I_\nu(|x|)) = |x|^{-\nu} I_\nu(|x|)$$

An easy computation shows that

$$(1.6) \quad Q = 1$$

on \mathbb{R}^d . Consider now a self-adjoint semigroup given by

$$(1.7) \quad \psi_t = \frac{1}{2} \Delta \psi - \frac{Q}{2} \psi$$

Its transition probability is

$$\hat{p}(x, y, t) = \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x-y|^2}{2t} - \frac{t}{2}}$$

On the other hand, let $F(x) = |x|^{-\nu} I_\nu(|x|)$ and $\psi = F\phi$. Notice that ψ satisfies (1.7) iff

$$(1.8) \quad \phi_t = \frac{1}{2} \Delta \phi + p \cdot \nabla \phi$$

The operator appearing in the right hand side of the above equality is precisely A from (1.5). So, due to the connection between the diffusion and the processes with killing [38, Ch. 8], Exercise 8.16, we have

$$q(x, y, t) = \frac{|y|^{-\nu} I_\nu(|y|)}{|x|^{-\nu} I_\nu(|x|)} \hat{p}(x, y, t)$$

where $q(x, y, t)$ is the transition probability for X_t , that is

$$\mathbb{P}(X_t \in A, X_0 = x) = \int_A q(x, y, t) dy.$$

Once again, the asymptotics (9.7.1) from [1] implies

$$(1.9) \quad q(x, y, t) \sim \frac{1}{(2\pi t)^{d/2}} \left(\frac{|x|}{|y|} \right)^{(d-1)/2} \exp \left(-\frac{|x-y|^2}{2t} - \frac{t}{2} + |y| - |x| \right)$$

as $|x|, |y|$ are large.

2. FROM ITÔ CALCULUS TO SPECTRAL PROPERTIES OF A SCHRÖDINGER OPERATOR

2.1. Feynman-Kac type formulae. We start this subsection by introducing some notations. Let H_V be a Schrödinger operator (0.1) and assume the potential V is nonnegative, continuous, and has a compact support. The Green's function $G = G_V$ of the operator H_V is defined by

$$(2.1) \quad ((H - k^2)^{-1}f)(x) = \int_{\mathbb{R}^d} G_V(x, y; k) f(y) dy$$

where $f \in L^2(\mathbb{R}^d)$, $k \in \mathbb{C}_+$, and $x, y \in \mathbb{R}^d$. Recall that for $V = 0$,

$$G_0(x, y; k) = C'_d (-ik)^\nu \frac{K_\nu(-ik|x-y|)}{|x-y|^\nu}$$

where $\nu = (d-2)/2$, $C'_d = 1/(2\pi)$ for $d = 2$, and $C'_d = 1/(4(2\pi)^{d-2})$ for $d \geq 3$. The asymptotics yields

$$G_0(x, y; k) \approx \tilde{C}_d \begin{cases} -\log|x-y|, & d = 2, \\ |x-y|^{-(d-2)}, & d \geq 3, \end{cases} \quad x \rightarrow y$$

where $\tilde{C}_d = 1/(2\pi)$ for $d = 2$, $\tilde{C}_d = \Gamma(d/2)/(2\pi^{d/2}(d-2))$ for $d \geq 3$. We also have

$$(2.2) \quad G_0(x, y; k) \approx C_d (-ik)^{\nu-1/2} \frac{e^{ik|x-y|}}{|x-y|^{(d-1)/2}}, \quad |x| \rightarrow \infty$$

Since V is compactly supported, we have the following relations (e.g., Denisov [16]):

- For a fixed x^0 , define the amplitude a_{x^0} as

$$(2.3) \quad G(x, x^0; k) \approx C_d (-ik)^{\nu-1/2} \frac{e^{ik|x|}}{|x|^{(d-1)/2}} a_{x^0}(\theta, k),$$

where $\theta = x/|x| \in \Sigma_1$ and $|x| \rightarrow \infty$, the constant C_d is from (2.2).

- Furthermore,

$$(2.4) \quad G(x, x^0; k) \approx C_d (-ik)^{\nu-1/2} \frac{e^{ik|x-x^0|}}{|x-x^0|^{(d-1)/2}} \beta_{x^0}(\theta, k),$$

where $k \in \mathbb{C}^+$ and $|x| \rightarrow \infty$.

- We have $|x| = \langle x^0, \theta \rangle + |x - x^0| + \bar{o}(1)$ as $|x| \rightarrow \infty$ and consequently

$$(2.5) \quad a_{x^0}(\theta, i) = \beta_{x^0}(\theta, i) e^{-ik\langle x^0, \theta \rangle}$$

- Consider the function $u(\cdot, k) = (H - k^2)^{-1}f$, where $f \in L^2(\mathbb{R}^d)$ and has a compact support. The function u has the following asymptotics

$$u(x, k) \approx C_d (-ik)^{\nu-1/2} \frac{e^{ik|x|}}{|x|^{(d-1)/2}} A_f(\theta, k)$$

as $|x| \rightarrow \infty$, $\theta = x/|x|$. For A_f , we have

$$(2.6) \quad \begin{aligned} A_f(\theta, k) &= \int e^{-ik\langle x^0, \theta \rangle} \beta_{x^0}(\theta, k) f(x^0) dx^0 \\ &= \int a_{x^0}(\theta, k) f(x^0) dx^0 \end{aligned}$$

The next result establishes the connections between the amplitude $a_{x^0} = a_{x^0}(\cdot, i)$ and the properties of certain Itô diffusions. More precisely, the average of a_{x^0} on the unit sphere Σ_1 is related to diffusion X_t (1.4) with radial drift (1.1). Diffusion G_t (2.8) is connected to the value $a_{x^0}(\theta)$, $\theta \in \Sigma_1$. Observe that the diffusions X_t and G_t are defined on purpose so that their generators match the Schrödinger operator in relations (2.11) and (2.13), respectively.

Let \mathbb{P}_{x^0} and \mathbb{E}_{x^0} be the probability measure and the expectation corresponding to the processes starting at $x^0 \in \mathbb{R}^d$, i.e. $X_0 = x^0$ or $G_0 = x^0$. We denote by $\{e^k\}_k$ be the standard basis in \mathbb{R}^d and, for $x \in \mathbb{R}^d$, we write $x = \sum_k x_k e^k = x_1 e^1 + x'$; so $x = (x_1, x')$.

Theorem 2.1. (*Feynman–Kac type formula*)

(1) *Let X_t be the solution to (1.4). Then we have*

$$(2.7) \quad \int_{\Sigma_1} a_{x^0}(\theta) d\theta = C_1 \mathbb{E}_{x^0} \left[\exp \left(-\frac{1}{2} \int_0^\infty V(X_\tau) d\tau \right) \right]$$

where the subscript x^0 means that the process starts at x^0 .

(2) *For $\theta \in \Sigma_1$, let $dG_t = \theta dt + dB_t$. Then*

$$(2.8) \quad a_{x^0}(\theta) = C_2 \mathbb{E}_{x^0} \left[\exp \left(-\frac{1}{2} \int_0^\infty V(G_\tau) d\tau \right) \right]$$

Proof. Let V have support inside the ball $\{|x| < \rho\}$. Take r and R such that $\rho \ll r \ll R$ and consider the solution to the following problem

$$(2.9) \quad -\Delta \psi + V\psi = -\psi + f, \quad \psi|_{\Sigma_R} = 0$$

where f is characteristic function of the spherical layer $\{r < |x| < r+1\}$. Then, we have two expressions for $\psi(x^0)$. On the one hand,

$$\psi(x^0) = \int_{|x| < R} G^{(R)}(x^0, y) f(y) dy$$

where $G^{(R)}(x, y)$ is the Green's function for the Schrödinger operator in $\{|x| < R\}$ with Dirichlet boundary condition. On the other hand, we substitute

$$(2.10) \quad \psi = F \phi$$

where, just as before, $F(x) = |x|^{-\nu} I_\nu(|x|)$. The asymptotics of I_ν near zero is discussed in the beginning of Section 1. It implies that F is infinitely smooth on \mathbb{R}^d and $F \geq C > 0$. So, recalling (1.8)

$$(2.11) \quad \frac{1}{2} \Delta \phi + p \nabla \phi - \frac{1}{2} V \phi = -\frac{1}{2} f F^{-1}$$

and $\phi = 0$ on Σ_R . We have the following representation [38, Ch. 9], Exercise 9.12,

$$\phi(x^0) = \mathbb{E}_{x^0} \left[\int_0^{T_R} \exp \left(-\frac{1}{2} \int_0^\tau V(X_s) ds \right) f_1(X_\tau) d\tau \right]$$

where $f_1(x) = \frac{1}{2} f F^{-1}$ and T_R is the standard stopping time, i.e., the random time when X_t hits the boundary Σ_R for the first time. The path X_t is the solution to stochastic differential equation (1.4). Thus we have the identity (2.12)

$$\frac{e^r}{r^{(d-1)/2}} \int_{r < |x| < r+1} G^{(R)}(x^0, y) f(y) dy = \mathbb{E}_{x^0} \left[\int_0^{T_R} \exp \left(-\frac{1}{2} \int_0^\tau V(X_s) ds \right) f_2(X_\tau) d\tau \right]$$

where

$$f_2(x) = f_1(x) \frac{e^r}{r^{(d-1)/2}}$$

It is well known that $G^{(R)}(x, y)$ tends to $G(x, y)$ uniformly for x, y from a fixed compact as $R \rightarrow \infty$. Therefore, asymptotics (2.3) yields

$$\lim_{r \rightarrow \infty} \lim_{R \rightarrow \infty} (\text{LHS of (2.12)}) = C \int_{\Sigma_1} a_{x^0}(\theta) d\theta$$

Since T_R goes to infinity as $R \rightarrow \infty$, we have

$$\begin{aligned} \mathbb{E}_{x^0} \left[\int_0^{T_R} \exp \left(-\frac{1}{2} \int_0^\tau V(X_s) ds \right) f_2(X_\tau) d\tau \right] &\longrightarrow \\ \mathbb{E}_{x^0} \left[\int_0^\infty \exp \left(-\frac{1}{2} \int_0^\tau V(X_s) ds \right) f_2(X_\tau) d\tau \right] \end{aligned}$$

Now, let us compute the limit of the last expression as $r \rightarrow \infty$. It makes sense to consider the stopping time $T_{r/2}$: the time when the path X_t hits the sphere $\Sigma_{r/2}$ for the first time. Hence,

$$\begin{aligned} \mathbb{E}_{x^0} \left[\int_0^\infty \exp \left(-\frac{1}{2} \int_0^\tau V(X_s) ds \right) f_2(X_\tau) d\tau \right] &= \\ \mathbb{E}_{x^0} \left[\exp \left(-\frac{1}{2} \int_0^{T_{r/2}} V(X_s) ds \right) \int_{T_{r/2}}^\infty \exp \left(-\frac{1}{2} \int_{T_{r/2}}^\tau V(X_s) ds \right) f_2(X_\tau) d\tau \right] \end{aligned}$$

By the strong Markov property [38, Sect. 7.2], the last expectation equals to

$$\mathbb{E}_{x^0} \left[\exp \left(-\frac{1}{2} \int_0^{T_{r/2}} V(X_s) ds \right) \mathbb{E}_{X_{T_{r/2}}} \left[\int_0^\infty \exp \left(-\frac{1}{2} \int_0^\tau V(\tilde{X}_s) ds \right) f_2(\tilde{X}_\tau) d\tau \right] \right]$$

The process \tilde{X}_t is the solution to (1.4) where $x^0 = X_{T_{r/2}}$.

As $r \rightarrow \infty$, the inner expectation tends to a constant independent of $X_{T_{r/2}}$ by (1.9). Since $T_{r/2}$ goes to infinity almost surely, the dominated convergence theorem yields (2.7).

The proof of the second claim of the theorem is similar. Without loss of generality, we assume $\theta = e^1$. Let f in (2.9) be given by

$$f = \chi_{\{r < |x| < r+1\}} \cdot \chi_{\{|x'| < \sqrt{r}\}}$$

We now take $F(x) = e^{x_1}$ in (2.10). Then, we have

$$(2.13) \quad \frac{1}{2}\Delta\phi + \phi_{x_1} - \frac{1}{2}V\phi = -\frac{1}{2}fe^{-x_1}$$

and

$$\phi(x^0) = \mathbb{E}_{x^0} \left[\int_0^{T_R} \exp\left(-\frac{1}{2} \int_0^\tau V(G_s) ds\right) f_1(G_\tau) d\tau \right]$$

where $f_1(x) = \frac{1}{2}f(x)e^{-x_1}$. We again have

(2.14)

$$\int_{r < |x| < r+1} e^r G^{(R)}(x^0, y) f(y) dy = \mathbb{E}_{x^0} \left[\int_0^{T_R} \exp\left(-\frac{1}{2} \int_0^\tau V(G_s) ds\right) f_2(G_\tau) d\tau \right]$$

and $f_2(x) = \frac{1}{2}f(x)e^{r-x_1}$. Take $R \rightarrow \infty$ first and then $r \rightarrow \infty$. Definition of a_{x^0} (2.3) implies that the left hand side of the above equality will converge to a multiple of $a_{x^0}(e^1)$. For the right hand side, we have

$$\lim_{R \rightarrow \infty} (\text{RHS of (2.14)}) = \mathbb{E}_{x^0} \left[\int_0^\infty \exp\left(-\frac{1}{2} \int_0^\tau V(G_s) ds\right) f_2(G_\tau) d\tau \right]$$

Let ϵ be small positive ($\epsilon < 1/2$ is enough) and t_{r^ϵ} be the first time when G_τ hits the plane $x_1 = r^\epsilon$. Then,

$$\begin{aligned} & \mathbb{E}_{x^0} \left[\int_0^\infty \exp\left(-\frac{1}{2} \int_0^\tau V(G_s) ds\right) f_2(G_\tau) d\tau \right] = \\ & \mathbb{E}_{x^0} \left[\exp\left(-\frac{1}{2} \int_0^{t_{r^\epsilon}} V(G_s) ds\right) \mathbb{E}_{G_{t_{r^\epsilon}}} \left[\int_0^\infty \exp\left(-\frac{1}{2} \int_0^\tau V(\tilde{G}_s) ds\right) f_2(\tilde{G}_\tau) d\tau \right] \right] \end{aligned}$$

The process \tilde{G}_t again solves the same equation $d\tilde{G}_t = e^1 dt + dB_t$ but with initial value $\tilde{G}_0 = G_{t_{r^\epsilon}}$.

Let $|G'_{t_{r^\epsilon}}| < r^\epsilon$. Then, the inner expectation converges to some $G_{t_{r^\epsilon}}$ -independent constant as $r \rightarrow \infty$. On the other hand, $\mathbb{P}(|G'_{t_{r^\epsilon}}| > r^\epsilon) \rightarrow 0$, so the remaining factor from the right hand side of the above equality tends to

$$\mathbb{E}_{x^0} \left[\exp\left(-\frac{1}{2} \int_0^\infty V(G_\tau) d\tau\right) \right]$$

□

Remark. Given any continuous nonnegative potential V , define truncations $V^{(n)}(x) = V(x) \cdot \mu_n(x)$ where $\mu_n(x)$ is smooth, equals to one on $|x| < n$ and to 0 on $|x| > n + 1$, and $0 \leq \mu_n(x) \leq 1$ everywhere. For each n , the above theorem applies. The monotonicity of Green's function in V and the monotone convergence theorem allow one to show that formula (2.7) is true for all continuous nonnegative potentials. The amplitude $a_{x^0}(\theta)$ is well-defined as the $\lim_{n \rightarrow \infty} a_{x^0}^{(n)}(\theta)$ where $a_{x^0}^{(n)}$ corresponds to $V^{(n)}$. Of course, in this case both expressions can be equal to 0.

2.2. Applications to the scattering theory of Schrödinger operators. We apply the methods of the previous subsection to the study of the a.c. spectrum of a Schrödinger operator. Notice that various results of a similar flavor were recently obtained in [15, 16].

Theorem 2.2. *Let V be any continuous nonnegative function. Assume that $f \in L^2(\mathbb{R}^d)$ is nonnegative and has a compact support. Let σ_f be the spectral measure of f with respect to H_V and σ'_f be the density of its a.c. part. Then we have*

$$(2.15) \quad \exp \left[\frac{1}{2\pi} \int_{\mathbb{R}} \frac{\log \sigma'_f(k^2)}{1+k^2} dk \right] \geq C_f \int f(x^0) \mathbb{E}_{x^0} \left[\exp \left(-\frac{1}{2} \int_0^\infty V(X_\tau) d\tau \right) \right] dx^0$$

where the constant $C_f > 0$ does not depend on V .

Proof. Suppose that V has a compact support. Recall definitions (2.3)-(2.6) from the beginning of this section. The function $\beta_{x^0}(\theta, k)$ is analytic in $k \in \mathbb{C}^+$ and has the the following asymptotics for large $\text{Im } k$

$$\beta_{x^0}(\theta, k) = 1 + \underline{O}(\text{Im } k)^{-1}$$

This follows from the analysis of the perturbation series for the resolvent, i.e.

$$G(x, y, k) = G_0(x, y, k) - \int G_0(x, s, k) V(s) G_0(s, y, k) ds + \dots$$

Notice also that the function $A_f(\theta, k)$, as a function in k , is analytic on \mathbb{C}^+ and is continuous up to $\mathbb{R} \setminus \{0\}$. The last property follows from the limiting absorption principle, see Agmon [2]. The representation

$$(2.16) \quad u = (-\Delta - k^2)^{-1}(f - Vu)$$

implies

$$(2.17) \quad |A_f(\theta, k)| \leq \frac{C_{f,V}}{\text{dist}(k^2, \mathbb{R}_+)^{1/2}}$$

for $k \rightarrow 0$. The asymptotics of β_{x^0} for large $\text{Im } k$ and (2.6) yield

$$(2.18) \quad |A_f(\theta, k)| < \exp(C_f \text{Im } k)(1 + \underline{O}(\text{Im } k)^{-1})$$

for $\text{Im } k \rightarrow +\infty$. The constant C_f depends on f only.

From (2.16), we also have the estimate for large k within the region $\{0 < \text{Im } k < C\}$

$$(2.19) \quad |A_f(\theta, k)| \leq \frac{C_{f,V}}{\text{dist}(k^2, \mathbb{R}_+)^{1/2}}$$

where the constant $C_{f,V}$ depends on V and f .

The following simple identity is true (e.g., Denisov [16])

$$(2.20) \quad \sigma'_f(k^2) = C|k|^{d-2} \|A_f(\theta, k)\|_{L^2(\Sigma_1)}^2,$$

here $k \in \mathbb{R}, k \neq 0$. It follows from the integration by parts and the limiting absorption principle [2]. In [16], this formula was proved for $d = 3$ but the same argument works for any d .

Now, observe that the function

$$g(k) = \log \|A_f(k, \theta) \exp(ikC_f)\|_{L^2(\theta \in \Sigma_1)}$$

is subharmonic on \mathbb{C}^+ (the constant C_f is chosen from (2.18)). Due to the properties of A_f listed above, we can apply the mean value inequality to g within the domain $\Omega_{\epsilon,L,M}$ bounded by the curves: $\gamma_1 = \{z : \text{Im } z = 0, \epsilon < |\text{Re } z| < L\}$, $\gamma_2 = \{z : \text{Im } z > 0, |z| = \epsilon\}$, $\gamma_3 = \{z : |\text{Re } z| = L, 0 < \text{Im } z < M\}$, $\gamma_4 = \{z : \text{Im } z = M, |\text{Re } z| < L\}$. Letting $\gamma = \bigcup_{j=1}^4 \gamma_j$ and $\omega_{\epsilon,L,M}$ be the harmonic measure of $\Omega_{\epsilon,L,M}$ aiming at i , we have

$$\int_{\gamma} g(k) \omega_{\epsilon,L,M}(k) d|k| \geq g(i)$$

Taking $L \rightarrow \infty$, then $M \rightarrow \infty$, and then $\epsilon \rightarrow 0$, we have

$$(2.21) \quad \frac{1}{\pi} \int_{\mathbb{R}} \frac{\log \|A_f(\theta, k)\|_{L^2(\Sigma_1)}}{1+k^2} dk \geq \log \|A_f(\theta, i)\|_{L^2(\Sigma_1)} - C_f$$

Each of these limits is justified by (2.17), (2.18), and (2.19).

On the other hand,

$$\|A_f(\theta, i)\|_{L^2(\Sigma_1)} \gtrsim \int_{\Sigma_1} A_f(\theta, i) d\theta = \int_{\mathbb{R}^3} \int_{\Sigma_1} f(x^0) a_{x^0}(\theta, i) d\theta dx^0$$

as follows from (2.6). Applying the first claim of Theorem 2.1 to the right hand side of the above relation and factorization (2.20) to the left hand side of (2.21), we come to

$$(2.22) \quad \exp \left[\frac{1}{2\pi} \int_{\mathbb{R}} \frac{\log \sigma'_f(k^2)}{1+k^2} dk \right] \geq C_f \int f(x^0) \mathbb{E}_{x^0} \left[\exp \left(-\frac{1}{2} \int_0^\infty V(X_\tau) d\tau \right) \right] dx^0$$

Now, consider any continuous nonnegative V and take truncations $V^{(n)} = V(x) \cdot \mu_n(x)$ defined in the remark after Theorem 2.1. For each n , (2.22) holds true. Let $\sigma_f^{(n)}$ be the spectral measure of f with respect to the Schrödinger operator with potential $V^{(n)}$. It is well-known that $\sigma_f^{(n)} \rightarrow \sigma_f$ in the weak-star topology. Therefore, the semicontinuity of the entropy (see Killip-Simon [28], for instance) applied to the left hand side of (2.22) and the monotone convergence theorem applied to its right hand side give (2.15). \square

As a corollary, we get that the absolutely continuous spectrum of H_V contains \mathbb{R}_+ if the potential V is summable over X_t with positive probability. Checking the last property for the concrete V might be difficult and should be done on a case-by-case basis. It is conceivable that one can handle the situation when V changes sign using the technique developed in [17] but

then the statements will be generic in coupling constant. The condition that f is nonnegative is not important and can be dropped.

The following is a simple corollary of Theorem 2.2, an estimate for the transition probability (1.9) and Jensen's inequality.

Corollary 2.1. *Let V and f be as in the previous theorem. Then,*

$$\int_{\mathbb{R}} \frac{\log \sigma'_f(k^2)}{1+k^2} dk \geq -C_{1,f} \int \frac{V(x)}{|x|^{d-1}+1} dx - C_{2,f}$$

This result is not new; see Laptev-Naboko-Safronov [34, 35] and Theorem 0.2.

Let $\Omega \subset \mathbb{R}^d$ be an unbounded domain. We have

Corollary 2.2. *Let V be as in Theorem 2.2, $V(x) = 0$ on Ω , and $\mathbb{P}_{x^0}(X_t \in \Omega, \forall t) > 0$. Then the a.c. spectrum of H is equal to \mathbb{R}_+ . Moreover,*

$$(2.23) \quad a_{x^0}(\theta) \gtrsim \mathbb{P}_{x^0}(G_t \in \Omega, \forall t)$$

where G_t is from (2.8).

3. THE MODIFIED HARMONIC MEASURE, CAPACITY AND THEIR PROPERTIES

The results of the previous section suggest that one needs to study functions a_{x^0} in different directions. One can use probabilistic or analytic methods for this purpose. We are going to consider an important case when $V = 0$ on Ω and one has no control on the size of V on $E = \Omega^c$. We only handle $d = 3$ in this section and explain how the results we obtain look like for $d = 2$. The situation when $d > 3$ can be treated similarly and will be discussed briefly in the Appendix B. Without loss of generality we can always assume that $\theta = \pm e^1$.

Let $\Omega \subset \mathbb{R}^3$ be a given domain with the compact $E = \Omega^c$. Take $\Gamma = \partial\Omega$ and consider the Dirichlet problem

$$(3.1) \quad \frac{1}{2}\Delta u + u_{x_1} = 0, \quad u|_{\Gamma} = f$$

where u decays at infinity, $f \in C(\Gamma)$. The solutions of Dirichlet problem for general elliptic equations on the arbitrary bounded domains go back to Püschel [41] and Oleinik [39].

The domains Ω we are interested in are unbounded but the same results easily follow by an approximation argument. Furthermore, it is proved in [41, 39] that the regular points for (3.1) coincide with the Wiener regular points (i.e., regular points with respect to Δ) and thus can be identified by the Wiener's test [21, Sect. III.7]. That said, one can fix any reference point $x^0 \in \Omega$ and consider the solution of (3.1) as a linear bounded functional Φ_{x^0} of f . By the Riesz representation theorem, we have

$$u(x^0) = \Phi_{x^0}(f) = \int_{\Gamma} f d\omega_{x^0}$$

The measure $\omega(x^0, A, \Omega)$, $A \subseteq \Gamma$ is called the modified harmonic measure of A . Clearly, any regular point of Γ belongs to the support of the modified

harmonic measure and the set of irregular points has capacity zero, see Mizuta [36, p. 136].

We will be mostly interested in the estimates on $\omega(\cdot, \Gamma, \Omega)$, which is the solution of (3.1) with $f = \chi_A$, $A \subseteq \Gamma$. It is known [39] that there exist domains Ω_n with piece-wise smooth boundaries such that $\overline{\Omega}_n \subset \Omega_{n+1}$, $\bigcup_n \Omega_n = \Omega$, and

$$(3.2) \quad \omega(x^0, \Gamma, \Omega) = \lim_{n \rightarrow \infty} \omega(x^0, \Gamma_n, \Omega_n), \quad \Gamma_n = \partial\Omega_n$$

This approximation allows us to assume the smoothness of the boundary later on.

The probabilistic meaning of ω is

$$\omega(x^0, A, \Omega) = \mathbb{P}_{x^0}(G_t \text{ hits } \Gamma \text{ for the first time at } A)$$

where $G_t = x^0 + t \cdot e^1 + B_t$, $x^0 \in \Omega$. If one solves

$$-\frac{1}{2}\Delta\psi = -\frac{1}{2}\psi, \quad x \in \Omega, \quad \psi|_{\Gamma} = \chi_A e^{x^1},$$

then

$$\omega(x) = \psi(x)e^{-x^1}$$

Thus, assuming that Ω has a piece-wise smooth boundary, the measure ω has piece-wise smooth density as well and it is given by

$$-e^{-x^1 + \xi^1} \frac{\partial}{\partial n_\xi} G(x; \xi),$$

where G is the Green's function for $2(-\Delta_D + 1)^{-1}$, Δ_D is the Laplacian with Dirichlet boundary condition on Γ , and $\xi = (\xi_1, \xi') \in \Gamma$.

The estimate from below for $1 - \omega(x^0, \Gamma, \Omega)$ is what we need to control in (2.23) in order to guarantee that the asymptotics of Green's function in the direction e^1 is comparable to the asymptotics of the free Green's function. The similar problem of visibility of infinity from the origin (for the standard Brownian motion) was recently considered in Carroll–Ortega–Cerdà [9].

In this section, we will also build the potential theory for the operators

$$L^\pm = L^{\pm e^1} = \frac{1}{2}\Delta \mp \partial_{x_1}$$

A particular attention is paid to the study of the corresponding capacity and harmonic measure. The definitions and ideas of the proofs are close in the spirit to the constructions from the potential theory for the elliptic and parabolic cases, e.g., Doob [18]. From this point of view, many results are rather standard. In the meantime, we feel like we have to write them up for the reader's convenience. Only Theorem 3.5 is substantially new and seems not to be known even for the parabolic capacities to the best of our knowledge.

Through the rest of the paper, the prefix “ L^\pm ” in front of adjectives will be systematically dropped, i.e., L^\pm -harmonic functions will be called just harmonic functions etc.

3.1. Potential theory: a special case. The content of this subsection follows the lines of the general potential theory as presented in Landkof [33], Hayman-Kennedy [24] and Garnett-Marshall [21].

Let $z = (z_1, z')$, $\xi = (\xi_1, \xi') \in \mathbb{R}^3$, and the reference direction be $+e^1$. We introduce the potential

$$(3.3) \quad K^-(z, \xi) = 2G_0(z, \xi)e^{\xi_1 - z_1}$$

where G_0 is Green's function for $(-\Delta + 1)^{-1}$ on \mathbb{R}^3 , that is

$$G_0(z, \xi) = G_0(z, \xi; i) = \frac{1}{4\pi} \frac{e^{-|z-\xi|}}{|z-\xi|}$$

Obviously, $G_0(z, 0) \sim \frac{1}{4\pi}|z|^{-1}$ as $|z| \rightarrow 0$. So, for small $|z - \xi|$, the potential $K(z, \xi)$ behaves like the standard elliptic potential for \mathbb{R}^3 and for $|z - \xi|$ large it is similar to the parabolic potential. Thus, we expect two regimes: the microscopic one will mimic the elliptic theory and macroscopic will have some resemblance to the parabolic case.

For the “dual” reference direction $-e^1$, the differential operator is L^+ and

$$K^+(z, \xi) = 2G_0(z, \xi)e^{z_1 - \xi_1}$$

It is important that

$$(3.4) \quad K^-(z, \xi) = K^+(\xi, z)$$

Let E be a compact subset of \mathbb{R}^3 and $\mathcal{P}(E)$ be the set of probability Borel measures on E . For $\mu \in \mathcal{P}(E)$, put

$$U_\mu^\pm(z) = \int_E K^\pm(z, \xi) d\mu(\xi)$$

to be the corresponding potential. Clearly, U_μ^\pm is lower semicontinuous. Consider

$$(3.5) \quad C^\pm(E) = \left(\inf_{\mu \in \mathcal{P}(E)} \sup_{z \in \mathbb{R}^3} U_\mu^\pm(z) \right)^{-1}$$

Definition 3.1. We call $C^\pm(E)$ the modified capacity of a set E in the direction $\mp e^1$.

It is clear that the capacity is translation-invariant but is not invariant under the rotation, in general. Since E is a compact, $C^\pm(E) = 0$ if and only if the elliptic capacity of E is zero as well (i.e., the polar sets in our case are the same as in the standard elliptic theory). Below, we mostly discuss the “-”-case; the “+”-case can be handled similarly.

The capacity can also be defined in the following way. Introduce the class of admissible measures $\mathcal{A}^-(E)$ as follows: $\nu \in \mathcal{A}^-(E)$ iff ν is positive measure supported on E and

$$\sup_{z \in \mathbb{R}^3} U_\nu^-(z) = 1$$

Then, we have

$$(3.6) \quad \sup_{\nu \in \mathcal{A}^-(E)} \nu(E) = C^-(E)$$

Assume $0 < C^-(E) < \infty$ and let $\{\nu_n\}$ be a maximizing sequence to (3.6). We denote one of its weak limits by ν^- . Then, $\nu^-(E) = C^-(E)$. We have ([21], Lemma 4.2)

$$U_{\nu^-}^-(z) \leq \liminf_{n \rightarrow \infty} U_{\nu_n}^-(z) \leq 1$$

and

$$\alpha = \sup_{z \in \mathbb{R}^3} U_{\nu^-}^-(z) \leq 1$$

Thus, $\hat{\nu} = \nu^-/\alpha \in \mathcal{A}^-(E)$ and $\hat{\nu}(E) = C^-(E)/\alpha \geq C^-(E)$ which means that $\alpha = 1$ and ν^- is a maximizer. Therefore, a minimizer μ^- for (3.5) exists and is equal to $\nu^-/C^-(E)$.

The main results of the elliptic theory are true in our case as well and we list some of them below for the reader's convenience. They are stated for C^- but their analogs hold with respect to any direction.

1. Monotonicity: for $E_1 \subseteq E_2$, then $C^-(E_1) \leq C^-(E_2)$.
2. Subadditivity: if $E = \bigcup_j E_j$, E_j are disjoint, then

$$C^-(E) \leq \sum_j C^-(E_j)$$

Indeed, we use (3.6). If ν^- is a maximizer in (3.6) for E and ν_j is its restriction to E_j , then

$$\nu_j(E_j) \leq C^-(E_j) \sup_z U_{\nu_j}^-(z) \leq C^-(E_j) \sup_z U_{\nu^-}^-(z) = C^-(E_j)$$

Hence

$$C^-(E) = \nu^-(E) = \sum_j \nu_j(E_j) \leq \sum_j C^-(E_j)$$

3. Macroscopic scale: let $T_h = [0, h^2] \times \Pi'_h$, $h > 1$, and $\Pi'_h = [0, h]^2$. Then

$$C^-(T_h) \sim h^2$$

Indeed, take $z_h = -h^2 e^1$. Then, for any $\mu \in \mathcal{P}(T_h)$, we have

$$U_\mu^-(z_h) \sim \frac{1}{h^2}$$

thus

$$\inf_\mu \sup_z U_\mu^-(z) \geq \frac{C_1}{h^2}$$

On the other hand, if μ is the normalized Lebesgue measure on T_h , we see

$$\sup_z U_\mu^-(z) \leq \frac{C_2}{h^2}$$

and thus

$$\inf_\mu \sup_z U_\mu^-(z) \leq \frac{C_2}{h^2}$$

4. Microscopic scale: the following proposition is immediate from the properties of the kernel K^- and the definition of the standard Wiener capacity $C_W(E)$ (i.e., the one related to Δ):

Proposition 3.1. *If $\text{diam}(E) \lesssim 1$, then $C^-(E) \sim C_W(E)$.*

So, for example, $C^-(B(z, r)) \sim r$, as $r \rightarrow 0$.

5. If G_n is a sequence of compacts, $G_{n+1} \subseteq G_n$, $\bigcap_n G_n = E$, then

$$(3.7) \quad \lim_{n \rightarrow \infty} C^-(G_n) = C^-(E)$$

Indeed, assume that ν_n is a maximizer in (3.6) for G_n . Then, one can find ν_{k_n} that converges weakly to ν supported on E . Since $U_{\nu_n}^-(x) \leq 1$, we have $U_\nu^-(x) \leq 1$ as well. Thus, $C^-(E) \geq \nu(E)$. On the other hand, $\nu(E) = \lim_n \nu_{k_n}(G_{k_n}) = \lim_n C^-(G_{k_n}) \geq C^-(E)$ since $C^-(G_n) \geq C^-(E)$ by monotonicity.

The last approximation result with G_n having the piece-wise smooth boundary allows one, like for modified harmonic measure, to reduce the analysis to the smooth case.

At last, let θ be a fixed vector from Σ_1 and $x = \theta r$, $|x| = r$. For any measure μ , we have

$$(3.8) \quad \begin{aligned} \lim_{r \rightarrow +\infty} r U_\mu^-(\theta r) &= 0, \quad \theta \neq -e^1, \\ \lim_{r \rightarrow +\infty} 2\pi r U_\mu^-(-e^1 r) &= \mu(E) \end{aligned}$$

3.2. More on modified harmonic measure and capacity. In this subsection, we will relate the capacity to the modified harmonic measure of a compact. We first assume that E has a piece-wise smooth boundary, e.g., it is a finite union of closed balls. Then the approximation results (3.2) and (3.7) will enable us to handle the case of any compact.

Take a compact E in the half-space $\{x = (x_1, x') : x_1 > 1\}$. Let α_z be the density of the harmonic measure for the half-space $\Pi_- = \{x : x_1 < 0\}$ and the reference point $z \in \Pi_-$, e.g.,

$$\mathbb{P}_z(G_t \text{ first hits the plane } \{x : x_1 = 0\} \text{ at set } B) = \int_B \alpha_z(y') dy'$$

where $B \subset \{x : x_1 = 0\}$. Then,

$$\omega^-(z, A, \Omega) = \int_{\{x_1=0\}} \alpha_z(y') \omega^-(y', A, \Omega) dy'$$

for any $A \subset \Gamma$. Then, introduce the balayage of the two-dimensional Lebesgue measure on $\{x_1 = 0\}$ to Γ ,

$$p^-(A) = \int_{\{x_1=0\}} \omega^-(y', A, \Omega) dy'$$

The explicit formula for α_z is easy to write down, namely

$$\alpha_z(y') = -\frac{1}{2\pi} e^{-z_1} \frac{\partial}{\partial y_1} \left(\frac{e^{-|z-y|}}{|z-y|} - \frac{e^{-|z_*-y|}}{|z_*-y|} \right) \Big|_{y_1=0}$$

where z_* is symmetric to z with respect to $\{y : y_1 = 0\}$. The above formulae imply that

$$\omega^-((-r, y'), A, \Omega) \approx \frac{1}{2\pi r} p^-(A)$$

as $r \rightarrow +\infty$ and y' is fixed. Notice that, due to our assumptions on Γ , p^- is a measure on Γ with piece-wise smooth density. It can be interpreted as follows: if $L^-u = 0$ on Ω , $u \in C(\bar{\Omega})$, and decays at infinity, then

$$(3.9) \quad \lim_{r \rightarrow +\infty} (2\pi r) u(-r, y') = \int_{\Gamma} u(\xi) dp^-(\xi)$$

Consider the following potential

$$U_{p^-}^+(z) = \int_E K^-(\xi, z) dp^-(\xi) \left(= \int_E K^+(z, \xi) dp^-(\xi) \right)$$

If z belongs to the interior of E , then $f(\xi) = K^-(\xi, z)$ satisfies $L^-f = 0$ outside E and therefore, referring to (3.9),

$$(3.10) \quad U_{p^-}^+(z) = \lim_{r \rightarrow +\infty} (2\pi r) K^-((-r, 0), z) = 1$$

which holds on the interior of E and, by continuity, on all of E . By the maximum principle for the operator $L^+ = \frac{1}{2}\Delta - \partial_{x_1}$ [33, Ch. 6], one has $\sup_{z \in \mathbb{R}^3} U_{p^-}^+(z) = 1$, and so

$$(3.11) \quad C^+(E) \geq p^-(E)$$

Similarly,

$$(3.12) \quad C^-(E) \geq p^+(E)$$

and, for any $z \in E$

$$U_{p^+}^-(z) = 1, \quad U_{p^-}^+(z) = 1$$

The duality argument then gives

$$p^+(E) = \int_E \left(\int_E K^-(\xi, z) dp^-(\xi) \right) dp^+(z) = p^-(E)$$

by Fubini. Moreover, one has

$$\begin{aligned} p^+(E) &\geq \int_E \left(\int_E K^-(\xi, z) d\nu^+(\xi) \right) dp^+(z) = \int_E \left(\int_E K^-(\xi, z) dp^+(z) \right) d\nu^+(\xi) \\ &= \nu^+(E) = C^+(E) \end{aligned}$$

Therefore, for E with piece-wise smooth boundary, we have

$$(3.13) \quad C^+(E) = C^-(E) = p^+(E) = p^-(E)$$

and p^+ is a minimizer in (3.6). Moreover, $U_{p^+}^-(x)$ is solution to the L^- -Dirichlet problem in Ω with boundary value $f = 1$.

Now, consider an arbitrary compact E . The measures p^\pm can be defined in the same way, they are both supported on Γ . Then, we have

Theorem 3.1.

i) For any compact E

$$C^+(E) = C^-(E) = p^+(E) = p^-(E)$$

ii) p^+ is a maximizer for (3.6) and $U_{p^+}^- = 1$ on the interior of E . Moreover, $U_{p^+}^-$ is the generalized solution to the Dirichlet problem

$$(3.14) \quad L^-U_{p^+}^- = 0, \quad U_{p^+}^-|_{\Gamma} = 1$$

the boundary condition being understood quasi-everywhere.

Proof. Consider a sequence of compacts G_n with piece-wise smooth boundaries that decreases to E as in (3.7). Obviously, p_n^+ converges weakly to p^+ . Moreover, $U_{p_n^+}^-(x) = 1$ on E and thus $U_{p^+}^-(x) = 1$ for any interior point $x \in E$ (if there is any). Since $U_{p_n^+}^-$ converges to $U_{p^+}^-$ on Ω and each $U_{p_n^+}^-$ solves the Dirichlet problem for G_n with boundary value $f = 1$, we get that $U_{p^+}^-$ is generalized solution to (3.14). Since $U_{p^+}^-$ is also quasicontinuous on \mathbb{R}^3 , we get $U_{p^+}^- = 1$ quasi-everywhere on Γ . Since we have $U_{p^+}^- \leq 1$ on \mathbb{R}^3 , p^+ is admissible. On the other hand, by (3.7) and (3.2), we get $C^-(E) = \lim_n C^-(G_n) = \lim_n p^+(G_n) = p^+(E)$. This shows that p^+ is a maximizer. \square

From now on, the symbol $C(E)$ will denote $C^-(E) = C^+(E)$.

The proof of the forthcoming uniqueness result relies on the following extended minimum principle.

Theorem 3.2. *Let O is bounded domain and u is bounded L^- -harmonic function on O . If S is polar subset of ∂O and $\liminf_{x \rightarrow \xi} u(x) \geq M$ for any $\xi \in \partial O, \xi \notin S$, then either $u(x) > M$ or $u(x) = M$ in O .*

The proof of this theorem is identical to the proof of Theorem 5.16 of [24, Ch. 5] as long as one has the following

Lemma 3.1. *Let S be a bounded polar set. Then there is a nonnegative L^- -superharmonic function u on \mathbb{R}^3 such that $u(x) = +\infty$ for any $x \in S$ and $u(x^0)$ is finite at a prescribed point $x^0 \notin S$.*

Proof. By [24], Theorem 5.11, and the Riesz representation theorem for superharmonic functions, there exists a compactly supported measure ν on \mathbb{R}^3 such that for the potential $\psi(x) = \int |x - y|^{-1} d\nu(y)$ equals $+\infty$ on S and is finite at x^0 . Consider the function

$$u(x) = \int K^-(x, y) d\nu(y) = C\psi(x) + \int \tilde{K}(x, y) d\nu(y)$$

where $\tilde{K}(\cdot, \cdot)$ is a continuous kernel. Clearly, u has the properties claimed in the lemma. \square

Theorem 3.3. *The measure p^+ is the unique maximizer in (3.6).*

Proof. Assume first that E does not have interior points. Take again G_n as in the proof of Theorem 3.1. Let \tilde{p} be another maximizer in (3.6). From Theorem 3.1, we have

$$\int dp_n^-(x) \int K^-(x, y) d\tilde{p}(y) = \int d\tilde{p}(y) \int K^+(y, x) dp_n^-(x) = C(E) = C(G_n) - \epsilon_n$$

where

$$\epsilon_n = C(G_n) - C(E) \rightarrow 0$$

The function

$$h_n = U_{p_n^+}^- - U_{\tilde{p}}^-$$

is L^- -harmonic on G_n^c and nonnegative there by the maximum principle since $U_{\tilde{p}}^-(x) \leq 1$, $U_{p_n^+}^-(x) = 1$ on $\Gamma_n = \partial G_n$ and they both decay at infinity.

Moreover,

$$\int h_n(x) dp_n^-(x) = \epsilon_n \rightarrow 0$$

Therefore,

$$\int_{\{x_1=0\}} h_n(x) dx' = \epsilon_n \rightarrow 0$$

and by Harnack principle, $h_n(x) \rightarrow 0$ on any compact in Ω . Since $U_{p_n^+}^-(x) \rightarrow U_{p^+}^-(x)$ on Ω , we have $U_{\tilde{p}}^-(x) = U_{p^+}^-(x)$ on Ω . This means that $U_{\tilde{p}}^-(x) = U_{p^+}^-(x) = 1$ quasi-everywhere on E since the potential $U_{\tilde{p}}^-$ is quasicontinuous on \mathbb{R}^3 [36, p. 73]. Then, $U_{\tilde{p}}^- = U_{p^+}^-$ a.e. on \mathbb{R}^3 . Taking Fourier transform, we get

$$(w^2 + 1)^{-1} \mathcal{F}(e^{x_1 \tilde{p}}) = (w^2 + 1)^{-1} \mathcal{F}(e^{x_1 p^+})$$

and thus the measures are equal.

Now, consider the general case when E has nonempty interior. Consider any open component in the interior and call it O . The boundary $\Gamma' = \partial O$ is a subset of Γ . In a similar way, we can prove that the potential $u = U_{\tilde{p}}^-$ is continuous and equals to 1 quasi-everywhere on Γ' . We also know that u is L^- -superharmonic on O and $0 \leq u(x) \leq 1$. Let us apply now the Theorem 3.2 to O and $u(x)$ with $M = 1$ and S taken as a set of points of Γ' at which $U_{\tilde{p}}^-$ is not continuous or its value $\neq 1$. Then, $U_{\tilde{p}}^- = 1$ on O and thus $\tilde{p} = p^+$ by the same argument. \square

In view of these results, we can call p^\pm the equilibrium measures.

Let $\mathbf{i} = (i_2, i_3) \in \mathbb{Z}^2$. For a given \mathbf{i} , $z \in \mathbb{R}^3$ and $T > 0$, put $\Pi_{\mathbf{i}}' = [i_2 T, (i_2 + 1)T] \times [i_3 T, (i_3 + 1)T]$ if $i_{2(3)} \geq 0$. The following proposition provides another useful relation between the capacity and harmonic measure.

Proposition 3.2. *Assume that T is large and $E \subset [T^2, 2T^2] \times \Pi_{\mathbf{i}}'$. Then*

$$\omega^-(0, E, \Omega) \sim \frac{C(E)}{T^2}$$

for small $|\mathbf{i}|$ and

$$\omega^-(0, E, \Omega) \lesssim \exp\left(-\frac{|\mathbf{i}|^2}{2 + \sqrt{4 + |\mathbf{i}|^2 T^{-2}}}\right) \frac{C(E)}{T^2}$$

in general.

Proof. We can always assume that ∂E is smooth. Then, consider

$$U^-(z) = \int K^-(z, \xi) dp^+(\xi)$$

We know that $L^-U^- = 0$ on Ω and $U^- = 1$ on ∂E . Therefore,

$$\omega^-(0, E, \Omega) = U^-(0) \sim \frac{p^+(E)}{T^2} = \frac{C(E)}{T^2}$$

for $|\mathbf{i}| < i_0$ and a simple computation yields

$$(3.15) \quad \omega^-(0, E, \Omega) \lesssim \exp\left(-\frac{|\mathbf{i}|^2}{2 + \sqrt{4 + |\mathbf{i}|^2 T^{-2}}}\right) \frac{C(E)}{T^2}$$

□

As the law of iterated logarithm suggests, the interesting range for $|\mathbf{i}|$ is $|\mathbf{i}| \leq C\sqrt{T \log \log T}$ with some C . Then, in (3.15), the weight is dominated by $\exp(-(\frac{1}{4} - \epsilon)|\mathbf{i}|^2)$, and $\epsilon > 0$ is arbitrary.

In some cases, the capacity is “almost additive”.

Proposition 3.3. *Let $E = \bigcup_j E_j$ such that disjoint sets $E_j \subset [0, T^2] \times \Pi'_{\mathbf{i}_j}$, $\mathbf{i}_j \in \mathbf{I}$, for some $T > 1$, and \mathbf{I} be finite. Then*

$$C(E) \gtrsim \sum_j C(E_j)$$

Proof. Let ν_j^- be a maximizer for E_j . Take $\nu = \sum \nu_j^-$. Then,

$$\nu(E) = \sum C(E_j)$$

On the other hand, we can estimate the potential in the following way. Take any point z in, say, E_1 . Then

$$U_\nu^-(z) = \sum_j U_{\nu_j^-}^-(z)$$

For $|\mathbf{i}_j - \mathbf{i}_1| \leq 2$ we use estimates $U_{\nu_j^-}^-(z) \leq 1$. For any \mathbf{i}_j , $|\mathbf{i}_j - \mathbf{i}_1| \geq 3$ and $z_{\mathbf{i}_j} = z + (\mathbf{i}_j - \mathbf{i}_1)T$, we have

$$K(z, \xi) \leq K(z_{\mathbf{i}_j}, \xi) e^{-\alpha|\mathbf{i}_j - \mathbf{i}_1|},$$

for any $\xi \in E_j$ and some universal $\alpha > 0$. Therefore,

$$U_{\nu_j^-}^-(z) \leq e^{-\alpha|\mathbf{i}_j - \mathbf{i}_1|}$$

and so

$$U_\nu^-(z) \leq C$$

This implies the required bound. □

3.3. Anisotropic Hausdorff content and capacity. Since we have to consider both microscopic and macroscopic regimes, one has to define the anisotropic Hausdorff content in the following way [21, App. D].

Let $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be some measure function, i.e. $h(0) = 0$, h is continuous and increasing. Set $\Pi'_r = [0, r] \times [0, r]$ for $r \geq 0$. We say that $H(r)$ is the characteristic set if it can be translated to a parallelepiped $[0, r^2] \times \Pi'_r$ if $r \geq 1$ and to a cube $[0, r] \times \Pi'_r$ if $r < 1$. We say that the size of $H(r)$ is r . Cover E by a countable number of $\{H(r_j)\}$. Then the Hausdorff content of E (with respect to h) is

$$(3.16) \quad M_h(E) = \inf \sum_j h(r_j)$$

where the infimum is taken over all coverings.

Take

$$(3.17) \quad h(r) = \begin{cases} r, & 0 < r \leq 1 \\ r^2, & r > 1 \end{cases}$$

Since the capacity is subadditive, one has a simple

Proposition 3.4. *For any compact set E and the measure function h given by (3.17), we have*

$$C(E) \lesssim M_h(E)$$

One can compare this to [21, App. D], Theorem D.1. This is a key estimate which will allow us to guarantee the wave propagation (see Section 4) in terms of the metric properties of the set of obstacles E . It seems possible to relate the capacity to the Hausdorff dimension or other metric properties of the set similarly to the elliptic/parabolic case (see, e.g. an excellent paper by Taylor-Watson [44]). We do not pursue this here.

Remark. If $d = 2$, we need to introduce the measure function differently, i.e.

$$(3.18) \quad h(r) = \begin{cases} |\log r|^{-1}, & 0 < r \leq 1/2 \\ Cr, & r > 1/2 \end{cases}, \quad C = \frac{2}{\log 2}$$

The proposition above and the results below then hold true.

3.4. How does the capacity change under projection? It is a well known fact in classical potential theory that the capacity (Wiener or logarithmic) can only decrease when the set is contracted. Analogous statements for harmonic measure are known as Beurling's projection theorem and Hall's lemma, see [21, Sect. 3.9]. We are interested in proving similar results for modified capacity. Naturally, the results will be anisotropic.

If $\text{diam}(E)$ is small then all results for the Wiener's capacity hold. It is the macroscopic regime that we are more interested in. Let Pr' be the projection on the plane $\{x_1 = 0\}$ and \aleph be any contraction in the x' -plane (i.e., the plane OX_2X_3). That is, under the action of \aleph , the x_1 coordinates of points in E stay the same, but the x' -sections of E are contracted. This makes the capacity of the set smaller, compare this to [21, Ch. 3], Theorem 4.5.

Theorem 3.4. *For any set E , we have*

$$(3.19) \quad C(\aleph E) \leq C(E)$$

Proof. We can assume that E has piece-wise smooth boundary. Consider any measure ν on E and take its projection under contraction, call it $\tilde{\nu}$.

$$\tilde{\nu}(\aleph E) = \nu(E)$$

We also have

$$K^-(z, \xi) \leq K^-(\aleph z, \aleph \xi)$$

for any $z, \xi \in E$ and so

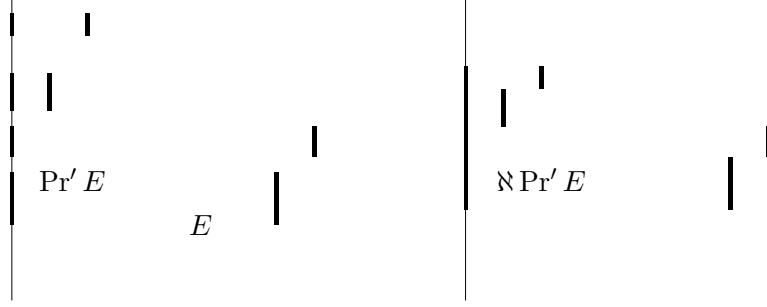
$$U_\nu^-(z) \leq U_{\tilde{\nu}}^-(\aleph z)$$

Now, take $\tilde{\nu}$ to be the maximizer for $C^-(\aleph E)$ and let ν be its preimage under \aleph such that $\nu(E) = \tilde{\nu}(\aleph E)$. Then,

$$U_\nu^-(z) \leq 1$$

so $C(E) \geq C(\aleph E)$ by definition. \square

An interesting question is what happens under the action of Pr' . We are interested in the case when E consists of the finite number of disjoint components with piece-wise smooth boundaries, e.g., a finite union of balls or parallelepipeds. The notation $|\text{Pr}' E|$ stands for Lebesgue measure of the

FIGURE 1. Action of contraction \aleph on $\text{Pr}' E$ ($d = 2$).

projection of E . Let \mathcal{C} be the set of all contractions in the plane $\{x_1 = 0\}$ that preserve the Lebesgue measure of $\text{Pr}' E$. Define

$$(3.20) \quad \tilde{D}(\text{Pr}' E) = \inf_{\aleph \in \mathcal{C}} \text{diam } \aleph(\text{Pr}' E)$$

Theorem 3.5. *Let $H_T = [0, T^2] \times \Pi'_T$, $T > 1$, $E \subset H_T$, and $\tilde{D}(\text{Pr}' E) > 2$. Then*

$$(3.21) \quad \frac{|\text{Pr}' E|}{\log \tilde{D}(\text{Pr}' E)} \lesssim C(E)$$

This inequality is sharp, at least for sets E with the property $\tilde{D}(\text{Pr}' E) \sim \text{diam } \text{Pr}' E$.

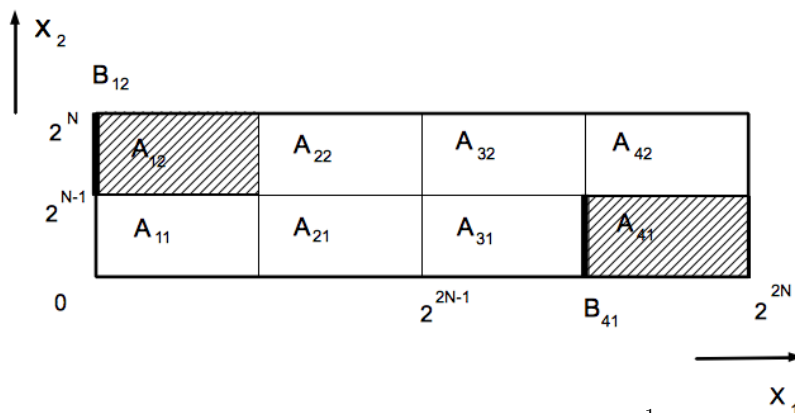
Proof. We start with a certain reduction. By the standard approximation argument, one can focus on the case when $E = \bigcup_{j=1}^N E_j$, where $E_j \subset \{x : x_1 = x_1^j\}$ is a (planar) compact and different $\text{Pr}' E_j$ are disjoint.

We also suppose that $\text{diam}(\text{Pr}' E) \lesssim \tilde{D}(\text{Pr}' E)$ since otherwise we can use the previous theorem. So, let $v = m_2|_E$, where m_2 is the (planar) Lebesgue measure. Of course, $v(E) = |\text{Pr}' E|$. Fix any $z = (z_1, z'_1) \in E$. That is, $z_1 = x_1^{(j_0)}$ for a $j_0 \in \{1, \dots, N\}$. For brevity, put E' (E'_j) to be the x' -projection of E (E_j , respectively) on $\{x : x_1 = x_1^{(j_0)}\}$ and $\tilde{d} = \tilde{D}(\text{Pr}' E)$. Notice that, up to a translation in the x' -direction, $E' \subset B'(0, \tilde{d})$. Then,

$$U_v^-(z) \lesssim 1 + \int_{B'(0, \tilde{d}) \setminus B'(0, 1)} \chi_{E'}(x') \frac{e^{-(|x'|^2 + h^2(x'))^{1/2} + h(x')}}{(|x'|^2 + h^2(x'))^{1/2}} dx' = 1 + I$$

where $h : E' \rightarrow \mathbb{R}_+$ is the step function parameterizing the set E , i.e. $h(x') = x_1^{(j)} - x_1^{(j_0)}$ for $x' \in \text{Pr}' E_j$. In I , the contribution from the set where $h(x') \leq 0$ is controlled by the absolute constant. Then,

$$\begin{aligned} I &\leq C + \int_{B'(0, \tilde{d}) \setminus B'(0, 1), h(x') > 0} \frac{e^{-(|x'|^2 + h^2(x'))^{1/2} + h(x')}}{(|x'|^2 + h^2(x'))^{1/2}} \chi_{E'}(x') dx' \\ &\leq C + \int_{B'(0, \tilde{d}) \setminus B'(0, 1), h(x') > 0} \frac{e^{-\frac{1}{2} \frac{|x'|^2}{(|x'|^2 + h^2(x'))^{1/2}}}}{(|x'|^2 + h^2(x'))^{1/2}} \chi_{E'}(x') dx' \end{aligned}$$

FIGURE 2. The first configuration \mathcal{C}^1 .

Let

$$\phi(x') = \frac{1}{2} \frac{|x'|^2}{(|x'|^2 + h^2(x'))^{1/2}}$$

and $Z_n = \{x' : n < 4\phi(x') \leq n + 1\}$, $n = 1, 2, \dots$. Then,

$$\begin{aligned} I &\lesssim 1 + \int_{x' \in B'(0, \tilde{d}) \setminus B'(0, 1), |x'|^2 \leq h(x')} (\dots) + \sum_n \int_{x' \in B'(0, \tilde{d}) \setminus B'(0, 1), x' \in Z_n} (\dots) \\ &\lesssim 1 + \int_{B'(0, \tilde{d}) \setminus B'(0, 1)} \frac{1}{|x'|^2} dx' + \sum_n \int_{B'(0, \tilde{d}) \setminus B'(0, 1)} e^{-n/4} \frac{n}{|x'|^2} dx' \lesssim 1 + \log \tilde{d} \end{aligned}$$

This estimate and the maximum principle prove inequality (3.21).

It is more interesting that this estimate is essentially sharp under assumptions mentioned in the statement of the theorem.

Let $T = 2^N$. We will take the set $E_T = A_T \times [0, T]$ where A_T is the subset of $\tilde{H}_T = [0, T^2] \times [0, T]$ constructed as follows. The set A_T will be chosen such that $\text{Pr}' E_T = [0, T] \times [0, T]$. Divide \tilde{H}_T into eight equal parts each translationally equivalent to $\tilde{H}_{T/2}$, see Figure 2. These parts are denoted by $A_{i_1 i_2}$, where i_k is the number of the part in the e^k -direction, $k = 1, 2$. That is, we have $i_1 = 1, \dots, 4$, $i_2 = 1, 2$. The left-hand sides of the rectangles $A_{i_1 i_2}$ (in the e^1 -direction) are denoted by $B_{i_1 i_2}$. Now, fix a configuration \mathcal{C}_1 of these rectangles; the subscript 1 indicates the first generation. For instance, let $\mathcal{C}^1 = \{A_{12}, A_{41}\}$. The rectangles from \mathcal{C}^1 are hatched on Figure 2.

We will properly scale and translate the configuration to get a “fractal dyadic-type” set. Dyadically partition every $A_{i_1 i_2}$ as we did it for \tilde{H}_T ; every part of $A_{i_1 i_2}$ is translationally invariant to $\tilde{H}_{T/2}$, see Figure 3. These parts are denoted by $A_{i_1 i_2; i'_1 i'_2}$, the corresponding left-hand sides are $B_{i_1 i_2; i'_1 i'_2}$. Place the shrunk and translated images of the rectangle from \mathcal{C}^1 inside each part of \mathcal{C}^1 . This is the configuration \mathcal{C}^2 , i.e. $\mathcal{C}_2 = \{A_{12; 12}, A_{12; 41}, A_{41; 12}, A_{41; 41}\}$.

Continue in the same manner up to \mathcal{C}^N . All rectangles from the N -th generation (and hence those appearing in \mathcal{C}^N) are the unit squares. Now we set

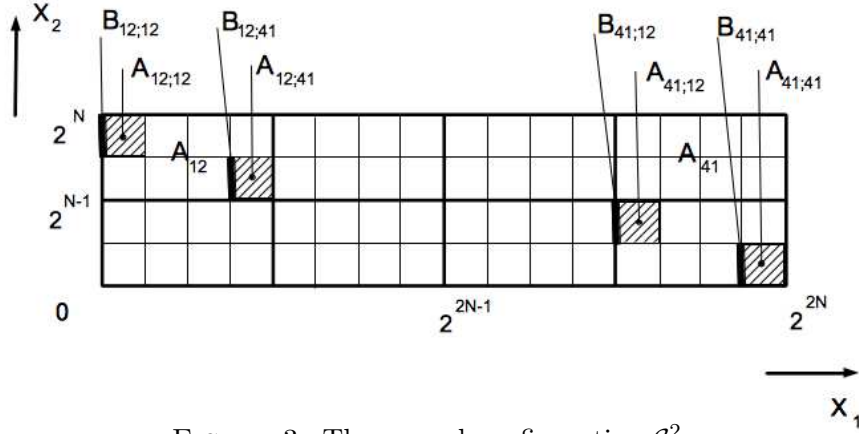


FIGURE 3. The second configuration \mathcal{C}^2 .

$$A_T = \bigcup_{A \in \mathcal{C}^N} \{ \text{the side } B \text{ corresponding to } A \}$$

and

$$(3.22) \quad E_T = A_T \times [0, T],$$

see Figure 4 (where $N = 2$ to keep the drawing reasonably simple). That is, every rectangle from E_T is of unit base and height $T = 2^N$, they are 2^N , and their common area is T^2 . This means exactly that $|\text{Pr}' E_T| = T^2$ and E_T satisfies the assumption $\tilde{D}(\text{Pr}' E) \sim \text{diam Pr}' E$ from the formulation of the theorem.

The next step of the proof is to estimate $C(E_T)$. First, let $\mu = m_2|_{E_T}$ where m_2 is the (planar) Lebesgue measure. Obviously, $\mu(E_T) = T^2$.

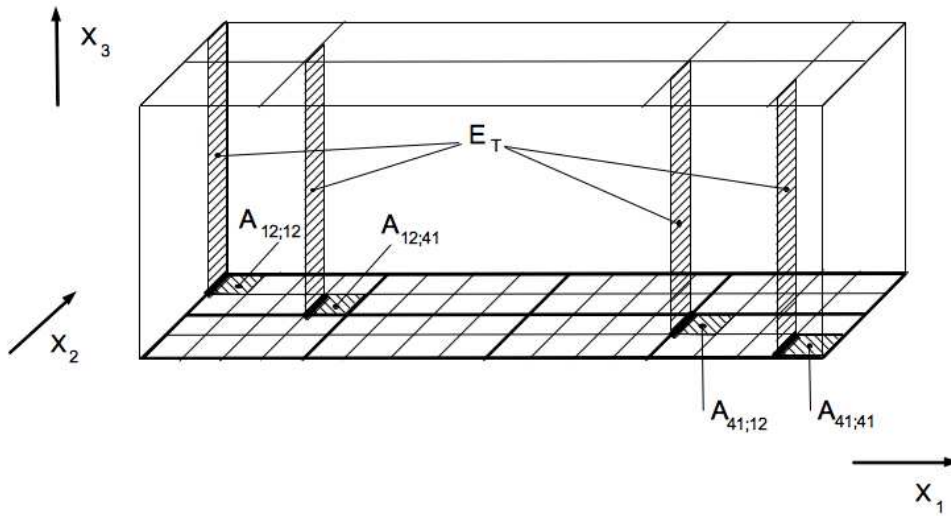


FIGURE 4. Construction of the set E_T .

As a preparation, we start proving that the set E_T can be partitioned so that

$$(3.23) \quad E_T = \bigcup_{j=0}^N \Omega_j$$

where Ω_j are disjoint collections of rectangles from (3.22) with the property

- $U_{\mu}^{-}(z) \geq \alpha j$ for $j = 0, \dots, N$ and α is a universal positive constant,
- $\#\Omega_j = \binom{N}{j}$, $\#(\cdot)$ of course denotes the cardinality of the set.

We explain the construction for \mathcal{C}^1 and \mathcal{C}^2 ; this will serve as a base of recursion, and the recursion itself will then follow quite simply.

For \mathcal{C}^1 , let $E^1 = E_{12} \cup E_{41}$, where $E_{i_1 i_2} = B_{i_1 i_2} \times [0, T]$. Set $\mu^1 = m_2|_{E^1}$, m_2 being the (planar) Lebesgue measure, and $\mu_{i_1 i_2}^1 = m_2|_{E_{i_1 i_2}}$. Using (3.3), we obtain that, for $z \in E_{12}$

$$U_{\mu^1}^{-} = U_{\mu_{12}^1}^{-} + U_{\mu_{41}^1}^{-} \geq U_{\mu_{12}^1}^{-} + \alpha \geq \alpha$$

where α is a positive constant. Obviously $U_{\mu^1}^{-} \geq 0$ for $z \in E_{41}$. So we get $\Omega_0^1 = \{E_{41}\}$ and $\Omega_1^1 = \{E_{12}\}$, see Figure 2.

For \mathcal{C}^2 , let $E^2 = E_{12;12} \cup E_{12;41} \cup E_{41;12} \cup E_{41;41}$, where $E_{i_1 i_2; i'_1 i'_2} = B_{i_1 i_2; i'_1 i'_2} \times [0, T]$. We write $E_{i_1 i_2}^2$ to denote the collection of the rectangles $E_{i_1 i_2; i'_1 i'_2}$ with their bases $B_{i_1 i_2; i'_1 i'_2}$ belonging to $A_{i_1 i_2}$. As before, $\mu^2 = m_2|_{E^2}$. The notation $\mu_{i_1 i_2; i'_1 i'_2}^2$ is self-obvious, and $\mu_{i_1 i_2}^2$ is the restriction of μ^2 to $E_{i_1 i_2}$. Once again, we have for $z \in E_{12}$

$$U_{\mu^2}^{-} = U_{\mu_{12}^2}^{-} + U_{\mu_{41}^2}^{-} \geq U_{\mu_{12}^2}^{-} + \alpha$$

On the other hand, repeat the estimate for $z \in E_{41;12}$ using the “self-similarity”. This yields

$$U_{\mu^2}^{-} \geq U_{\mu_{41}^2}^{-} = U_{\mu_{41;12}^2}^{-} + U_{\mu_{41;41}^2}^{-} \geq U_{\mu_{41;12}^2}^{-} + \alpha \geq \alpha$$

The same applies for $U_{\mu_{12}^2}^{-}$ as well. So, summing up, we get:

- $U_{\mu^2}^{-} \geq 0$, for $z \in E_{41;41}$,
- $U_{\mu^2}^{-} \geq U_{\mu_{41;12}^2}^{-} + U_{\mu_{41;41}^2}^{-} \geq U_{\mu_{41;12}^2}^{-} + \alpha \geq \alpha$, for $z \in E_{41;12}$
- $U_{\mu^2}^{-} \geq U_{\mu_{12;41}^2}^{-} + U_{\mu_{41}^2}^{-} \geq U_{\mu_{12;41}^2}^{-} + \alpha \geq \alpha$, for $z \in E_{12;41}$
- $U_{\mu^2}^{-} \geq U_{\mu_{12;12}^2}^{-} + U_{\mu_{12;41}^2}^{-} + U_{\mu_{41}^2}^{-} \geq U_{\mu_{12;12}^2}^{-} + 2\alpha \geq 2\alpha$, for $z \in E_{12;12}$.

Hence $\Omega_0^2 = \{E_{41;41}\}$, $\Omega_1^2 = \{E_{41;12}, E_{12;41}\}$, and $\Omega_2^2 = \{E_{12;12}\}$, see Figure 3.

Continue similarly for higher generations \mathcal{C}^n , $n = 0, 1, \dots, N$. By recursion, $\#\Omega_j^n = \binom{n}{j}$, $\#\Omega_j^{n+1} = \#\Omega_j^n + \#\Omega_{j-1}^n = \binom{n+1}{j}$, and

$$E_T = \bigcup_{j=0}^N \Omega_j^N,$$

which is exactly a partition with properties stated in (3.23).

Notice that

$$\sum_{j: |j - N/2| \lesssim \sqrt{N}} \binom{N}{j} > \delta 2^N, \quad \delta > 0$$

by Stirling's formula.

Second, we are going to assign E_T a slightly different measure μ^* so that $\mu \leq \mu^*$. We take $\mu^* = \mu$ for obstacles in the groups Ω_j with $j \geq L = \epsilon N$, $\epsilon < 1/2$ and

$$\mu^* = N\mu$$

on E^j for $j < L$. Notice that we now have

$$U_{\mu^*}^-(z) \gtrsim N = \log T$$

for all $z \in E_T$ by construction (on Ω_0 one can obviously improve the bound to $U_{\mu}^-(z) > C > 0$). On the other hand,

$$\sum_{j < L} \binom{N}{j} \lesssim N \binom{N}{L}$$

and, by Stirling's formula once again,

$$\binom{N}{L} \lesssim \frac{N^{-1/2}}{x^N}, \quad x = \epsilon^\epsilon (1 - \epsilon)^{1-\epsilon} > 1/2$$

since $\epsilon < 1/2$. Therefore, $\mu^*(E_T) \sim T^2$. Furthermore, $U_{p^+}^-(z) \lesssim U_{\mu^*}^-(z)/\log T$ for any $z \in E_T$, and, by the maximum principle, for all $z \in \mathbb{R}^3$. Then, taking $z = (-r, 0, 0)$, $r \rightarrow +\infty$ and recalling (3.9), (3.10), we come to

$$C^-(E_T) \lesssim \frac{T^2}{\log T}$$

which finishes the proof. \square

When we estimate the modified capacity of the set E_T , we are essentially in the macroscopic regime where the parabolic approximation takes place.

Consequently, it is instructive to compare this situation to the results obtained by Taylor-Watson [44]. Let $(t, x') \in \mathbb{R} \times \mathbb{R}^d$ and

$$W(t, x') = \begin{cases} \frac{1}{(4\pi t)^{d/2}} \exp\left(-\frac{|x'|^2}{4t}\right), & t > 0 \\ 0, & t \leq 0 \end{cases}, \quad R_\alpha(x') = \frac{1}{|x'|^\alpha}$$

be the heat (also called parabolic) and the α -Riesz kernels, respectively. Recall that the parabolic and α -Riesz potentials are defined as

$$v(t, x') = \int_E W(t - s, x' - y') d\mu(s, y'), \quad v_\alpha(x') = \int_{E'} R_\alpha(x' - y') d\nu(y')$$

where the measures μ, ν are supported on E and E' with $E \subset \mathbb{R}^{d+1}$ and $E' \subset \mathbb{R}^d$, respectively. The parabolic capacity C_P and the α -Riesz capacity C_α are then defined in the standard way as in (3.5).

Theorem 1 from [44] says that for a $E \subset \mathbb{R}^{d+1}$, the parabolic capacity $C_P(E)$ is bounded above by an appropriate Hausdorff content $M_h(E)$ similar to (3.16). This theorem is of course parallel to our Proposition 3.4. The next step is to see when the parabolic capacity C_P can be expressed in terms of various metric quantities of certain nature. For instance, Theorem 5 from [44] says that for a parabolically separated $E \subset \mathbb{R}^{d+1}$, $C_P(E) > 0$ if and only if $C_{d/2}(\text{Pr } E) > 0$. The set E is parabolically separated, if $|x' - y'|^2 \leq C|t - s|$ for any $(t, x'), (s, y') \in E$. Example 3 to Theorem 5 in [44] shows that the inverse to the above Theorem 1 is false even for parabolically separated sets,

i.e. a Cantor-type set E with the following properties is constructed ($d = 1$, but the construction is valid for $d \geq 1$):

- $E \subset [0, 1]^2$,
- $|\Pr E| = 0, |\Pr' E| = 1$,
- $C_P(E) = 0$.

This is equivalent to say that $\mathbb{P}((t, B_t) \text{ hits } E) = 0$. Ideologically, our construction from the second part of Theorem 3.5 is close to this example. Its further consequences are discussed in Section 4.

The theorem below is the specification of Theorem 3.5 to the case $d = 2$. In this situation, we always have $\tilde{D}(\Pr_y E) = |\Pr_y E|$ (see (3.20) for the definition of \tilde{D}). Therefore,

Theorem 3.6. *Let $E \subset [0, T^2] \times [0, T]$, $T > 1$, and $|\Pr_y E| > 2$. Then*

$$\frac{|\Pr_y E|}{\log |\Pr_y E|} \lesssim C(E)$$

and this inequality is sharp.

Figures 1 and 2 illustrate the right choice of contraction \aleph and the construction of the set E_T .

3.5. How does the capacity of a set depend on the speed of the drift? The Itô diffusion G_t introduced in Theorem 2.1, (2) (see also (2.8)), was designed to handle the asymptotics of $G(x, 0; i)$ as $|x| \rightarrow \infty$. It is clear that the same constructions go through for any $ik \in i\mathbb{R}_+$ with the only difference that the kernels K^\pm defining the potential theory are

$$K_k^\mp(x, y) = \frac{1}{2\pi} \frac{e^{-k(|x-y| \pm (x_1 - y_1))}}{|x - y|}$$

where $x, y \in \mathbb{R}^3$. So let $k \in \mathbb{R}^+$ be fixed. Recall the definitions and notations from Sections 3.1, 3.2. Take a compact set E . Let μ_k be a maximizer for C_k , i.e. the capacity constructed with respect to K_k^- . That is,

$$\sup_x \int K_k^-(x, y) d\mu_k(y) = 1, \quad \mu_k(E) = C_k(E)$$

The monotonicity of the kernel K_k^- in k implies that $C_k(E)$ grows in k .

4. SOME APPLICATIONS

We can now state some results which are direct corollaries of what we have proved so far. We only treat $d = 2$, the general case can be handled similarly. Consider

$$H = H_V = -\Delta + V,$$

where $V = 0$ on Ω . We also assume that $V \geq 0$, V is measurable and locally bounded. These conditions are sufficient to define H as a self-adjoint operator.

The first application is a relatively simple geometric test that guarantees the presence of the a.c. spectrum. Denote $E = \Omega^c$. We introduce the following sets via polar coordinates/complex notation:

$$Q_{n,j} = \left\{ z = re^{i\theta}, 4^n \leq r \leq 4^{n+1}, \frac{2\pi j}{2^n} \leq \theta \leq \frac{2\pi(j+1)}{2^n} \right\}$$

where $n = 0, 1, 2, \dots$, and $j = 0, 1, \dots, 2^n - 1$. These curvilinear rectangles have (polar) “lengths” and “heights” comparable to 4^n and 2^n , respectively. So, they roughly resemble the characteristic rectangles discussed in Sections 3.2, 3.4. Notice also that the angular projections of $Q_{n+1,j}$ for various j is the dyadic collection of arcs which is the refinement of arcs from the previous n -th generation. Define

$$E_{n,j} = E \cap Q_{n,j}$$

For each n and j , introduce the Hausdorff content in the direction $\theta_{n,j} = (2\pi j)2^{-n}$ (see Section 3.3) and compute $M_{\theta_{n,j},h}(E_{n,j})$. Notice that good upper bounds for these quantities are sufficient to handle the convergence of series (4.1) appearing below.

Let $d_{n,j}(\theta)$ be the arclength distance from $e^{i\theta}$ to the arc

$$I_{n,j} = [2\pi j 2^{-n}, 2\pi(j+1) 2^{-n}]$$

as a set on the unit circle \mathbb{T} . Now, we can introduce

$$(4.1) \quad \phi(\theta) = \sum_{n=0}^{\infty} \sum_{j=0}^{2^n-1} M_{\theta_{n,j},h}(E_{n,j})(2^{-n} + \sin |\theta - \theta_{n,j}|) \exp\left(-4^n(1 - \cos d_{n,j}(\theta))\right)$$

Theorem 4.1. *If $\phi(\theta)$ is finite on a set of positive measure, then $\sigma_{ac}(H) = \mathbb{R}_+$.*

Proof. By the standard trace-class perturbation argument, $\sigma_{ac}(H) = \sigma_{ac}(-\Delta + V \cdot \chi_{|x|>R})$ for any $R > 0$. Thus, we can always assume that

$$V(x) = 0, \quad |x| < 4^{n_0}$$

for any given $n_0 > 0$. Let $\phi_{n_0}(\theta)$ be the remainder of series (4.1)

$$\phi_{n_0}(\theta) = \sum_{n=n_0}^{\infty} \sum_{j=0}^{2^n-1} M_{\theta_{n,j},h}(E_{n,j})(2^{-n} + \sin |\theta - \theta_{n,j}|) \exp\left(-4^n(1 - \cos d_{n,j}(\theta))\right)$$

By Egorov’s theorem on convergence, one can find a measurable set $\Omega_{n_0} \subseteq \mathbb{T}$ such that $|\Omega_{n_0}| > \delta > 0$ uniformly in n_0 and $\phi_{n_0}(\theta) < \epsilon_{n_0}$ on Ω_{n_0} , where $\epsilon_{n_0} \rightarrow 0$ as $n_0 \rightarrow \infty$.

It is sufficient to show that, for large n_0 , we have

$$\inf_{\theta \in \Omega_{n_0}} a_N(\theta) > \frac{1}{2}$$

uniformly in N , where a_N is amplitude (2.3) corresponding to the truncated potential $V_N(x) = V \cdot \chi_{4^{n_0} < |x| < 4^{N+1}}$ and $x^0 = 0$.

Fix $\theta \in \Omega_{n_0}$. By (2.23), we need to prove

$$\mathbb{P}\left(G_t^\theta \text{ does not hit } E \cap \{4^{n_0} \leq |z| \leq 4^{N+1}\}\right) > \frac{1}{2}$$

uniformly in N . Instead, we will show that

$$\mathbb{P}\left(G_t^\theta \text{ hits } E \cap \{4^{n_0} \leq |z| \leq 4^{N+1}\}\right)$$

is small uniformly in N provided n_0 is large enough. Here, $G_t^\theta = te^{i\theta} + B_t$. Obviously,

$$\mathbb{P}\left(G_t^\theta \text{ hits } E \cap \{4^{n_0} \leq |z| \leq 4^{N+1}\}\right) \leq \sum_{n=n_0}^N \sum_{j=0}^{2^n-1} \mathbb{P}\left(G_t^\theta \text{ hits } E_{n,j}\right)$$

Fix n and consider $E_{n,j}$. Let $d\nu_{n,j}$ be the maximizer in the definition of $C_\theta(E_{n,j})$. Then,

$$\mathbb{P}\left(G_t^\theta \text{ hits } E_{n,j}\right) = \int K_\theta^-(0, \xi) d\nu_{n,j}(\xi) \lesssim 2^{-n} \int e^{-|\xi|(1-\cos\alpha(\xi, \theta))} d\nu_{n,j}(\xi)$$

where $\alpha(\xi, \theta)$ is the angle between ξ and θ . We always have $C_\theta(E_{n,j}) \lesssim 4^n$. If j is such that, say, $\alpha(\xi, \theta) > \frac{1}{10}$, then

$$(4.2) \quad 2^{-n} \int e^{-|\xi|(1-\cos\alpha(\xi, \theta))} d\nu_{n,j}(\xi) \lesssim 2^{-n} e^{-4^{n-1}} 4^n$$

Hence, we are only interested in small angles $\alpha(\xi, \theta)$ for which

$$(4.3) \quad 2^{-n} \int e^{-|\xi|(1-\cos\alpha(\xi, \theta))} d\nu_{n,j}(\xi) \lesssim 2^{-n} \exp\left(-4^n(1 - \cos d_{n,j}(\theta))\right) C_\theta(E_{n,j})$$

Proposition 3.4 yields

$$C_\theta(E_{n,j}) \lesssim M_{\theta,h}(E_{n,j})$$

Now, obvious geometric considerations based on rotation of the coverings give

$$M_{\theta,h}(E_{n,j}) \lesssim M_{\theta_{n,j},h}(1 + 2^n \sin |\theta_{n,j} - \theta|)$$

So, summation gives

$$\mathbb{P}\left(G_t^\theta \text{ hits } E \cap \{4^{n_0} \leq |z| \leq 4^{N+1}\}\right) \lesssim \phi_{n_0}(\theta) + \sum_{n=n_0}^{\infty} 4^n e^{-4^{n-1}} \rightarrow 0$$

for $n_0 \rightarrow \infty$ uniformly in N . \square

Remark. The following estimate on $\phi(\theta)$ is straightforward

$$\phi(\theta) \lesssim \sum_{n=n_0}^{\infty} 2^{-n} \sum_j M_{\theta_{n,j},h}(E_{n,j})(1 + k(n, j, \theta)) \exp\left(-\alpha k^2(n, j, \theta)\right) + \bar{o}(1)$$

where $0 \leq \alpha < \frac{1}{2}$ and $k(n, j, \theta)$ is the number of the full dyadic intervals on \mathbb{T} between θ and $I_{n,j}$. An advantage of Theorem 4.1 is its relative simplicity since any reasonable covering of E provides an upper bound for the Hausdorff content.

The second application deals with the case when the Hausdorff content is too rough an instrument to estimate the capacity. We are going to construct a set E such that any ray issued from the origin intersects it infinitely many times and yet the associated Schrödinger operator has the a.c. spectrum that is equal to \mathbb{R}_+ .

Example. Consider some large $T = 2^n$ and the annulus $\{T^2 \leq |z| \leq 2T^2\}$. Take intervals $I_j = [(2\pi j)T^{-1}, 2\pi(j+1)T^{-1})$, $j = 0, 1, \dots, T-1$, and sets $Q_j = \{z = re^{i\theta} : T^2 \leq r \leq 2T^2, \theta \in I_j\}$. In each Q_j , place a set E_j in a way similar to that one of example given in the proof of Theorem 3.5. The

only difference now is that we dyadically divide the angle and the radius. Also, instead of vertical segments of unit length we take the corresponding sectorial rectangles of the size comparable to one. In this construction, all E_j can be obtained from, say, E_0 by rotations by angles $(2\pi j)T^{-1}$, $j = 1, \dots, T-1$. Finally, set $E_T = \bigcup_j E_j$.

Lemma 4.1. *As $T \rightarrow \infty$, we have*

$$\delta_T = \sup_{\theta \in [0, 2\pi)} \mathbb{P} \left(G_t^\theta = (te^{i\theta} + B_t) \text{ hits } E \right) \rightarrow 0$$

Sketch of the proof. Take an arbitrary $\theta \in \mathbb{T}$. For each interval I_j , let $k(j, \theta)$ be the total number of intervals $\{I_m\}$ between θ and I_j . Estimating just like in (4.2) and (4.3), we have

$$(4.4) \quad \mathbb{P} \left(G_t^\theta \text{ hits } \bigcup_{j: k(j, \theta) > k_0} E_j \right) \lesssim e^{-CT^2} + \sum_{k(j, \theta) \geq k_0} e^{-Ck^2(j, \theta)}$$

On the other hand, for any fixed value $k(j, \theta)$, the bounds from the proof of Theorem 3.5 apply as long as T is large enough and one has

$$\lim_{T \rightarrow \infty} C_\theta(E_j)/T \rightarrow 0$$

That, in turn, implies

$$\mathbb{P}(G_t^\theta \text{ hits } E_j) \rightarrow 0, \text{ as } T \rightarrow \infty$$

Then (4.4) finishes the proof. \square

Now, it is enough to take T_k , $\lim_{k \rightarrow \infty} T_k = \infty$ with the property

$$\sum_k \delta_{T_k} < 1/2$$

and consider $V \geq 0$ supported on $E = \bigcup_k E_{T_k}$. Then we have

$$\sup_{\theta \in [0, 2\pi)} \mathbb{P} \left(te^{i\theta} + B_t \text{ hits } E \right) < 1/2$$

and thus $\sigma_{ac}(H) = \mathbb{R}_+$. Notice that by construction any ray issued from the origin intersects the support of V infinitely many times.

APPENDIX A: THE UPPER BOUNDS ON THE HARMONIC MEASURE

In this Appendix, we adjust the well-known Carleman estimates on harmonic measure to our case. These bounds can be used to prove that the amplitude $a_{x_0} = 0$ and so they do not have immediate consequences for the study of spectral types. Nevertheless, we prove them to emphasize the “parabolic nature” of the domains for which the transition from $a_{x_0} = 0$ to positive values occurs.

Consider the following problem. Let $\Omega \subset \mathbb{R}^3$ be a bounded connected domain (but not necessarily simply connected) with smooth boundary. Assume also that $B(0, 1) \subset \Omega$. Then, define $\Omega_R = \Omega \cap \{x : -1 < x_1 < R\}$ for R large, and let $\Gamma_R = \partial\Omega_R$. We consider the modified harmonic measure $\omega(\cdot, A, \Omega_R)$ (3.1), but with respect to the bounded Ω_R and $A \subset \Gamma_R \cap \{x : x_1 = R\}$. Recall the probabilistic interpretation of $\omega(0, A, \Omega_R)$: it is the probability

of a random trajectory $G_t = 0 + t \cdot e^1 + B_t$ to hit the boundary Γ_R for the first time at A . Above, B_t is the three-dimensional Brownian motion. We are interested in the case when Ω is elongated in the e^1 -direction. To make the explanation more intuitive, we denote the first component of the vector $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ by t , $t = x_1$. As before $x = (t, x')$, $x' \in \mathbb{R}^2$.

We start with some notations. Let Θ be a (x' -planar) bounded open set with piece-wise smooth boundary and

$$\mathcal{W}(\Theta) = \{f \in W^{1,2}(\Theta) : f|_{\partial\Theta} = 0, f \neq 0\}$$

Consider the operator $-\Delta_D$ where Δ_D is the Laplacian on Θ with Dirichlet boundary condition on $\partial\Theta$. It is well known that its spectrum is discrete, $\sigma(-\Delta_D) = \{\lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots\}$. The value $\lambda_1 > 0$ is called principal eigenvalue and we denote it by $\lambda = \lambda_\Theta$. The classical Courant-Hilbert principle [11, Sect. 4.2, 6.7.9] says that

$$(4.5) \quad \min_{f \in \mathcal{W}(\Theta)} \frac{\int_{\Theta} |\nabla f|^2 dy'}{\int_{\Theta} f^2 dy'} = \lambda_\Theta$$

Denote the section of Ω_R by the plane $\{x : t = s\}$ by Θ_s . Clearly, Θ_s is a union of a finite number of connected components Θ_s^j , $j = 1, \dots, N$. Let

$$\lambda(s) = \min_j \lambda_{\Theta_s^j}$$

Then, for $f \in \mathcal{W}(\Theta_s)$,

$$(4.6) \quad \begin{aligned} \int_{\Theta_s} |\nabla' f|^2 dy' &\geq \sum_j \int_{\Theta_s^j} |\nabla' f|^2 dy' \geq \lambda \sum_j \int_{\Theta_s^j} f^2 dy' \\ &= \lambda \int_{\Theta_s} f^2 dy' \end{aligned}$$

where ∇' is the gradient with respect to x_2, x_3 .

Theorem 4.2. *We have*

$$(4.7) \quad \omega(0, A, \Omega_R) \leq C_1 |A|^{1/2} e^R \left(1 + \int_0^R \exp \left(2 \int_0^t \sqrt{\lambda(u) + 1} du \right) dt \right)^{-1/2}$$

In particular,

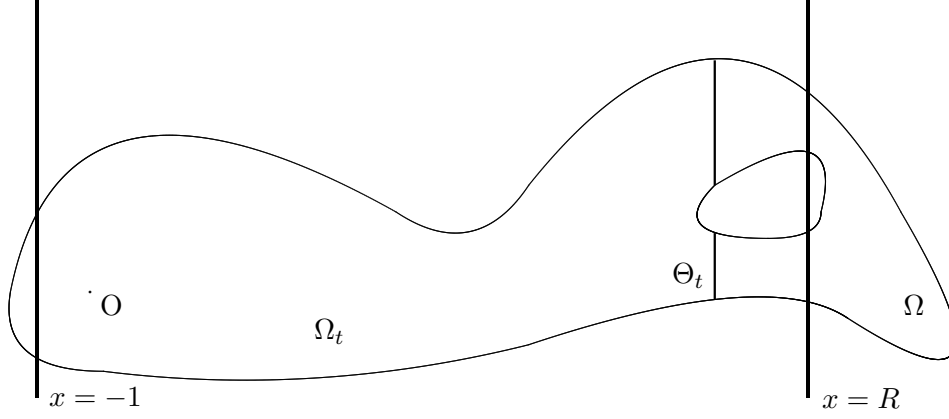
$$(4.8) \quad \omega(0, A, \Omega_R) \leq C_2 |A|^{1/2} \exp \left(R - \int_0^{R-1} \sqrt{\lambda(u) + 1} du \right)$$

Proof. The proof essentially follows Carleman [8], Haliste [23]; see also [21, Appendix G]. Recall that $\Omega_s = \Omega_R \cap \{x : t < s\}$. The function $u(z) = \omega(z, A, \Omega_R)$ satisfies

$$\Delta u + 2u_{x_1} = 0, \quad u|_{\Gamma} = \chi_A$$

Consider

$$(4.9) \quad \phi(t) = \int_{\Theta_t} u^2(t, y') dy', \quad A(t) = \int_{\Omega_t} |\nabla u(s, y')|^2 ds dy'$$

FIGURE 5. Carleman's method ($d = 2$).

Applying Gauss-Ostrogradsky formula to $u \cdot \nabla u$ on Ω_t , we have the following relations

$$A(t) = \int_{\Theta_t} u(t, y') u_t(t, y') dy' + \int_{\Theta_t} u^2(t, y') dy' = \phi'(t)/2 + \phi(t)$$

and

$$A'(t) = \int_{\Theta_t} |\nabla u(t, y')|^2 dy' = \int_{\Theta_t} (u_{x_1}^2(t, y') + u_{x_2}^2(t, y') + u_{x_3}^2(t, y')) dy'$$

Recalling the definition of λ and (4.6), we see

$$\begin{aligned} \int_{\Theta_t} (u_{x_2}^2(t, y') + u_{x_3}^2(t, y')) dy' &= \int_{\Theta_t} |\nabla' u|^2 dy' \\ &\geq \lambda(t) \int_{\Theta_t} u^2 dy' = \lambda(t) \phi(t) \end{aligned}$$

By Cauchy-Schwarz,

$$\phi'(t) \leq 2\sqrt{\phi(t)} \left(\int_{\Theta_t} u_{x_1}^2(t, y') dy' \right)^{1/2}$$

Thus, we have the following differential inequality

$$\phi'' + 2\phi' \geq 2\lambda(t)\phi + \frac{\phi'^2}{2\phi}$$

Make the substitution $\phi = \varphi e^{-2t}$. Then,

$$\varphi'' \geq 2(\lambda + 1)\varphi + \frac{\varphi'^2}{2\varphi}$$

This inequality implies $\varphi'' > 0$ for $t \in (-1, R)$ and, since $\varphi(-1) = 0$ and $\varphi(R) > 0$, we must have $\varphi'(t) > 0$. After multiplying by $2/\varphi$ and complementing to a full square, we obtain

$$\left(\frac{\varphi''}{\varphi'}\right)^2 - \left(\frac{\varphi''}{\varphi'} - \frac{\varphi'}{\varphi}\right)^2 \geq 4(\lambda + 1)$$

Since $\varphi''/\varphi' > 0$, we have $\varphi'' \geq 2\sqrt{\lambda + 1}\varphi'$. Set $\mu = 2\sqrt{\lambda + 1}$ and

$$\psi(t) = \int_{-1}^t \exp\left(\int_{-1}^u \mu(s) ds\right) du$$

for the sake of brevity. One has $\psi'' = \mu\psi'$. Hence,

$$\left(\log \frac{\varphi'}{\psi'}\right)' = \frac{\varphi''}{\varphi'} - \frac{\psi''}{\psi'} \geq 0$$

so the function φ'/ψ' is non-decreasing and, for $-1 < x < t$

$$\varphi'(x)\psi'(t) \leq \varphi'(t)\psi'(x)$$

Integrating in x and t from -1 , we see that $\varphi(x)\psi(t) \leq \varphi(t)\psi(x)$. Putting this a bit differently,

$$\phi(x) \leq \frac{\psi(x)}{\psi(t)}\phi(t)e^{2(t-x)}$$

Since $\phi(R) = |A|$, we have

$$\phi(0) \lesssim |A|e^{2R} \left(1 + \int_0^R \exp\left(2 \int_0^t \sqrt{\lambda + 1} ds\right) dt\right)^{-1}$$

Recalling (4.9) and Harnack's principle,

$$\omega(0, A, \Omega_R) \lesssim |A|^{1/2} e^R \left(1 + \int_0^R \exp\left(2 \int_0^t \sqrt{\lambda + 1} ds\right) dt\right)^{-1/2}$$

As for the second inequality claimed in the theorem, we only need to use

$$\int_0^R \exp\left(2 \int_0^t \sqrt{\lambda + 1} ds\right) dt \geq \exp\left(2 \int_0^{R-1} \sqrt{\lambda + 1} ds\right)$$

□

Remark. The case of a wide long strip shows that the first estimate is essentially sharp.

Remark. For $d = 2$, the bound on the harmonic measure is the same but the principal eigenvalue equals to

$$\lambda(t) = \left(\frac{\pi}{l(t)}\right)^2$$

where $l(t)$ is the length of the longest interval among intervals forming Θ_t . If $d = 3$ and Θ_t is simply-connected, the principal eigenvalue is controlled by the inradius: $\lambda(t) \sim r^{-2}(t)$. In general, if $d > 3$ or $d = 3$ and Θ_t is not simply-connected, one can use $\lambda(t) \geq h^2(t)/4$ where $h(t)$ is the Cheeger's constant of Θ_t [22]. Unfortunately, it is neither easy to compute nor to estimate in practice.

APPENDIX B: THE LOWER BOUNDS ON HARMONIC MEASURE

In this section, we will estimate the harmonic measure from below. Recall that a set $A \subset \mathbb{R}^d$ is Steiner symmetric with respect to a $(d-1)$ -dimensional plane p , if $s \cap A$ is an interval symmetric with respect to p for any straight line s perpendicular to p provided $s \cap A \neq \emptyset$, see Pólya-Szegő [40]. We say that A is Steiner symmetric (without mentioning p) if A is Steiner symmetric with respect to coordinate planes.

Let Θ be a Steiner symmetric domain with piece-wise smooth boundary lying in the x' -plane, and $\tilde{\Theta}_t = \tilde{k}(t) \cdot \Theta$, where $\tilde{k} : \mathbb{R} \rightarrow \mathbb{R}_+$ is a monotonically decreasing smooth function, $\tilde{k}(0) = 1$. Denote by $\lambda = \lambda(0)$ the principal eigenvalue of the operator $-\Delta_D$ on Θ . Let ψ be the corresponding eigenfunction normalized by $\psi(0, 0) = 1$.

Lemma 4.2. *Let $\Omega = \{(t, x') : 0 < t < \infty, x' \in \tilde{\Theta}_t\}$ and u solve the following equation*

$$u_t = \frac{1}{2} \Delta u$$

on Ω . Above, the boundary conditions are $u(0, x') = 1, x' \in \Theta$, and $u|_{\partial\tilde{\Theta}_t} = 0, t > 0$. Then

$$u(t, 0, 0) \geq \exp\left(-\frac{\lambda}{2} \int_0^t \frac{d\tau}{\tilde{k}^2(\tau)}\right)$$

Proof. Take the trial function

$$(4.10) \quad \tilde{u}(t, x') = \psi\left(\frac{x'}{\tilde{k}(t)}\right) \exp\left(-\frac{\lambda}{2} \int_0^t \frac{d\tau}{\tilde{k}^2(\tau)}\right)$$

Since $\psi = 0$ on $\partial\Theta$, the function $\tilde{u}(t, \cdot)$ vanishes on $\partial\tilde{\Theta}_t$ for any t . Furthermore, we have by [23], Lemma 7.3, that $x' \cdot \nabla\psi \leq 0$ and, in particular, ψ has its maximum at the origin. Consequently, $\tilde{u}(t, \cdot)$ also attains its maximum at the origin (of the x' -plane); the maximum is easy to compute using (4.10). Moreover,

$$\frac{1}{2} \Delta \tilde{u} - \tilde{u}_t = \frac{\tilde{k}'}{\tilde{k}^2} (x' \cdot \nabla\psi) \exp\left(-\frac{\lambda}{2} \int_0^t \frac{d\tau}{\tilde{k}^2(\tau)}\right) \geq 0$$

By the maximum principle for (parabolic) subharmonic functions ([32], Theorem 2.1), we get $\tilde{u} \leq u$, and the bound is proved. \square

Let us now change the set up a little: for Θ as above, define $\Theta_t = k(t) \cdot \Theta$, where $k : \mathbb{R} \rightarrow \mathbb{R}_+$ is monotonically increasing smooth function, $k(0) = 1$.

Theorem 4.3. *We have*

$$(4.11) \quad \omega(0, \Theta_R, \Omega_R) \geq C(\delta) \exp\left(-\frac{\lambda}{2} \int_2^R \frac{1}{k^2(t-t^\delta)} dt\right)$$

where $\lambda = \lambda(0)$ is the principal eigenvalue of $-\Delta_D$ on Θ and $\frac{1}{2} < \delta < 1$ is a fixed parameter.

Proof. Recall the probabilistic interpretation of $\omega(0, \Theta_R, \Omega_R)$. We write $G_t = t \cdot e^1 + B(t) = (t + B_1(t), B'(t))$ and B_j are independent one-dimensional

Brownian motions (warning: $B'(t) = (B_2(t), B_3(t))$, and not a derivative). Then,

$$\omega(0, \Theta_R, \Omega_R) \gtrsim \mathbb{P}_{B'}\left(B'(t) \in \Theta_{k(t-t^\delta)}, \forall t > 2 \mid E\right) \mathbb{P}_{B_1}(E)$$

where E is an event $\{|B_1(t)| < t^\delta \text{ for all } t > 2\}$. By the law of iterated logarithm, $\mathbb{P}_{B_1}(E) > 0$ and thus the problem is reduced to estimating the solution for the parabolic equation from the above lemma. Apply it to the function $\tilde{k}(t) = k(t-t^\delta)$ and make the change of variable $t \mapsto R-t, t \in (0, R)$; the theorem follows. \square

Remark. Comparison of (4.11) to (4.8) shows that the estimates are sharp in a certain sense for the Steiner symmetric domains monotonically opening at infinity. The case when the scaling function k is decreasing can be handled by adjusting the methods of [23], Theorem 7.1.

If the function k is not monotonic, proving the lower bounds for the harmonic measure is an interesting problem.

The estimate that we just obtained is not so sharp if one tries to prove that the harmonic measure is positive uniformly in R . For example, it does not allow one to recover the law of iterated logarithm. Instead, one can use estimates from Theorem 4.1. As an alternative to this method, we suggest the following approach.

We want to obtain the bound from below on $\omega(0, A, \Omega_R)$ which is uniform in R . Let us first consider domains embedded in \mathbb{R}^2 , i.e. $\Omega = \{(t, x) : t \in (-1, \infty), |x| < \theta(t)\}$ where x is scalar and θ is a positive smooth function. The boundary of Ω is denoted by Γ . Let $G_t = (t + B_1(t), B_2(t))$, where B_j are independent one-dimensional Brownian motions. We now want to find the conditions on θ which guarantee that $\mathbb{P}(G_t \text{ does not hit } \Gamma \forall t) > 0$.

For this purpose, we will use the argument from the proof of the law of iterated logarithm [37], Theorem 5.1. Take any $\epsilon > 0$ and $q > 1$ and define

$$C_q = 2 \log q, \quad l_n = \sqrt{(C_q + \epsilon)q^{n+1} \log_q n},$$

For a given q , we introduce the following characteristic parameters of θ :

$$\kappa_n = \min_{[q^n, q^{n+1}]} \theta(t)$$

where $n = 1, 2, \dots$

Theorem 4.4. *If*

$$\sum_n \frac{q^{n/2}}{\kappa_n} \exp\left(-\frac{\kappa_n^2}{2q^{n+1}}\right) < \infty$$

then $\mathbb{P}(G_t \text{ does not hit } \Gamma \forall t) > 0$.

Proof. Assume that this is not the case. Then, for any large $T > 0$, the trajectory $G'_t = (T + t + B_1(t), B_2(t))$ hits Γ almost surely (warning: once again, G'_t is not a derivative in time). Take T large and introduce the geometric decomposition in time $I_n = [q^n, q^{n+1}), n \in \mathbb{N}$. We also need

$$J_n = [q^n + l_n, q^{n+1} - l_n], \quad D_n = [q^n - l_{n-1}, q^n + l_n]$$

Consider the random numbers

$$\alpha_n = \max_{J_n} |B_1(t)|, \quad \beta_n = \max_{J_n} |B_2(t)|, \quad \gamma_n = \max_{D_n} |B_1(t)|, \quad \delta_n = \max_{D_n} |B_2(t)|$$

We have

$$(4.12) \quad \begin{aligned} \mathbb{P}(G'_t \text{ hits } \Gamma) &\leq \mathbb{P}(G'_t \text{ hits } \Gamma \text{ for some } t \in [0, q^{n_0}]) \\ &+ \sum_{n > n_0} \mathbb{P}(G'_t \text{ hits } \Gamma \text{ for some } t \in J_n) \\ &+ \sum_{n > n_0} \mathbb{P}(G'_t \text{ hits } \Gamma \text{ for some } t \in D_n) \end{aligned}$$

Recall that [37], Theorem 2.18, for $a > 0$,

$$\mathbb{P}(\max_{t \in [0, T]} B(t) > a) = 2\mathbb{P}(B(T) > a),$$

Then

$$\begin{aligned} \mathbb{P}(G'_t \text{ hits } \Gamma \text{ for some } t \in J_n) &= \mathbb{P}(G'_t \text{ hits } \Gamma \text{ for some } t \in J_n | E_n) \mathbb{P}(E_n) \\ &+ \mathbb{P}(G'_t \text{ hits } \Gamma \text{ for some } t \in J_n | E_n^c) \mathbb{P}(E_n^c) \end{aligned}$$

where E_n is the event that $\{\alpha_n < l_n\}$. An easy computation shows

$$(4.13) \quad \mathbb{P}(E_n^c) \lesssim \exp\left(-(\log q + \epsilon/2) \log_q n\right) \in \ell^1$$

and the estimate on β_n gives

$$\mathbb{P}(G'_t \text{ hits } \Gamma \text{ for some } t \in J_n | E_n) \lesssim \frac{q^{n/2}}{\kappa_n} \exp\left(-\frac{\kappa_n^2}{2q^{n+1}}\right) \in \ell^1$$

by assumption.

Similarly, the estimates for γ_n and δ_n yield

$$\begin{aligned} \mathbb{P}(G'_t \text{ hits } \Gamma \text{ for some } t \in D_n) &\lesssim \nu_n + \frac{\sqrt{q^n + 2l_n}}{\kappa_{n-1}} \exp\left(-\frac{\kappa_{n-1}^2}{2(q^n + 2l_n)}\right) \\ &+ \frac{\sqrt{q^n + 2l_n}}{\kappa_n} \exp\left(-\frac{\kappa_n^2}{2(q^n + 2l_n)}\right) \end{aligned}$$

where $\nu_n \in \ell^1$ as in (4.13). The last two terms are in ℓ^1 as well by our assumption. Thus, one can choose n_0 large enough to guarantee that the second and the third terms in the right hand side of (4.12) are arbitrarily small. Then, as long as n_0 is fixed, the first term can be made arbitrarily small by taking T large, and the theorem is proved. \square

Remark. If $\theta(t) = \sqrt{\gamma t \log \log t}$, then the law of iterated logarithm shows our result is sharp in some sense (choose $\gamma < 2$ and $q = 1 + \epsilon$, $\epsilon > 0$ small enough).

We now turn to the three-dimensional case and consider $G(t) = t \cdot e^1 + B(t) = (t + B_1(t), B_2(t), B_3(t))$, and

$$\Omega = \{(t, x_2, x_3) : t \in (-1, \infty), |x_2| < \theta_2(t), |x_3| < \theta_3(t)\},$$

$\theta_{2,3}$ being positive smooth functions. For $q_2, q_3 > 1$, define

$$\kappa_{j,n} = \min_{[q_j^n, q_j^{n+1}]} \theta_j(t)$$

where $j = 2, 3$, and $n = 1, 2, \dots$

The proof of Theorem 4.5 is based on Theorem 4.4 and the independence of Brownian motions B_2 and B_3 .

Theorem 4.5. *If*

$$\sum_n \frac{q_j^{n/2}}{\kappa_{j,n}} \exp\left(-\frac{\kappa_{j,n}^2}{2q_j^{n+1}}\right) < \infty$$

for $j = 2, 3$, then $\mathbb{P}(G_t \text{ does not hit } \Gamma \forall t) > 0$.

APPENDIX C: THE CASE $d > 3$

Let, as before, $x = (x_1, x') \in \mathbb{R}^d$ and $x' \in \mathbb{R}^{d-1}$. The Green's function of $(-\Delta + 1)^{-1}$ on $L^2(\mathbb{R}^d)$ is given by

$$G_0(x, y) = C_d |x - y|^{-\nu} K_\nu(|x - y|)$$

and

$$G_0(x, y) \approx \begin{cases} C_d^1 e^{-|x-y|} |x - y|^{-(d-1)/2}, & |x - y| \rightarrow \infty, \\ C_d^2 |x - y|^{-(d-2)}, & |x - y| \rightarrow 0, \end{cases}$$

see Section 2.1 for the values of the constants $C_d^{1,2}$. Once again, introduce potentials

$$K^-(z, \xi) = 2G_0(z, \xi)e^{\xi_1 - z_1}, \quad K^+(z, \xi) = K^-(\xi, z)$$

and compare this to (3.3). The corresponding capacity $C = C^\pm$ is then defined as in (3.5) or (3.6).

As for Hausdorff content, proceed like in Section 3.3. Set $\Pi_r = [0, r]^d$ for $0 < r \leq 1$, and $\Pi_r = [0, r^2] \times [0, r]^{d-1}$ for $r > 1$. We say that $H(r)$ is the characteristic set if it can be translated into the parallelepiped Π_r . The Hausdorff content $M_h(E)$ of a set $E \subset \mathbb{R}^d$ is defined by relation (3.16) with

$$h(r) = \begin{cases} r^{d-2}, & 0 < r \leq 1 \\ r^{d-1}, & r > 1 \end{cases}$$

Proposition 4.1. *For any set $E \subset \mathbb{R}^d$ and the above measure function h , we have*

$$C(E) \lesssim M_h(E)$$

Furthermore, the definitions of contractions \aleph and Pr' do not change. Obviously, for $E \subset \mathbb{R}^d$, the expression $|\text{Pr}' E|$ refers now to $(d-1)$ -dimensional volume etc. With these conventions, the formulations of Theorems 3.4, 3.5, 4.2 are the same word-for-word.

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