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On the extinction of Continuous State Branching Processes with catastrophes

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Abstract

We consider continuous state branching processes (CSBP's) with additional multiplicative jumps modeling dramatic events in a random environment. These jumps are described by a Lévy process with bounded variation paths. We construct the associated class of processes as the unique solution of a stochastic differential equation. The quenched branching property of the process allows us to derive quenched and annealed results and make appear new asymptotic behaviors. We characterize the Laplace exponent of the process as the solution of a backward ordinary differential equation and establish when it becomes extinct. For a class of processes for which extinction and absorption coincide (including the α -stable CSBP's plus a drift), we determine the speed of extinction. Four regimes appear, as in the case of branching processes in random environment in discrete time and space. The proofs rely on a fine study of the asymptotic behavior of exponential functionals of Lévy processes. Finally, we apply these results to a cell infection model and determine the mean speed of propagation of the infection.

Key words. Continuous State Branching Processes, Lévy processes, Poisson Point Processes, Random Environment, Extinction, Long time behavior

A.M.S. Classification. 60J80, 60J25, 60G51, 60H10, 60G55, 60K37.

1 Introduction

Continuous state branching processes (CSBP's) are the analogues of Galton-Watson (GW) processes in continuous time and continuous state space. They have been introduced by Jirina [Jir58] and studied by many authors included Bingham [Bin76], Grey

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[Gre74], Grimvall [Gri74], Lamperti [Lam67a, Lam67b], to name but a few. A CSBP $Z = (Z_t, t \geq 0)$ is a strong Markov process taking values in $[0, \infty]$ and satisfying the branching property. The states 0 and ∞ are absorbing and $(\mathbb{P}_x, x > 0)$ denotes the law of Z starting from x . Lamperti has proved in [Lam67b] that there is a bijection between CSBP's and scaling limits of GW processes. Thus they may model the evolution of renormalized large populations evolving during a large time window.

The branching property implies that the Laplace transform of Z_t is of the form

$$\mathbb{E}_x \left[\exp(-\lambda Z_t) \right] = \exp\{-xu_t(\lambda)\}, \quad \text{for } \lambda \geq 0,$$

for some non negative function u_t . According to Silverstein [Sil68], this function is determined by the integral equation

$$\int_{u_t(\lambda)}^{\lambda} \frac{1}{\psi(u)} du = t,$$

where ψ is the branching mechanism. We assume here that Z has a finite first moment, so that we have the following classical representation

$$\psi(\lambda) = -g\lambda + \sigma^2\lambda^2 + \int_0^\infty \left(e^{-\lambda x} - 1 + \lambda x \right) \mu(dx), \quad (1)$$

where $g \in \mathbb{R}$, $\sigma \geq 0$ and μ is a σ -finite measure on $(0, \infty)$ such that $\int_{(0, \infty)} (x \wedge x^2) \mu(dx)$ is finite. The CSBP is then characterized by the triplet (g, σ, μ) and can also be defined as the unique non-negative strong solution of a stochastic differential equation. More precisely, from [FL10],

$$Z_t = Z_0 + \int_0^t gZ_s ds + \int_0^t \sqrt{2\sigma^2 Z_s} dB_s + \int_0^t \int_0^\infty \int_0^{Z_{s-}} z \tilde{N}_0(ds, dz, du), \quad (2)$$

where B is a standard Brownian motion, $N_0(ds, dz, du)$ is a Poisson random measure with intensity $ds\mu(dz)du$ independent of B , and \tilde{N}_0 is the compensated measure of N_0 .

The stable case with an additional drift is given by $\psi(\lambda) = g\lambda + c\lambda^{1+\beta}$, with β in $(0, 1]$. It corresponds to the CSBP's that one can obtain by scaling limits of GW processes with a fixed reproduction law. It is of special interest in this paper since the Laplace exponent can be computed and used to derive asymptotic results for more general processes.

In this work, we take into account catastrophes which occur randomly and kill each individual with some probability (depending on the catastrophe). In the scaling limit of large population (the continuous state setting), it amounts to let the process make a negative jump and multiply its value by a random fraction. The process we obtain is still Markovian if the catastrophes happen following a time homogeneous Poisson Point Process. We show that conditionally on the times and the effects of the catastrophes, the process satisfies the branching property. Thus, it yields a particular class of CSBP's in random environment, which can also be obtained as scaling limit of GW in random environment (see [BS47]). Such processes are motivated in particular by a cell division model, see [BT11] and Section 5.

We can also consider positive jumps, accounting for immigration events proportional to the size of the population. One motivation comes from the aggregation behavior of some species. We refer to Chapter 12 in [DGC08] for adaptive explanations of these

aggregation behaviors, or [RLF⁺13] which shows that aggregation behaviors may result from manipulation by parasites to increase their transmission. For convenience, we still call these dramatic events catastrophes.

The process Y that we consider in this paper is then called a *CSBP with catastrophes*. Informally, it can be defined as follows. The process Y follows the SDE (2) between the catastrophes, which are given by the jumps of a Lévy process with bounded variations paths. Thus the time when the catastrophes occur may have accumulation points but the mean effect of the catastrophes has a finite first moment. When a catastrophe with effect m_t occurs at time t , we have

$$Y_t = m_t \cdot Y_{t-}.$$

We defer the formal definitions to Section 2. We also note that Brockwell has considered birth and death branching processes with other kind of catastrophes, see e.g. [Bro85].

We first check that the CSBP's with catastrophes are well defined as the solution of the stochastic differential equation (5). We characterize their Laplace exponent via an ordinary differential equation (Theorem 1), which allows us to describe their long time behavior. In particular, we prove that the extinction criterion of the CSBP with catastrophes Y is given by the sign of $\mathbb{E}(g + \sum_{s \leq 1} \log m_s)$. We also establish a central limit theorem under some moment assumption when the process survives (Corollary 3).

We then focus on the case when Y is a stable CSBP. Here, the extinction and absorption events coincide, which means that $\{\lim_{t \rightarrow \infty} Y_t = 0\} = \{\exists t \geq 0, Y_t = 0\}$. We prove that the speed of extinction is directly related to the asymptotic behavior of exponential functionals of Lévy processes (Proposition 4). More precisely, we show that the extinction probability of a stable CSBP with catastrophes can be expressed in the following way:

$$\mathbb{P}(Y_t > 0) = \mathbb{E} \left[F \left(\int_0^t e^{-\beta K_s} ds \right) \right],$$

where $\beta \in (0, 1]$, F is a function with a particular asymptotic behavior and $K_t := gt + \sum_{s \leq t} \log m_s$ is a Lévy process with bounded variation paths that does not go to ∞ and has some exponential positive moments.

We establish the asymptotic behavior of these quantities (Theorem 7) and find four different regimes. They depend on the shape of the Laplace exponent of K , i.e. the rate of growth g of the CSBP and the law of the catastrophes. The asymptotic behavior of exponential functionals of Lévy processes drifting to $+\infty$ has been deeply studied by many authors, see for instance Bertoin and Yor [BY05] and references therein. Up to our knowledge, the other cases have been studied only by Carmona et al. [CPY97] (see Lemma 4.7) but their result focuses on one single regime.

Our result is closely related to the discrete framework via asymptotic behavior of functionals of random walks. More precisely, we use in our arguments local limit theorems for semi direct product [LPP97, GL01] and some analytical results on random walks [Koz76, Hir98], see Section 4.

Actually, such asymptotic behaviors were known for branching processes in random environments in discrete time and space (see e.g. [GL01, GKV03, AGKV05]). In the particular case $\beta = 1$, the connection will be specified in the Remarks 2 and 3.

From the speed of extinction in the stable case, we can deduce the speed of extinction for a large class of CSBP's with catastrophes, when extinction and absorption coincide (Corollary 6). The generalization of the results to Lévy processes with infinite variations

raises many difficulties, among which the existence of the process Y and the adaptation of the approximation methods. The particular case when $\mu = 0$ and the environment K is given by a Brownian motion has been handled in [BH12]. Their proof relies on the explicit law of $\int_0^t \exp(-\beta K_s) ds$ and the authors have obtained similar asymptotic regimes.

Finally, we apply our results to a cell infection model introduced in [BT11] (Section 5). In this model, the infection in a cell line is given by a Feller diffusion with catastrophes. We derive here the different possible speeds of propagation of the infection.

More generally, these results can be related to discussions in ecology about the role of environmental and demographical stochasticities. Such topics are fundamental in conservation biology, as discussed for instance in Chapter 1 in [LES03]. Indeed, the survival of the population may be either due to the randomness of the individual reproduction, which are specified in our model by the parameters σ and μ of the CSBP, or to the randomness (rate, size) of the catastrophes due to the environment. For a study of relative effects of environmental and demographical stochasticities, one can read [Lan93] and references therein.

The remainder of the paper is structured as follows. In the next section, we define and study the CSBP's with catastrophes. Section 3 is devoted to their speed of extinction. In Section 4 we study the asymptotic behavior of exponential functionals of Lévy processes with bounded variation paths, which is the key result to get the different extinction regimes. In Section 5, we apply our results to a cell infection model. Section 6 contains some technical results used in the proofs and deferred for the convenience of the reader.

2 CSBP with catastrophes

We consider a CSBP $Z = (Z_t, t \geq 0)$ defined by (2) and characterized by the triplet (g, σ, μ) , where we recall that μ satisfies

$$\int_0^\infty (x \wedge x^2) \mu(dx) < \infty. \quad (3)$$

The catastrophes are independent of the process Z and are given by a Poisson random measure $N_1 = \sum_{i \in I} \delta_{m_{t_i}, t_i}$ on $[0, \infty) \times [0, \infty)$ with intensity $dt\nu(dx)$ such that

$$0 < \int_{(0, \infty)} (1 \wedge |x - 1|) \nu(dx) < \infty. \quad (4)$$

The jump process

$$\Delta_t = \int_0^t \int_{(0, \infty)} \log(x) N_1(ds, dx) = \sum_{s \leq t} \log(m_s)$$

is thus a Lévy process with bounded variation paths, which is non identically zero.

The CSBP (g, σ, μ) with catastrophes ν is defined as the solution of the following stochastic differential equation:

$$\begin{aligned} Y_t = Y_0 + \int_0^t g Y_s ds + \int_0^t \sqrt{2\sigma^2 Y_s} dB_s &+ \int_0^t \int_{[0, \infty)} \int_0^{Y_{s-}} z \tilde{N}_0(ds, dz, du) \\ &+ \int_0^t \int_{[0, \infty)} (m - 1) Y_{s-} N_1(ds, dm), \end{aligned} \quad (5)$$

where $Y_0 > 0$ a.s. Denoting by $\mathcal{BV}(\mathbb{R}_+)$ the set of càdlàg functions on \mathbb{R}^+ with bounded variations, we have the following result of existence and unicity:

Theorem 1. *The stochastic differential equation (5) has a unique non-negative strong solution Y for any $g \in \mathbb{R}$, $\sigma \geq 0$, μ and ν satisfying the conditions (3) and (4), respectively.*

Then, the process $Y = (Y_t, t \geq 0)$ is a càdlàg Markov process satisfying the branching property conditionally on $\Delta = (\Delta_t, t \geq 0)$ and its generator is given by

$$\begin{aligned} \mathcal{A}f(x) &= gf'(x) + \sigma^2 x f''(x) + \int_0^\infty (f(xz) - f(x)) \nu(dz) \\ &\quad + \int_0^\infty (f(x+z) - f(x) - zf'(x)) x \mu(dz). \end{aligned} \quad (6)$$

Moreover, for every $t \geq 0$,

$$\mathbb{E}_y \left[\exp \left\{ -\lambda \exp \left\{ -gt - \Delta_t \right\} Y_t \right\} \middle| \Delta \right] = \exp \left\{ -yv_t(0, \lambda, \Delta) \right\} \quad a.s.,$$

where for every $(\lambda, \delta) \in (\mathbb{R}_+, \mathcal{BV}(\mathbb{R}_+))$, $v_t : s \in [0, t] \mapsto v_t(s, \lambda, \delta)$ is the unique solution of the following backward differential equation :

$$\frac{\partial}{\partial s} v_t(s, \lambda, \delta) = e^{gs + \delta s} \psi_0(e^{-gs - \delta s} v(s, \lambda, \delta)), \quad v_t(t, \lambda, \delta) = \lambda, \quad (7)$$

and

$$\psi_0(\lambda) = \psi(\lambda) - \lambda \psi'(0) = \sigma^2 \lambda^2 + \int_0^\infty (e^{-\lambda x} - 1 + \lambda x) \mu(dx). \quad (8)$$

Proof. Under Lipschitz conditions, the existence and uniqueness of strong solutions for the stochastic differential equations are classical results (see [IW89]). Here the result is a consequence of Proposition 2.2 and Theorems 3.2 and 5.1 in [FL10]. By Itô's formula (see for instance [IW89] Th.5.1), the solution $(Y_t, t \geq 0)$ solves the following martingale problem. For every $f \in C_b^2(\mathbb{R}_+)$,

$$\begin{aligned} f(Y_t) &= f(Y_0) + \text{loc. mart.} + g \int_0^t f'(Y_s) Y_s ds \\ &\quad + \sigma^2 \int_0^t f''(Y_s) Y_s ds + \int_0^t \int_0^\infty Y_s (f(Y_s + z) - f(Y_s) - f'(Y_s)z) \mu(dz) ds \\ &\quad + \int_0^t \int_0^\infty (f(zY_{s-}) - f(Y_{s-})) \nu(dz) ds, \end{aligned}$$

where the local martingale is given by

$$\begin{aligned} &\int_0^t f'(Y_s) \sqrt{2\sigma^2 Y_s} dB_s + \int_0^t \int_0^\infty (f(zY_{s-}) - f(Y_{s-})) \tilde{N}_1(ds, dz) \\ &\quad + \int_0^t \int_0^\infty \int_0^{Y_{s-}} (f(Y_{s-} + z) - f(Y_{s-})) \tilde{N}_0(ds, dz, du), \end{aligned} \quad (9)$$

and \tilde{N}_1 is the compensated measure of N_1 . From pathwise uniqueness, we deduce that the solution of (5) is a strong Markov process whose generator is given by (6).

The branching property of Y , conditionally on Δ , is inherited from the branching property of the CSBP and the fact that the additional jumps are multiplicative.

To prove the second part of the theorem, let us now work conditionally on Δ . By applying Itô's formula to the process $\tilde{Z}_t = Y_t \exp(-gt - \Delta_t)$, we obtain:

$$\tilde{Z}_t = Y_0 + \int_0^t e^{-gs - \Delta_s} \sqrt{2\sigma^2 Y_s} dB_s + \int_0^t \int_0^\infty \int_0^{Y_{s-}} e^{-gs - \Delta_{s-} - z} \tilde{N}_0(ds, dz, du),$$

and \tilde{Z} is a local martingale conditionally on Δ . A new application of Itô's formula ensures that for every $F \in C_b^{1,2}$, $F(t, \tilde{Z}_t)$ is also a local martingale if and only if for every $t \geq 0$,

$$\begin{aligned} & \int_0^t \frac{\partial^2}{\partial x^2} F(s, \tilde{Z}_s) \sigma^2 \tilde{Z}_s e^{-gs - \Delta_s} ds + \int_0^t \frac{\partial}{\partial t} F(s, \tilde{Z}_s) ds \\ & + \int_0^t \int_0^\infty \tilde{Z}_s \left(\left[F(s, \tilde{Z}_s + ze^{-gs - \Delta_s}) - F(s, \tilde{Z}_s) \right] e^{gs + \Delta_s} - \frac{\partial}{\partial x} F(s, \tilde{Z}_s) z \right) \mu(dz) ds = 0. \end{aligned} \quad (10)$$

In the vein of [IW89, BT11], we choose $F(s, x) := \exp\{-xv_t(s, \lambda, \Delta)\}$, where $v_t(s, \lambda, \Delta)$ is differentiable with respect to the variable s , non negative and such that $v_t(t, \lambda, \Delta) = \lambda$, for $\lambda \geq 0$. The function F is bounded, so that $(F(s, \tilde{Z}_s), 0 \leq s \leq t)$ will be a martingale if and only if for every $s \in [0, t]$

$$\frac{\partial}{\partial s} v_t(s, \lambda, \Delta) = e^{gs + \Delta_s} \psi_0(e^{-gs - \Delta_s} v_t(s, \lambda, \Delta)), \quad \text{a.s.},$$

where ψ_0 is defined in (8). Proposition 15 in Section 6 ensures that a.s. the solution of this backward differential equation exists and is unique, which essentially comes from the local Lipschitz property of ψ_0 and bounded variation paths of Δ . Then the process $(\exp\{-\tilde{Z}_s v_t(s, \lambda, \Delta)\}, 0 \leq s \leq t)$ is a martingale conditionally on Δ and hence

$$\mathbb{E}_y \left[\exp \left\{ -\tilde{Z}_t v_t(t, \lambda, \Delta) \right\} \middle| \Delta \right] = \mathbb{E}_y \left[\exp \left\{ -\tilde{Z}_0 v_t(0, \lambda, \Delta) \right\} \middle| \Delta \right] \quad \text{a.s.},$$

which implies

$$\mathbb{E}_y \left[\exp \left\{ -\lambda \tilde{Z}_t \right\} \middle| \Delta \right] = \exp \left\{ -y v_t(0, \lambda, \Delta) \right\} \quad \text{a.s.}, \quad (11)$$

and ends up the proof. \square

Referring to Theorem 7.2 in [Kyp06], we recall that a Lévy process has three possible asymptotic behaviors: either it drifts to ∞ , $-\infty$, or oscillates a.s. When it has a finite first moment, the sign of its expectation yields the regimes above. We extend this classification to CSBP with catastrophes.

Corollary 2. *We have the three following regimes.*

- i) If $(\Delta_t + gt)_{t \geq 0}$ drifts to $-\infty$, then $\mathbb{P}(Y_t \rightarrow 0 \mid \Delta) = 1$ a.s.
- ii) If $(\Delta_t + gt)_{t \geq 0}$ oscillates, then $\mathbb{P}(\liminf_{t \rightarrow \infty} Y_t = 0 \mid \Delta) = 1$ a.s.
- iii) If $(\Delta_t + gt)_{t \geq 0}$ drifts to $+\infty$ and there exists $\varepsilon > 0$, such that

$$\int_0^\infty x \log(1+x)^{1+\varepsilon} \mu(dx) < \infty, \quad (12)$$

then $\mathbb{P}(\liminf_{t \rightarrow \infty} Y_t > 0 \mid \Delta) > 0$ a.s. and there exists a non negative finite r.v. W such that

$$e^{-gt - \Delta_t} Y_t \xrightarrow[t \rightarrow \infty]{} W \quad \text{a.s.}, \quad \{W = 0\} = \left\{ \lim_{t \rightarrow \infty} Y_t = 0 \right\}.$$

Remark 1. In the regime (ii), one can have $\liminf_{t \rightarrow \infty} Y_t = 0$ a.s. but Y_t a.s. does not tend to zero. For example, if $\mu = 0$ and $\sigma = 0$, then $Y_t = \exp(gt + \Delta_t)$ and $\limsup_{t \rightarrow \infty} Y_t = \infty$.

Assumption (iii) of the corollary does not imply that $\{\lim_{t \rightarrow \infty} Y_t = 0\} = \{\exists t : Y_t = 0\}$. Thus, the case $\mu(dx) = x^{-2} \mathbf{1}_{[0,1]}(x) dx$ inspired by Neveu CSBP yields $\psi(u) \sim u \log u$ as $u \rightarrow \infty$. Then, according to Remark 2.2 in [Lam08], $\mathbb{P}(\exists t : Y_t = 0) = 0$ and $0 < \mathbb{P}(\lim_{t \rightarrow \infty} Y_t = 0) < 1$.

Proof. We use (10) with $F(s, x) = x$ to get that $\tilde{Z} = (Y_t \exp(-gt - \Delta_t) : t \geq 0)$ is a non negative local martingale. Thus it is a non negative supermartingale and it converges a.s. to a non negative finite random variable W . It ends up the proof of (i-ii).

In the case when $(\Delta_t + gt)_{t \geq 0}$ goes to $+\infty$, we prove that $\mathbb{P}(W > 0 \mid \Delta) > 0$ a.s. According to Lemma 17 in Section 6, the assumptions of (iii) ensure the existence of a non negative increasing function h on \mathbb{R}^+ such that for all $\lambda > 0$,

$$\psi_0(\lambda) \leq \lambda h(\lambda) \quad \text{and} \quad c(\Delta) := \int_0^\infty h(e^{-gt + \Delta_t}) dt < \infty \quad \text{a.s.}$$

For every $(t, \lambda) \in (\mathbb{R}_+^*)^2$, the solution v_t of (7) increases on $[0, t]$. Thus for all $s \in [0, t]$, $v_t(s, 1, \Delta) \leq 1$, and

$$\begin{aligned} \psi_0(e^{-gs - \Delta_s} v_t(s, 1, \Delta)) &\leq e^{-gs - \Delta_s} v_t(s, 1, \Delta) h(e^{-gs - \Delta_s} v_t(s, 1, \Delta)) \\ &\leq e^{-gs - \Delta_s} v_t(s, 1, \Delta) h(e^{-gs - \Delta_s}) \quad \text{a.s.} \end{aligned}$$

Then (7) gives

$$\frac{\partial}{\partial s} v_t(s, 1, \Delta) \leq v_t(s, 1, \Delta) h(e^{-gs - \Delta_s}).$$

This implies

$$-\ln(v_t(0, 1, \Delta)) \leq \int_0^t h(e^{-gs - \Delta_s}) ds \leq c(\Delta) < \infty \quad \text{a.s.}$$

Hence, for every $t \geq 0$, $v_t(0, 1, \Delta) \geq \exp(-c(\Delta)) > 0$, and conditionnally on Δ there exists a positive lower bound for $v_t(0, 1, \Delta)$. Finally from (11),

$$\mathbb{E}_y(\exp(-\lambda W) \mid \Delta) = \exp\left(-y \lim_{t \rightarrow \infty} v_t(0, 1, \Delta)\right) < 1$$

and $\mathbb{P}(W > 0 \mid \Delta) > 0$ a.s.

Moreover, since Y satisfies the branching property conditionally on Δ , we can show (see Lemma 18 in Section 6) that

$$\{W = 0\} = \left\{ \lim_{t \rightarrow \infty} Y_t = 0 \right\} \quad \text{a.s.},$$

which completes the proof. \square

We now derive a central limit theorem in the supercritical regime:

Corollary 3. Assume that $(\Delta_t + gt)_{t \geq 0}$ drifts to $+\infty$ and (12) is satisfied. Then, under the additional assumption

$$\int_{(0, e^{-1}] \cup [e, \infty)} (\log x)^2 \nu(dx) < \infty, \quad (13)$$

conditionally on $\{W > 0\}$,

$$\frac{\log(Y_t) - \mathbf{m}t}{\rho\sqrt{t}} \xrightarrow[t \rightarrow \infty]{d} \mathcal{N}(0, 1),$$

where \xrightarrow{d} means convergence in distribution,

$$\mathbf{m} := g + \int_{\{|\log(x)| \geq 1\}} \log(x) \nu(dx) < \infty, \quad \rho^2 := \int_0^\infty (\log x)^2 \nu(dx) < \infty,$$

and $\mathcal{N}(0, 1)$ denotes a centered gaussian random variable with variance equals 1.

Proof. We use a central limit theorem for the Lévy process $(gt + \Delta_t, t \geq 0)$ under the assumption (13) which relies on Theorem 3.5 in Doney and Maller [DM02]. Letting the details be deferred to Section 6.4, we get

$$\frac{gt + \Delta_t - \mathbf{m}t}{\rho\sqrt{t}} \xrightarrow[t \rightarrow \infty]{d} \mathcal{N}(0, 1). \quad (14)$$

From Corollary 2 part *iii*), on the event $\{W > 0\}$, we get

$$\log(Y_t) - (gt + \Delta_t) \xrightarrow[t \rightarrow \infty]{a.s.} \log(W) \in (-\infty, \infty),$$

and we conclude by using (14). □

3 Speed of extinction of CSBP with catastrophes

In this section, we first focus on the stable CSBP's with growth $g \in \mathbb{R}$. Then we derive a similar result for another class of CSBP's.

3.1 The stable case

We assume in this section that

$$\psi(\lambda) = g\lambda + c_+ \lambda^{\beta+1}, \quad (15)$$

for some $\beta \in (0, 1]$, $c_+ > 0$ and g in \mathbb{R} .

If $\beta = 1$ (Feller diffusion), then $\mu = 0$ and the CSBP Z follows the continuous diffusion

$$Z_t = Z_0 + \int_0^t gZ_s ds + \int_0^t \sqrt{2\sigma^2 Z_s} dB_s, \quad t \geq 0.$$

If $\beta \in (0, 1)$, then $\sigma = 0$ and $\mu(dx) = c(\beta + 1)x^{-\beta} dx / \Gamma(1 - \beta)$. The process has positive jumps with infinite intensity [Lam07]. Moreover,

$$Z_t = Z_0 + \int_0^t gZ_s ds + \int_0^t Z_{s-}^{1/(\beta+1)} dX_s, \quad t \geq 0,$$

where X is a $(\beta + 1)$ -stable spectrally positive Lévy process.

For the stable CSBP with catastrophes, the backward differential Equation (7) can be solved and we get

Proposition 4. *For all $x_0 > 0$ and $t \geq 0$:*

$$\mathbb{P}_{x_0}(Y_t > 0 \mid \Delta) = 1 - \exp \left\{ -x_0 \left(c_+ \beta \int_0^t e^{-\beta(\Delta_s + gs)} ds \right)^{-1/\beta} \right\} \quad \text{a.s.} \quad (16)$$

Moreover,

$$\mathbb{P}_{x_0}(\text{there exists } t > 0, Y_t = 0 \mid \Delta) = 1 \quad \text{a.s.},$$

if and only if the process $(\Delta_t + gt, t \geq 0)$ does not go to $+\infty$.

Proof. We solve equation (7) with $\psi(\lambda) = g\lambda + c_+\lambda^{\beta+1}$. Since $\psi_0(\lambda) = c_+\lambda^{\beta+1}$, a direct integration yields

$$v_t(u, \lambda, \Delta) = \left[c_+ \beta \int_u^t e^{-\beta(\Delta_s + gs)} ds + \lambda^{-\beta} \right]^{-1/\beta},$$

so that

$$\mathbb{E}_{x_0} \left[e^{-\lambda \bar{Z}_t} \mid \Delta \right] = \exp \left\{ -x_0 \left(c_+ \beta \int_0^t e^{-\beta(\Delta_s + gs)} ds + \lambda^{-\beta} \right)^{-1/\beta} \right\} \quad \text{a.s.} \quad (17)$$

The expression for the absorption probability is a direct application of (17). Letting λ go to ∞ , we have

$$\mathbb{P}_{x_0}(Y_t = 0 \mid \Delta) = \exp \left\{ -x_0 \left(c_+ \beta \int_0^t e^{-\beta(\Delta_s + gs)} ds \right)^{-1/\beta} \right\} \quad \text{a.s.}$$

As $\mathbb{P}_{x_0}(\exists t \geq 0 : Y_t = 0 \mid \Delta) = \lim_{t \rightarrow \infty} \mathbb{P}_{x_0}(Y_t = 0 \mid \Delta)$ a.s.,

$$\mathbb{P}_{x_0}(\exists t \geq 0 : Y_t = 0 \mid \Delta) = \exp \left\{ -x_0 \left(c_+ \beta \int_0^\infty e^{-\beta(\Delta_s + gs)} ds \right)^{-1/\beta} \right\} \quad \text{a.s.}$$

Finally, according to Theorem 1 in [BY05], $\int_0^\infty \exp(-\beta(\Delta_s + gs)) ds = \infty$ a.s. if and only if the process $(\Delta_t + gt, t \geq 0)$ does not go to $+\infty$. It completes the proof. \square

In what follows, we assume that the Laplace exponent of the Lévy process Δ is well-defined for some positive real number, i.e.

$$\phi(\lambda) = \log \mathbb{E}[e^{\lambda \Delta_1}] < \infty \quad \text{for } \lambda \in [0, \theta_{max}), \quad (18)$$

where $\theta_{max} = \sup\{\lambda > 0, \phi(\lambda) < \infty\}$. In other words, $\int_{[e, \infty)} x^\lambda \nu(dx) < \infty$ if $\lambda \in [0, \theta_{max})$.

We recall that Δ_t/t converges a.s. to $\phi'(0)$ and that $g + \phi'(0)$ is negative in the subcritical case. Proposition 4 then yields the asymptotic behavior of the quenched survival probability :

$$e^{-gt - \Delta_t} \mathbb{P}_{x_0}(\exists t \geq 0; Y_t = 0 \mid \Delta) \sim x_0 \left(c_+ \beta \int_0^t e^{\beta(\Delta_t + gt - \Delta_s - gs)} ds \right)^{-1/\beta} \quad (t \rightarrow \infty),$$

which converges in distribution to a positive finite limit proportional to x_0 . Then,

$$\frac{1}{t} \log \mathbb{P}_{x_0}(\exists t \geq 0; Y_t = 0 \mid \Delta) \rightarrow g + \phi'(0) \quad (t \rightarrow \infty)$$

in probability.

Additional work is required to get the asymptotic behavior of the annealed survival probability, for which different regimes appear:

Proposition 5. *We assume that ν , ϕ and ψ satisfy (13), (18) and (15) respectively.*

a/ If $\phi'(0) + g < 0$ (subcritical case) and $\theta_{max} > 1$, then

(i) If $\phi'(1) + g < 0$ (strongly subcritical regime), then there exists $c_1 := d_1(\beta, \nu)c_+^{-1/\beta} > 0$ such that for every $x_0 > 0$,

$$\mathbb{P}_{x_0}(Y_t > 0) \sim c_1 x_0 e^{t(\phi(1)+g)}, \quad \text{as } t \rightarrow \infty.$$

(ii) If $\phi'(1) + g = 0$ (intermediate subcritical regime), then there exists $c_2 := d_2(\beta, \nu)c_+^{-1/\beta} > 0$ such that for every $x_0 > 0$,

$$\mathbb{P}_{x_0}(Y_t > 0) \sim c_2 x_0 t^{-1/2} e^{t(\phi(1)+g)}, \quad \text{as } t \rightarrow \infty.$$

(iii) If $\phi'(1) + g > 0$ (weakly subcritical regime) and $\theta_{max} > \beta + 1$, then for every $x_0 > 0$, there exists $c_3 := c_3(x_0, \psi, \nu) > 0$ such that

$$\mathbb{P}_{x_0}(Y_t > 0) \sim c_3 t^{-3/2} e^{t(\phi(\tau)+g\tau)}, \quad \text{as } t \rightarrow \infty,$$

where τ is the root of $\phi' + g$ on $]0, 1[$: $\phi(\tau) + g\tau = \min_{0 < s < 1} \{\phi(s) + gs\}$.

b/ If $\phi'(0) + g = 0$ (critical case) and $\theta_{max} > \beta$, then for every $x_0 > 0$, there exists $c_4 := c_4(x_0, \psi, \nu) > 0$ such that

$$\mathbb{P}_{x_0}(Y_t > 0) \sim c_4 t^{-1/2}, \quad \text{as } t \rightarrow \infty.$$

Proof. From Proposition 4 we know that

$$\mathbb{P}_{x_0}(Y_t > 0) = 1 - \mathbb{E} \left[\exp \left\{ -x_0 \left(c_+ \beta \int_0^t e^{-\beta(\Delta_s + gs)} ds \right)^{-1/\beta} \right\} \right] = \mathbb{E} \left[F \left(\int_0^t e^{-\beta K_s} ds \right) \right],$$

where $F(x) = 1 - \exp\{-x_0(c_+\beta x)^{-1/\beta}\}$ and $K_s = \Delta_s + gs$. The function F satisfies Assumption (23) required in Theorem 7 stated and proved in the next Section. Hence Proposition 5 follows from a direct application of this Theorem. \square

In the case of CSBP's without catastrophes ($\nu = 0$), the subcritical regime is reduced to (i), and the critical case differs from b/, since then the asymptotic behavior is given by $1/t$.

In the strongly and intermediate subcritical cases (i) and (ii), $\mathbb{E}(Y_t)$ gives the good exponential rate for the decrease of the survival probability, $\phi(1)+g$. Moreover the probability of non-extinction is proportional to the initial state x_0 of the population. We refer to the proof of Lemma 10 and Section 4.4 for details.

In the weakly subcritical case (iii), the exponential rate of decrease of the survival probability is $\phi(\tau) + g\tau$, which is strictly smaller than $\phi(1) + g$. In fact, as it appears in the proof of Theorem 7, the quantity which determines the scale of the asymptotic behavior in all cases is $\mathbb{E}[\exp(\inf_{s \in [0, t]} (\Delta_s + gs))]$. Let us note also that c_3 and c_4 should not be proportional to x_0 . We refer to [Ban09] for a result in this vein for discrete branching processes in random environment.

More generally, the results stated above can be compared to the results which appear in the literature of discrete (time and space) branching processes in random environment (BPRE), see e.g. [GL01, GKV03, AGKV05]. A BPRE $(X_n, n \in \mathbb{N})$ is an integer valued branching process, specified by a sequence of generating functions $(f_n, n \in \mathbb{N})$. Conditionally on the environment, individuals reproduce independently of each other and the offsprings of an individual at generation n has generating function f_n . We present briefly the results of Theorem 1.1 in [GK00] and Theorems 1.1, 1.2 and 1.3 in [GKV03]. To lighten the presentation, we do not specify here the moment conditions.

In the *subcritical case*, i.e. $\mathbb{E}[\log f'_0(1)] < 0$, we have the following three asymptotic regimes when n goes to ∞ ,

$$\mathbb{P}(X_n > 0) \sim ca_n, \quad \text{as } n \rightarrow \infty,$$

for some positive constant c and

$$a_n = \mathbb{E}[f'_0(1)]^n, \quad a_n = n^{-1/2} \mathbb{E}[f'_0(1)]^n \quad \text{and} \quad a_n = n^{-3/2} \left(\min_{0 < s < 1} \mathbb{E}[(f'_0(1))^s] \right)^n,$$

if respectively $\mathbb{E}[f'_0(1) \log f'_0(1)]$ is negative, zero or positive.

In the *critical case*, i.e. $\mathbb{E}[\log f'_0(1)] = 0$, we have

$$\mathbb{P}(X_n > 0) \sim cn^{-1/2}, \quad \text{as } n \rightarrow \infty,$$

for some positive constant c .

In the particular case $\beta = 1$, these results on BPRE and the approximation technics implemented in Section 4 can be used to achieve the proof of Proposition 5 result above. We refer to remarks 2 and 3 for details.

Finally, in the continuous framework, such results have been established for Feller diffusion ($\beta = 1$) whose drift varies following a Brownian motion (see [BH12]). Thus K is a Brownian motion plus a drift and the authors use an explicit formula for Laplace transform of exponential functionals of Brownian motion, which seems not possible for the Lévy processes K considered here.

3.2 Beyond the stable case.

In this section, we prove a similar result to Theorem 5 for CSBP's with catastrophes when the branching mechanism ψ_0 is not stable. For technical reasons, we assume that the Brownian coefficient is positive and the associated Lévy measure μ satisfies a second moment condition. It allows us to obtain the following result from the Feller diffusion case in Theorem 5, i.e $\beta = 1$.

Corollary 6. *Assume that (18) holds and*

$$\int_{(0, \infty)} x^2 \mu(dx) < \infty, \quad \sigma^2 > 0, \quad \int_{(0, \infty)} (\log x)^2 \nu(dx) < \infty.$$

a/ If $\phi'(0) + g < 0$ and $\theta_{max} > 1$, then

(i) If $\phi'(1) + g < 0$, there exist $0 < c_1 \leq c'_1 < \infty$ such that for every x_0 ,

$$c_1 x_0 e^{t(\phi(1)+g)} \leq \mathbb{P}_{x_0}(Y_t > 0) \leq c'_1 x_0 e^{t(\phi(1)+g)} \quad (t \geq 0).$$

(ii) If $\phi'(1) + g = 0$, there exist $0 < c_2 \leq c'_2 < \infty$ such that for every x_0 ,

$$c_2 x_0 t^{-1/2} e^{t(\phi(1)+g)} \leq \mathbb{P}_{x_0}(Y_t > 0) \leq c'_2 x_0 t^{-1/2} e^{t(\phi(1)+g)} \quad (t \geq 0).$$

(iii) If $\phi'(1) + g > 0$ and $\theta_{max} > \beta + 1$, for every x_0 , there exist $0 < c_3(x_0) \leq c'_3(x_0) < \infty$ such that

$$c_3(x_0) t^{-3/2} e^{t(\phi(\tau)+g\tau)} \leq \mathbb{P}_{x_0}(Y_t > 0) \leq c'_3(x_0) t^{-3/2} e^{t(\phi(\tau)+g\tau)} \quad (t \geq 0),$$

where τ is the root of $\phi' + g$ on $]0, 1[$.

b/ If $\phi'(0) + g = 0$ and $\theta_{max} > \beta$, then for every x_0 , there exist $0 < c_4(x_0) < c'_4(x_0)' < \infty$ such that

$$c_4(x_0) t^{-1/2} \leq \mathbb{P}_{x_0}(Y_t > 0) \leq c'_4(x_0) t^{-1/2} \quad (t \geq 0).$$

Let us note that the assumption $\sigma^2 > 0$ is only required for the upper bounds.

Proof. We recall that the branching mechanism associated with the CSBP Z satisfies (1) for every $\lambda \geq 0$. So for every $\lambda \geq 0$,

$$2\sigma^2 \leq \psi''(\lambda) = 2\sigma^2 + \int_{(0,\infty)} x^2 e^{-\lambda x} \mu(dx).$$

Since $c := \int_0^\infty x^2 \mu(dx) < \infty$, then ψ'' is continuous over $[0, \infty)$. By Taylor Lagrange Theorem, we get for every $\lambda \geq 0$, $\psi_-(\lambda) \leq \psi(\lambda) \leq \psi_+(\lambda)$, where

$$\psi_-(\lambda) = \lambda\psi'(0) + \sigma^2\lambda^2 \quad \text{and} \quad \psi_+(\lambda) = \lambda\psi'(0) + (\sigma^2 + c/2)\lambda^2.$$

We first consider the case $\nu(0, \infty) < \infty$, so that Δ has a finite number of jumps on each compact interval a.s. We introduce the CSBP's with catastrophes Y_- and Y_+ which are both associated with the same catastrophes as Y , Δ , but respectively with the CSBP $(g, \sigma^2, 0)$ and $(g, \sigma^2 + c/2, 0)$. We denote $u_{-,t}$ and $u_{+,t}$ their Laplace exponent, i.e. for all $(\lambda, t) \in \mathbb{R}_+^2$,

$$\mathbb{E}\left[\exp\{-\lambda Y_t^-\}\right] = \exp\{-u_{-,t}(\lambda)\}, \quad \mathbb{E}\left[\exp\{-\lambda Y_t^+\}\right] = \exp\{-u_{+,t}(\lambda)\}.$$

Thus conditionally on Δ , for every time t such that $\Delta_t = \Delta_{t-}$, by Theorem 1,

$$u'_{-,t}(\lambda) = -\psi_-(u_{-,t}), \quad u'_{+,t}(\lambda) = -\psi_+(u_{+,t}), \quad u'_t(\lambda) = -\psi(u_t).$$

Moreover for every t such that $\theta_t = \exp\{\Delta_t - \Delta_{t-}\} > 0$, it is clear that

$$\frac{u_{-,t}(\lambda)}{u_{-,t-}(\lambda)} = \frac{u_t(\lambda)}{u_{t-}(\lambda)} = \frac{u_{+,t}(\lambda)}{u_{+,t-}(\lambda)} = \theta_t,$$

and $u_{-,0}(\lambda) = u_0(\lambda) = u_{+,0}(\lambda) = \lambda$. So for all t, λ ,

$$u_{+,t}(\lambda) \leq u(t, \lambda) \leq u_{-,t}(\lambda).$$

The generalization of this inequality to the case $\nu(0, \infty) \in [0, \infty]$ can be achieved by successive approximations. We defer the technical details to Section 6.6.

This implies, taking $\lambda \rightarrow \infty$, that

$$\mathbb{P}(Y_t^+ > 0) \leq \mathbb{P}(Y_t > 0) \leq \mathbb{P}(Y_t^- > 0).$$

The result follows from the asymptotic behavior of $\mathbb{P}(Y_t^- > 0)$ and $\mathbb{P}(Y_t^+ > 0)$, which are given by Proposition 5. \square

4 Local limit theorem for some functionals of Lévy processes

We have proved in Proposition 4 that the probability that a stable CSBP with catastrophes becomes extinct at time t equals the expectation of a functional of a Lévy process. We now prove the key result of the paper. It deals with the asymptotic behavior of the mean of some Lévy functionals.

More precisely, we are interested in the asymptotic behavior at infinity of

$$a_F(t) := \mathbb{E} \left[F \left(\int_0^t \exp\{-\beta K_s\} ds \right) \right],$$

where K is a Lévy process with bounded variation paths and F belongs to a particular class of functions on \mathbb{R}_+ . We will focus on functions which decrease polynomially at infinity (with exponent $-1/\beta$). The motivations come from the previous Section and our aim application is the proof of Proposition 5.

Thus, we consider a Lévy process $K = (K_t, t \geq 0)$ of the form

$$K_t = \gamma t + \sigma_t^{(+)} - \sigma_t^{(-)}, \quad t \geq 0, \quad (19)$$

where γ is a real constant, $\sigma^{(+)}$ and $\sigma^{(-)}$ are two independent subordinators without drift. We denote by Π , $\Pi^{(+)}$ and $\Pi^{(-)}$ the associated Lévy measures of K , $\sigma^{(+)}$ and $\sigma^{(-)}$, respectively.

Let us also define the Laplace exponents of K , $\sigma^{(+)}$ and $\sigma^{(-)}$ by

$$\phi_K(\lambda) = \log \mathbb{E}[e^{\lambda K_1}], \quad \phi_+(\lambda) = \log \mathbb{E}[e^{\lambda \sigma_1^{(+)}}] \quad \text{and} \quad \phi_-(\lambda) = \log \mathbb{E}[e^{-\lambda \sigma_1^{(-)}}], \quad (20)$$

and assume that

$$\theta_{max} = \sup \left\{ \lambda \in \mathbb{R}^+, \int_{[1, \infty)} e^{\lambda x} \Pi^{(+)}(dx) < \infty \right\} > 0. \quad (21)$$

From the Lévy-Khintchine formula, we deduce

$$\phi_K(\lambda) = \gamma \lambda + \int_{(0, \infty)} (e^{\lambda x} - 1) \Pi^{(+)}(dx) + \int_{(0, \infty)} (e^{-\lambda x} - 1) \Pi^{(-)}(dx).$$

Finally, we assume that $\mathbb{E}(K_1^2) < \infty$, which is equivalent to

$$\int_{(-\infty, \infty)} x^2 \Pi(dx) < \infty. \quad (22)$$

Theorem 7. *We assume that (19), (21) and (22) hold.*

Let $\beta \in (0, 1]$ and F be a positive non increasing function such that for $x \geq 0$

$$F(x) = C_F(x+1)^{-1/\beta} \left[1 + (1+x)^{-\varsigma} h(x) \right], \quad (23)$$

where $\varsigma \geq 1$, C_F is a positive constant, and h is a Lipschitz function which is bounded.

a/ If $\phi'_K(0) < 0$

(i) If $\theta_{max} > 1$ and $\phi'_K(1) < 0$, there exists a positive constant c_1 such that

$$a_F(t) \sim c_1 e^{t\phi_K(1)}, \quad \text{as } t \rightarrow \infty.$$

(ii) If $\theta_{max} > 1$ and $\phi'_K(1) = 0$, there exists a positive constant c_2 such that

$$a_F(t) \sim c_2 t^{-1/2} e^{t\phi_K(1)}, \quad \text{as } t \rightarrow \infty.$$

(iii) If $\theta_{max} > \beta + 1$ and $\phi'_K(1) > 0$, there exists a positive constant c_3 such that

$$a_F(t) \sim c_3 t^{-3/2} e^{t\phi_K(\tau)}, \quad \text{as } t \rightarrow \infty,$$

where τ is the root of ϕ'_K on $]0, 1[$.

b/ If $\theta_{max} > \beta$ and $\phi'_K(0) = 0$, there exists a positive constant c_4 such that

$$a_F(t) \sim c_4 t^{-1/2}, \quad \text{as } t \rightarrow \infty.$$

This result generalizes Lemma 4.7 in Carmona et al. [CPY97] in the case when the process K has bounded variation paths. More precisely, the authors in [CPY97] only provide a precise asymptotic behavior in the case when $\phi'_K(1) < 0$.

The assumption on the tail of F as $x \rightarrow \infty$ is finely used to get the asymptotic behavior of $a_F(t)$. Lemma 20 gives the properties of F required in the proof.

The strongly subcritical case (case (i)) is proven using a continuous time change of measure (Section 4.4). For the other cases, we divide the proof in three steps. The first step (Lemma 8) consists in discretizing the integral $\int_0^t \exp(-\beta K_s) ds$ by introducing the random variables

$$A_{p,q} = \sum_{i=0}^p \exp\{-\beta K_{i/q}\} = \sum_{i=0}^p \prod_{j=0}^{i-1} \exp\left\{-\beta(K_{(j+1)/q} - K_{j/q})\right\} \quad ((p, q) \in \mathbb{N} \times \mathbb{N}^*). \quad (24)$$

Secondly (Lemmas 10, 11 and 12), we study the asymptotic behavior of the discretized expectation :

$$F_{p,q} := \mathbb{E}\left[F\left(A_{p,q}/q\right)\right] \quad (q \in \mathbb{N}^*), \quad (25)$$

when p goes to infinity. This step relies on Theorem 2.1 in [GL01], which is a limit theorem for random walks on an affine group and generalizes theorems A and B in [LPP97].

Finally (Sections 4.3 and 4.4), we prove that the limit of $F_{[qt],q}$, when $q \rightarrow \infty$, and $a_F(t)$ both have the same asymptotic behavior when t goes to infinity.

4.1 Discretization of the Lévy process

The following result, which is a direct consequence from the definition of Lévy processes, allows us to concentrate our attention on $A_{p,q}$, which was defined in (24).

Lemma 8. *Let $t \geq 1$ and $q \in \mathbb{N}^*$. There exist two random variables $C_{[qt],q}$ and $D_{[qt]-1,q}$ such that*

$$\frac{1}{q}e^{-\beta|\gamma|/q}D_{[qt]-1,q} \leq \int_0^t e^{-\beta K_s} ds \leq \frac{1}{q}e^{\beta|\gamma|/q}C_{[qt],q},$$

and for every $(p,q) \in \mathbb{N} \times \mathbb{N}^*$,

$$D_{p,q} \stackrel{(d)}{=} U_{1/q}^\beta A_{p,q} \quad \text{and} \quad C_{p,q} \stackrel{(d)}{=} V_{1/q}^\beta A_{p,q},$$

where the couple of random variables $(U_{1/q}, V_{1/q})$ is independent of $A_{p,q}$ and distributed as $(\exp(-\sigma_{1/q}^{(+)}) , \exp(\sigma_{1/q}^{(-)}))$.

Proof. Let (p,q) be in $\mathbb{N} \times \mathbb{N}^*$ and s in $[p/q, (p+1)/q]$. Then

$$K_s \leq K_{p/q} + |\gamma|/q + [\sigma_{(p+1)/q}^{(+)} - \sigma_{p/q}^{(+)}] \quad \text{and} \quad K_s \geq K_{p/q} - |\gamma|/q - [\sigma_{(p+1)/q}^{(-)} - \sigma_{p/q}^{(-)}]. \quad (26)$$

Now introduce

$$K_{p/q}^{(1)} = K_{p/q} + [\sigma_{(p+1)/q}^{(+)} - \sigma_{p/q}^{(+)}] - \sigma_{1/q}^{(+)} \quad \text{and} \quad K_{p/q}^{(2)} = K_{p/q} - [\sigma_{(p+1)/q}^{(-)} - \sigma_{p/q}^{(-)}] + \sigma_{1/q}^{(-)}.$$

Then, we have for all $(p,q) \in \mathbb{N} \times \mathbb{N}^*$

$$(K_0, K_{1/q}, \dots, K_{p/q}) \stackrel{(d)}{=} (K_0^{(1)}, K_{1/q}^{(1)}, \dots, K_{p/q}^{(1)}) \stackrel{(d)}{=} (K_0^{(2)}, K_{1/q}^{(2)}, \dots, K_{p/q}^{(2)}).$$

Moreover, $(K_0^{(1)}, K_{1/q}^{(1)}, \dots, K_{p/q}^{(1)})$ is independent of $\sigma_{1/q}^{(+)}$ and $(K_0^{(2)}, K_{1/q}^{(2)}, \dots, K_{p/q}^{(2)})$ is independent of $\sigma_{1/q}^{(-)}$. Finally, the inequalities in (26) lead to,

$$\frac{1}{q}e^{-\beta(|\gamma|/q + \sigma_{1/q}^{(+)})} \sum_{i=0}^{[qt]-1} e^{-\beta K_{i/q}^{(1)}} \leq \int_0^t e^{-\beta K_s} ds \leq \frac{1}{q}e^{\beta(|\gamma|/q + \sigma_{1/q}^{(-)})} \sum_{i=0}^{[qt]} e^{-\beta K_{i/q}^{(2)}},$$

which ends the proof. \square

4.2 Asymptotical behavior of the discretized process

We first recall Theorem 2.1 in [GL01], in the case where the test functions do not vanish. This result is the key to obtain the asymptotic behavior of the discretized process.

Theorem 9 (Giuvarc'h, Liu 01). *Let $(a_n, b_n)_{n \geq 0}$ be a $(\mathbb{R}_+^*)^2$ valued sequence of iid random variables such that $\mathbb{E}[\log(a_0)] = 0$. Assume that $b_0/(1-a_0)$ is not a.s. constant and define $A_0 = 1$, $A_n = \prod_{k=0}^{n-1} a_k$ and $B_n = \sum_{k=0}^{n-1} A_k b_k$, for $n \geq 1$. Let η, κ, λ be three positive numbers such that $\kappa < \lambda$, and $\tilde{\phi}$ and $\tilde{\psi}$ be two positive continuous functions on \mathbb{R}_+ such that they do not vanish and for a constant $C > 0$ and for every $a > 0$, $b \geq 0$, $b' \geq 0$, we have*

$$\tilde{\phi}(a) \leq Ca^\kappa, \quad \tilde{\psi}(b) \leq \frac{C}{(1+b)^\lambda}, \quad \text{and} \quad |\tilde{\psi}(b) - \tilde{\psi}(b')| \leq C|b - b'|^\eta.$$

Moreover, assume that

$$\mathbb{E}[a_0^\kappa] < \infty, \quad \mathbb{E}[a_0^{-\eta}] < \infty, \quad \mathbb{E}[b_0^\eta] < \infty \quad \text{and} \quad \mathbb{E}[a_0^{-\eta} b_0^{-\eta}] < \infty.$$

Then there exist two positive constants $c(\tilde{\phi}, \tilde{\psi})$ and $c(\tilde{\psi})$ such that

$$\lim_{n \rightarrow \infty} n^{3/2} \mathbb{E} \left[\tilde{\phi}(A_n) \tilde{\psi}(B_n) \right] = c(\tilde{\phi}, \tilde{\psi}) \quad \text{and} \quad \lim_{n \rightarrow \infty} n^{1/2} \mathbb{E} \left[\tilde{\psi}(B_n) \right] = c(\tilde{\psi}).$$

Recall the definition of $A_{p,q}$ and $F_{p,q}$ in (23) to (25). The three following Lemmas study the asymptotic behavior of their expectations in the regimes (ii), (iii) and b/.

Lemma 10. *Assume that $|\phi'_K(0+)| < \infty$, $\theta_{max} > 1$ and $\phi'_K(1) = 0$. Then there exists a positive and finite constant $c_2(q)$ such that,*

$$F_{p,q} \sim C_F c_2(q) (p/q)^{-1/2} e^{(p/q)\phi_K(1)}, \quad \text{as } p \rightarrow \infty, \quad (27)$$

and

$$\mathbb{E} \left[(A_{p,q}/q)^{-1/\beta} \right] \sim c_2(q) (p/q)^{-1/2} e^{(p/q)\phi_K(1)}, \quad \text{as } p \rightarrow \infty. \quad (28)$$

Proof. Let introduce the exponential change of measure known as the Escheer transform

$$\left. \frac{d\mathbb{P}^{(\lambda)}}{d\mathbb{P}} \right|_{\mathcal{F}_t} = e^{\lambda K_t - \phi_K(\lambda)t} \quad \text{for } \lambda \in [0, \theta_{max}[, \quad (29)$$

where $(\mathcal{F}_t)_{t \geq 0}$ is the natural filtration generated by K which is naturally completed.

The following equality in law

$$A_{p,q} = e^{-\beta K_{p/q}} \left(\sum_{i=0}^p e^{\beta(K_{p/q} - K_{i/q})} \right) \stackrel{(d)}{=} e^{-\beta K_{p/q}} \left(\sum_{i=0}^p e^{\beta K_{i/q}} \right),$$

leads to $e^{-(p/q)\phi_K(1)} \mathbb{E} \left[A_{p,q}^{-1/\beta} \right] = \mathbb{E}^{(1)} \left[\tilde{A}_{p,q}^{-1/\beta} \right]$, where $\tilde{A}_{p,q} = \sum_{i=0}^p e^{\beta K_{i/q}}$. Let $\varepsilon > 0$ such that (46) holds and observe that $\tilde{A}_{p,q} \geq 1$ p.s. for every (p, q) in $\mathbb{N} \times \mathbb{N}^*$. Thus,

$$\mathbb{E}^{(1)} \left[\tilde{A}_{p,q}^{-(1+\varepsilon)/\beta} \right] \leq \mathbb{E}^{(1)} \left[\tilde{A}_{p,q}^{-1/\beta} \right] \leq \mathbb{E}^{(1)} \left[\inf_{i \in [0, p] \cap \mathbb{N}} e^{-K_{i/q}} \right].$$

Since $\phi'_K(1) = 0$ and $\mathbb{E}[K_{1/q}^2] < \infty$, Theorem A in [Koz76] yields

$$\mathbb{E}^{(1)} \left[\inf_{i \in [0, p] \cap \mathbb{N}} e^{-K_{i/q}} \right] \sim C_q (p/q)^{-1/2}, \quad \text{as } p \rightarrow \infty,$$

where C_q is a positive finite constant. We define for $z \geq 1$,

$$D_q(z, p) = (p/q)^{1/2} \mathbb{E}^{(1)} \left[\tilde{A}_{p,q}^{-z/\beta} \right].$$

Note that there exists $p_0 \in \mathbb{N}$ such that for $p \geq p_0$, $D_q(1, p) \leq 2C_q$.

Our aim is to prove that $D_q(1, p)$ converges to a finite positive constant $d_2(q)$. Then, we introduce an arbitrary $x \in (0, (C_F/M)^{1/\varepsilon} q^{-1/\beta})$ and apply Theorem 9 with

$$\tilde{\psi}(z) = F(z), \quad \tilde{\phi}(z) = z^{1/(2\beta)}, \quad (\eta, \kappa, \lambda) = (1, 1/(2\beta), 1/\beta).$$

We observe that F is a Lipschitz function and under the probability measure $\mathbb{P}^{(1)}$, $(a_n, b_n)_{n \geq 0} = (\exp(\beta(K_{(n+1)/q} - K_{n/q})), x^{-\beta} q^{-1})_{n \geq 0}$ is an i.i.d. sequence of random variables such that $\mathbb{E}^{(1)}[\log(a_0)] = 0$, as $\phi'_K(1) = 0$. A simple computation gives

$$\mathbb{E}^{(1)}[a_0^{-1}] = e^{(\phi_K(1-\beta) - \phi_K(1))/q} < \infty.$$

So the moment conditions of Theorem 9 are satisfied. We apply the result with

$$B_n = q^{-1} x^{-\beta} \sum_{i=0}^{n-1} e^{\beta K_{i/q}}, \quad n \in \mathbb{N}^*.$$

We get the existence of a positive real number $b(q, x)$ such that

$$(p/q)^{1/2} \mathbb{E}^{(1)} \left[F \left(x^{-\beta} \tilde{A}_{p,q}/q \right) \right] \rightarrow b(q, x), \quad \text{as } p \rightarrow \infty.$$

Now, we define $\liminf_{n \rightarrow \infty} D_q(1, n) = \underline{D}_q$ and $\limsup_{n \rightarrow \infty} D_q(1, n) = \overline{D}_q$. Taking expectation in (46) yields

$$\left| (p/q)^{1/2} \mathbb{E}^{(1)} \left[F \left(x^{-\beta} \tilde{A}_{p,q}/q \right) \right] - C_F x q^{1/\beta} D_q(1, p) \right| \leq M x^{1+\varepsilon} q^{(1+\varepsilon)/\beta} D_q(1 + \varepsilon, p).$$

Let n_k and m_k be two increasing subsequences in \mathbb{N} such that,

$$D(1, n_k) \xrightarrow[k \rightarrow \infty]{} \underline{D}_q \quad \text{and} \quad D_q(1, m_k) \xrightarrow[k \rightarrow \infty]{} \overline{D}_q.$$

Since $D_q(z, p)$ is decreasing with respect to z , we have for all k in \mathbb{N} ,

$$(C_F x q^{1/\beta} + M x^{1+\varepsilon} q^{(1+\varepsilon)/\beta}) D_q(1, n_k) \geq (n_k/q)^{1/2} \mathbb{E}^{(1)} \left[F \left(x^{-\beta} \tilde{A}_{n_k,q}/q \right) \right],$$

and

$$(C_F x q^{1/\beta} - M x^{1+\varepsilon} q^{(1+\varepsilon)/\beta}) D_q(1, m_k) \leq (m_k/q)^{1/2} \mathbb{E}^{(1)} \left[F \left(x^{-\beta} \tilde{A}_{m_k,q}/q \right) \right].$$

This implies, taking $k \rightarrow \infty$, that

$$\underline{D}_q \geq \frac{b(q, x)}{C_F x q^{1/\beta} + M x^{1+\varepsilon} q^{(1+\varepsilon)/\beta}} > 0, \quad \overline{D}_q \leq \frac{b(q, x)}{C_F x q^{1/\beta} - M x^{1+\varepsilon} q^{(1+\varepsilon)/\beta}} < \infty,$$

and

$$\overline{D}_q - \underline{D}_q \leq \frac{4MC_q x^\varepsilon q^{(1+\varepsilon)/\beta}}{C_F}.$$

Finally, letting $x \rightarrow 0$, we get that $D_q(1, p)$ converges to a finite positive constant $d_2(q)$, which gives (28).

Using (46), we get

$$\mathbb{E} \left| F_{p,q} - C_F (A_{p,q}/q)^{-1/\beta} \right| \leq \mathbb{E} \left[(A_{p,q}/q)^{-(1+\varepsilon)/\beta} \right],$$

so (27) will be proved as soon as

$$\mathbb{E} \left[A_{p,q}^{-(1+\varepsilon)/\beta} \right] = o \left(\mathbb{E} \left[A_{p,q}^{-1/\beta} \right] \right), \quad \text{as } p \rightarrow \infty.$$

From the Escheer transform (29), with $\lambda = 1 + \varepsilon$, and the independent increments of K , we have

$$\begin{aligned} \mathbb{E}\left[A_{p,q}^{-(1+\varepsilon)/\beta}\right] &= e^{(p/q)\phi_K(1)} \mathbb{E}^{(1)}\left[\left(\sum_{i=0}^p e^{-\beta K_{i/q}}\right)^{-\varepsilon/\beta} \left(\sum_{i=0}^p e^{\beta(K_{p/q}-K_{i/q})}\right)^{-1/\beta}\right] \\ &\leq e^{(p/q)\phi_K(1)} \mathbb{E}^{(1)}\left[\inf_{0 \leq i \leq \lfloor p/3 \rfloor} e^{\varepsilon K_{i/q}} \inf_{\lfloor 2p/3 \rfloor \leq j \leq p} e^{-(K_{p/q}-K_{j/q})}\right] \\ &= e^{(p/q)\phi_K(1)} \mathbb{E}^{(1)}\left[\inf_{0 \leq i \leq \lfloor p/3 \rfloor} e^{\varepsilon K_{i/q}}\right] \mathbb{E}^{(1)}\left[\inf_{0 \leq j \leq \lfloor p/3 \rfloor} e^{-K_{j/q}}\right]. \end{aligned}$$

Using (22), we observe that $\mathbb{E}^{(1)}(K_{1/q}) = 0$ and $\mathbb{E}^{(1)}(K_{1/q}^2) < \infty$. We can then apply Theorem A in [Koz76] to the random walks $(-K_{i/q})_{i \geq 1}$ and $(\varepsilon K_{i/q})_{i \geq 1}$. Therefore, there exists $c > 0$ such that

$$\mathbb{E}\left[A_{p,q}^{-(1+\varepsilon)/\beta}\right] \leq (c/p)e^{(p/q)\phi_K(1)} = o\left(\mathbb{E}\left[A_{p,q}^{-1/\beta}\right]\right), \quad \text{as } p \rightarrow \infty.$$

Taking $c_2(q) = d_2(q)q^{1/\beta}$ leads to the result. \square

Remark 2. In the particular case when $\beta = 1$, it is enough to apply Theorem 1.2 in [GKV03] to a geometric BPRE $(X_n, n \geq 0)$ whose p.g.f.'s satisfy

$$f_n(s) = \sum_{k=0}^{\infty} p_n q_n^k s^k = \frac{p_n}{1 - q_n s},$$

with $1/p_n = 1 + \exp\{\beta(K_{(n+1)/q} - K_{n/q})\}$, $q_n = 1 - p_n$. Using $\mathbb{E}(A_{p,q}^{-1}) = \mathbb{P}(X_p > 0)$ and $\log f'_0(1) = K_{1/q}$, allows to get the asymptotic behavior of $\mathbb{E}(A_{p,q}^{-1})$ from the speed of extinction of BPRE in the case of geometric reproduction law (but we need the extra assumption $\phi_K(2) < \infty$).

Recall that τ is the root of ϕ'_K on $]0, 1[$, i.e. $\phi_K(\tau) = \min_{0 < s < 1} \phi_K(s)$.

Lemma 11. *Assume that $\phi'_K(0+) < 0$, $\phi'_K(1) > 0$ and $\beta + 1 < \theta_{max}$. Then there exist two positive and finite constants $d(q)$ and $c_3(q)$ such that*

$$F_{p,q} \sim c_3(q)(p/q)^{-3/2} e^{(p/q)\phi_K(\tau)}, \quad \text{as } p \rightarrow \infty \quad (30)$$

and

$$\mathbb{E}\left[(A_{p,q}/q)^{-1/\beta}\right] \sim d(q)(p/q)^{-3/2} e^{(p/q)\phi_K(\tau)}, \quad \text{as } p \rightarrow \infty. \quad (31)$$

Proof. First we apply Theorem 9 where, for $z \geq 0$,

$$\tilde{\psi}(z) = F(z), \quad \tilde{\phi}(z) = z^{\tau/\beta}, \quad (\eta, \kappa, \lambda) = (1, \tau/\beta, 1/\beta).$$

Again F is a Lipschitz function, and under the probability measure $\mathbb{P}^{(\tau)}$, $(a_n, b_n)_{n \geq 0} = (\exp(-\beta(K_{(n+1)/q} - K_{n/q})), q^{-1})_{n \geq 0}$, is an i.i.d. sequence of random variables such that $\mathbb{E}^{(\tau)}[\log(a_0)] = 0$, as $\phi'_K(\tau) = 0$. The moment conditions

$$\mathbb{E}^{(\tau)}[a_0^{\tau/\beta}] = e^{-\phi_K(\tau)/q} < \infty \quad \text{and} \quad \mathbb{E}^{(\tau)}[a_0^{-1}] = e^{(\phi_K(\beta+\tau) - \phi_K(\tau))/q} < \infty,$$

enable us to apply Theorem 9. In this case,

$$B_n = q^{-1} \sum_{i=0}^{n-1} e^{-\beta K_{i/q}}, \quad n \in \mathbb{N}^*$$

Then there exists $c_3(q) > 0$ such that

$$\mathbb{E} [F(A_{p,q}/q)] e^{-(p/q)\phi_K(\tau)} = \mathbb{E}^{(\tau)} [F(A_{p,q}/q) e^{-\tau K_{p/q}}] \sim c_3(q)(p/q)^{-3/2},$$

as $p \rightarrow \infty$. This gives (30).

To prove

$$\mathbb{E} \left[(A_{p,q}/q)^{-1/\beta} \right] \sim d(q)(p/q)^{-3/2} e^{\frac{p}{q}\phi_K(\tau)}, \quad \text{as } p \rightarrow \infty$$

for $d(q) > 0$, we follow the same arguments as those used in the proof of Lemma 10. In other words, we define for $z \geq 1$,

$$D_q(z, p) = (p/q)^{3/2} e^{-(p/q)\phi_K(\tau)} \mathbb{E} \left[A_{p,q}^{-z/\beta} \right],$$

which is decreasing with respect to z . We obtain the same type of inequalities as in Lemma 10, for the random variable A instead of \tilde{A} .

Again we take $\varepsilon > 0$ such that (46) holds. Then Lemma 7 in [Hir98] yields the existence of $C_q > 0$ such that for p large enough,

$$\mathbb{E} \left[A_{p,q}^{-(1+\varepsilon)/\beta} \right] \leq \mathbb{E} \left[A_{p,q}^{-1/\beta} \right] \leq \mathbb{E} \left[\inf_{i \in [0, p] \cap \mathbb{N}} e^{-K_{i/q}} \right] \sim C_q (p/q)^{-3/2} e^{(p/q)\phi_K(\tau)}.$$

Finally, we use Theorem 9 to get $0 < \liminf_{n \rightarrow \infty} D_q(1, n) = \limsup_{n \rightarrow \infty} D_q(1, n) < \infty$, which ends the proof. \square

Lemma 12. *Assume that $\phi'_K(0+) = 0$ and $\beta < \theta_{max}$. Then there exist two positive and finite constants $b(q)$ and $c_4(q)$ such that*

$$F_{p,q} \sim c_4(q)(p/q)^{-1/2}, \quad \text{as } p \rightarrow \infty, \quad (32)$$

and

$$\mathbb{E} \left[(A_{p,q}/q)^{-1/\beta} \right] \sim b(q)(p/q)^{-1/2}, \quad \text{as } p \rightarrow \infty. \quad (33)$$

Proof. The proof is almost the same as for Lemma 11. We first apply Theorem 9 to the same function $\tilde{\psi}$ and sequence $(a_n, b_n)_{n \geq 0}$ defined in Lemma 11 but with the probability measure \mathbb{P} instead of $\mathbb{P}^{(\tau)}$. Then, we get

$$\mathbb{E} \left[F(A_{p,q}/q) \right] \sim c_4(q)(p/q)^{-1/2}, \quad \text{as } p \rightarrow \infty.$$

We then define for $z \geq 1$,

$$D_q(z, p) = (p/q)^{1/2} \mathbb{E} \left[A_{p,q}^{-z/\beta} \right],$$

and from Theorem A in [Koz76] and Theorem 9, we obtain that $D_q(1, p)$ has a positive finite limit when p goes to infinity. \square

4.3 From the discretized process to the continuous process

Up to now, the asymptotic behavior of the processes was depending on the step size $1/q$. By letting q tend to infinity, we obtain our results in continuous time. Recalling the notations (27) to (33), we prove the following limits :

Lemma 13. *There exist five positive finite constants b , d , c_2 , c_3 and c_4 such that*

$$(b(q), d(q), c_2(q), c_3(q), c_4(q)) \longrightarrow (b, d, c_2, c_3, c_4), \quad \text{as } q \rightarrow \infty. \quad (34)$$

Proof. Let us first prove the convergence of $d(q)$. From Lemma 8, we know that for every $n \in \mathbb{N}^*$

$$e^{\frac{\phi_-(1)-|\gamma|}{q}} \mathbb{E} \left[\left(A_{nq,q}/q \right)^{-1/\beta} \right] \leq \mathbb{E} \left[\left(\int_0^n e^{-\beta K_u} du \right)^{-1/\beta} \right] \leq e^{\frac{\phi_+(1)+|\gamma|}{q}} \mathbb{E} \left[\left(A_{nq-1,q}/q \right)^{-1/\beta} \right].$$

A direct application of Lemma 21 with

$$a(q) = d(q), \quad c^-(q) = e^{(\phi_-(1)-|\gamma|)/q}, \quad \text{and} \quad c^+(q) = e^{(\phi_+(1)+|\gamma|)/q},$$

yields that $d(q)$ converges as $q \rightarrow \infty$ to a finite constant d .

Similar arguments lead to the convergence of $b(q)$ and we now prove the convergence of $c_2(q)$, $c_3(q)$ and $c_4(q)$. Again the proofs of the three cases are very similar, so we only prove the second one. From Lemmas 8 and 11, we know that for every $(n, q) \in (\mathbb{N}^*)^2$,

$$\mathbb{E} \left[F \left(e^{\beta|\gamma|/q} V_{1/q}^\beta A_{nq,q}/q \right) \right] \leq a_F(n) \leq \mathbb{E} \left[F \left(e^{-\beta|\gamma|/q} U_{1/q}^\beta A_{nq-1,q}/q \right) \right].$$

Using (47), we obtain

$$\begin{aligned} F_{nq,q} + M \mathbb{E} \left[e^{-|\gamma|/q} V_{1/q}^{-1} - 1 \right] \mathbb{E} \left[\left(\frac{A_{nq,q}}{q} \right)^{-\frac{1}{\beta}} \right] \\ \leq \frac{a_F(n)}{n^{-3/2} e^{n\phi_K(\tau)}} \leq \\ F_{nq-1,q} + M \mathbb{E} \left[e^{|\gamma|/q} U_{1/q}^{-1} - 1 \right] \mathbb{E} \left[\left(\frac{A_{nq-1,q}}{q} \right)^{-\frac{1}{\beta}} \right]. \end{aligned}$$

We divide now by $n^{-3/2} \exp(n\phi_K(\tau))$ so the result comes from Lemmas 11, 21 and Equation (47) with

$$a(q) = c_3(q), \quad c^-(q) = 1 - \frac{Md(q)(e^{(\phi_-(1)-|\gamma|)/q} - 1)}{c_3(q)} \quad c^+(q) = 1 + \frac{Md(q)(e^{(\phi_+(1)+|\gamma|)/q} - 1)}{c_3(q)}.$$

□

4.4 Proof of Theorem 7

Proof of Theorem 7 a/ (i). Recall from Lemma II.2 in [BLG00] that the process $(K_t - K_{(t-s)^-}, 0 \leq s \leq t)$ has the same law as $(K_s, 0 \leq s \leq t)$. Then

$$\int_0^t e^{-\beta K_s} ds = \int_0^t e^{-\beta K_{(t-s)^-}} ds = e^{-\beta K_t} \int_0^t e^{\beta K_t - \beta K_{(t-s)^-}} ds \stackrel{(d)}{=} e^{-\beta K_t} \int_0^t e^{\beta K_s} ds.$$

We first note that for every $q \in \mathbb{N}^*$ and $t \geq 2/q$, Lemma 8 leads to

$$\begin{aligned} \mathbb{E} \left[\left(\int_0^t e^{-\beta K_s} ds \right)^{-1/\beta} \right] &\leq \mathbb{E} \left[\left(\int_0^{2/q} e^{-\beta K_s} ds \right)^{-1/\beta} \right] \\ &\leq q^{1/\beta} e^{|\gamma|/q} \mathbb{E} \left(U_{1/q}^{-1/\beta} A_{1,q}^{-1/\beta} \right) \\ &= q^{1/\beta} \exp \left(\frac{\phi_K(1) + |\gamma| + \phi_K^+(1)}{q} \right) < \infty, \end{aligned}$$

where ϕ_K^+ was defined in (20). Hence using (29), with $\lambda = 1$, we have

$$\mathbb{E} \left[\left(\int_0^t e^{-\beta K_s} ds \right)^{-1/\beta} \right] = \mathbb{E} \left[e^{Kt} \left(\int_0^t e^{\beta K_s} ds \right)^{-1/\beta} \right] = e^{t\phi_K(1)} \mathbb{E}^{(1)} \left[\left(\int_0^t e^{\beta K_s} ds \right)^{-1/\beta} \right].$$

The above identity implies that the decreasing function $t \mapsto \mathbb{E}^{(1)}[(\int_0^t e^{\beta K_s} ds)^{-1/\beta}]$ is finite for all $t > 0$. So it converges to a non negative and finite limit c_1 , as t increases. This limit is positive, since under the probability $\mathbb{P}^{(1)}$, K is still a Lévy process with negative mean $\mathbb{E}^{(1)}(K_1) = \phi'_K(1)$ and according to Theorem 1 in [BY05], we have

$$\int_0^\infty e^{\beta K_s} ds < \infty, \quad \mathbb{P}^{(1)\text{-a.s.}}$$

It remains to prove that

$$a_F(t) \sim C_F \mathbb{E} \left[\left(\int_0^t e^{-\beta K_s} ds \right)^{-1/\beta} \right], \quad \text{as } t \rightarrow \infty. \quad (35)$$

Recall that $\theta_{max} > 1$ and $\phi'_K(1) < 0$. So we can choose $\varepsilon > 0$ such that (46) holds, $1 + \varepsilon < \theta_{max}$, $\phi_K(1 + \varepsilon) < \phi_K(1)$ and $\phi'_K(1 + \varepsilon) < 0$. Then

$$\left| F \left(\int_0^t e^{-\beta K_s} ds \right) - \left(\int_0^t e^{-\beta K_s} ds \right)^{-1/\beta} \right| \leq M \left(\int_0^t e^{-\beta K_s} ds \right)^{-(1+\varepsilon)/\beta}.$$

Thus, we just need to show that

$$\mathbb{E} \left[\left(\int_0^t e^{-\beta K_s} ds \right)^{-(1+\varepsilon)/\beta} \right] = o(e^{t\phi_K(1)}), \quad \text{as } t \rightarrow \infty.$$

It is achieved by a new change of measure (29), with $\lambda = 1 + \varepsilon$,

$$\begin{aligned} \mathbb{E} \left[\left(\int_0^t e^{-\beta K_s} ds \right)^{-(1+\varepsilon)/\beta} \right] &= \mathbb{E} \left[e^{(1+\varepsilon)Kt} \left(\int_0^t e^{\beta K_s} ds \right)^{-(1+\varepsilon)/\beta} \right] \\ &= e^{t\phi_K(1+\varepsilon)} \mathbb{E}^{(1+\varepsilon)} \left[\left(\int_0^t e^{\beta K_s} ds \right)^{-(1+\varepsilon)/\beta} \right]. \end{aligned}$$

Again using Lemma 8, we obtain for $t \geq q/2$,

$$\mathbb{E} \left[\left(\int_0^t e^{-\beta K_s} ds \right)^{-\frac{1+\varepsilon}{\beta}} \right] \leq q^{(1+\varepsilon)/\beta} \exp \left(\frac{\phi_K(1 + \varepsilon) + |\gamma|(1 + \varepsilon) + \phi_K^+(1 + \varepsilon)}{q} \right) < \infty,$$

which ensures that the decreasing function $t \mapsto \mathbb{E}^{(1+\varepsilon)}[(\int_0^t \exp(\beta K_s) ds)^{-(1+\varepsilon)/\beta}]$ is finite for all $t > 0$ and gives the result. \square

Remark 3. In the particular case when $\beta = 1$, it is enough to apply Theorem 1.1 in [GKV03] to the geometric BPRE $(X_n, n \geq 0)$ defined in Remark 2 to obtain the result.

Proof of Theorem 7 a/ (ii), (iii), and b/. The proofs are similar for the two regimes and we only focus on the proof of the regime in a/(iii).

Let $\varepsilon > 0$ and $q \in \mathbb{N}^*$ such that $q \geq 1/\varepsilon$ and $(1 - \varepsilon)c_3 \leq c_3(q) \leq (1 + \varepsilon)c_3$. Then for every $t \geq 1$, the monotonicity of F yields

$$\mathbb{E}\left[F(C_{\lfloor qt \rfloor, q} e^{\beta|\gamma|/q}/q)\right] \leq a_F(t) \leq \mathbb{E}\left[F(D_{\lfloor qt \rfloor - 1, q} e^{-\beta|\gamma|/q}/q)\right].$$

Applying (47), we obtain :

$$\begin{aligned} \left| \mathbb{E}\left[F(C_{\lfloor qt \rfloor, q} e^{\beta|\gamma|/q}/q)\right] - F_{\lfloor qt \rfloor, q} \right| &\leq (1 - e^{-\varepsilon(|\gamma| - \phi_-(1))}) M \mathbb{E}\left[(A_{\lfloor qt \rfloor, q}/q)^{-1/\beta}\right], \\ \left| \mathbb{E}\left[F(D_{\lfloor qt \rfloor - 1, q} e^{-\beta|\gamma|/q}/q)\right] - F_{\lfloor qt \rfloor - 1, q} \right| &\leq (e^{\varepsilon(|\gamma| + \phi_+(1))} - 1) M \mathbb{E}\left[(A_{\lfloor qt \rfloor - 1, q}/q)^{-1/\beta}\right]. \end{aligned}$$

When t goes to infinity, we can bound both terms by

$$h(\varepsilon)t^{-3/2}e^{t\phi_K(\tau)} = \left[2Md(e^{\varepsilon(|\gamma| + \phi_+(1))} - e^{-\varepsilon(|\gamma| - \phi_-(1))})e^{-\varepsilon\phi_K(\tau)}\right]t^{-3/2}e^{t\phi_K(\tau)} \quad (36)$$

where ϕ_- and ϕ_+ are defined in (20), and $h(\varepsilon)$ goes to 0 with ε . On the other hand, for t large enough

$$(1 - 2\varepsilon)c_3t^{-3/2}e^{t\phi_K(\tau)} \leq F_{\lfloor qt \rfloor, q} \leq a_F(t) \leq F_{\lfloor qt \rfloor - 1, q} \leq (1 + 2\varepsilon)c_3t^{-3/2}e^{t\phi_K(\tau)},$$

which ends the proof of Theorem 7. \square

5 Application to a cell division model

When the reproduction law has a finite second moment, the scaling limit of the GW process is a Feller diffusion with growth g and diffusion part σ^2 . It yields the stable case with $\beta = 1$ and additional drift term g . Such process is also the scaling limit of birth and death processes. It gives a natural model for populations which die and multiply fast, randomly, without interaction. Such a model is considered in [BT11] for parasites growing in dividing cells. The cell divides at constant rate r and a random fraction Θ in $(0, 1)$ of parasites goes in the first daughter cell, whereas the rest goes in the second daughter cell. Following the infection in a cell line, the parasites grow as a Feller diffusion process and undergo a catastrophe when the cell divides. We denote by N_t and N_t^* the numbers of cells and infected cells at time t , respectively. We say that the cell population recovers when the asymptotic proportion of contaminated cells vanishes. If there is one infected cell at time 0, $\mathbb{E}[N_t] = e^{rt}$ and $\mathbb{E}[N_t^*] = e^{rt}\mathbb{P}(Y_t > 0)$, where

$$Y_t = 1 + \int_0^t gY_s ds + \int_0^t \sqrt{2\sigma^2 Y_s} dB_s + \int_0^t \int_0^1 (\theta - 1)Y_{s-} \rho(ds, d\theta). \quad (37)$$

Here B is a Brownian motion and $\rho(ds, d\theta)$ a Poisson Point measure with intensity $2rds\mathbb{P}(\Theta \in d\theta)$. Let us note that the intensity of ρ is twice the cell division rate. This bias follows from the fact that if we pick an individual at random at time t , we are more likely to choose a lineage in which many division events have occurred. Hence the ancestral lineages from typical individuals at time t have a division rate $2r$.

Theorem 5 and Corollary 2 directly ensure the following result.

Corollary 14. *a/ We assume that $g < 2r\mathbb{E}[\log(1/\Theta)]$. Then there exist positive constants c_1, c_2, c_3 such that*

(i) If $g < 2r\mathbb{E}[\Theta \log(1/\Theta)]$, then

$$\mathbb{E}[N_t^*] \sim c_1 e^{gt}, \quad \text{as } t \rightarrow \infty.$$

(ii) If $g = 2r\mathbb{E}[\Theta \log(1/\Theta)]$, then

$$\mathbb{E}[N_t^*] \sim c_2 t^{-1/2} e^{gt}, \quad \text{as } t \rightarrow \infty.$$

(iii) If $g > 2r\mathbb{E}[\Theta \log(1/\Theta)]$, then

$$\mathbb{E}[N_t^*] \sim c_3 t^{-3/2} e^{\alpha t}, \quad \text{as } t \rightarrow \infty.$$

where $\alpha = \min_{\lambda \in [0,1]} \{g\lambda + 2r(\mathbb{E}[\Theta^\lambda] - 1/2)\} < g$.

b/ We assume now $g = 2r\mathbb{E}[\log(1/\Theta)]$, then there exists $c_4 > 0$ such that,

$$\mathbb{E}[N_t^*] \sim c_4 t^{-1/2} e^{rt}, \quad \text{as } t \rightarrow \infty.$$

c/ Finally, if $g > 2r\mathbb{E}[\log(1/\Theta)]$, then there exists $0 < c_5 < 1$ such that,

$$\mathbb{E}[N_t^*] \sim c_5 e^{rt}, \quad \text{as } t \rightarrow \infty.$$

Hence if $g > 2r\mathbb{E}[\log(1/\Theta)]$ (supercritical case c/), the mean number of infected cells is equivalent to the mean number of cells. In the critical case (b/), there are a bit less infected cells, owing to the additional square root term. In the strongly subcritical regime (a/ (i)), the mean number of infected cells is of the same order as the number of parasites. It let us think that parasites do not accumulate in some infected cells. The asymptotic behavior in the two remaining cases is more complex.

We stress the fact that fixing the growth rate g of parasites and the cell division rate r , but making the law of the repartition Θ vary, changes the asymptotic behavior of the number of infected cells. For example, if we focus on random variables Θ satisfying $\mathbb{P}(\Theta = \theta) = \mathbb{P}(\Theta = 1 - \theta) = 1/2$ for a given $\theta \in]0, 1/2]$, the different regimes can be described easily (see figure 1).

If $g/r > \log 2$, the cell population either recovers or not, depending on the asymmetry of the parasite sharing. If $g/r \leq \log 2/2$, the cell population recovers but the speed of recovery increases with respect to the asymmetry of the parasite sharing, as soon as the weakly subcritical regime is reached.

Such phenomena were known in the discrete time, discrete space framework (see [Ban08]), but the boundaries between the regimes are not the same, due to the bias in division rate in the continuous setting.

Moreover, we note that if $g/r \in (\log 2/2, \log 2)$, then parasites are in the weakly subcritical regime whatever the distribution of Θ on $]0, 1[$. This phenomenon also only occurs in the continuous setting.

6 Auxiliary results

This section is devoted to the technical results which are necessary for the previous proofs.

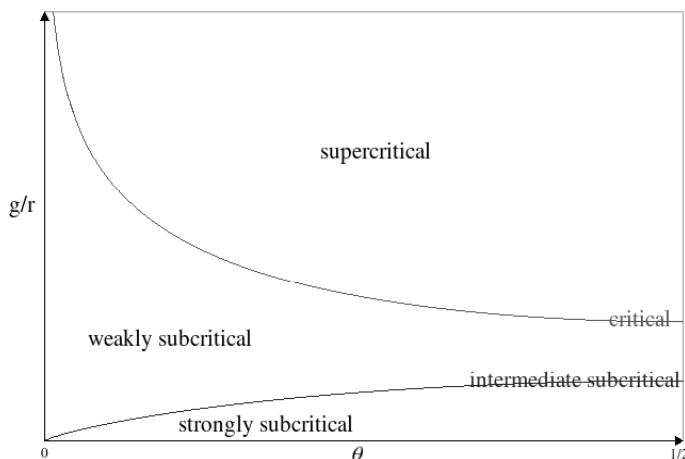


Figure 1: Extinction regimes in the case $\mathbb{P}(\Theta = \theta) = \mathbb{P}(\Theta = 1 - \theta) = 1/2$. Boundaries between the different regimes are given by $g/r = -\log(\theta(1 - \theta))$ (supercritical and subcritical) and $g/r = -\theta \log \theta - (1 - \theta) \log(1 - \theta)$ (strongly and weakly subcritical)

6.1 Existence and uniqueness of the backward ordinary differential equation

The Laplace exponent of \tilde{Z} in Theorem 1 is the solution of a backward ODE. The existence and uniqueness of this latter are stated and proved below.

Proposition 15. *Let δ be in $\mathcal{BV}(\mathbb{R}^+)$. Then the backward ordinary differential equation (7) admits a unique solution.*

The proof relies on a classical approximation of the solution of (7) and the Cauchy-Lipschitz Theorem. When there is no accumulation of jumps, the latter ensures the existence and uniqueness of the solution between two successive jump times of δ . The problem remains on the times where accumulation of jumps occurs. Let us define the family of functions δ^n by deleting the small jumps of δ ,

$$\delta_t^n = \delta_t - \sum_{s \leq t} \left(\delta_s - \delta_{s-} \right) \mathbf{1}_{\{|\delta_s - \delta_{s-}| < 1/n\}}.$$

We note that ψ_0 is continuous, and $s \mapsto e^{gs + \delta_s^n}$ is piecewise C^1 on \mathbb{R}^+ with a finite number of discontinuities. From the Cauchy-Lipschitz Theorem, for every $n \in \mathbb{N}^*$ we can define a solution $v_t^n(\cdot, \lambda, \delta)$ continuous with càdlàg first derivative of the backward differential equation:

$$\frac{\partial}{\partial s} v_t^n(s, \lambda, \delta) = e^{gs + \delta_s^n} \psi_0(e^{-gs - \delta_s^n} v_t^n(s, \lambda, \delta)), \quad 0 \leq s \leq t, \quad v_t^n(t, \lambda, \delta) = \lambda.$$

We want to show that the sequence $(v_t^n(\cdot, \lambda, \delta))_{n \geq 1}$ converges to a function $v_t(\cdot, \lambda, \delta)$ solution of (7). This follows from the next result. Let t be fixed and

$$S := \sup_{s \in [0, t], n \in \mathbb{N}^*} \left\{ e^{gs + \delta_s^n}, e^{-gs - \delta_s^n} \right\}. \quad (38)$$

Lemma 16. *For every $\lambda > 0$,*

(i) we have

$$I := \inf_{0 \leq s \leq t, 1 \leq n} v_t^n(s, \lambda, \delta) > 0, \quad (39)$$

(ii) there exists a positive finite constant C such that for all $IS^{-1} < \eta \leq \kappa \leq \lambda S$,

$$0 \leq \psi_0(\kappa) - \psi_0(\eta) \leq C(\kappa - \eta). \quad (40)$$

Proof. First, we observe that S is finite. Now, using that $x \mapsto e^{-x} + x$ is increasing on \mathbb{R}_+^* and the Taylor-Lagrange's formula to $x \mapsto e^{-x}$, we can check that for all $0 < \eta < \kappa$ and $x \geq 0$

$$0 \leq \frac{e^{-\kappa x} - e^{-\eta x} + (\kappa - \eta)x}{(\kappa - \eta)x} \leq 1 + \frac{\kappa - \eta}{2\eta e}. \quad (41)$$

In order to prove (i), we first note that $\psi_0(\lambda) \geq 0$ for all $\lambda \geq 0$. Moreover, if there exists $0 \leq s_0 < t$, such that $v_t^n(s_0, \lambda, \delta) = 0$, then $v_t^n(s, \lambda, \delta)$ equals 0 for every $s \in [s_0, t]$ since $\psi_0(0) = 0$. Hence $v_t^n(\cdot, \lambda, \delta)$ is non decreasing and

$$v_t^n(s, \lambda, \delta) \in (0, \lambda] \quad \text{for all } s \in [0, t] \text{ and } n \geq 1. \quad (42)$$

Moreover ψ_0 is increasing and for all $s \in [0, t]$, and $n \geq 1$:

$$e^{gs + \delta_s^n} \psi_0(e^{-gs - \delta_s^n} v_t^n(s, \lambda, \delta)) \leq S \psi_0(S v_t^n(s, \lambda, \delta)),$$

where S was defined in (38). On the other hand, note that for all $x \geq 0$,

$$0 \leq e^{-x} - 1 + x \leq x \wedge x^2. \quad (43)$$

Then for all $0 \leq v \leq \lambda S$, we get

$$\psi_0(v) \leq \left[\lambda S \left(\int_0^1 x^2 \mu(dx) + \sigma^2 \right) + \int_1^\infty x \mu(dx) \right] v := Bv.$$

Putting all the pieces together, we deduce that for all $0 \leq s \leq t$,

$$\frac{\partial}{\partial s} v_t^n(s, \lambda, \delta) \leq SB v_t^n(s, \lambda, \delta),$$

and since $v_t^n(t, \lambda, \delta) = \lambda$, we have

$$v_t^n(s, \lambda, \delta) \geq \lambda e^{SB(s-t)}, \quad \text{for all } s \in [0, t].$$

We get (i) by defining $I = \lambda e^{-SBt}$.

Finally, we note that for all $IS^{-1} < \eta < \kappa \leq \lambda S$,

$$\begin{aligned} & \psi_0(\kappa) - \psi_0(\eta) \\ &= \sigma^2(\kappa^2 - \eta^2) + \int_1^\infty (e^{-\kappa x} - e^{-\eta x} + (\kappa - \eta)x) \mu(dx) \\ & \quad + (\kappa - \eta) \int_0^1 x(1 - e^{-\eta x}) \mu(dx) + \int_0^1 (e^{-(\kappa-\eta)x} - 1 + (\kappa - \eta)x) e^{-\eta x} \mu(dx) \\ & \leq \sigma^2(\kappa^2 - \eta^2) + (\kappa - \eta) \left(1 + \frac{(\kappa - \eta)}{2\eta e} \right) \int_1^\infty x \mu(dx) \\ & \quad + (\kappa - \eta) \eta \int_0^1 x^2 \mu(dx) + (\kappa - \eta)^2 \int_0^1 x^2 \mu(dx) \\ & \leq \left[2\lambda S \sigma^2 + 2\lambda S \int_0^1 x^2 \mu(dx) + \left(1 + \frac{\lambda S^2}{2Ie} \right) \int_1^\infty x \mu(dx) \right] (\kappa - \eta), \end{aligned}$$

which proves part (ii). \square

We can now prove the result of existence and uniqueness.

Proof of Proposition 15. We now prove that $(v_t^n(s, \lambda, \delta), s \in [0, t])_{n \geq 0}$ is a Cauchy sequence. For sake of simplicity, we denote $v^n(s) = v_t^n(s, \lambda, \delta)$, and for all $v \geq 0$:

$$\psi^n(s, v) = e^{gs + \delta_s^n} \psi_0(e^{-gs - \delta_s^n} v) \quad \text{and} \quad \psi^\infty(s, v) = e^{gs + \delta_s} \psi_0(e^{-gs - \delta_s} v).$$

We have for any $0 \leq s \leq t$ and $m, n \geq 1$:

$$\begin{aligned} |v^n(s) - v^m(s)| &= \left| \int_s^t \psi^n(u, v^n(u)) du - \int_s^t \psi^m(u, v^m(u)) du \right| \\ &\leq \int_s^t (R^n(u) + R^m(u)) du + \int_s^t \left| \psi^\infty(u, v^n(u)) - \psi^\infty(u, v^m(u)) \right| du, \end{aligned} \quad (44)$$

where for any $u \in [0, t]$,

$$\begin{aligned} R^n(u) &= \left| \psi^n(u, v^n(u)) - \psi^\infty(u, v^n(u)) \right| \\ &\leq e^{gs + \delta_s^n} \left| \psi_0(e^{-gs - \delta_s^n} v^n(u)) - \psi_0(e^{-gs - \delta_s} v^n(u)) \right| + e^{gs} \psi_0(e^{-gs - \delta_s} v^n(u)) \left| e^{\delta_s^n} - e^{\delta_s} \right|. \end{aligned}$$

Moreover, from (38) to (40), we obtain

$$\begin{aligned} R^n(u) &\leq SC\lambda \left| e^{-\delta_s^n} - e^{-\delta_s} \right| + e^{gt} \psi_0(\lambda S) \left| e^{\delta_s^n} - e^{\delta_s} \right| \\ &\leq \left(SC\lambda + e^{gt} \psi_0(\lambda S) \right) \sup_{s \in [0, t]} \left\{ \left| e^{-\delta_s^n} - e^{-\delta_s} \right|, \left| e^{\delta_s^n} - e^{\delta_s} \right| \right\} := s_n, \end{aligned}$$

which implies

$$\sup \left\{ \int_s^t R^n(u) du, s \in [0, t] \right\} \leq t s_n \xrightarrow{n \rightarrow \infty} 0.$$

Using similar arguments as above, we get from (40),

$$\left| \psi^\infty(u, v^n(u)) - \psi^\infty(u, v^m(u)) \right| \leq CS^2 \left| v^n(u) - v^m(u) \right|.$$

From (44), we use Gronwall's Lemma (see Lemma 3.2 in [Dyn91] or Lemma 4.6 in [BS47]) with

$$R_{m,n}(s) = \int_s^t R^n(u) du + \int_s^t R^m(u) du,$$

to deduce that for all $0 \leq s \leq t$,

$$\left| v^n(s) - v^m(s) \right| \leq R_{m,n}(s) + CS^2 e^{CS^2(t-s)} \int_s^t R_{m,n}(s) ds.$$

Hence for every $n_0 \in \mathbb{N}^*$,

$$\sup_{m, n \geq n_0, s \in [0, t]} \left| v^n(s) - v^m(s) \right| \leq t \left[1 + CS^2 e^{CS^2 t} \right] \sup_{m, n \geq n_0} (s_n + s_m).$$

Thus $(v^n(s), s \in [0, t])_{n \geq 0}$ is a Cauchy sequence under the uniform norm. Then there exists v a continuous function on $[0, t]$ such that $v^n \rightarrow v$, as n goes to ∞ . Now we prove

that v is solution of the equation (7). By continuity, v satisfies (38) and (40). Then for any $s \in [0, t]$ and $n \in \mathbb{N}^*$:

$$\begin{aligned} & \left| v(s) - \int_s^t \psi^\infty(s, v(s)) ds - \lambda \right| \\ & \leq \left| v(s) - v^n(s) \right| + \int_s^t \left| \psi^\infty(s, v(s)) - \psi^n(s, v(s)) \right| ds + \int_s^t \left| \psi^n(s, v(s)) - \psi^n(s, v^n(s)) \right| ds \\ & \leq ts_n + (1 + CS^2) \sup \left\{ \left| v(s) - v^n(s) \right|, s \in [0, t] \right\}. \end{aligned}$$

so that letting $n \rightarrow \infty$ yields

$$\left| v(s) - \int_s^t \psi^\infty(s, v(s)) ds - \lambda \right| = 0.$$

It proves that v is solution of (7). The uniqueness follows from Gronwall's lemma. \square

6.2 An upper bound for ψ_0

The study of the Laplace exponent of \tilde{Z} in Corollary 2 requires a fine control of the branching mechanism ψ_0 .

Lemma 17. *Assume that the process $(gs + \Delta_s, s \geq 0)$ goes to $+\infty$ a.s. There exists a non negative increasing function h on \mathbb{R}^+ such that for every $\lambda \geq 0$*

$$\psi_0(\lambda) \leq \lambda h(\lambda) \quad \text{and} \quad \int_0^\infty h(e^{-(gt+\Delta t)}) dt < \infty.$$

Proof. The inequality (43) implies that for every $\lambda \geq 0$,

$$\begin{aligned} \psi_0(\lambda) & \leq \sigma^2 \lambda^2 + \int_0^\infty (\lambda^2 x^2 \mathbf{1}_{\{\lambda x \leq 1\}} + \lambda x \mathbf{1}_{\{\lambda x > 1\}}) \mu(dx) \\ & \leq \left(\sigma^2 + \int_0^1 x^2 \mu(dx) \right) \lambda^2 + \lambda^2 \mathbf{1}_{\{\lambda < 1\}} \int_1^{1/\lambda} x^2 \mu(dx) + \lambda \int_{1/\lambda}^\infty x \mu(dx). \end{aligned}$$

Now, using condition (12) we obtain the existence of a positive constant c such that

$$\lambda \int_{1/\lambda}^\infty x \mu(dx) \leq \lambda \log(1 + 1/\lambda)^{-(1+\epsilon)} \int_{1/\lambda}^\infty x \log(1 + x)^{1+\epsilon} \mu(dx) \leq c \lambda \log(1 + 1/\lambda)^{-(1+\epsilon)}.$$

Next, let us introduce the following function f :

$$f(x) = x^{-1} \log(1 + x)^{1+\epsilon}, \quad \text{for } x \in [1, \infty).$$

By differentiation, we check that there exists a positive real number A such that f is decreasing on $[A, \infty)$. Therefore, for every $\lambda < 1/A$,

$$\begin{aligned} \int_1^{1/\lambda} \lambda^2 x^2 \mu(dx) & \leq \lambda^2 \int_A^{1/\lambda} x^2 \mu(dx) \\ & \leq \lambda \log(1 + 1/\lambda)^{-(1+\epsilon)} f(1/\lambda) \int_A^{1/\lambda} \frac{x \log(1 + x)^{1+\epsilon}}{f(x)} \mu(dx) \\ & \leq \lambda \log(1 + 1/\lambda)^{-(1+\epsilon)} \int_A^{1/\lambda} x \log(1 + x)^{1+\epsilon} \mu(dx) \end{aligned}$$

Adding that for $\lambda \in [1, 1/A]$, $\int_1^{1/\lambda} \lambda^2 x^2 \mu(dx) \leq \lambda^2 \int_1^A x^2 \mu(dx) \leq \lambda^2 A \int_1^\infty x \mu(dx)$ and using again condition (12), we deduce that there exists a positive constant c' such that for every $\lambda \geq 0$,

$$\psi_0(\lambda) \leq c' \left(\lambda^2 + \lambda \log(1 + 1/\lambda)^{-(1+\varepsilon)} \right).$$

Since λ^2 is negligible with respect to $\lambda \log(1 + 1/\lambda)^{-(1+\varepsilon)}$ when λ is near 0 or infinity, we conclude that there exists a positive constant c'' such that

$$\psi_0(\lambda) \leq c'' \lambda \log(1 + 1/\lambda)^{-(1+\varepsilon)}.$$

Defining the function $h(x) = c'' x \log(1 + 1/x)^{-(1+\varepsilon)}$, for $x > 0$, we get:

$$0 \leq \int_0^\infty h\left(e^{-(gt+\Delta_t)}\right) dt \leq c'' \int_0^\infty (gt + \Delta_t)^{-(1+\varepsilon)} dt,$$

which is finite since the process $(gs + \Delta_s, s \geq 0)$ goes linearly to $+\infty$ a.s. More precisely, one can find $\eta > 0$ such that $(gs + \Delta_s - \eta s : s \geq 0)$ has positive expectation for $s = 1$, which ensures that it goes to ∞ a.s. and there exists $L > -\infty$ a.s. such that $gs + \Delta_s \geq L + \eta s$ a.s. This ends the proof. \square

6.3 Extinction versus explosion

Let us here check that Y_t can be properly renormalized as $t \rightarrow \infty$ on the non-extinction event. We use a classical branching argument.

Lemma 18. *Let Y be a non negative Markov process satisfying the branching property.*

Assume also that there exists a positive function a_t such that for every $x_0 > 0$, there exists a non negative finite random variable W such that

$$a_t Y_t \xrightarrow[t \rightarrow \infty]{} W \quad a.s., \quad \mathbb{P}_{x_0}(W > 0) > 0, \quad a_t \xrightarrow[t \rightarrow \infty]{} 0.$$

Then

$$\{W = 0\} = \left\{ Y_t \xrightarrow[t \rightarrow \infty]{} 0 \right\} \quad \mathbb{P}_{x_0} \quad a.s.$$

Proof. First, we prove that

$$\mathbb{P}_{x_0}(\limsup_{t \rightarrow \infty} Y_t = \infty \mid \limsup_{t \rightarrow \infty} Y_t > 0) = 1. \quad (45)$$

Let $0 < x \leq x_0 \leq A$ be fixed. As $a_t \rightarrow 0$ and $\mathbb{P}_x(W > 0) > 0$, there exists $t_0 > 0$ such that $\alpha := \mathbb{P}_x(Y_{t_0} \geq A) > 0$. By the branching property, the process is stochastically monotone as a function of its initial value. Thus, for every $y \geq x$ (including $y = x_0$),

$$\mathbb{P}_y(Y_{t_0} \geq A) \geq \alpha > 0.$$

Let us define the following stopping times

$$T_0 := 0, \quad T_{i+1} = \inf\{t \geq T_i + t_0 : Y_t \geq x\} \quad (i \geq 0)$$

For any $i \in \mathbb{N}^*$, by strong Markov property

$$\mathbb{P}_{x_0}(Y_{T_i+t_0} \geq A \mid (Y_t : t \leq T_i), T_i < \infty) \geq \alpha.$$

Conditionally on $\{\limsup_{t \rightarrow \infty} Y_t > x\}$, the stopping times T_i are finite a.s. and for all $0 < x \leq x_0 \leq A$,

$$\mathbb{P}_{x_0}(\forall i \geq 0 : Y_{T_i+t_0} < A, \limsup_{t \rightarrow \infty} Y_t > x) = 0.$$

Then, $\mathbb{P}_{x_0}(\limsup_{t \rightarrow \infty} Y_t < \infty, \limsup_{t \rightarrow \infty} Y_t > x) = 0$. Adding that $\{\limsup_{t \rightarrow \infty} Y_t > 0\} = \cup_{x \in (0, x_0]} \{\limsup_{t \rightarrow \infty} Y_t > x\}$ yields (45).

Let us now consider the stopping times $T_n = \inf\{t \geq 0 : Y_t \geq n\}$. We get by the strong Markov property and branching property,

$$\mathbb{P}_{x_0}(W = 0; T_n < \infty) = \mathbb{E}_{x_0}(\mathbf{1}_{T_n < \infty} \mathbb{P}_{Y_{T_n}}(W = 0)) \leq \mathbb{P}_n(a_t Y_t \xrightarrow[t \rightarrow \infty]{} 0) = \mathbb{P}_1(a_t Y_t \xrightarrow[t \rightarrow \infty]{} 0)^n,$$

which goes to zero as $n \rightarrow \infty$, since $\mathbb{P}_1(a_t Y_t \xrightarrow[t \rightarrow \infty]{} 0) = \mathbb{P}_1(W = 0) < 1$. Then,

$$0 = \mathbb{P}_{x_0}(W = 0; \forall n : T_n < \infty) = \mathbb{P}_{x_0}(W = 0, \limsup_{t \rightarrow \infty} Y_t = \infty) = \mathbb{P}_{x_0}(W = 0, \limsup_{t \rightarrow \infty} Y_t > 0),$$

where the last identity comes from (45). This completes the proof. \square

6.4 A Central limit theorem

We need the following central limit theorem of Lévy processes in Corollary 3.

Lemma 19. *Under the assumption (13) we have*

$$\frac{gt + \Delta_t - mt}{\rho\sqrt{t}} \xrightarrow[t \rightarrow \infty]{d} N(0, 1).$$

Proof. For simplicity, let η be the image measure of ν under the mapping $x \mapsto e^x$. The assumption (13) is equivalent to $\int_{|x| \geq 1} x^2 \eta(dx) < \infty$, or $\mathbb{E}[\Delta_1^2] < \infty$.

We define $T(x) = \eta((-\infty, -x)) + \eta((x, \infty))$ and $U(x) = 2 \int_0^x y T(y) dy$, and assume that $T(x) > 0$ for all $x > 0$. According to Theorem 3.5 in Doney and Maller [DM02] there exist two functions $a(t), b(t) > 0$ such that

$$\frac{gt + \Delta_t - a(t)}{b(t)} \xrightarrow[t \rightarrow \infty]{d} N(0, 1), \quad \text{if and only if} \quad \frac{U(x)}{x^2 T(x)} \xrightarrow[x \rightarrow \infty]{} \infty.$$

If the above condition is satisfied, then b is regularly varying with index 1/2 and it may be chosen to be strictly increasing to ∞ as $t \rightarrow \infty$. Moreover $b^2(t) = tU(b(t))$ and $a(t) = tA(b(t))$, where

$$A(x) = g + \int_{\{|z| < 1\}} z \eta(dz) + \eta((1, \infty)) - \eta((-\infty, -1)) + \int_1^x (\eta((y, \infty)) - \eta((-\infty, -y))) dy.$$

Note that under our assumption $x^2 T(x) \rightarrow 0$, as $x \rightarrow \infty$. Moreover, note

$$U(x) = x^2 T(x) + \int_{(-x, 0)} z^2 \eta(dx) + \int_{(0, x)} z^2 \eta(dx),$$

and

$$A(x) = g + \int_{\{|z| < x\}} z \eta(dz) + x (\eta((x, \infty)) - \eta((-\infty, -x))).$$

Hence assumption (13) implies that

$$U(x) \xrightarrow{x \rightarrow \infty} \int_{(-\infty, \infty)} z^2 \eta(dz) = \rho^2, \quad A(x) \xrightarrow{x \rightarrow \infty} g + \int_{\mathbb{R}} z \eta(dz) = \mathbf{m},$$

Therefore, we deduce $U(x)/(x^2 T(x)) \rightarrow \infty$ as $x \rightarrow \infty$, $b(t) \sim \rho \sqrt{t}$ and $a(t) \sim \mathbf{m}t$, as $t \rightarrow \infty$.

Now assume that $T(x) = 0$, for x large enough. Define

$$\Psi(\lambda, t) = -\log \mathbb{E} \left[\exp \left\{ i\lambda \left(\frac{gt + \Delta_t - a(t)}{b(t)} \right) \right\} \right],$$

where the functions $a(t)$ and $b(t)$ are defined as above. Hence, we can write

$$\begin{aligned} \Psi(\lambda, t) &= t \int_{\{|x| < b(t)\}} \left(1 - e^{\frac{i\lambda}{b(t)}x} + \frac{i\lambda}{b(t)}x + \frac{(i\lambda)^2}{2b^2(t)}x^2 \right) \eta(dx) + t \int_{\{|x| \geq b(t)\}} \left(1 - e^{\frac{i\lambda}{b(t)}x} \right) \eta(dx) \\ &\quad - \frac{t(i\lambda)^2}{2b^2(t)} \int_{\{|x| < b(t)\}} x^2 \eta(dx) + i\lambda t \left(\eta(b(t), \infty) - \eta(-\infty, -b(t)) \right). \end{aligned}$$

Since $T(x) = 0$ for all x large, $b(t) \rightarrow \infty$ and $t^{-1}b^2(t) \rightarrow \rho$, as $t \rightarrow \infty$, therefore

$$\Psi(\lambda, t) \xrightarrow{t \rightarrow \infty} \frac{\lambda^2}{2},$$

which implies the result thanks to Lévy's Theorem. \square

6.5 Two technical Lemmas

We now give two technical lemmas, useful in the proofs of Section 4.

Lemma 20. *Assume that F satisfies (23). Then there exist two positive constants η and M such that for all (x, y) in \mathbb{R}_+^2 and ε in $[0, \eta]$,*

$$\left| F(x) - C_F x^{-1/\beta} \right| \leq M x^{-(1+\varepsilon)/\beta}, \quad (46)$$

$$\left| F(x) - F(y) \right| \leq M \left| x^{-1/\beta} - y^{-1/\beta} \right|. \quad (47)$$

Proof. To obtain (46), it is enough to choose $\varepsilon \leq 1$ as we assume in (23) that $\varsigma \geq 1$.

To prove (47), we first define the function $\tilde{h} : x \in \mathbb{R}^+ \mapsto (1+x)^{1-\varsigma} h(x)$ and let $0 \leq x \leq y$. Then,

$$\begin{aligned} \frac{F(x) - F(y)}{C_F} &\leq \left((x+1)^{-1/\beta} - (y+1)^{-1/\beta} \right) + (1+y)^{-1/\beta-1} \left| \tilde{h}(x) - \tilde{h}(y) \right| \\ &\quad + \left| \tilde{h}(x) \right| \left((1+x)^{-1/\beta-1} - (1+y)^{-1/\beta-1} \right). \end{aligned} \quad (48)$$

As $\beta \in (0, 1]$, we have the following inequalities :

$$\begin{aligned} (1+x)^{-1/\beta-1} - (1+y)^{-1/\beta-1} &\leq (1+y)^{-1/\beta-1} \left(\left(\frac{1+y}{1+x} \right)^{1/\beta} - 1 \right) \left(\frac{1+y}{1+x} + 1 \right) \\ &\leq (1+y)^{-1/\beta} \left(\left(\frac{y}{x} \right)^{1/\beta} - 1 \right) \frac{1}{1+y} \cdot 2 \frac{1+y}{1+x} \\ &\leq 2 \left(x^{-1/\beta} - y^{-1/\beta} \right). \end{aligned}$$

Moreover, the Mean Value Theorem applied to the function $z \in \mathbb{R}_+ \mapsto (z+1)^{-1/\beta}$ on $[x, y]$ ensures that

$$\frac{1}{\beta}(y+1)^{-1/\beta-1}(y-x) \leq (x+1)^{-1/\beta} - (y+1)^{-1/\beta}.$$

Now, we denote by k the Lipschitz constant of h . The equation (48) finally gives

$$0 \leq F(x) - F(y) \leq C_F(1 + 2\|h\|_\infty + k\beta)(x^{-1/\beta} - y^{-1/\beta}).$$

which ends up the proof. \square

Lemma 21. *Assume that the positive sequences $(a_{n,q})_{(n,q) \in \mathbb{N}^2}$, $(a'_{n,q})_{(n,q) \in \mathbb{N}^2}$ and $(b_n)_{n \in \mathbb{N}}$ satisfy for every $(n, q) \in \mathbb{N}^2$:*

$$a_{n,q} \leq b_n \leq a'_{n,q},$$

and that there exist three sequences $(a(q))_{q \in \mathbb{N}}$, $(c^-(q))_{q \in \mathbb{N}}$ and $(c^+(q))_{q \in \mathbb{N}}$ such that

$$\lim_{n \rightarrow \infty} a_{n,q} = c^-(q)a(q), \quad \lim_{n \rightarrow \infty} a'_{n,q} = c^+(q)a(q), \quad \text{and} \quad \lim_{q \rightarrow \infty} c^-(q) = \lim_{q \rightarrow \infty} c^+(q) = 1$$

Then there exists a positive finite constant a such that

$$\lim_{q \rightarrow \infty} a(q) = \lim_{n \rightarrow \infty} b_n = a$$

Proof. Letting n go to infinity, we have for every $q \in \mathbb{N}$

$$\limsup b_n \leq c^+(q)a(q) \quad \text{and} \quad c^-(q)a(q) \leq \liminf b_n.$$

Then letting q go to infinity, we obtain

$$\limsup b_n \leq \liminf a(q) \quad \text{and} \quad \limsup a(q) \leq \liminf b_n,$$

which ends the proof. \square

6.6 Approximations of the survival probability for $\nu(0, \infty) = \infty$

Finally, we end the proof of Corollary 6 in the case where $\nu(0, \infty) = \infty$.

End of proof of Corollary 6. We let $A^{\varepsilon_1, \varepsilon_2} = (0, 1 - \varepsilon_1) \cup (1 + \varepsilon_2, \infty)$, where $0 < 1 - \varepsilon_1 < 1 < 1 + \varepsilon_2$ and define the Poisson random measure $N_1^{\varepsilon_1, \varepsilon_2}$ as the restriction of N_1 to $A^{\varepsilon_1, \varepsilon_2} \times \mathbb{R}^+$. We denote by $d\nu^{\varepsilon_1, \varepsilon_2}(dx)$ its intensity measure, where $\nu^{\varepsilon_1, \varepsilon_2}(dx) = \mathbf{1}_{\{x \in A^{\varepsilon_1, \varepsilon_2}\}} \nu(dx)$, and the corresponding Lévy process $\Delta^{\varepsilon_1, \varepsilon_2}$ defined by

$$\Delta_t^{\varepsilon_1, \varepsilon_2} = \int_0^t \int_{(0, \infty)} \log(x) N_1^{\varepsilon_1, \varepsilon_2}(ds, dx).$$

We also consider the CSBP's $Y^{\varepsilon_1, \varepsilon_2}$ (resp $Y^{\varepsilon_1, \varepsilon_2, -}$ and $Y^{\varepsilon_1, \varepsilon_2, +}$) with branching mechanism ψ (resp. ψ_- and ψ_+) and catastrophes $\Delta^{\varepsilon_1, \varepsilon_2}$ via (5). Since $\nu^{\varepsilon_1, \varepsilon_2}(0, \infty) < \infty$, from the first step we have $u_{+,t}^{\varepsilon_1, \varepsilon_2}(\lambda) \leq u^{\varepsilon_1, \varepsilon_2}(t, \lambda) \leq u_{-,t}^{\varepsilon_1, \varepsilon_2}(\lambda)$, where as expected $\mathbb{E}[\exp\{-\lambda Y_t^{\varepsilon_1, \varepsilon_2, *}\}] = \exp\{-u_{*,t}^{\varepsilon_1, \varepsilon_2}(\lambda)\}$ for each $* \in \{+, \emptyset, -\}$.

Similarly, let $A^{\varepsilon_1} = (0, 1 - \varepsilon_1) \cup (1, \infty)$ and define the Poisson random measure $N_1^{\varepsilon_1}$ as the restriction of N_1 to $A^{\varepsilon_1} \times \mathbb{R}^+$ with intensity measure $dt\nu^{\varepsilon_1}(dx)$, where $\nu^{\varepsilon_1}(dx) = \mathbf{1}_{\{x \in A^{\varepsilon_1}\}}\nu(dx)$. Let us fix t in \mathbb{R}_+^* , and define Y^{ε_1} the unique strong solution of

$$Y_t^{\varepsilon_1} = Y_0 + \int_0^t gY_s^{\varepsilon_1} ds + \int_0^t \sqrt{2\sigma^2 Y_s^{\varepsilon_1}} dB_s + \int_0^t \int_{[0, \infty)} \int_0^{Y_{s-}^{\varepsilon_1}} z \tilde{N}_0(ds, dz, du) + \int_0^t \int_{[0, \infty)} (z - 1) Y_{s-}^{\varepsilon_1} N_1^{\varepsilon_1}(ds, dz). \quad (49)$$

We already know from Theorem 1 that Equation (49) has a unique non negative strong solution. Moreover, from Theorem 5.5 in [FL10] and the fact that $N_1^{\varepsilon_1}$ has the same jumps as $N_1^{\varepsilon_1, \varepsilon_2}$ plus additional jumps greater than one, we know that

$$Y_t^{\varepsilon_1, \varepsilon_2} \leq Y_t^{\varepsilon_1}, \quad \text{a.s.}$$

Using assumption (4), we can apply Gronwall Lemma to the non negative function $t \mapsto \mathbb{E}[Y_t^{\varepsilon_1} - Y_t^{\varepsilon_1, \varepsilon_2}]$ and obtain

$$\mathbb{E}\left[|Y_t^{\varepsilon_1, \varepsilon_2} - Y_t^{\varepsilon_1}|\right] \xrightarrow{\varepsilon_2 \rightarrow 0} 0.$$

Adding that $Y^{\varepsilon_1, \varepsilon_2}$ is decreasing with ε_2 , we finally get, $Y_t^{\varepsilon_1, \varepsilon_2} \xrightarrow{\text{a.s.}} Y_t^{\varepsilon_1}$, as $\varepsilon_2 \rightarrow 0$. Using similar arguments as above for $Y^{\varepsilon_1, \varepsilon_2, +}$ and $Y^{\varepsilon_1, \varepsilon_2, -}$, we deduce

$$u_{+,t}^{\varepsilon_1}(\lambda) \leq u^{\varepsilon_1}(t, \lambda) \leq u_{-,t}^{\varepsilon_1}(\lambda).$$

Letting ε_1 go to 0 completes the proof. \square

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