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# CONVERGENCE TO THE EQUILIBRIUM IN A LOTKA-VOLTERRA ODE COMPETITION SYSTEM WITH MUTATIONS

JÉRÔME COVILLE AND FRÉDÉRIC FABRE

ABSTRACT. In this paper we are investigating the long time behaviour of the solution of a mutation competition model of Lotka-Volterra's type. Our main motivation comes from the analysis of the Lotka-Volterra's competition system with mutation which simulates the demo-genetic dynamics of diverse virus in their host :

$$\frac{dv_i(t)}{dt} = v_i \left[ r_i - \frac{1}{K} \Psi_i(v) \right] + \sum_{j=1}^N \mu_{ij} (v_j - v_i).$$

In a first part we analyse the case where the competition terms  $\Psi_i$  are independent of the virus type  $i$ . In this situation and under some rather general assumptions on the functions  $\Psi_i$ , the coefficients  $r_i$  and the mutation matrix  $\mu_{ij}$  we prove the existence of a unique positive globally stable stationary solution i.e. the solution attracts the trajectory initiated from any nonnegative initial datum. Moreover the unique steady state  $\bar{v}$  is strictly positive in the sense that  $\bar{v}_i > 0$  for all  $i$ . These results are in sharp contrast with the behaviour of Lotka-Volterra without mutation term where it is known that multiple non negative stationary solutions exist and an exclusion principle occurs (i.e For all  $i \neq i_0, \bar{v}_i = 0$  and  $\bar{v}_{i_0} > 0$ ). Then we explore a typical example that has been proposed to explain some experimental data. For such particular models we characterise the speed of convergence to the equilibrium. In a second part, under some additional assumption, we prove the existence of a positive steady state for the full system and we analyse the long term dynamics. The proofs mainly rely on the construction of a relative entropy which plays the role of a Lyapunov functional.

*Keywords:* Demo-genetic dynamics, Lotka-Volterra competition system with mutation, equilibria, Relative entropy, Global stability.

*2010 Mathematics Subject Classification:* 34A34, 34A40, 34D05, 34D23, 92D15, 92D25.

## 1. INTRODUCTION

In this paper we are investigating the long time behaviour of the solution of some models that have been recently used in epidemiology. Our analysis focuses on a Lotka-Volterra competition system with mutation which basically simulates the demo-genetic dynamic of a genetically diverse virus population in its hosts, highlighting the numerous links existing between ecological and within-host infection dynamics. Such type of model has been proposed to explain some experimental data e.g.[16, 22, 37]. To be more specific the demo-genetic dynamic is modelled by  $N$  ordinary differential equations which simulates at host scale the dynamics of  $v_i(t)$  the number of virus particles of genotype  $i$  at time  $t$  :

$$(1.1) \quad \frac{dv_i(t)}{dt} = v_i \left[ r_i - \frac{1}{K} \Psi_i(v) \right] + \sum_{j=1}^N \mu_{ij} (v_j - v_i)$$

where  $r_i, K$  and  $\mu_{ij}$  represent respectively the growth rate for each genotype, the total carrying capacity of the host and a nonnegative matrix characterising the rate of mutation from a virus of genotype  $i$  to a virus of genotype  $j$ . For each  $i$ ,  $\Psi_i(v) : \mathbb{R}^N \rightarrow \mathbb{R}$  is a locally Lipschitz application describing the intensity of the interaction between a virus of genotype  $i$  with all its competitors.

The mutation term of the system can also be interpreted as a dispersal term. Indeed, mutation naturally corresponds to dispersal into the discrete space of genotype. But the mutation term can

also handle dispersal between physical patches. With this in mind, the above system of equation can also be used to model the demo-genetic dynamic of a diverse virus population in structured hosts, each host tissue types being virus "habitats" connected to each others by dispersal via fluid flow (e.g. [38]). Ways of derive results for this interpretation are discussed in the biological comments subsection.

In what follows we will always make the following assumptions on  $r_i$ ,  $\Psi_i$  and  $\mu_{ij}$

$$(1.2) \quad \left\{ \begin{array}{l} \text{For all } i, r_i > 0, \\ \text{The matrix } (\mu_{ij}) \text{ is nonnegative symmetric and irreducible} \\ \Psi_i(v) \in C_{loc}^{0,1}(\mathbb{R}^N, \mathbb{R}), \Psi_i(0) = 0 \\ \Psi_i \text{ is monotone increasing with respect to the natural order of } \mathbb{R}^N \end{array} \right.$$

Furthermore we will assume that for all  $i$  there exists positive constants  $R_i, k_i, c_i$  with  $k_i > 0$  so that the function  $\Psi_i$  satisfies for all  $v \in \mathbb{R}^{N,+} \setminus Q_{R_i}(0)$ ,

$$(1.3) \quad c_i \left( \sum_{j=1}^N v_j \right)^{k_i} \leq \Psi_i(v),$$

where  $Q_{R_i}(0)$  denotes the ball of radius  $R_i$  and centred at 0 associated to the  $l^1$  norm.

A typical example of model satisfying our assumption is given by

$$\frac{dv_i(t)}{dt} = v_i \left[ r_i - \frac{1}{K} \sum_{j=1}^4 \beta_{ij} v_j \right] + Mv$$

with

$$M := \begin{pmatrix} (1-\mu)^2 - 1 & \mu(1-\mu) & \mu(1-\mu) & \mu^2 \\ \mu(1-\mu) & (1-\mu)^2 - 1 & \mu^2 & \mu(1-\mu) \\ \mu(1-\mu) & \mu^2 & (1-\mu)^2 - 1 & \mu(1-\mu) \\ \mu^2 & \mu(1-\mu) & \mu(1-\mu) & (1-\mu)^2 - 1 \end{pmatrix},$$

where  $\mu$  is a parameter giving the point mutation rate per replication cycle and per nucleotide. This mutation matrix corresponds to a viral population composed of 4 variants differing only by one or two substitutions involved in adaptative proprieties (e.g. pathogenicity). The interactions terms  $\Psi_i := \sum_{j=1}^4 \beta_{ij} v_j$  can handle a wide range of possible inter-specific (inter-variants) competition rates between any pairs of virus variants. This particular structure has been used recently to model the adaptation of plant virus to resistance genes[21, 22]. This particular form of competition is commonly used to model virus evolution,[16, 32, 37].

Without the mutation's matrix  $(\mu_{ij})$ , the system (1.1) reduces to a classical competition system in the sense of Hirsch [26, 27, 28]

$$(1.4) \quad \frac{dv_i(t)}{dt} = v_i \left[ r_i - \frac{1}{K} \Psi_i(v) \right].$$

Such systems has been intensively studied and many aspects are now well understood see for example [1, 5, 6, 13, 15, 19, 24, 26, 27, 28, 29, 30, 31, 33, 39] and references therein. In particular, the existence of stationary solution and the asymptotic behaviour of the solution has been obtained in [12, 13, 31, 39]. Those systems are characterised by the existence of at least as many equilibrium states that the number of competing species (or genotypes) involved. In addition, the dynamics exhibit a competitive exclusion principle which state that the fittest species initially present will overcome all the other ones.

When the mutations's matrix  $(\mu_{ij})$  is non trivial, the system (1.1) does not fall into Hirsch's definition of competitive system and less results are known. If for reasonably smooth interaction functions, the existence of solution of (1.1) defined for all times is not an issue, the existence of a non trivial stationary solution and the analysis of the asymptotic behaviour are challenging questions.

Most of the known results concerns either particular interaction functions  $\Psi_i$  for which the existence of steady states and their local stability are investigated [4, 15, 24, 29] or for some ODE's systems (1.1) where the mutation matrix  $\mu_{ij}$  is considered as a small parameter. In the latter the system (1.1) is seen as a perturbation of (1.4) and analysed using perturbative techniques [6, 7, 8]. Recently, there has been an intense activity on continuous trait version of (1.1) where some of the techniques can be used to obtain the existence of locally stable steady states for (1.1) see for example [3, 7, 8, 9, 10, 11, 12, 13, 18, 19, 31, 39, 40, 41]. However, to our knowledge there is no results on global stability of the steady states for systems like(1.1) neither for its continuous trait version.

In this direction, our first results concern the systems (1.1) where the competition terms  $\Psi_i$  are independent of  $i$ . A typical case is  $\Psi_i(v) = \sum_{j=1}^N v_j$  which corresponds to a situation where inter- and intra-species (genotypes) competitions are equals (i.e. blind and uniform competition between variants, see Lafforgue et al. [32]). In this situation one can show that there exists a unique positive stationary solution of (1.1)  $\bar{v} \in \mathbb{R}^N$ , which attracts all the trajectories initiated from any nonnegative and non zero initial data. Namely we show that

**Theorem 1.1.** *Assume that the interaction  $\Psi_i$  is independent of  $i$  and is satisfying the assumptions (1.2)–(1.3), then there exists a unique positive stationary solution  $\bar{v}$  to (1.1). Moreover for any nonnegative initial datum  $v(0)$  not identically zero, the corresponding solution  $v(t)$  of (1.1) converges to  $\bar{v}$ .*

A case of particular interest is when the interactions  $\Psi_i$  take the following form  $\Psi_i(v) = \sum_{j=1}^N r_j v_j$ . This particular structure of interaction was initially introduced on a theoretical ground by Sole et al. [44] to model the competition between viruses. Recently, this form of interactions has been used to explain experimental results of virus evolution [16, 22, 37]. Sole et al. showed that Eigen's model of molecular quasi-species [20] was to a large extent equivalent to the Lotka-Volterra competition equations under this assumption.

For this type of interaction, we can rewrite the system (1.1) as follows

$$(1.5) \quad \frac{dv_i(t)}{dt} = v_i \left[ r_i - \frac{1}{K} \sum_{j=1}^N r_j v_j \right] + \sum_{j=1}^N \mu_{ij} (v_j - v_i).$$

For (1.5) besides the asymptotic behaviour of the solution obtained as an application of Theorem 1.1 we can precise the speed of convergence to the equilibrium. Furthermore we can give an estimate of the time to reach near the equilibrium. In epidemiology, this type of informations is of practical use for building tractable nested models. Nested models are a class of model which explicitly links the relationships between processes at different levels of biological organization. They are often used to study the pathogen evolution by linking the disease dynamic of within- and between-host, see Mideo et al. [36]. Their formalisation becomes more simple when the within-host pathogen dynamic is faster than the between-hosts dynamic. Indeed, in such cases, using for example slow-fast reduction techniques commonly used in ecology [2], the within-host dynamic can be approximated by its equilibrium, see for example [23].

More precisely, we show that

**Theorem 1.2.** *Assume that the interactions  $\Psi_i$  take the form  $\Psi_i(v) = \sum_{j=1}^N r_j v_j$  then for any nonnegative initial datum  $v(0)$  not identically zero, the solution  $v(t)$  of (1.5) converges exponentially fast to its unique equilibrium. That is to say there exists two positive constants  $C_1$  and  $C_2$  so that*

$$\|v - \bar{v}\|_{\infty} \leq C_1 e^{-C_2 t}.$$

Next, we analyse the situation where the functions  $\Psi_i$  are not reduced to a single function. From a biological point of view, this situation could appears as, often, genotypes (species) exhibit particular association patterns (see for example [45] in the case of plant viruses). A way to model such phenomena is to take  $\Psi_i$  of the form  $\Psi_i(v) = \sum_{j=1}^N \alpha_{i,j} r_j v_j$  where  $\alpha_{i,j}$  are crowing index [34, 25]. This index is equal to 1 when the 2 species are distributed independently and ranges from 0 (complete avoidance) to a large constant (near overlap) according to patterns of species association.

In this general context, our first result concerns the existence of a positive stationary solution for the system (1.1) assuming we have the following extra condition

$$(1.6) \quad \forall i \quad \sum_{j=1}^N \mu_{ij} \leq \frac{r_i}{2}.$$

This condition makes sense in our application framework since mutation rates (expressed as substitutions per nucleotide per cell infection) range from  $10^{-8}$  to  $10^{-6}$  for DNA viruses and from  $10^{-6}$  to  $10^{-3}$  for RNA viruses [43] whereas growth rates  $r_i$  are of the order of the unit or higher. For example, the overall growth rate of the RNA virus VSV was estimated to 0.6 virus/hour [17]. Each cell infected by a single VSV particle produces from 50 to 8000 progeny virus particle/ cell infection [47].

Under the extra assumption (1.6) we prove

**Theorem 1.3.** *Assume that  $r_i, \Psi_i$  and  $\mu_{ij}$  satisfy the assumptions (1.2)–(1.3). Assume further that (1.6) holds. Then there exists a positive stationary solution  $\bar{v}$  to the system (1.1).*

Lastly, we obtain the convergence to the equilibria for some general interactions  $\Psi_i$ . Namely, we show that

**Theorem 1.4.** *Assume that  $r_i, \Psi_i$  and  $\mu_{ij}$  satisfy (1.2)–(1.3). Assume further that (1.6) holds and  $\Psi_i(v) = \alpha(v) + \epsilon\psi_i(v)$  with  $\psi_i \in C_{loc}^1$  uniformly bounded and  $\alpha \in C_{loc}^1$  satisfying (1.2)–(1.3) is so that  $\forall x \in \mathbb{R}^N, \nabla\alpha(x) > 0$ . Then there exists  $\epsilon_0$  so that, for all  $\epsilon \leq \epsilon_0$ , the positive stationary solution  $\bar{v}_\epsilon$  of (1.1) attracts all the possible trajectories initiated from any non zero and nonnegative initial data.*

Note that  $C_{loc}^1$  perturbation of the particular interaction function  $\alpha(v) = \sum_{j=1}^N r_j v_j$  satisfied the assumptions of the above Theorem.

### 1.1. General remarks.

Before going to the proofs, we want to make some general remarks and comments.

*Mathematical comments :* First as remarked in [13], we can interpret the system of equations (1.5) as a discrete version of the continuous model

$$(1.7) \quad \frac{\partial v(t, x)}{\partial t} = v(t, x) \left[ r(x) - \frac{1}{K} \int_{\mathbb{R}^n} r(y)v(t, dy) \right] + \int_{\mathbb{R}^n} \mu(x, y)v(t, dy) - \mu(x)v(t, x)$$

by posing for each sub population corresponding to a typical trait  $x_i \in \mathbb{R}^d$ ,

$$v(t, x) = \sum_i^N v_i(t)\delta_{x_i}, \quad r(x_i) = r_i, \quad \mu(x_i, x_j) = \mu_{ij} \text{ and } \mu(x) = \int_{\mathbb{R}^n} \mu(x, y)dy.$$

Recently there have been a lot of works dealing with (1.7) and generalisation of it, see for example [3, 5, 7, 8, 9, 10, 11, 13, 12, 18, 19, 31, 39, 40, 41] and references therein. A large part of the analysis are concerning (1.7) in absence of mutation (ie  $\mu \equiv 0$ ) or in the limit  $\mu \rightarrow 0$ . In the latter case, much have been done in developing a constrained Hamilton-Jacobi approach to analyse the long time behaviour of positive solution of this type of models see for instance [3, 19]. Other approaches based on semigroup theory have also been developed to analyse the asymptotic behaviour and local stability of the stationary solution of (1.7) see [7, 8]. Although some of the techniques developed in this two frameworks may be adapted to analyse the system (1.1) most of them fail when we try to prove the global stability of the stationary states. To tackle this difficulty we construct a set of Lyapunov functionals for the solution of (1.1) when the interactions  $\Psi_i$  are independent of  $i$ . Properly used these Lyapunov functionals enable us to analyse the asymptotic behaviour of the solution of (1.1) in this particular situation. From this analysis, we derive some consideration on the asymptotic behaviour of the solution of (1.1) in a general situation. The Lyapunov functionals are constructed in the spirit of the relative entropy introduced for linear parabolic operators in [35]. Even if our problem is nonlinear such type of relative entropy can still be constructed. These is an interesting new feature of nonlinear dissipative system.

It is worth noticing that a similar construction can be made for (1.7) giving us access to a simple way of analysing the asymptotic behaviour and the global stability of steady solution of (1.7), see [14].

Along some of the proofs we notice that the existence of a steady state in Theorem 1.1 and Theorem 1.3 can be generalised to situation where the mutation matrix  $(\mu_{ij})$  is not symmetric. Indeed, when the interactions  $\Psi_i$  are independent of  $i$  (Theorem 1.1), the construction of a unique stationary solution relies only on the Perron-Frobenius Theorem which holds true for non symmetric matrices. However, for the general case (Theorem 1.3) to obtain the existence of a steady state, we do require that the condition (1.6) is replaced by

$$(1.8) \quad \forall i \quad \sum_{j=1}^N \left( \frac{\mu_{ij} + \mu_{ji}}{2} \right) \leq \frac{r_i}{3}.$$

*Biological comments* : First, we emphasize the biological interpretation of our result and particularly the role of the mutation term. Indeed, under biological compatible assumptions concerning the competition  $\psi_i$ , the mutation matrix  $(\mu_{ij})$  and the growth rate  $r_i$  we have shown that, in sharp contrast with the classical results known for the Lotka-Volterra system without mutation, the mutation term deeply changes the dynamics of the system. On one hand, the mutation term stabilizes the dynamic of system by reducing the number of equilibrium up to a single equilibrium and on the other hand, the mutation term precludes the competitive exclusion principle to occur.

Second, we emphasize the biological relevance of relaxing some hypothesis regarding (i) the monotony of  $\Psi_i$  and (ii) the symmetry of the mutation matrix  $\mu_{ij}$  to study viral demo-genetics dynamics in structured hosts. To illustrate this point, let consider 2 patches  $p_1$  and  $p_2$ , and 2 virus genotypes  $v_1$  and  $v_2$ . The demo-genetics dynamics of the viral population in this system can be modelled by

$$\frac{dw}{dt} = Mw + Rw - \Psi(w)w$$

where  $w$  is the vector  $(v_{1,p_1}, v_{2,p_1}, v_{1,p_2}, v_{2,p_2})$  and  $R, M$  and  $\Psi(w)$  are the following matrices  $R := (r_i \delta_{ij})$ ,

$$M := \begin{pmatrix} \mu & 1-\mu & 0 & 0 \\ 1-\mu & \mu & 0 & 0 \\ 0 & 0 & \mu & 1-\mu \\ 0 & 0 & 1-\mu & \mu \end{pmatrix} + \begin{pmatrix} -d_1 & 0 & d_2 & 0 \\ 0 & -d_1 & 0 & d_2 \\ d_1 & 0 & -d_2 & 0 \\ 0 & d_1 & 0 & -d_2 \end{pmatrix},$$

$$\Psi(w) := \begin{pmatrix} r_1 v_{1,p_1} + r_2 v_{2,p_1} & 0 & 0 & 0 \\ 0 & r_1 v_{1,p_1} + r_2 v_{2,p_1} & 0 & 0 \\ 0 & 0 & r_1 v_{1,p_2} + r_2 v_{2,p_2} & 0 \\ 0 & 0 & 0 & r_1 v_{1,p_2} + r_2 v_{2,p_2} \end{pmatrix}.$$

In structured hosts some tissue types often act as virus "sources" and others are "sinks", creating an asymmetry in the exchange between patches. In the above example this implies that  $d_1 > d_2$  if the patch  $p_1$  is a "source" and the patch  $p_2$  is a "sink", making the mutation matrix  $M$  non-symmetric. Moreover, in this example since the competition takes place only inside a given patch, we can check that the two monotone interaction functionals involved  $\Psi_1(v) := r_1 v_{1,p_1} + r_2 v_{2,p_1}$  and  $\Psi_2(v) := r_1 v_{1,p_2} + r_2 v_{2,p_2}$  do not satisfy the monotone properties (1.2).

Such type of structure, inducing asymmetries in the exchanges and some weak interaction functionals, are expected to play a role in constraining or facilitating adaptive evolution of viruses in heterogeneous host environment [38]. It is then relevant to study extension of our results in a context of more general assumptions on  $\mu_{ij}$  and  $\Psi_i$ .

*Organization of the Paper* : In Section 2 we start by establishing some preliminaries results about the system (1.1). We derive the relative entropy identities in 3 Then in Section 4 we analyse in details the system (1.1) for a particular type of interactions and we prove the Theorem 1.1. We show Theorem 1.2 in Section 5. Finally, we are proving the Theorems 1.3 and 1.4 in Sections 6 and 7.

## 2. GLOBAL FACTS ON THE SYSTEM (1.1)

In this section we establish some useful properties of solution of (1.1) and prove the existence of a positive global in time solution of the system (1.1), that for convenience we rewrite

$$(2.1) \quad \frac{dv}{dt} = A(\Psi(t))v$$

where  $A(\Psi(t))$  is the following matrix:

$$A := \begin{pmatrix} (r_1 - \frac{1}{K}\Psi_1(t)) - \mu_1 + \mu_{11} & & \mu_{1j} \\ & \ddots & \\ \mu_{ij} & & (r_N - \frac{1}{K}\Psi_N(t)) - \mu_N + \mu_{NN} \end{pmatrix},$$

with  $\Psi_i(t) := \Psi_i(v(t))$  and  $\mu_i = \sum_{j=1}^N \mu_{ij}$ . Since the function  $\Psi_i$  are locally Lipschitz the local existence of a solution of (2.1) is a straightforward application of the Cauchy-Lipschitz Theorem. To obtain a global solution starting with non-negative initial data, we need more *a priori* estimates on the solutions  $(v_i)_{i \in \{1, \dots, N\}}$ . Let us first show that the system (2.1) preserve the positivity.

### 2.1. Positivity.

**Lemma 2.1** (Positivity). *The system (2.1) preserves the positivity. That is to say that if  $v(0)$  is a non-negative initial value, then the solution  $v$  of (2.1) stays nonnegative. Moreover, for all  $i$   $v_i(t) > 0$  for all times  $t \in (0, T]$ .*

#### Proof:

We argue by contradiction and assume that  $v$  changes its sign. Let  $t_1 \in (0, T]$  be the first time where  $v_i(t_1) = 0$  for some  $i$ ,  $v(t) > 0$  for all  $t \in (0, t_1)$  and there exists  $t > t_1$  so that  $v(t) < 0$ .  $t_1 > 0$  is well defined since  $v$  changes its sign and  $v_i(0) \geq 0$  for all  $i$ ,  $v_i(0) > 0$  for some  $i$  and  $\lim_{t \rightarrow 0^+} \frac{dv_i(t)}{dt} > 0$ . Now at the time  $t_1$ , we have

$$0 \geq \frac{dv_i(t_1)}{dt} = \sum_j \mu_{ij} v_j(t_1) \geq 0.$$

Therefore for all  $i$   $v_i(t_1) = 0 = \frac{dv_i(t_1)}{dt}$  and by the Cauchy-Lipschitz Theorem we deduce that for all  $i$  and all  $t \in [t_1, T]$   $v_i(t) = 0$ . Thus  $v \geq 0$  in  $(0, T]$  which contradicts that  $v$  changes its sign.

Next we show that  $v_i(t) > 0$  for all times  $t \in (0, T]$ . From the above argumentation,  $v$  is nonnegative for all times. To show that  $v$  stays positive for all time, we can see that it is enough to show that  $\mathcal{N}(t) = \sum_{i=1}^N v_i(t) > 0$  for all times  $t \in [0, T]$ . Indeed, if there is  $t_1 \in (0, T]$  so that  $t_1$  is the first time where for some  $i$ ,  $v_i(t_1) = 0$  then arguing as above we see that for all  $i$  and  $\mathcal{T} \geq t_1$   $v_i(t) = 0$  so  $\mathcal{N}(t) = 0$  for all  $t \geq t_1$ .

Now, let us denote  $Q(0, 1)$  the following set

$$Q(0, 1) := \left\{ x \in \mathbb{R}^N \mid \sum_{i=1}^N |x_i| \leq 1 \right\}.$$

Since  $\Psi_i$  is locally Lipschitz, we have on  $Q(0, 1)$ ,

$$(2.2) \quad \forall i, \exists \kappa_i \text{ so that } \Psi_i(x) \leq \kappa_i \sum_{i=1}^N |x_i|.$$

Let  $\kappa_0 := \sum_{i=1}^N \kappa_i$  and  $\delta$  a positive constant so that  $\delta < \min \left\{ 1, \frac{\mathcal{N}(0)}{2}, \frac{r_{min}}{2\kappa_0} \right\}$ . We will prove that  $\mathcal{N} \geq \delta$  for all times in  $(0, T]$ . We argue by contradiction and assume there exists  $t_1 < t_2$  so that  $\mathcal{N}(t_1) = \delta$  and  $\mathcal{N}(t) \leq \delta$  in  $(t_1, t_2]$ . Using that the  $v_i$  are non negative functions and (2.2), from our assumption we deduce that on  $[t_1, t_2]$  and for all  $i$ ,

$$\frac{dv_i}{dt} \geq v_i (r_{min} - \kappa_i \mathcal{N}) + \sum_{j=1}^N \mu_{ij} (v_j - v_i).$$

By summing over all the possible  $i$  we end up with the following equation

$$(2.3) \quad \frac{d\mathcal{N}}{dt} \geq \mathcal{N}(r_{min} - \mathcal{N}\kappa_0),$$

$$(2.4) \quad \mathcal{N}(t_1) = \delta.$$

Using the comparison principle on the latter equation, we achieve on  $[t_1, t_2]$ ,  $\mathcal{N} \geq \tilde{\mathcal{N}}$  where  $\tilde{\mathcal{N}}$  is the solution of the logistic type equation (2.3). By construction we have  $\tilde{\mathcal{N}} > \delta$  for all times  $t > t_1$ . Thus we get the contradiction  $\delta < \mathcal{N} \leq \delta$ . Hence  $\mathcal{N} > \delta$  for all  $t \in (0, T]$ .  $\square$

*Remark 2.2.* Of the above proof, we also have a bound from below for  $\mathcal{N}$ . Namely, we have for all  $t$

$$\mathcal{N}(t) \geq \min \left\{ 1, \frac{\mathcal{N}(0)}{2}, \frac{r_{min}}{2\kappa_0} \right\}.$$

## 2.2. Existence of a global solution.

Next we show the following estimate an

**Lemma 2.3.** *Let  $v(t)$  be a solution of the Cauchy problem (2.1), then there exists two constants  $C_0$  and  $C_1$  and a positive vector  $v_p$  so that  $v(t) \leq C_0 e^{C_1 t} v_p$*

**Proof:**

From Lemma 2.1 we know that the  $v_i$  are non negative functions. Therefore by (2.1) we can see that the  $v_i$  satisfy

$$(2.5) \quad \frac{dv_i}{dt} \leq r_i v_i + \sum_{j=1}^N \mu_{ij} (v_j - v_i).$$

Let us denote  $M$  and  $R$  the two following matrices

$$M := \begin{pmatrix} -\mu_1 + \mu_{11} & & \mu_{1j} \\ & \ddots & \\ \mu_{ij} & & -\mu_N + \mu_{NN} \end{pmatrix}, \quad R := \begin{pmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_N \end{pmatrix}.$$

We can then rewrite the inequalities (2.5) as follows

$$(2.6) \quad \frac{dv}{dt} \leq (R + M)v.$$

By choosing  $\bar{\mu} = \sup_{i \in \{1, \dots, N\}} \mu_i$ , we can see that  $R + M + \bar{\mu} Id$  is nonnegative matrix. Since  $R + M + \bar{\mu} Id$  is also irreducible, by the Perron-Frobenius Theorem  $R + M + \bar{\mu} Id$  posses a unique principal eigenpair  $(\nu_p, v_p)$  so that  $v_p$  is a positive vector, i.e. there exists  $(\nu_p, v_p)$  so that  $v_p > 0$  and

$$(R + M + \bar{\mu} Id)v_p = \nu_p v_p.$$

Note that  $e^{(\nu_p - \bar{\mu})t} \|v(0)\|_{\infty} v_p$  satisfies

$$\frac{du}{dt} = (R + M)u.$$

So, by the Cauchy-Lipschitz Theorem, from (2.6) we deduce that  $v \leq e^{(\nu_p - \bar{\mu})t} \|v(0)\|_{\infty} v_p$  for all times  $t$ .  $\square$



The existence of a global in time solution for the system (2.1) is then a consequence of the Cauchy-Lipschitz Theorem and the above a priori estimates.

*Remark 2.4.* By adapting some ideas in [39] and for a particular type of  $\Psi_i$ , we can derive an explicit formula for the solution of (2.1). Indeed when the interaction terms take the form  $\Psi_i(v) = \alpha(v) = \sum_{j=1}^N \alpha_j v_j$  for all  $i$  then the solution  $v_i(t)$  can be expressed by the formula

$$(2.7) \quad v_i(t) = \frac{(e^{(R+M)t}v(0))_i}{1 + \sum_{j=1}^N \alpha_j \int_0^t (e^{(R+M)s}v(0))_j ds}.$$

To obtain this formula we argue as in [39] (Chapter 2 Section 2.1) and we start by introducing the functions

$$\alpha(t) := \sum_{j=1}^N \alpha_j v_j \quad \text{and} \quad V_i(t) := e^{\int_0^t \alpha(s) ds} v_i(t).$$

We remark that the  $V_i$  satisfy the linear equation

$$(2.8) \quad \frac{dV_i(t)}{dt} = r_i V_i + \sum_{j=1}^N \mu_{ij} (V_j - V_i).$$

Thus  $V_i(t) := (e^{(R+M)t}v(0))_i$  and  $v_i(t)$  is implicitly given by the formula

$$v_i(t) = e^{-\int_0^t \alpha(s) ds} (e^{(R+M)t}v(0))_i.$$

Now let us evaluate the term  $e^{-\int_0^t \alpha(s) ds}$ . By differentiating  $e^{-\int_0^t \alpha(s) ds}$  one has

$$\frac{d}{dt} (e^{\int_0^t \alpha(s) ds}) = \alpha(t) e^{\int_0^t \alpha(s) ds} = \sum_{j=1}^N \alpha_j V_j(t).$$

Therefore we have

$$e^{\int_0^t \alpha(s) ds} = 1 + \int_0^t \sum_{j=1}^N \alpha_j V_j(s) ds = 1 + \sum_{j=1}^N \alpha_j \int_0^t (e^{(R+M)s}v(0))_j ds.$$

Hence

$$v_i(t) = \frac{(e^{(R+M)t}v(0))_i}{1 + \sum_{j=1}^N \alpha_j \int_0^t (e^{(R+M)s}v(0))_j ds}.$$

□

### 3. RELATIVE ENTROPY IDENTITIES.

Here, we prove the following general principle which give us access to some useful identifies that we constantly use along this paper.

**Theorem 3.1.** *Assume that  $r_i, (\mu_{ij}), \Psi_i$  satisfies (1.2)– (1.3). Let  $v$  and  $\bar{v}$  be respectively a positive solution and a positive stationary solution of (1.1), and let  $H$  be a smooth (at least  $C^1$ ) function. Then the function  $\mathcal{H}(v) := \sum_{i=1}^N \bar{v}_i^2 H\left(\frac{v_i}{\bar{v}_i}\right)$  satisfies*

$$(3.1) \quad \frac{d\mathcal{H}(t)}{dt} = -\mathcal{D}(v) + \frac{1}{K} \sum_{i=1}^N \bar{v}_i H' \left( \frac{v_i}{\bar{v}_i} \right) \Gamma_i v_i$$

where

$$\mathcal{D}(v) := \sum_{i,j=1}^N \mu_{ij} \bar{v}_i \bar{v}_j \left[ H\left(\frac{v_j}{\bar{v}_j}\right) - H\left(\frac{v_i}{\bar{v}_i}\right) \right] + \sum_{i,j=1}^N \mu_{ij} \bar{v}_i \bar{v}_j H'\left(\frac{v_i}{\bar{v}_i}\right) \left[ \frac{v_i}{\bar{v}_i} - \frac{v_j}{\bar{v}_j} \right]$$

$$\Gamma_i := \Psi_i(\bar{v}) - \Psi_i(v)$$

**Proof:**

By (1.1), for all  $i$  we have

$$(3.2) \quad \frac{dv_i}{dt} = \left( r_i v_i - \frac{1}{K} \Psi_i(\bar{v}) v_i + \sum_{j=1}^N \mu_{ij} (v_j - v_i) \right) + \frac{1}{K} \Gamma_i v_i$$

Using that  $\bar{v}$  is a stationary solution, we have for all  $i$

$$\left( r_i - \frac{1}{K} \Psi_i(\bar{v}) \right) \bar{v}_i = - \sum_{j=1}^N \mu_{ij} (\bar{v}_j - \bar{v}_i),$$

and we can rewrite the above equation as follows

$$\frac{dv_i}{dt} = \sum_{j=1}^N \mu_{ij} \left( v_j - \frac{\bar{v}_j v_i}{\bar{v}_i} \right) + \frac{1}{K} \Gamma_i v_i.$$

By multiplying the above equality by  $\bar{v}_i H'\left(\frac{v_i}{\bar{v}_i}\right)$  and by summing over all  $i$  we achieve

$$\sum_{i=1}^N \bar{v}_i H'\left(\frac{v_i}{\bar{v}_i}\right) \frac{dv_i}{dt} = \frac{1}{K} \sum_{i=1}^N \bar{v}_i H'\left(\frac{v_i}{\bar{v}_i}\right) \Gamma_i v_i + \sum_{i=1}^N \bar{v}_i H'\left(\frac{v_i}{\bar{v}_i}\right) \sum_{j=1}^N \mu_{ij} \left( v_j - \frac{\bar{v}_j v_i}{\bar{v}_i} \right).$$

Thus we have

$$\frac{d\mathcal{H}(t)}{dt} = \frac{1}{K} \sum_{i=1}^N \bar{v}_i H'\left(\frac{v_i}{\bar{v}_i}\right) \Gamma_i v_i - \sum_{i,j=1}^N \mu_{ij} \bar{v}_i \bar{v}_j H'\left(\frac{v_i}{\bar{v}_i}\right) \left( \frac{v_i}{\bar{v}_i} - \frac{v_j}{\bar{v}_j} \right).$$

Hence we have

$$\frac{d\mathcal{H}(t)}{dt} = \frac{1}{K} \sum_{i=1}^N \bar{v}_i H'\left(\frac{v_i}{\bar{v}_i}\right) \Gamma_i v_i - \mathcal{D}(v).$$

since by symmetry of  $\mu_{ij}$ ,

$$\sum_{i,j=1}^N \mu_{ij} \bar{v}_i \bar{v}_j \left[ H\left(\frac{v_j}{\bar{v}_j}\right) - H\left(\frac{v_i}{\bar{v}_i}\right) \right] = 0.$$

□

As immediate corollary of the Theorem 3.1, we have the following identities that we constantly used along this paper,

**Lemma 3.2.** *Assume that  $r_i, (\mu_{ij}), \Psi_i$  satisfies (1.2)– (1.3). Let  $v$  and  $\bar{v}$  be respectively a positive solution and a positive stationary solution of (1.1), then we have*

(i)

$$\frac{d}{dt} \left( \sum_{i=1}^N v_i \bar{v}_i \right) = \frac{1}{K} \sum_{i=1}^N (\Psi_i(\bar{v}) - \Psi_i(v)) v_i \bar{v}_i$$

(ii)

$$\frac{d}{dt} \left( \sum_{i=1}^N v_i^2 \right) = - \sum_{i,j=1}^N \mu_{ij} \bar{v}_i \bar{v}_j \left( \frac{v_j}{\bar{v}_j} - \frac{v_i}{\bar{v}_i} \right)^2 + \frac{2}{K} \sum_{i=1}^N v_i^2 (\Psi_i(\bar{v}) - \Psi_i(v)).$$

**Proof:**

(i) and (ii) can be obtained straightforwardly from the Theorem 3.1. Indeed, by using  $H(s) = s$  in Theorem 3.1, and by observing that from the symmetry of  $\mu_{ij}$ ,

$$\sum_{i,j=1}^N \mu_{ij} \bar{v}_i \bar{v}_j \left[ \frac{v_i}{\bar{v}_i} - \frac{v_j}{\bar{v}_j} \right] = 0,$$

we easily get

$$\frac{d\mathcal{H}(t)}{dt} = \frac{1}{K} \sum_{i=1}^N \Gamma_i \bar{v}_i v_i,$$

which proves (i) since for the function  $H(s) = s$ , we have  $\mathcal{H}(t) = \sum_{i=1}^N \bar{v}_i v_i$ . We can obtain (ii) in a similar way by using the function  $H(s) = s^2$  in the Theorem 3.1 and by observing that

$$\sum_{i,j=1}^N \mu_{ij} \bar{v}_i \bar{v}_j \left( 2 \frac{v_i}{\bar{v}_i} \right) \left[ \frac{v_i}{\bar{v}_i} - \frac{v_j}{\bar{v}_j} \right] = \sum_{i,j=1}^N \mu_{ij} \bar{v}_i \bar{v}_j \left[ \frac{v_i}{\bar{v}_i} - \frac{v_j}{\bar{v}_j} \right]^2.$$

□

#### 4. THE SPECIAL CASE $\Psi_i(\cdot) = \alpha(\cdot)$ :

In this section we analyse in details the asymptotic behaviour of a positive solution of (1.1) when the competition functional  $\Psi_i$  is independent of  $i$ , i.e for all  $i$ ,  $\Psi_i(v) = \alpha(v)$  where  $\alpha$  is a function from  $\mathbb{R}^N \rightarrow \mathbb{R}$  which satisfies (1.2)–(1.3). As we expressed in Theorem 1.1 that we recall below, in this situation the system (1.1) has a unique positive stationary solution which attracts all the trajectories initiated from any nonnegative and non zero initial data. More precisely, we prove

**Theorem 4.1.** *Assume that for all  $i$  the function  $\Psi_i(\cdot) = \alpha(\cdot)$ , then there exists a unique stationary solution  $\bar{v}$  of (2.1). Moreover, for all nonnegative and non zero initial datum  $v(0)$ , the corresponding solution  $v(t)$  converges to  $\bar{v}$ .*

To prove the Theorem, we first analyse the existence of stationary solution

##### 4.1. Study of the existence of equilibria.

Recall that we look for a stationary solution of

$$(4.1) \quad \frac{dv}{dt} = A(\alpha(t))v.$$

Therefore if there exists a stationary equilibria for the system (4.1) the  $v_i$  must satisfies the following equations:

$$(4.2) \quad A(\bar{\alpha})v = 0$$

$$(4.3) \quad \bar{\alpha} = \alpha(v)$$

where  $A(\bar{\alpha})$  is the matrix

$$A(\bar{\alpha}) := \begin{pmatrix} (r_1 - \frac{\bar{\alpha}}{K}) - \mu_1 + \mu_{11} & & \mu_{1j} \\ & \ddots & \\ \mu_{ij} & & (r_N - \frac{\bar{\alpha}}{K}) - \mu_N + \mu_{NN} \end{pmatrix}.$$

Note that we can rewrite the matrix  $A(\bar{\alpha}) := (R - (\frac{\bar{\alpha}}{K}) Id + M)$  where  $R$  and  $M$  are matrices defined in Section 2.

Therefore, a solution  $v$  to (4.2) is a solution to

$$(4.4) \quad (M + R)v = \left( \frac{\bar{\alpha}}{K} \right) v.$$

Let us now establish some important property of the equilibrium.

**Lemma 4.2.** *If  $v$  is a nonnegative stationary solution of (4.1), then either  $v \equiv 0$  or  $v > 0$  (i.e.  $\forall i, v_i > 0$ ).*

**Proof:**

First, we observe that 0 is a solution of the problem (4.2). Now let  $v$  be nonnegative stationary solution of (4.1) so that  $v_i = 0$  for some  $i$ . From (4.4) we see that

$$(M + Rv)_i = \left(\frac{\bar{\alpha}}{K}\right) v_i = 0.$$

Therefore

$$0 = r_i v_i + \sum_{j=1}^N \mu_{ij} (v_j - v_i) = \sum_{j=1}^N \mu_{ij} v_j.$$

Thus  $v_j = 0$  for all  $j$  since by assumption  $v_j \geq 0$  and  $(\mu_{ij})$  is irreducible. Hence any nonnegative stationary solution is either positive or the zero solution.  $\square$

Observe that from the above Lemma and (4.4), one can see that  $v$  is a positive eigenvector of the matrix  $M + R$  associated to the eigenvalue  $\frac{\bar{\alpha}}{K}$ . Now we are in position to prove that there exists a unique positive stationary solution to (4.4).

**Lemma 4.3.** *There exists a unique  $(\bar{\alpha}, \bar{v})$  solution of the equations (4.2) and (4.3). Moreover  $v$  satisfies  $\alpha(v) = \nu_p$ .*

**Proof:**

By choosing  $\bar{\mu} = \sup_{i \in \{1, \dots, N\}} \mu_{ii}$ , we can see that  $R + M + \bar{\mu}$  is nonnegative matrix. Since  $R + M + \bar{\mu}Id$  is also irreducible, by the Perron-Frobenius Theorem  $R + M + \bar{\mu}Id$  posses a unique principal eigenpair  $(\nu_p, v_p)$  so that  $v_p$  is a positive vector, i.e. there exists  $(\nu_p, v_p)$  so that  $v_p > 0$  and

$$(4.5) \quad (R + M + \bar{\mu}Id)v_p = \nu_p v_p.$$

Moreover, the linear subspace associated to the eigenvalue  $\nu_p$  is one dimensional [46]. So without any loss of generality we can assume that  $\sum_{i=1}^N (v_p)_i^2 = 1$ . From the equation (4.5) we deduce that the vector  $v_p$  is a positive eigenvector of the matrix  $M + R$  associated with the eigenvalue  $\lambda_p := (\nu_p - \bar{\mu})$ . By construction one can see that  $\lambda_p$  is the unique eigenvalue of the matrix  $M + R$  associated with a positive eigenvector. A quick computation shows that  $\lambda_p = (\nu_p - \bar{\mu}) > 0$ . Indeed, if not we have

$$(R + M)v_p \leq 0.$$

Thus for all  $i \in \{1, \dots, N\}$  we have

$$r_i (v_p)_i + \sum_{j=1}^N \mu_{ij} ((v_p)_j - (v_p)_i) \leq 0.$$

Let  $(v_p)_{i_0} := \min_{i \in \{1, \dots, N\}} (v_p)_i$  then for  $(v_p)_{i_0}$  we have

$$\sum_{j=1}^N \mu_{ij} ((v_p)_j - (v_p)_{i_0}) \geq 0.$$

Since  $R$  is a positive matrix we achieve the contradiction

$$0 < r_{i_0} (v_p)_{i_0} + \sum_{j=1}^N \mu_{ij} ((v_p)_j - (v_p)_{i_0}) \leq 0.$$

Now from (4.4) we deduce that there exists an unique positive  $\bar{\alpha}$  so that  $\frac{\bar{\alpha}}{K} = \lambda_p$ . Let us now construct our solution. Note that for any  $\mu \in \mathbb{R}$ , the vector  $\mu v_p$  is also a solution to (4.4) with the eigenvalue  $\lambda_p$ . So to obtain a solution  $\bar{v}$  to (4.2) and (4.3) we only have to adjust  $\mu$  in such a way that  $\alpha(\mu v_p) = \bar{\alpha}$ . This is always possible for a unique  $\mu$  since  $\alpha(0) = 0$ ,  $\lim_{\mu \rightarrow \infty} \alpha(\mu v_p) = +\infty$  and  $\alpha$  is an increasing function.  $\square$

#### 4.2. Convergence to the unique equilibrium.

Let us look at the convergence of  $v(t)$  toward its equilibrium. Let us first establish some useful identities.

**Lemma 4.4.** *Let us denote  $(\bar{\alpha}, \bar{v})$  the stationary solution constructed above. Let  $v$  be a solution of (4.1) then  $v$  satisfies the following identities*

(i)

$$\frac{d}{dt} \left( \sum_{i=1}^N v_i \bar{v}_i \right) = \frac{1}{K} (\bar{\alpha} - \alpha(t)) \left( \sum_{i=1}^N v_i \bar{v}_i \right).$$

(ii)

$$\frac{d}{dt} \sum_{i=1}^N (v_i)^2 = - \sum_{i,j=1}^N \mu_{ij} \bar{v}_i \bar{v}_j \left( \frac{v_j}{\bar{v}_j} - \frac{v_i}{\bar{v}_i} \right)^2 + \frac{2}{K} (\bar{\alpha} - \alpha(t)) \sum_{i=1}^N (v_i)^2.$$

**Proof:**

Since here for all  $i$ ,  $\Psi_i = \alpha$ , from Lemma 3.2 we deduce that

$$\begin{aligned} \frac{d}{dt} \left( \sum_{i=1}^N v_i \bar{v}_i \right) &= \frac{1}{K} \left( \sum_{i=1}^N (\alpha(\bar{v}) - \alpha(v(t))) v_i \bar{v}_i \right), \\ \frac{d}{dt} \sum_{i=1}^N (v_i)^2 &= - \sum_{i,j=1}^N \mu_{ij} \bar{v}_i \bar{v}_j \left( \frac{v_j}{\bar{v}_j} - \frac{v_i}{\bar{v}_i} \right)^2 + \frac{2}{K} (\bar{\alpha} - \alpha(t)) \sum_{i=1}^N (\alpha(\bar{v}) - \alpha(v)) (v_i)^2. \end{aligned}$$

Thus (i) and (ii) hold true. □

From the above Lemma we can derive a useful Lyapunov functional.

**Lemma 4.5.** *Let  $\bar{v}$  be the positive stationary solution of (4.1). For any positive solution  $v$  of (4.1), let us denote  $\mathcal{E}$ ,  $\beta$  and  $F$  the following quantities*

$$\mathcal{E}(v) := \sum_{i=1}^N (v_i)^2, \quad \beta(v) := \sum_{i=1}^N v_i \bar{v}_i, \quad F(v) := \log \left[ \frac{\mathcal{E}(v)}{(\beta(v))^2} \right].$$

Then  $F \geq \log\left(\frac{1}{\sup_i \bar{v}_i}\right)$  and for any positive solution  $v$  of (4.1) we have

$$\frac{d}{dt} F(v) = - \frac{1}{\mathcal{E}(v)} \sum_{i,j=1}^N \mu_{ij} \bar{v}_j \bar{v}_i \left( \frac{v_j}{\bar{v}_j} - \frac{v_i}{\bar{v}_i} \right)^2 < 0.$$

**Proof:**

First let us show that  $F$  is bounded from below. By construction and using a standard convexity argument, we see that  $\beta^2(v) \leq (\sup_i \bar{v}_i) \mathcal{E}(v)$ . So from the monotonicity of the log, we conclude that  $F(v) \geq \log\left(\frac{1}{\sup_i \bar{v}_i}\right)$ .

Now let us show that  $F$  is non increasing. To do so let us compute  $\frac{d}{dt} F(v)$ . From the definition of  $F$  we see that

$$\frac{d}{dt} F(v) = \frac{d_t \mathcal{E}(v)}{\mathcal{E}(v)} - 2 \frac{d_t \beta(v)}{\beta(v)},$$

where  $d_t$  denotes  $\frac{d}{dt}$ . By Lemmas 4.4 we have

$$\begin{aligned} \frac{d_t \mathcal{E}(v)}{\mathcal{E}(v)} &= - \frac{1}{\mathcal{E}(v)} \sum_{i,j=1}^N \mu_{ij} \bar{v}_j \bar{v}_i \left( \frac{v_j}{\bar{v}_j} - \frac{v_i}{\bar{v}_i} \right)^2 + \frac{2}{K} (\bar{\alpha} - \alpha(t)) \\ \frac{d_t \beta(v)}{\beta(v)} &= \frac{1}{K} (\bar{\alpha} - \alpha(t)). \end{aligned}$$

Thus,

$$\frac{d}{dt}F(v) = -\frac{1}{\mathcal{E}(v)} \sum_{i,j=1}^N \mu_{ij} \bar{v}_j \bar{v}_i \left( \frac{v_j}{\bar{v}_j} - \frac{v_i}{\bar{v}_i} \right)^2 \leq 0.$$

□

Next we derive some *a priori* estimates on the solution  $v$  of (4.1) from the previous Lemmas. Namely, we show that

**Lemma 4.6.** *Let  $v$  be a positive solution of (4.1), then there exists  $C_1$  so that*

$$\mathcal{E}(v) + \beta(v) \leq C_1.$$

**Proof:**

Observe that from Lemma 4.5, to obtain the bound it is sufficient to have a uniform bound on  $\beta$ . Indeed, since  $F$  is decreasing in time, we have for all times

$$\mathcal{E}(v) \leq \beta^2(t) \frac{\mathcal{E}(v(0))}{\beta^2(v(0))}.$$

Now recall that by Lemma 4.4,  $\beta(v)$  satisfies the following equation

$$\frac{d\beta(t)}{dt} = \frac{1}{K}(\bar{\alpha} - \alpha(t))\beta(t).$$

Since  $\alpha$  satisfies the assumptions (1.2)–(1.3), there exists  $R_\alpha, c_\alpha, k_\alpha$  so that for all  $x \in \mathbb{R}^{N,+} \setminus Q_{R_\alpha}(0)$ ,

$$(4.6) \quad c_\alpha \left( \sum_{i=1}^N x_i \right)^{k_\alpha} \leq \alpha(x).$$

Assume that for some  $t > 0$ ,  $v(t) \in \mathbb{R}^{N,+} \setminus \bar{Q}_{R_\alpha}(0)$  otherwise we are done since

$$\beta(v) \leq \left( \max_{i \in \{1, \dots, N\}} \bar{v}_i \right) \sum_{j=1}^N |v_j|.$$

Let  $\Sigma$  be

$$\Sigma := \{t \in \mathbb{R}^+ \mid \mathcal{N}(t) > R_\alpha\}.$$

From (4.6) and using that the  $v_i$  are non negative, we see that for all  $t \in \Sigma$

$$\alpha(t) \geq c_\alpha \mathcal{N}(t)^{k_\alpha} \geq c_\alpha \left( \frac{1}{\bar{v}_{max}} \right)^{k_\alpha} \beta(t)^{k_\alpha},$$

where  $\bar{v}_{max} := \max_{i \in \{1, \dots, N\}} \bar{v}_i$ . Therefore, on  $\Sigma$  we have

$$\frac{d\beta(t)}{dt} \leq \frac{1}{K} (\bar{\alpha} - \tilde{c}_0 \beta^{k_\alpha}(t)) \beta(t).$$

Using the logistic character of the above equation, we can check that

$$\beta(v(t)) \leq \sup \left\{ \beta(v(0)), \max_{x \in \bar{Q}_{R_\alpha}(0)} \beta(x), \left( \frac{\bar{\alpha}}{\tilde{c}_0} \right)^{\frac{1}{k_\alpha}} \right\}.$$

□

We are now in position to prove the convergence of  $v$  toward its equilibrium.

**Lemma 4.7.** *Let  $(\bar{v}_i)_{i=1 \dots N}$  be the unique stationary solution of (4.1). Then for any non negative initial datum  $(v_i(0))_{i=1 \dots N}$  not identically zero, the corresponding solution  $(v_i(t))_{i=1 \dots N}$  of (4.1) converges to  $(\bar{v}_i)_{i=1 \dots N}$  as  $t$  goes to infinity.*

**Proof:**

For simplicity we denote  $\langle, \rangle$  the standard scalar product in  $\mathbb{R}^N$ .

Now, since  $\bar{v} \neq 0$  and for all times  $t$ ,  $v(t) = (v_1, \dots, v_N)$  is a vector of  $\mathbb{R}^N$ , we can write  $v(t) := \lambda(t)\bar{v} + h(t)$  with for all  $t$ ,  $\langle h(t), \bar{v} \rangle = 0$ . Substituting  $v$  by this decomposition in (4.1), it follows that

$$(4.7) \quad \lambda'(t)\bar{v} + \frac{dh(t)}{dt} = \lambda(t)A(\alpha(t))\bar{v} + A(\alpha(t))h,$$

$$(4.8) \quad = \frac{1}{K}(\bar{\alpha} - \alpha(t))\lambda(t)\bar{v} + A(\alpha(t))h.$$

Therefore, we have

$$\langle \lambda'(t)\bar{v} + \frac{dh(t)}{dt}, h \rangle = \langle \frac{1}{K}(\bar{\alpha} - \alpha(t))\bar{v} + A(\alpha(t))h, h \rangle.$$

Thus

$$\langle \frac{dh(t)}{dt}, h \rangle = \frac{1}{2} \frac{d\mathcal{E}(h)}{dt} = \langle A(\alpha(t))h, h \rangle.$$

By following the computation developed for the proof of (ii) in Lemma 4.4, we see that

$$\frac{d\mathcal{E}(h)}{dt} = - \sum_{i,j=1}^N \mu_{ij} \bar{v}_i \bar{v}_j \left( \frac{h_j}{\bar{v}_j} - \frac{h_i}{\bar{v}_i} \right)^2 + \frac{2}{K}(\bar{\alpha} - \alpha(t))\mathcal{E}(h).$$

Since  $\mathcal{E}(h) \geq 0$  for all times, we will analyse separately two situations: Either  $\mathcal{E}(h(t)) > 0$  for all times  $t$  or there exists  $t_0 \in \mathbb{R}$  so that  $\mathcal{E}(h(t_0)) = 0$ . In the latter case, from the above equation we see that we must have  $\mathcal{E}(h(t)) = 0$  for all  $t \geq t_0$  and so for all  $t \geq t_0$ , we must have  $v(t) = \lambda(t)\bar{v}$ . Hence from (4.8) we are reduced to analyse the following ODE equation

$$\lambda'(t) = \frac{\lambda(t)}{K}(\bar{\alpha} - \tilde{\alpha}(\lambda(t)))$$

where  $\tilde{\alpha}$  is the increasing locally Lipschitz function defined by  $\tilde{\alpha}(s) := \alpha(s\bar{v})$ . Note that since  $\lambda(t) \langle \bar{v}, \bar{v} \rangle = \beta(v) > 0$ , we have  $\lambda(t) \geq 0$  for all times  $t$ . The above ODE is of logistic type with non negative initial datum therefore by a standard argumentation we see that  $\lambda(t)$  converges to  $\bar{\lambda} > 0$  where  $\bar{\lambda}$  is the unique solution of  $\tilde{\alpha}(\bar{\lambda}) = \bar{\alpha}$ . By construction we have  $\tilde{\alpha}(1) = \bar{\alpha}$ , so we deduce that  $\bar{\lambda} = 1$ . Hence, in this situation,  $v$  converges to  $\bar{v}$  as time goes to infinity.

In the other situation,  $\mathcal{E}(h(t)) > 0$  for all  $t$  and we claim that

**Claim 4.8.**  $\mathcal{E}(h(t)) \rightarrow 0$  as  $t \rightarrow +\infty$ .

Assume the Claim holds true then we can conclude the proof by arguing as follows. From the decomposition  $v(t) = \lambda(t)\bar{v} + h(t)$ , we can express the function  $\beta(v(t))$  by  $\beta(v(t)) = \langle v, \bar{v} \rangle = \lambda(t) \langle \bar{v}, \bar{v} \rangle$ . Therefore from Lemma 4.4 we deduce that

$$(4.9) \quad \lambda'(t) = \frac{1}{K}(\bar{\alpha} - \alpha(\lambda(t)\bar{v} + h(t)))\lambda(t).$$

Now by using  $\mathcal{E}(h) \rightarrow 0$ , we deduce that  $h \rightarrow 0$  as  $t \rightarrow \infty$  and from (4.9) we are reduced to analyse the ODE

$$\begin{aligned} \lambda'(t) &= \frac{1}{K}(\bar{\alpha} - \tilde{\alpha}(\lambda(t)))\lambda(t) + \lambda(t)(\tilde{\alpha}(\lambda(t)\bar{v}) - \alpha(\lambda(t)\bar{v} + h(t))) \\ &= \frac{1}{K}(\bar{\alpha} - \tilde{\alpha}(\lambda(t)))\lambda(t) + \lambda(t)o(1) \end{aligned}$$

where  $|o(1)| = |\tilde{\alpha}(\lambda(t)\bar{v}) - \alpha(\lambda(t)\bar{v} + h(t))| \leq C\sqrt{\mathcal{E}(h)} \rightarrow 0$  as  $t \rightarrow \infty$ . As before we can conclude that  $\lambda(t) \rightarrow 1$  and  $v$  converges to  $\bar{v}$ . □

**Proof of Claim 4.8:**

Since  $\mathcal{E}(h(t)) > 0$  for all  $t$ , as in Lemma 4.5 we have

$$(4.10) \quad \frac{d}{dt} \log \left[ \frac{\mathcal{E}(h)}{(\beta(v))^2} \right] = -\frac{1}{\mathcal{E}(h)} \sum_{i,j=1}^N \mu_{ij} \bar{v}_i \bar{v}_j \left( \frac{h_j}{\bar{v}_j} - \frac{h_i}{\bar{v}_i} \right)^2.$$

Thus the function  $\tilde{F} := \log \left[ \frac{\mathcal{E}(h)}{(\beta(v))^2} \right]$  is a decreasing smooth function.

First we observe that the claim is proved if there exists a sequence  $(t_n)_{n \in \mathbb{N}}$  converging to infinity so that  $\mathcal{E}(h(t_n)) \rightarrow 0$ . Indeed, assume such sequence exists and let  $(s_k)_{k \in \mathbb{N}}$  be a sequence converging to  $+\infty$ . Then there exists  $k_0$  and a subsequence  $(t_{n_k})_{k \in \mathbb{N}}$  of  $(t_n)_{n \in \mathbb{N}}$  so that for all  $k \geq k_0$ , we have  $s_k \geq t_{n_k}$ . Therefore from the monotonicity of  $\tilde{F}$  we have for all  $k \geq k_0$

$$\log \left[ \frac{\mathcal{E}(h(s_k))}{(\beta(v(s_k)))^2} \right] \leq \log \left[ \frac{\mathcal{E}(h(t_{n_k}))}{(\beta(v(t_{n_k})))^2} \right].$$

By letting  $k$  to infinity in the above inequality, we deduce that

$$\lim_{k \rightarrow \infty} \log \left[ \frac{\mathcal{E}(h(s_k))}{(\beta(v(s_k)))^2} \right] = -\infty,$$

which implies that  $\mathcal{E}(h(s_k)) \rightarrow 0$ , since by Lemma 4.6  $(\beta(v(t_k)))_{k \in \mathbb{N}}$  is uniformly bounded. The sequence  $(s_k)_{k \in \mathbb{N}}$  being chosen arbitrarily this implies that  $\mathcal{E}(h(t)) \rightarrow 0$  as  $t \rightarrow +\infty$ .

Let us now prove that such sequence  $(t_n)_{n \in \mathbb{N}}$  exists. We argue by contradiction and assume that  $\inf_{t \in \mathbb{R}^+} \mathcal{E}(h(t)) > 0$ . Therefore from the monotonicity and the smoothness of  $\tilde{F}$  we deduce that there is  $c_0 \in \mathbb{R}$  so that

$$\tilde{F}(h(t)) \rightarrow c_0 \quad \text{and} \quad \frac{d}{dt} \tilde{F}(h(t)) \rightarrow 0 \quad \text{as} \quad t \rightarrow +\infty.$$

Thus by Lemma 4.6 and (4.10) it follows that

$$(4.11) \quad \lim_{t \rightarrow \infty} \sum_{i,j=1}^N \mu_{ij} \bar{v}_j \bar{v}_i \left( \frac{h_j}{\bar{v}_j} - \frac{h_i}{\bar{v}_i} \right)^2 = 0.$$

From the *a priori* estimates of Lemma 4.6, there exists a sequence  $t_n \rightarrow \infty$  so that for all  $i$   $h_i(t_n) \rightarrow \tilde{h}_i$ . Passing to the limit along this sequence in the equation (4.11) it yields

$$0 = \sum_{i,j=1}^N \mu_{ij} \bar{v}_j \bar{v}_i \left( \frac{\tilde{h}_j}{\bar{v}_j} - \frac{\tilde{h}_i}{\bar{v}_i} \right)^2.$$

By using the irreducibility assumption on the nonnegative matrix  $\mu_{ij}$  and the positivity of the quantities  $\bar{v}_i$ , one can deduce from the above equality that we must have for all  $i$  and  $j$

$$\frac{\tilde{h}_j}{\bar{v}_j} = \frac{\tilde{h}_i}{\bar{v}_i}.$$

Thus if we set  $\lambda := \frac{\tilde{h}_1}{\bar{v}_1}$  we have  $\tilde{h} = \lambda \bar{v}$ . So by using that  $\langle h, \bar{v} \rangle = 0$  for all time it follows that  $\lambda = 0$ . Hence, we get the contradiction

$$0 < \inf_{t \in \mathbb{R}} \mathcal{E}(h(t)) \leq \mathcal{E}(h(t_n)) \rightarrow 0.$$

□



## 5. A CASE OF INTEREST

In this section we analyse more precisely the dynamics of the solution  $v$  of (1.1) when the interactions  $\Psi_i$  take the form  $\Psi_i(v) := \sum_{j=1}^N r_j v_j$ . We prove the Theorem 1.2 that we recall below.

**Theorem 5.1.** *Assume that the interactions  $\Psi_i$  take the form  $\Psi_i(v) = \sum_{j=1}^N r_j v_j$  then for any nonnegative initial datum  $v(0)$  not identically zero, the solution  $v(t)$  of (1.5) converges exponentially fast to its unique equilibrium. That is to say there exists two positive constants  $C_1$  and  $C_2$  so that*

$$\|v - \bar{v}\|_\infty \leq C_1 e^{-C_2 t}.$$

Before proving this Theorem we establish two auxiliary results that for convenience we present in two separate subsections. We prove Theorem 1.2 at the end of this section.

## 5.1. Study of the evolution of the total population.

Let us denote  $\mathcal{N}(t) = \sum_{i=1}^N v_i$  the total population. A straightforward computation shows that for the interactions  $\Psi_i$  of the form  $\Psi_i(v) := \sum_{j=1}^N r_j v_j$  we see that  $\mathcal{N}(t)$  satisfies the equation :

$$(5.1) \quad \frac{d\mathcal{N}}{dt} = \alpha(t) \left(1 - \frac{\mathcal{N}}{K}\right),$$

which written with the new variable  $P(t) = K - \mathcal{N}(t)$  takes the form

$$(5.2) \quad \frac{dP}{dt} = -\frac{\alpha(t)}{K} P.$$

The dynamic of the above equation is strongly related to the behaviour of  $\alpha(t)$  and we have

$$(5.3) \quad |P(t)| = |P_0| e^{-\frac{1}{K} \int_0^t \alpha(s) ds}.$$

**Lemma 5.2.** *For any nonnegative initial datum  $v(0)$  not identically zero,  $\mathcal{N}(t)$  converges exponentially fast toward its unique equilibrium  $K$ . Moreover  $\mathcal{N}$  satisfies identically*

$$(5.4) \quad \min\{\mathcal{N}_{min}(t), \mathcal{N}_{max}(t)\} \leq \mathcal{N}(t) \leq \max\{\mathcal{N}_{min}(t), \mathcal{N}_{max}(t)\},$$

where  $\mathcal{N}_{min}$  and  $\mathcal{N}_{max}$  are the solutions of the logistic equations:

$$\begin{aligned} \frac{du}{dt} &= \xi^\pm u \left(1 - \frac{u}{K}\right) \\ u(0) &= \mathcal{N}(0) \end{aligned}$$

with respectively  $\xi^- = \min\{r_1, \dots, r_N\}$  and  $\xi^+ = \max\{r_1, \dots, r_N\}$

**Proof:**

Assume for the moment that (5.4) holds then the convergence exponentially fast to  $K$  is a straightforward consequence of (5.3). Indeed by (5.4) we deduce that  $R_{min} \min(\mathcal{N}_{min}, \mathcal{N}_{max}) \leq \alpha(s) \leq R_{max} \min(\mathcal{N}_{min}, \mathcal{N}_{max})$  where  $R_{max} := \max\{r_1, \dots, r_N\}$  and  $R_{min} := \min\{r_1, \dots, r_N\}$ . Therefore  $\alpha(s) > C_0$ , since  $\mathcal{N}_{min}$  and  $\mathcal{N}_{max}$  converge to  $K$ .

To obtain (5.4) we investigate the following three cases,  $\mathcal{N}(0) = K$ ,  $\mathcal{N}(0) < K$  and  $\mathcal{N}(0) > K$ .

In the first case  $\mathcal{N}(0) = K$ , we see that (5.4) holds true trivially, since  $\mathcal{N}_{max} = \mathcal{N}_{min} = \mathcal{N}(t) \equiv K$  for all  $t$ . Let us now investigate the two other situations. The argumentation in both situation being similar we expose only the case  $\mathcal{N}(0) < K$ .

In this situation,  $\mathcal{N}_{min}$  and  $\mathcal{N}_{max}$  being the solutions of logistic equations they are increasing functions. Moreover we have  $\mathcal{N}_{min}(t) \leq \mathcal{N}_{max}(t) < K$  for all  $t$ . On another hand, by continuity of  $\mathcal{N}(t)$ , there exists also  $t_1 > 0$  so that  $\mathcal{N}(t) < K$  on  $[0, t_1)$ . Furthermore on  $(0, t_1)$  we can see that  $\mathcal{N}(t)$  satisfies the following differential inequalities

$$(5.5) \quad \frac{d\mathcal{N}}{dt} \geq R_{min} \mathcal{N} \left(1 - \frac{\mathcal{N}}{K}\right)$$

$$(5.6) \quad \frac{d\mathcal{N}}{dt} \leq R_{max} \mathcal{N} \left(1 - \frac{\mathcal{N}}{K}\right).$$

From the above differentials inequalities, by comparing  $\mathcal{N}$ ,  $\mathcal{N}_{\min}$  and  $\mathcal{N}_{\max}$  via the Cauchy Lipschitz Theorem, we obtain  $\mathcal{N}_{\min} \leq \mathcal{N} \leq \mathcal{N}_{\max}$  for all  $t \in [0, t_1)$ . Since  $\mathcal{N}_{\max}(t_1) < K$ , we can bootstrap the above argument and show that (5.4) holds true for all  $t$ .  $\square$

## 5.2. A useful functional inequality.

Next we establish a useful functional inequality satisfied by vectors  $h \in \bar{v}^\perp$  where  $\bar{v}^\perp$  denotes the linear subspace of  $\mathbb{R}^N$  orthogonal to  $\bar{v}$ .

**Lemma 5.3.** *There exists  $C_1 > 0$  so that for all  $h \in \bar{v}^\perp$*

$$C_1 \mathcal{E}(h) \leq \sum_{i,j=1}^N \mu_{ij} \bar{v}_i \bar{v}_j \left( \frac{h_j}{\bar{v}_j} - \frac{h_i}{\bar{v}_i} \right)^2.$$

Moreover  $C_1 = \lambda_2$  where  $\lambda_2$  is the minimal eigenvalue strictly positive of the linear eigenvalue problem

$$h_i \sum_{j=1}^N \mu_{ij} \frac{\bar{v}_j}{\bar{v}_i} - \sum_{j=1}^N \mu_{ij} h_j = \lambda h_i.$$

**Proof :**

Let  $\mathcal{I}$  be the following Rayleigh quotient

$$\mathcal{I}(h) := \frac{1}{\mathcal{E}(h)} \sum_{i,j=1}^N \mu_{ij} \bar{v}_i \bar{v}_j \left( \frac{h_j}{\bar{v}_j} - \frac{h_i}{\bar{v}_i} \right)^2.$$

Observe that the first part of the Lemma is proved if we show that

$$(5.7) \quad \inf_{h \in \bar{v}^\perp} \mathcal{I}(h) > 0,$$

or equivalently

$$(5.8) \quad \inf_{h \in \bar{v}^\perp, \mathcal{E}(h)=1} \mathcal{I}(h) = \inf_{h \in \bar{v}^\perp} \mathcal{I}(h) > 0,$$

since for all real  $\mu$ ,  $\mathcal{I}(h) = \mathcal{I}(\mu h)$ .

To obtain (5.7), we argue by contradiction and assume that  $\inf_{h \in \bar{v}^\perp} \mathcal{I}(h) = 0$ . By (5.8) we can take  $(h_n)_{n \in \mathbb{N}}$  a minimising sequence so that  $h_n \in \bar{v}^\perp$ ,  $\mathcal{E}(h_n) = 1$  for all  $n$ . Since  $\{x \in \mathbb{R}^N \mid x \in \bar{v}^\perp, \mathcal{E}(x) = 1\}$  is a closed bounded set,  $(h_n)_{n \in \mathbb{N}}$  is uniformly bounded and we can extract a subsequence  $(h_{n_k})_{k \in \mathbb{N}}$  which converges to  $\bar{h} \in \bar{v}^\perp$  with  $\mathcal{E}(\bar{h}) = 1$ . Passing to the limit along this subsequence, we see that  $\mathcal{I}(\bar{h}) = 0$  which combined with  $\bar{h} \in \bar{v}^\perp$  implies that  $\bar{h} = 0$ . Thus we get the contradiction  $0 = \mathcal{E}(\bar{h}) = 1$ . Hence (5.7) holds true.

Let  $C_1$  be defined by  $C_1 = \inf_{h \in \bar{v}^\perp, \mathcal{E}(h)=1} \mathcal{I}(h)$ , let us try to compute  $C_1$ . By construction  $C_1$  is the result of a constrained minimisation problem. Therefore, by the standard optimization Theory [42], the minimizers must satisfy the following Euler-Lagrange equations

$$(5.9) \quad (D - \widetilde{M})h = \lambda h,$$

$$(5.10) \quad \mathcal{E}(h) = 1,$$

$$(5.11) \quad \langle h, \bar{v} \rangle = 0,$$

where  $\lambda \in \mathbb{R}$  is a Lagrange Multiplier to be determined and  $\widetilde{M}$  and  $D$  are the following matrices

$$\widetilde{M} := \begin{pmatrix} \mu_{11} & & \mu_{1j} \\ & \ddots & \\ \mu_{ij} & & \mu_{NN} \end{pmatrix}, \quad D := \begin{pmatrix} \frac{1}{\bar{v}_1} \sum_{j=1}^N \mu_{1j} \bar{v}_j & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\bar{v}_N} \sum_{j=1}^N \mu_{Nj} \bar{v}_j \end{pmatrix}.$$

By taking  $k > 0$  large enough, the irreducible matrix  $\widetilde{M} - D + kId$  becomes non-negative and so  $\widetilde{M} - D + kId$  is a symmetric positive definite matrix. As a consequence, the eigenvectors associated to the eigenvalue  $\nu_i$  of  $\widetilde{M} - D + kId$  form an orthogonal basis of  $\mathbb{R}^N$ , i.e.  $\mathbb{R}^N = \bigoplus^\perp E_{\nu_i}$ , where  $E_{\nu_i}$  denotes the eigenspace associated to the eigenvalue  $\nu_i$ .

Since  $\widetilde{M} - D + kId$  is non negative and irreducible by the Perron-Frobenius Theorem there exists a unique eigenvalue, says  $\nu_1$  associated with a positive eigenvector  $\phi_1$ . Furthermore  $\nu_1$  is the largest eigenvalue and is algebraically simple. By a direct computation one can see that  $(\widetilde{M} - D + kId)\bar{v} = k\bar{v}$ , thus  $\nu_1 = k$  and  $\phi_1 = \gamma\bar{v}$ . From the properties of  $\nu_1$  we have  $E_{\nu_1} = \text{lin}(\bar{v})$ , for all  $i \neq 1, \nu_i < \nu_1 = k$  and  $\bar{v}^\perp = \bigoplus_{i \neq 1}^\perp E_{\nu_i}$ .

By construction, the  $\lambda_i := \nu_i - k \leq 0$  are the eigenvalues of  $\widetilde{M} - D$  and we can see that for all  $h \in \bar{v}^\perp, \mathcal{I}(h) = 1$  we have

$$\mathcal{I}(h) = {}^t h(\widetilde{M} - D)h \geq \min_{i \neq 1} \{-\lambda_i\}.$$

Let  $\lambda_2 < 0$  be second largest eigenvalue of  $\widetilde{M} - D$  and  $\phi_2$  an associated eigenvector. By normalising  $\phi_2$  properly and since  $\phi_2 \in \bar{v}^\perp$  a straightforward computation shows that

$$\mathcal{I}(\phi_2) = {}^t \phi_2(D - \widetilde{M})\phi_2 = -\lambda_2 = \min_{i \neq 1} \{-\lambda_i\}.$$

Hence,  $\min_{h \in \bar{v}^\perp} \mathcal{I}(h) = -\lambda_2$ . □

### 5.3. Asymptotic behaviour of the solution.

We are now in position to obtain the exponential rate of convergence for any solution  $v$  of (1.5).

#### Proof of Theorem 1.2:

First we claim that

$$(5.12) \quad \sum_{i=1}^N \bar{v}_i = K.$$

Indeed from Lemma 4.3 we deduce that  $\bar{v} = \frac{K\lambda_p}{\sum_{i=1}^N r_i(v_p)_i} v_p$  where  $v_p$  is the positive normalised eigenvector associated to the principal eigenvalue  $\lambda_p$  of the matrix  $R + M$ . A straightforward computation shows that

$$\sum_{i=1}^N r_i(v_p)_i = \sum_{i=1}^N ((R + M)v_p)_i = \lambda_p \sum_{i=1}^N (v_p)_i.$$

Since  $\lambda_p > 0$  it follows that

$$\frac{\sum_{i=1}^N (v_p)_i}{\sum_{i=1}^N r_i(v_p)_i} = \frac{1}{\lambda_p}.$$

Thus

$$\frac{K\lambda_p}{\sum_{i=1}^N r_i(v_p)_i} \sum_{i=1}^N (v_p)_i = K.$$

Now recall that in the proof of Lemma 4.7 from the orthogonal decomposition  $v(t) = \lambda(t)\bar{v} + h(t)$  we have

$$\frac{d\mathcal{E}(h)}{dt} = - \sum_{i,j=1}^N \mu_{ij} \bar{v}_i \bar{v}_j \left( \frac{h_j}{\bar{v}_j} - \frac{h_i}{\bar{v}_i} \right)^2 + \frac{2}{K} (\bar{\alpha} - \alpha(t)) \mathcal{E}(h).$$

By Lemma 5.3 we obtain

$$\frac{d\mathcal{E}(h)}{dt} \leq -C_1 \mathcal{E}(h) + \frac{2}{K} (\bar{\alpha} - \alpha(t)) \mathcal{E}(h).$$

Now arguing as in the proof of Lemma 4.5, we end up with

$$\frac{d}{dt} \log \left( \frac{\mathcal{E}(h)}{\beta(v)^2} \right) \leq -C_1.$$

Therefore, thanks to Lemma 4.6 we deduce that

$$(5.13) \quad \mathcal{E}(h) \leq \left( \frac{\mathcal{E}(h(0))}{\beta(v(0))^2} \right) e^{-C_1 t} \beta^2(v) \leq C_2 e^{-C_1 t}.$$

Finally we get the exponential rate of convergence, by observing that

$$\mathcal{N}(t) = \langle v, 1 \rangle = \lambda(t) \langle \bar{v}, 1 \rangle + \langle h, 1 \rangle,$$

which thanks to (5.3) implies that

$$\lambda(t) = \frac{K}{\langle \bar{v}, 1 \rangle} + \frac{\langle h, 1 \rangle}{\langle \bar{v}, 1 \rangle} + \frac{|\mathcal{N}(0) - K| e^{-\frac{1}{K} \int_0^t \alpha(s) ds}}{\langle \bar{v}, 1 \rangle}.$$

By Lemma 5.2 and the estimates (5.13), (5.12), using standard norm estimates, we have

$$\begin{aligned} \|v - \bar{v}\|_\infty &\leq \left| \frac{\langle h, 1 \rangle}{K} + \frac{|\mathcal{N}(0) - K| e^{-\frac{1}{K} \int_0^t \alpha(s) ds}}{K} \right| \|\bar{v}\|_\infty + N\mathcal{E}(h) \\ &\leq \frac{|\mathcal{N}(0) - K| \|\bar{v}\|_\infty e^{-\frac{1}{K} \int_0^t \alpha(s) ds}}{K} + \left( 1 + \frac{\|\bar{v}\|_\infty}{K} \right) \tilde{C}_2 e^{-C_1 t}. \end{aligned}$$

□

## 6. THE GENERAL CASE: THE STATIONARY SOLUTION

In this section we investigate the existence of a positive stationary solution of (2.1) under the additional condition (1.6) on the matrices  $R$  and  $M$  that we recall below,

$$\forall i \quad \sum_{j=1}^N \mu_{ij} \leq \frac{r_i}{2}.$$

This assumption has for consequence that the matrix  $R + M$  is positive definite. Indeed, we have

$$\begin{aligned} {}^t h(R + M)h &= \sum_{i=1}^N r_i h_i^2 - \frac{1}{2} \sum_{i,j=1}^N \mu_{ij} (h_i - h_j)^2 \geq 2 \sum_{i,j=1}^N \mu_{ij} h_i^2 - \frac{1}{2} \sum_{i,j=1}^N \mu_{ij} (h_i - h_j)^2 \\ &\geq \sum_{i,j=1}^N (\mu_{ij} (h_i^2 + h_j^2 - \frac{1}{2}(h_i - h_j)^2)) \\ &\geq \frac{1}{2} \left( \sum_{i,j=1}^N \mu_{ij} (h_i + h_j)^2 \right). \end{aligned}$$

Thus  $\text{kern}(R + M) = \{0\}$  and the matrix  $R + M$  is invertible. Moreover from the last inequality we see that there exists positive constants  $c_0$  and  $C_0$  so that

$$(6.1) \quad c_0 \langle u, u \rangle_{R+M} \leq \mathcal{E}(u) \leq C_0 \langle u, u \rangle_{R+M},$$

where  $\langle u, u \rangle_{R+M} := {}^t u(R + M)u$ .

Let  $\Xi(v)$  be the diagonal matrix defined by

$$(\Xi(v))_{ij} = \delta_{ij} \Psi_i(v).$$

With this notation, a positive stationary solution of (2.1) is then a non negative solution of the following problem:

$$(6.2) \quad (R + M)v = \Xi(v)v$$

Note that when  $\Xi(v)$  can be written as  $\Xi(v) = \alpha(v)Id$ , the construction of a positive solution has already been made in Section 4. So in the later, we will assume that  $\Xi(v)$  cannot be written as  $\Xi(v) = \alpha(v)Id$ . It is worth mentioning that in this situation the method used in Section 4 does not work and we have to use another strategy.

Let  $T$  be the following map

$$T : \mathbb{R}^N \rightarrow \mathbb{R}^N$$

$$v \mapsto T v := (R + M)^{-1}[\Xi(v)v]$$

Since  $R + M$  is invertible,  $T$  is well defined and one can easily check that a positive solution of (6.2) is a positive fixed point of the map  $T$ . To check that  $T$  has a positive fixed point we use a degree argument.

Let  $\Psi_i(v)^s$  and  $\Xi(v)^s$  defined by

$$\Psi_i^s := s\Psi_i + (1-s)\Psi_1, \quad (\Xi^s(v))_{ij} := \delta_{ij}\Psi_i^s(v),$$

we consider the homotopy  $H \in C([0, 1] \times \mathbb{R}^N, \mathbb{R}^N)$  defined by

$$\begin{aligned} H : [0, 1] \times \mathbb{R}^N &\rightarrow \mathbb{R}^N \\ (s, v) &\mapsto H(s, v) := (R + M)^{-1}[\Psi^s(v)v]. \end{aligned}$$

One can see that  $H(1, \cdot) = T$  and  $H(0, \cdot) = T_0$  where  $T_0$  corresponds to the map

$$T_0 : \mathbb{R}^N \rightarrow \mathbb{R}^N$$

$$v \mapsto T_0 v := \Psi_1(v)(R + M)^{-1}v.$$

Note that there exists a unique positive fixed point to  $T_0$  which can be constructed by arguing as in Section 4.

The next step in this degree argument is to obtain for all  $s$ , a good a priori estimates on the fixed point of the map  $H(s, v)$ , i.e. a good estimate on the positive solutions of the following problem:

$$(6.3) \quad (R + M)V = \Psi^s(V)V.$$

In this direction we show the following:

**Lemma 6.1.** *Let  $V$  be a non negative solution of (6.3). Then either  $V \equiv 0$  or  $V > 0$  and there exists  $\bar{c}_1$  and  $\bar{C}_1$  independent of  $s$  so that*

$$\bar{c}_1 \leq \sum_i V_i \leq \bar{C}_1.$$

**Proof:**

To obtain that  $V$  is either positive or  $V = 0$  we can argue as in the proof of Lemma 2.1. So assume that there exists a  $i_0$  so that  $V_{i_0} = 0$ . By construction  $V_{i_0}$  is a minimum of the  $V_i$  and from the equation satisfied by  $V_{i_0}$  we get

$$0 \leq \sum_{j=1}^N \mu_{i_0 j} (V_j - V_{i_0}) = 0.$$

Therefore  $V_j = V_{i_0}$  for all  $j$  where  $\mu_{i_0 j} \neq 0$ . Since  $M$  is irreducible there exists  $j \neq i_0$  so that  $\mu_{i_0 j} \neq 0$ . Let  $\rho := \{k \mid V_k = 0\}$  then  $i_0$  and all  $j$  so that  $\mu_{i_0 j} \neq 0$  belongs to  $\rho$ . In the previous argument, by replacing  $i_0$  by any  $k \in \rho$ , we see that all  $j$  so that  $\mu_{kj} \neq 0$  belongs to the set  $\rho$ . By iterating enough times the above argument and using the irreducibility of the matrix  $M$  we can see that  $\rho = \{1, \dots, N\}$  so  $V_i = V_{i_0} = 0$  for all  $i$ . Therefore a nonnegative solution of (6.3) is either  $V \equiv 0$  or  $V > 0$ .

Now let us assume that  $V > 0$ . Recall that by (6.1) there exists positive constants  $c_0$  and  $C_0$  so that for all  $u \in \mathbb{R}^N$

$$c_0 < u, u >_{R+M} \leq \mathcal{E}(u) \leq C_0 < u, u >_{R+M}.$$

So for a solution  $V$  of (6.3) one has

$$\begin{aligned} \langle V, V \rangle_{R+M} &= \sum_i \Psi_i^s(V) V_i^2 \geq \left( s \min_{i \in \{2, \dots, N\}} \Psi_i(V) + \Psi_1(V) \right) \langle V, V \rangle \\ &\geq c_0 \Psi_1(V) \langle V, V \rangle_{R+M}, \end{aligned}$$

and we also get

$$\langle V, V \rangle_{R+M} \leq C_0 \left( s \max_{i \in \{2, \dots, N\}} \Psi_i(V) + \Psi_1(V) \right) \langle V, V \rangle_{R+M}.$$

Therefore we have

$$(6.4) \quad \Psi_1(V) \leq \frac{1}{c_0},$$

$$(6.5) \quad \frac{1}{C_0} \leq s \max_{i \in \{2, \dots, N\}} \Psi_i(V) + \Psi_1(V).$$

Now thanks to the assumptions (1.2)–(1.3) made on the functions  $\Psi_i$ , there exists  $R_1, c_1, k_1$  and  $N$  positive constants  $\kappa_i$  so that :

$$(6.6) \quad \text{For all } x \in \mathbb{R}^{N,+} \setminus Q_{R_1}(0), \quad c_1 \left( \sum_{i=1}^N x_i \right)^{k_1} \leq \Psi_1(x),$$

$$(6.7) \quad \text{For all } i, \text{ and for all } x \in Q_{R_1}(0), \quad \Psi_i(x) \leq \kappa_i \sum_{j=1}^N |x_j|.$$

By combining (6.4),(6.6), (6.5) and (6.7) we deduce that

$$\begin{aligned} \mathcal{N} &= \sum_{j=1}^N V_j \leq \sup \left\{ \left( \frac{1}{c_0 c_1} \right)^{\frac{1}{k_1}}, R_1 \right\}, \\ \mathcal{N} &= \sum_{j=1}^N V_j \geq \min \left\{ R_1, \frac{1}{C_0(\kappa_1 + \sup_i \kappa_i)} \right\}. \end{aligned}$$

□

### 6.1. Computation of the degree.

We are now in position to prove the existence of a positive solution to the equation (6.2) by means of the computation of the topological degree of  $T - id$  on a well chosen set  $\Omega \subset \mathbb{R}^{N,+}$ . Now we take two positive constants  $c_2$  and  $C_2$  so that  $c_2 < \bar{c}_1$  and  $C_2 > \bar{C}_1$  where  $\bar{c}_1$  and  $\bar{C}_1$  are the constants obtained in Lemma 6.1. Let  $\Omega$  be the following open set

$$\Omega := \left\{ v \in \mathbb{R}^{N,+} \mid c_2 \leq \sum_{i=1}^N v_i \leq C_2 \right\}$$

and let us compute  $\deg(T - Id, \Omega, 0)$ .

By Lemma 6.1 for all  $s \in [0, 1]$  in  $H(s, v) - v \neq 0$  on  $\partial\Omega$ . Therefore using that  $H(., .)$  is an homotopy, we conclude that  $\deg(T - Id, \Omega, 0) = \deg(H(1, .) - Id, \Omega, 0) = \deg(H(0, .) - Id, \Omega, 0)$ . By construction, from Section 4.1, one can check that  $\deg(H(0, .) - Id, \Omega, 0) \neq 0$  since the map  $T_0$  has an unique stable and positive fixed point. Thus  $\deg(T - Id, \Omega, 0) \neq 0$  which shows that  $T$  has a fixed point in  $\Omega$ .

□

## 7. THE GENERAL CASE: ASYMPTOTIC BEHAVIOUR

In this section we prove Theorem 1.4. That is to say, under the extra assumption (1.6) we analyse the asymptotic behaviour of the solution  $v(t)$  when for all  $i$  the interaction  $\Psi_i$  can be expressed like:  $\Psi_i(v) = \alpha(v) + \epsilon \psi_i(v)$  with  $\psi_i$  uniformly bounded. To obtain the asymptotic behaviour in this case, we follow the strategy developed in Subsection 4.2. Namely, we start by showing some *a priori* estimates on the solution, then we analyse the convergence by means of a Lyapunov functional. For convenience we dedicate a subsection to each essential part of the proof.

### 7.1. A priori estimate.

We start by establishing some useful differential inequalities. Namely we show that

**Lemma 7.1.** *Assume that  $r_i, (\mu_{ij}), \Psi_i$  satisfies (1.2), (1.3) and (1.6). Assume further that  $\Psi_i(v) = \alpha(v) + \epsilon\psi_i(v)$  with  $\psi_i$  uniformly bounded. Then there exists  $\epsilon_0 > 0$  so that for all  $0 \leq \epsilon \leq \epsilon_0$  there exists  $\omega^+ \in \mathbb{R}^N, \omega^+$  positive and a positive real  $\gamma$  so that*

(i)

$$\begin{aligned} \frac{d \sum_{i=1}^N \bar{\omega}_i^+ v_i}{dt} &\leq \frac{1}{K} (\alpha(\bar{\omega}^+) - \alpha(v)) \sum_{j=1}^N \bar{\omega}_j^+ v_j \\ \frac{d \sum_{i=1}^N \bar{\omega}_i^+ v_i}{dt} &\geq \frac{1}{K} (\alpha(\gamma \bar{\omega}^+) - \alpha(v)) \sum_{j=1}^N \bar{\omega}_j^+ v_j \end{aligned}$$

(ii)

$$\begin{aligned} \frac{d\mathcal{E}(v)}{dt} &\leq - \sum_{i,j=1}^N \mu_{ij} \bar{\omega}_i^+ \bar{\omega}_j^+ \left( \frac{v_i}{\bar{\omega}_i^+} - \frac{v_j}{\bar{\omega}_j^+} \right)^2 + \frac{2}{K} (\alpha(\omega^+) - \alpha(v)) \mathcal{E}(v) \\ \frac{d\mathcal{E}(v)}{dt} &\geq - \sum_{i,j=1}^N \mu_{ij} \bar{\omega}_i^+ \bar{\omega}_j^+ \left( \frac{v_i}{\bar{\omega}_i^+} - \frac{v_j}{\bar{\omega}_j^+} \right)^2 + \frac{2}{K} (\alpha(\gamma \bar{\omega}^+) - \alpha(v)) \mathcal{E}(v) \end{aligned}$$

(iii)

$$\begin{aligned} \frac{d}{dt} \log \left[ \frac{\mathcal{E}(v)}{(\sum_{i=1}^N \bar{\omega}_i^+ v_i)^2} \right] &\leq - \frac{1}{\mathcal{E}(v)} \sum_{i,j=1}^N \mu_{ij} \bar{\omega}_i^+ \bar{\omega}_j^+ \left( \frac{v_i}{\bar{\omega}_i^+} - \frac{v_j}{\bar{\omega}_j^+} \right)^2 + \frac{2}{K} (\alpha(\bar{\omega}^+) - \alpha(\gamma \bar{\omega}^+)) \\ \frac{d}{dt} \log \left[ \frac{\mathcal{E}(v)}{(\sum_{i=1}^N \bar{\omega}_i^+ v_i)^2} \right] &\geq - \frac{1}{\mathcal{E}(v)} \sum_{i,j=1}^N \mu_{ij} \bar{\omega}_i^+ \bar{\omega}_j^+ \left( \frac{v_i}{\bar{\omega}_i^+} - \frac{v_j}{\bar{\omega}_j^+} \right)^2 + \frac{2}{K} (\alpha(\gamma \bar{\omega}^+) - \alpha(\bar{\omega}^+)) \end{aligned}$$

#### Proof:

First, we observe that (iii) can be straightforwardly obtained by combining (i) and (ii). So we deal only with (i) and (ii).

Let us denote  $\sigma := \epsilon \|\psi\|_\infty$ . Then since  $v$  is positive, from (1.1) it follows that

$$\begin{aligned} \frac{dv_i}{dt} &\leq (r_i + \sigma - \alpha(v))v_i + \sum_{j=1}^N \mu_{ij}(v_j - v_i), \\ \frac{dv_i}{dt} &\geq (r_i - \sigma - \alpha(v))v_i + \sum_{j=1}^N \mu_{ij}(v_j - v_i). \end{aligned}$$

Let  $\bar{\omega}^+$  and  $\bar{\omega}^-$  be the stationary solutions of the corresponding equations

$$\begin{aligned} \frac{d\omega_i^+}{dt} &= (r_i + \sigma - \alpha(\omega^+))\omega_i^+ + \sum_{j=1}^N \mu_{ij}(\omega_j^+ - \omega_i^+), \\ \frac{d\omega_i^-}{dt} &= (r_i - \sigma - \alpha(\omega^-))\omega_i^- + \sum_{j=1}^N \mu_{ij}(\omega_j^- - \omega_i^-). \end{aligned}$$

Now, let us fix  $\epsilon$  small enough so that  $\bar{\omega}^\pm$  exists. Then by arguing as in the proof of Lemma 3.2, we obtain

$$\begin{aligned} \frac{d}{dt} \sum_{i=1}^N \bar{\omega}_i^+ v_i &\leq \frac{1}{K} (\alpha(\bar{\omega}^+) - \alpha(v)) \sum_{j=1}^N \bar{\omega}_j^+ v_j, \\ \frac{d}{dt} \sum_{i=1}^N \bar{\omega}_i^- v_i &\geq \frac{1}{K} (\alpha(\bar{\omega}^-) - \alpha(v)) \sum_{j=1}^N \bar{\omega}_j^- v_j, \\ \frac{d\mathcal{E}(v)}{dt} &\leq - \sum_{i,j=1}^N \mu_{ij} \bar{\omega}_i^+ \bar{\omega}_j^+ \left( \frac{v_i}{\bar{\omega}_i^+} - \frac{v_j}{\bar{\omega}_j^+} \right)^2 + \frac{2}{K} (\alpha(\bar{\omega}^+) - \alpha(v)) \mathcal{E}(v), \\ \frac{d\mathcal{E}(v)}{dt} &\geq - \sum_{i,j=1}^N \mu_{ij} \bar{\omega}_i^- \bar{\omega}_j^- \left( \frac{v_i}{\bar{\omega}_i^-} - \frac{v_j}{\bar{\omega}_j^-} \right)^2 + \frac{2}{K} (\alpha(\bar{\omega}^-) - \alpha(v)) \mathcal{E}(v). \end{aligned}$$

Note that by Lemma 4.3,  $\bar{\omega}^\pm$  are positive eigenvectors of the matrices  $R + M \pm \sigma Id$ . Thus  $\bar{\omega}$  are eigenvectors of the matrix  $R + M$  associated to the principal eigenvalue of  $R + M$ . Since  $R + M$  is irreducible, the eigenspace associated to the principal eigenvalue is unidimensional. So, we have  $\bar{\omega}^+ = \gamma \bar{\omega}^-$  for some positive  $\gamma$ . Hence (i) and (ii) hold true since

$$\sum_{i,j=1}^N \mu_{ij} \bar{\omega}_i^- \bar{\omega}_j^- \left( \frac{v_i}{\bar{\omega}_i^-} - \frac{v_j}{\bar{\omega}_j^-} \right)^2 = \sum_{i,j=1}^N \mu_{ij} \bar{\omega}_i^+ \bar{\omega}_j^+ \left( \frac{v_i}{\bar{\omega}_i^+} - \frac{v_j}{\bar{\omega}_j^+} \right)^2.$$

□

Next, we derive some *a priori* estimates for the solution  $v$  for an interaction  $\Psi$  as in Theorem 1.4.

**Lemma 7.2.** *Assume that  $r_i, (\mu_{ij}), \Psi_i$  satisfies (1.2), (1.3) and (1.6). Assume further that  $\Psi_i(v) = \alpha(v) + \epsilon \psi_i(v)$  with  $\psi_i$  uniformly bounded. Then there exists  $\bar{c}_1 < \bar{C}_1$  and  $\epsilon_1$  so that for all  $0 \leq \epsilon \leq \epsilon_1$  and for any positive stationary solution  $\bar{v}_\epsilon$  of (1.1), we have*

$$\bar{c}_1 \leq \sum_{i=1}^N \bar{v}_{\epsilon,i} < \bar{C}_1.$$

Moreover,  $\bar{v}_\epsilon$  satisfies  $\bar{\omega}^- \leq \bar{v}_\epsilon \leq \bar{\omega}^+$ . Furthermore, for any nonnegative initial datum  $v_i(0)$  not identically zero there exists two constants  $\bar{c}_2(v(0)), \bar{C}_2(v(0))$  so that for all  $0 \leq \epsilon \leq \epsilon_1$  the solution  $v_\epsilon$  satisfies for all  $t$ ,

$$\bar{c}_2 \leq \beta(v_\epsilon(t)) := \sum_{i=1}^N \bar{v}_{\epsilon,i} v_{\epsilon,i}(t) \leq \bar{C}_2.$$

**Proof:**

Let us first observe that for  $\epsilon \leq \epsilon_0$  by replacing  $v_\epsilon$  by  $\bar{v}_\epsilon$  in (i) of Lemma 7.1, we get

$$\begin{aligned} 0 &\leq \frac{1}{K} (\alpha(\bar{\omega}^+) - \alpha(\bar{v}_\epsilon)) \sum_{j=1}^N \bar{\omega}_j^+ \bar{v}_{\epsilon,j}, \\ 0 &\geq \frac{1}{K} (\alpha(\gamma \bar{\omega}^+) - \alpha(\bar{v}_\epsilon)) \sum_{j=1}^N \bar{\omega}_j^+ \bar{v}_{\epsilon,j}. \end{aligned}$$

From the latter inequalities, by using the positivity of  $\bar{v}_\epsilon, \bar{\omega}^+$  and the monotonicity of the function  $\alpha$ , we get

$$(7.1) \quad \gamma \bar{\omega}^+ \leq \bar{v}_\epsilon \leq \bar{\omega}^+.$$

From the proof of Lemma 4.3, we also see that

$$(7.2) \quad \alpha(\bar{\omega}^+) = K(\lambda_p + \sigma), \quad \alpha(\bar{\omega}^-) = K(\lambda_p - \sigma),$$



where  $\lambda_p$  is the positive principal eigenvalue of the matrix  $R+M$ . By using the monotonicity of  $\alpha$ , we deduce that the maps  $\sigma \mapsto \omega^\pm$  are monotone. Moreover, we have  $0 \leq \alpha(\bar{\omega}^+) - \alpha(\bar{\omega}^-) \leq 2\sigma K$ .

To obtain  $\bar{c}_1$  and  $\bar{C}_1$  we argue as follows. Let us fix  $\epsilon_1 := \min\{\epsilon_0, \frac{\lambda_p}{4\|\psi\|_\infty}\}$ . Then by (7.2), for all  $\epsilon \in [0, \epsilon_1]$  we have

$$(7.3) \quad \alpha(\bar{\omega}^+) \leq \alpha(\omega_{\epsilon_1}^+) = \frac{5K\lambda_p}{4}, \quad \alpha(\bar{\omega}^-) \geq \alpha(\omega_{\epsilon_1}^-) = \frac{3K\lambda_p}{4}.$$

Now thanks to the assumptions (1.2)–(1.3) satisfied by  $\alpha$ , there exists  $R_\alpha, c_\alpha, k_\alpha$  and  $\kappa_\alpha$  so that :

$$(7.4) \quad \text{For all } x \in \mathbb{R}^{N,+} \setminus Q_{R_\alpha}(0), \quad c_\alpha \left( \sum_{i=1}^N x_i \right)^{k_\alpha} \leq \alpha(x),$$

$$(7.5) \quad \text{For all } i, \text{ and for all } x \in Q_{R_\alpha}(0), \quad \alpha(x) \leq \kappa_\alpha \sum_{j=1}^N |x_j|.$$

So for  $\epsilon \in [0, \epsilon_1]$ , by combining (7.2), (7.3), (7.4) and (7.5) we achieve

$$(7.6) \quad \sum_{j=1}^N \bar{v}_{\epsilon,j} \leq \sup \left\{ \left( \frac{5K\lambda_p}{4c_\alpha} \right)^{\frac{1}{k_\alpha}}, R_\alpha \right\} =: \bar{C}_1,$$

$$(7.7) \quad \sum_{j=1}^N \bar{v}_{\epsilon,j} \geq \min \left\{ R_\alpha, \frac{3K\lambda_p}{4\kappa_\alpha} \right\} =: \bar{c}_1.$$

Next, we derive an uniform upper bound for  $\beta(v_\epsilon)$  when  $\epsilon \in [0, \epsilon_1]$ . In the sequel of this proof, for convenience we drop the subscript  $\epsilon$  on  $v$ .

First, we observe that by Lemma 3.2 we have

$$\begin{aligned} \frac{d}{dt} \left( \sum_{i=1}^N v_i \bar{v}_i \right) &= \frac{1}{K} \sum_{i=1}^N (\Psi_i(\bar{v}) - \Psi_i(v)) v_i \bar{v}_i, \\ &= \left( \frac{1}{K} (\alpha(\bar{v}) - \alpha(t)) \right) \sum_{i=1}^N v_i \bar{v}_i + \frac{\epsilon}{K} \sum_{i=1}^N (\psi_i(\bar{v}) - \psi_i(v)) v_i \bar{v}_i. \end{aligned}$$

Since the functions  $\psi_i$  are uniformly bounded, we achieve

$$\frac{d}{dt} \left( \sum_{i=1}^N v_i \bar{v}_i \right) \leq \left( \frac{1}{K} (\alpha(\bar{v}) + 2\epsilon\|\psi\|_\infty - \alpha(t)) \right) \sum_{i=1}^N v_i \bar{v}_i.$$

By using (7.1) and (7.3), it follows

$$\frac{d}{dt} \left( \sum_{i=1}^N v_i \bar{v}_i \right) \leq \frac{1}{K} \left[ \frac{7K\lambda_p}{4} - \alpha(t) \right] \sum_{i=1}^N v_i \bar{v}_i.$$

Again using that  $\alpha$  satisfies the assumptions (1.2)–(1.3), there exists  $R_\alpha, c_\alpha, k_\alpha$  so that for all  $x \in \mathbb{R}^{N,+} \setminus Q_{R_\alpha}(0)$ ,

$$(7.8) \quad c_\alpha \left( \sum_{i=1}^N x_i \right)^{k_\alpha} \leq \alpha(x).$$

Now, let us assume that for some  $t > 0$ ,  $v(t) \in \mathbb{R}^{N,+} \setminus \bar{Q}_{R_\alpha}(0)$  otherwise the proof is done since we have

$$\beta(v) \leq \left( \max_{i \in \{1, \dots, N\}} \bar{v}_i \right) \sum_{j=1}^N |v_j| \leq \bar{C}_1 R_\alpha,$$

where  $\bar{C}_1$  is the bound obtained above. Let  $\Sigma$  be the following set

$$\Sigma := \{t \in \mathbb{R}^+ \mid \mathcal{N}(t) > R_\alpha\}.$$

From (7.8), (7.6) and using that the  $v_i$  are non negative, we see that for all  $t \in \Sigma$

$$\alpha(t) \geq c_\alpha \mathcal{N}(t)^{k_\alpha} \geq c_\alpha \left(\frac{1}{\bar{C}_1}\right)^{k_\alpha} \beta(t)^{k_\alpha}.$$

Therefore, with  $\tilde{c}_0 := c_\alpha \left(\frac{1}{\bar{C}_1}\right)^{k_\alpha}$  we have for  $t \in \Sigma$

$$\frac{d\beta(t)}{dt} \leq \frac{1}{K} \left( \frac{7K\lambda_p}{4} - \tilde{c}_0 \beta^{k_\alpha}(t) \right) \beta(t).$$

Using the logistic character of the above equation, we can check that

$$\beta(v(t)) \leq \sup \left\{ \beta(v(0)), \sup_{x \in Q_{R_\alpha}(0)} \beta(x), \left( \frac{7K\lambda_p}{4\tilde{c}_0} \right)^{\frac{1}{k_\alpha}} \right\}.$$

Thus, by using (7.6), we achieve for all  $\epsilon \in [0, \epsilon_1]$

$$\beta(v_\epsilon(t)) \leq \bar{C}_2 := \sup \left\{ \beta(v(0)), \bar{C}_1 R_\alpha, \bar{C}_1 \left( \frac{7K\lambda_p}{4c_\alpha} \right)^{\frac{1}{k_\alpha}} \right\}.$$

To obtain the lower bound for  $\beta(v_\epsilon)$  we argue as follows. By (7.7), for all  $\epsilon \in [0, \epsilon_1]$  we have

$$(7.9) \quad \beta(v_\epsilon) \geq \min_{i \in \{1, \dots, N\}} \bar{v}_{\epsilon, i} \sum_{i=1}^N v_{\epsilon, i} \geq \frac{1}{N} \sum_{i=1}^N \bar{v}_{\epsilon, i} \sum_{i=1}^N v_{\epsilon, i} \geq \frac{\bar{c}_1}{N} \sum_{i=1}^N v_{\epsilon, i}.$$

Therefore to obtain a uniform lower bound for  $\beta(v_\epsilon)$  it is enough to obtain a uniform lower bound for  $\mathcal{N}_\epsilon := \sum_{i=1}^N v_{\epsilon, i}$ . From (1.1) by summing over all  $i$  and by using the definition of  $\Psi$  and the boundedness of the  $\psi_i$  we deduce that  $\mathcal{N}_\epsilon$  satisfies the following inequality

$$\frac{d\mathcal{N}_\epsilon}{dt} \geq \frac{1}{K} (\alpha(\bar{v}_\epsilon) - 2\epsilon \|\psi\|_\infty - \alpha(t)) \mathcal{N}_\epsilon.$$

Thanks to (7.1) and (7.3), we have

$$\frac{d\mathcal{N}_\epsilon}{dt} \geq \frac{1}{K} \left( \frac{K\lambda_p}{4} - \alpha(t) \right) \mathcal{N}_\epsilon.$$

By reproducing the argumentation of the proof of Lemma 2.1 and by using Remark 2.2 we can check that

$$(7.10) \quad \mathcal{N}_\epsilon(t) \geq \min \left\{ 1, \frac{K\lambda_p}{4\kappa_1}, \frac{\mathcal{N}(0)}{2} \right\},$$

where  $\kappa_1$  denotes the Lipschitz constant of the function  $\alpha$  in the unit cube. Hence, by collecting (7.9) -(7.10) we achieve for all  $\epsilon \leq \epsilon_1$  and all  $t > 0$ ,

$$\beta(v_\epsilon(t)) \geq \frac{\bar{c}_1}{N} \min \left\{ 1, \frac{K\lambda_p}{4\kappa_1}, \frac{\sum_{j=1}^N v_j(0)}{2} \right\} =: \bar{c}_2.$$

□

*Remark 7.3.* Note that from the above argumentation, using the Logistic character of the equations, we can get that for all  $\epsilon \leq \epsilon_1$  and all initial data  $v(0) \geq 0$ , there exists  $t_0$  so that for all  $t \geq t_0$  we have

$$\frac{1}{2} \frac{\bar{c}_1}{N} \min \left\{ 1, \frac{K\lambda_p}{4\kappa_1} \right\} \leq \beta(v) \leq 2 \sup \left\{ \bar{C}_1 R_\alpha, \bar{C}_1 \left( \frac{7K\lambda_p}{4c_\alpha} \right)^{\frac{1}{k_\alpha}} \right\}.$$

Lastly, we obtain some uniform control on a continuous set of homeomorphisms

$$\tilde{\Psi}_v(s) := \sum_{i=1}^N \Psi_i(sv) v_i^2$$

where  $v \in U \subset \mathbb{R}^{N,+}$ .

Namely, we show that

**Lemma 7.4.** *Assume that  $r_i, (\mu_{ij}), \Psi_i$  satisfies (1.2), (1.3) and (1.6). Assume further that  $\Psi_i(v) = \alpha(v) + \epsilon \psi_i(v)$  with  $\alpha \in C_{loc}^1$  satisfying (1.2), (1.3) and  $\psi_i \in C_{loc}^1$  uniformly bounded. Then there exists  $\epsilon_2$  and  $\tau_0 > 0$  so that for all  $\epsilon \leq \epsilon_2$  and for all  $\bar{v}_\epsilon$  stationary solution of (1.1) we have*

$$\bar{c}_3 \leq \tilde{\Psi}_{\bar{v}_\epsilon}(1) \leq \bar{C}_3 \leq 2\bar{C}_3 \leq \tilde{\Psi}_{\bar{v}_\epsilon}(1 + \tau_0).$$

Moreover there exists  $\epsilon_3$  and  $k > 0$  so that for all  $\epsilon \leq \epsilon_3$  we have for all  $\bar{v}_\epsilon$  stationary solution of (1.1) and  $t, s \in (0, 1 + \tau_0)$

$$|t - s| \leq k |\tilde{\Psi}_{\bar{v}_\epsilon}(t) - \tilde{\Psi}_{\bar{v}_\epsilon}(s)|.$$

**Proof:**

Recall that from the proof of Lemma 7.2 for all  $\epsilon \leq \epsilon_1$ , for any stationary solution of (1.1)  $\bar{v}_\epsilon$ , we have  $\bar{\omega}^- \leq \bar{v}_\epsilon \leq \bar{\omega}^+$ . So, for all  $\epsilon \leq \epsilon_1$  and for all stationary solution of (1.1) we have

$$< \bar{v}_\epsilon, \bar{v}_\epsilon > (\alpha(\bar{\omega}^-) - \sigma) \leq \tilde{\Psi}(\bar{v}_\epsilon) \leq (\alpha(\bar{\omega}^+) + \sigma) < \bar{v}_\epsilon, \bar{v}_\epsilon >,$$

where  $\sigma = \epsilon \|\psi\|_\infty$ . Let us fix  $\epsilon_2 := \min\{\epsilon_1, \frac{\lambda_p}{4K\|\psi\|_\infty}\}$ . From (7.3) and Lemma 7.2 we get

$$\bar{c}_3 := \frac{\bar{c}_1^2 K \lambda_p}{2N} \leq \tilde{\Psi}_{\bar{v}_\epsilon}(1) \leq \frac{3\bar{C}_1^2 K \lambda_p}{2} =: \bar{C}_3,$$

for all  $\epsilon \leq \epsilon_2$  and for any stationary solution of (1.1).

By using to the monotonicity of the map  $\sigma \rightarrow \bar{\omega}^\pm$  and Lemma 7.2, we also get that for all  $s_0 > 0$  and for any stationary solution  $\bar{v}_\epsilon$ ,

$$s_0 \bar{\omega}_{\epsilon_1}^- \leq s_0 \bar{v}_\epsilon \leq s_0 \bar{\omega}_{\epsilon_1}^+.$$

The latter inequalities imply that we have for all  $\epsilon \leq \epsilon_2$  and for any stationary solution  $\bar{v}_\epsilon$ ,

$$\tilde{\Psi}_{\bar{v}_\epsilon}(s_0 \bar{v}_\epsilon) \geq \alpha(s_0 \bar{\omega}_{\epsilon_1}^-) - \frac{\lambda_p}{4K}.$$

Let us fix  $s_0$  such that

$$\alpha(s_0 \bar{\omega}_{\epsilon_1}^-) - \frac{\lambda_p}{4K} = 2\bar{C}_3.$$

This is always possible since  $\alpha$  is monotone increasing and  $\lim_{\mu \rightarrow \infty} \alpha(\mu \bar{\omega}_{\epsilon_1}^-) = +\infty$ . By construction,  $s_0 > 1$ , since  $\tilde{\Psi}_{\bar{v}_\epsilon}$  is monotone increasing and we have

$$\tilde{\Psi}_{\bar{v}_\epsilon}(s_0 \bar{v}_\epsilon) \geq 2\tilde{\Psi}_{\bar{v}_\epsilon}(\bar{v}_\epsilon).$$

Let us denote  $\tau_0 := s_0 - 1$ .

Now since for each  $i$  the function  $\Psi_i$  satisfies (1.2), (1.3) and that  $\alpha$  and the  $\psi_i$  are  $C_{loc}^1(\mathbb{R}^N)$ , the function  $\tilde{\Psi}_{\bar{v}_\epsilon}(s) := \sum_{i=1}^N \Psi_i(s\bar{v}_\epsilon) \bar{v}_{\epsilon,i}^2$  is  $C_{loc}^1(\mathbb{R})$  and monotone increasing. Therefore, for any stationary solution of (1.1)  $\bar{v}_\epsilon$ , the function  $\tilde{\Psi}_{\bar{v}_\epsilon}$  is a  $\mathbb{R}^+$  homeomorphism.

Next, we check that for a fixed  $\epsilon$  and a fixed stationary solution  $\bar{v}_\epsilon$ , the homeomorphism  $\tilde{\Psi}_{\bar{v}_\epsilon}$  is a  $C^1$  diffeomorphism on  $(0, 1 + \tau_0) \rightarrow \tilde{\Psi}_{\bar{v}_\epsilon}((1 - \tau_0, 1 + \tau_0))$ . Thanks to the Inverse Function Theorem, to show that  $\tilde{\Psi}_{\bar{v}_\epsilon}$  is a local  $C^1$  diffeomorphism it is sufficient to prove that for all  $s \in (0, 1 + \tau_0)$ ,  $\tilde{\Psi}'_{\bar{v}_\epsilon}(s) \neq 0$ . By a straightforward computation we have:

$$(\Psi_{\bar{v}_\epsilon})'(s) = \langle \nabla \alpha(s\bar{v}_\epsilon), \bar{v}_\epsilon \rangle + \epsilon \sum_{i=1}^N \langle \nabla \psi_i(s\bar{v}_\epsilon), \bar{v}_\epsilon \rangle \bar{v}_{\epsilon,i}^2.$$

For all  $s \in (0, 1 + \tau_0)$ , and for any stationary solution  $\bar{v}_\epsilon \in Q_{(1+\tau_0)\bar{c}_1}(0)$  we have:

$$\begin{aligned} |\nabla\psi_i(s\bar{v}_\epsilon)| &\leq \sup_{v \in Q_{(1+\tau_0)\bar{c}_1}(0)} |\nabla\psi_i(v)| := \zeta_1, \\ \nabla\alpha(s\bar{v}_\epsilon) &\leq \sup_{v \in Q_{(1+\tau_0)\bar{c}_1}(0)} \nabla\alpha(v) := \zeta_2 > 0, \\ \nabla\alpha(s\bar{v}_\epsilon) &\geq \inf_{v \in Q_{(1+\tau_0)\bar{c}_1}(0)} \nabla\alpha(v) := \zeta_3 > 0. \end{aligned}$$

Therefore, by Lemma 7.2 we deduce that

$$(\Psi_{\bar{v}_\epsilon})'(s) \geq \frac{\bar{c}_1^3}{N} \zeta_3 - \epsilon \zeta_1 \bar{C}_1^3.$$

By choosing  $\epsilon \leq \epsilon_3 := \min \left\{ \epsilon_2, \frac{\bar{c}_1^3 \zeta_3}{2N \zeta_1 \bar{C}_1^3} \right\}$ , we get for all  $\epsilon, s \in (0, 1 + \tau_0)$  and for any stationary solution of (1.1)  $\bar{v}_\epsilon$ ,

$$0 < \frac{\bar{c}_1^3 \zeta_3}{2N} \leq (\Psi_{\bar{v}_\epsilon})'(s) \leq \bar{C}_1^3 \left( \zeta_2 + \frac{\zeta_3}{2N} \right).$$

From the the latter *a priori* bounds, we see that for all  $\epsilon \leq \epsilon_3, s \in (0, 1 + \tau_0)$  and for any stationary solution of (1.1)  $\bar{v}_\epsilon$ , we get the following *a priori* estimate

$$\frac{1}{\bar{C}_1^3 \left( \zeta_2 + \frac{\zeta_3}{2N} \right)} \leq (\Psi_{\bar{v}_\epsilon}^{-1})'(s) = \frac{1}{(\Psi_{\bar{v}_\epsilon})'[(\Psi_{\bar{v}_\epsilon}^{-1})(s)]} \leq \frac{2N}{\bar{c}_1^2 \zeta_3}.$$

Hence, we deduce that for all  $\epsilon \leq \epsilon_3, s, t \in (0, 1 + \tau_0)$  and for any stationary solution of (1.1)  $\bar{v}_\epsilon$ , we have

$$|s - t| \leq k |\tilde{\Psi}_{\bar{v}_\epsilon}(s) - \tilde{\Psi}_{\bar{v}_\epsilon}(t)|,$$

with

$$k := \frac{2N}{\bar{c}_1^2 \zeta_3}.$$

□

## 7.2. Asymptotic Behaviour.

We are now in position to obtain the asymptotic behaviour of the solution  $v_\epsilon(t)$  as  $t$  goes to  $+\infty$  for  $\epsilon \in [0, \epsilon^*]$ , where  $\epsilon^*$  is to be determined later on.

Let us fix  $\epsilon \in [0, \epsilon_1]$  where  $\epsilon_1$  is obtained in Lemma 7.2 and let  $\bar{v}_\epsilon$  be a stationary solution of (1.1). For simplicity we denote  $\langle, \rangle$  the standard scalar product in  $\mathbb{R}^N$ .

As in the proof of Lemma 4.7, we start by observing that since  $\bar{v}_\epsilon \neq 0$  we can write  $v_\epsilon(t) := \lambda(t)\bar{v}_\epsilon + h(t)$  with for all  $t, \langle h, \bar{v}_\epsilon \rangle = 0$ . In the sequel of this subsection, for more clarity in the presentation we drop the subscript  $\epsilon$  on  $v$  and  $\bar{v}$ .

First, we note that from this decomposition we can derive the following equalities:

$$(7.11) \quad \lambda \langle \bar{v}, \bar{v} \rangle = \sum_{i=1}^N v_i \bar{v}_i,$$

$$(7.12) \quad \frac{d\mathcal{E}(h)}{dt} = \frac{d\mathcal{E}(v)}{dt} - 2\lambda\lambda' \langle \bar{v}, \bar{v} \rangle.$$

By Lemma 7.2 and (7.11), we can check that for all  $\epsilon \leq \epsilon_1$  and for all  $t > 0$ , we have

$$(7.13) \quad \frac{\bar{c}_2}{\bar{C}_1^2} \leq \lambda(t) \leq \frac{N\bar{C}_2}{\bar{c}_1^2}.$$

Similarly, by using (7.13), Lemma 7.2 and  $\sum_{i=1}^N |h_i| \leq \lambda(t) \sum_{i=1}^N \bar{v}_i + \sum_{i=1}^N v_i$ , we deduce from  $\mathcal{E}(v) = \mathcal{E}(h) + \lambda^2 \mathcal{E}(\bar{v})$  that

$$(7.14) \quad \mathcal{E}(h) \leq \frac{N\bar{C}_2^2}{\bar{c}_1^2}(N+1),$$

$$(7.15) \quad \sum_{i=1}^N |h_i| \leq \frac{N\bar{C}_2}{\bar{c}_1} \left(1 + \frac{\bar{C}_1}{\bar{c}_1}\right).$$

By (7.13),(7.14),(7.15), Lemma 7.2, and by using the Lipschitz regularity of  $\Psi_i$  and the Cauchy-Schwartz inequality we can check that for some constant  $C > 0$  independent of  $\epsilon$  and  $v$

$$(7.16) \quad \sum_{i=1}^N |\Psi_i(\lambda(t)\bar{v}) - \Psi_i(v(t))| \bar{v}_i^2 \leq C\sqrt{\mathcal{E}(h)},$$

$$(7.17) \quad \sum_{i=1}^N |(\psi_i(\bar{v}) - \psi_i(v))\bar{v}_i h_i| \leq C\sqrt{\mathcal{E}(h)}.$$

Indeed by (7.13)–(7.15) and Lemma 7.2,  $\lambda\bar{v}$  and  $h$  are uniformly bounded and we have

$$\sum_{i=1}^N |\psi_i(\bar{v}) - \psi_i(v)| |h_i| \bar{v}_i \leq \sum_{i=1}^N \kappa_i \left( \sum_{j=1}^N |\bar{v}_j - v_j| \right) |h_i| \bar{v}_i,$$

where  $\kappa_i$  are the Lipschitz constant of  $\psi_i$  on the set  $Q_{R_0}(0)$  with  $R_0 := \frac{N\bar{C}_2\bar{C}_1}{\bar{c}_1^2} \left(1 + 2\frac{\bar{C}_1}{\bar{c}_1}\right)$ . From the decomposition of  $v$  and by using the Cauchy-Schwartz inequality, we get

$$(7.18) \quad \sum_{i=1}^N |(\psi_i(\bar{v}) - \psi_i(\lambda(t)\bar{v} + h))\bar{v}_i h_i| \leq \bar{\kappa} \left( |1 - \lambda(t)| \sum_{j=1}^N \bar{v}_j + \sum_{j=1}^N |h_j| \right) \sum_{i=1}^N |h_i| \bar{v}_i$$

$$(7.19) \quad \leq N\bar{\kappa} \sqrt{\mathcal{E}(h)\mathcal{E}(\bar{v})} \left[ |1 - \lambda(t)| \sqrt{\mathcal{E}(\bar{v})} + \sqrt{\mathcal{E}(h)} \right],$$

where  $\bar{\kappa} := \sup_{i \in \{1, \dots, N\}} \kappa_i$ . Therefore, by (7.14) and Lemma 7.2, the inequality (7.17) holds true for some positive constant  $C$  independent of  $\epsilon \leq \epsilon_1$ . A similar argumentation holds to get (7.16).

Next, we show that

**Lemma 7.5.** *There exists  $\epsilon^* \leq \min\{\epsilon_0, \epsilon_1, \epsilon_3\}$ , so that for all  $\epsilon \leq \epsilon^*$ ,  $\mathcal{E}(h_\epsilon(t)) \rightarrow 0$  as  $t \rightarrow +\infty$ .*

Assume the lemma holds true, then we can conclude the proof of Theorem 1.4 by arguing as follows.

By combining (7.11) and Lemma 3.2, we achieve

$$(7.20) \quad \lambda'(t) \langle \bar{v}, \bar{v} \rangle = \frac{d}{dt} \sum_{i=1}^N v_i \bar{v}_i = \frac{1}{K} \sum_{i=1}^N \Gamma_i v_i \bar{v}_i.$$

Now by using  $\mathcal{E}(h) \rightarrow 0$ , we deduce that  $h \rightarrow 0$  as  $t \rightarrow \infty$  and from (7.23) we are reduced to analyse the ODE

$$\lambda'(t)\mathcal{E}(\bar{v}) = \frac{\lambda(t)}{K} \sum_{i=1}^N (\Psi_i(\bar{v}) - \Psi_i(\lambda(t)\bar{v})) \bar{v}_i^2 + o(1),$$

where

$$o(1) := \sum_{i=1}^N (\Psi_i(\lambda(t)\bar{v}) - \Psi_i(v)) \bar{v}_i v_i,$$

which by (7.16) and (7.17) satisfies

$$(7.21) \quad |o(1)| \leq \frac{C(1+\epsilon)}{K} \sqrt{\mathcal{E}(h)} \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

By Lemma 7.2, for all  $\epsilon \leq \epsilon^* \leq \epsilon_1$  we have  $\bar{c}_2 \leq \sum_{i=1}^N \bar{v}_i v_i \leq \bar{C}_2$  with  $\bar{c}_2$  and  $\bar{C}_2$  independent of  $\epsilon$ . Therefore by (7.11)  $\frac{\bar{c}_1}{\bar{\mathcal{E}}(\bar{v})} \leq \lambda(t) \leq \frac{\bar{C}_2}{\bar{\mathcal{E}}(\bar{v})}$  for all  $t > 0$ . Thus  $\lambda$  satisfies an ODE of the form

$$\lambda'(t) = \frac{\lambda(t)}{K\mathcal{E}(\bar{v})}(\theta - \tilde{\Psi}(\lambda(t))) + \frac{\lambda(t)}{K\mathcal{E}(\bar{v})}o(1),$$

which can be rewritten

$$(7.22) \quad \lambda'(t) = \frac{\lambda(t)}{K\mathcal{E}(\bar{v})}(\theta + o(1) - \tilde{\Psi}(\lambda(t))),$$

where  $\theta := \sum_{i=1}^N \Psi_i(\bar{v})\bar{v}_i^2$  and  $\tilde{\Psi}(s) := \sum_{i=1}^N \Psi_i(s\bar{v})\bar{v}_i^2$ . By construction,  $\tilde{\Psi} \in C_{loc}^1(\mathbb{R})$  and is monotone increasing. The above ODE is of logistic type with a perturbation  $o(1) \rightarrow 0$  with a non negative initial datum. Therefore, by a standard argumentation, we see that  $\lambda(t)$  converges to  $\bar{\lambda} > 0$  where  $\bar{\lambda}$  is the unique solution of  $\tilde{\Psi}(\bar{\lambda}) = \theta$ . By construction, we have  $\tilde{\Psi}(1) = \theta$ , so we deduce that  $\bar{\lambda} = 1$ . Hence,  $\lambda(t) \rightarrow 1$  and we can conclude that  $v$  converges to  $\bar{v}$ .  $\square$

Let us now turn our attention to the proof of the Lemma 7.5.

**Proof of Lemma 7.5:**

First, let us denote  $\Gamma_i := \Psi_i(\bar{v}) - \Psi_i(v)$ . Then, by combining (7.11), (7.12) and Lemma 3.2, we achieve

$$(7.23) \quad \lambda'(t) < \bar{v}, \bar{v} > = \frac{d}{dt} \sum_{i=1}^N v_i \bar{v}_i = \frac{1}{K} \sum_{i=1}^N \Gamma_i v_i \bar{v}_i$$

and

$$\begin{aligned} \frac{d\mathcal{E}(h)}{dt} &= - \sum_{i,j=1}^N \mu_{ij} \bar{v}_i \bar{v}_j \left( \frac{h_j}{\bar{v}_j} - \frac{h_j}{v_j} \right)^2 + \frac{2}{K} \sum_{i=1}^N \Gamma_i h_i^2 + \frac{2\lambda}{K} \sum_{i=1}^N \Gamma_i h_i \bar{v}_i, \\ &= - \sum_{i,j=1}^N \mu_{ij} \bar{v}_i \bar{v}_j \left( \frac{h_j}{\bar{v}_j} - \frac{h_j}{v_j} \right)^2 + \frac{2}{K} \sum_{i=1}^N \Gamma_i v_i h_i. \end{aligned}$$

Therefore using the definition of  $\Psi_i$  and with the notation  $\gamma_i := \psi_i(\bar{v}) - \psi_i(v)$ , we have

$$\begin{aligned} \frac{d\mathcal{E}(h)}{dt} &= - \sum_{i,j=1}^N \mu_{ij} \bar{v}_i \bar{v}_j \left( \frac{h_j}{\bar{v}_j} - \frac{h_j}{v_j} \right)^2 + \frac{2}{K} (\alpha(\bar{v}) - \alpha(v)) \mathcal{E}(h) \\ &\quad + \frac{2\epsilon}{K} \sum_{i=1}^N \gamma_i h_i^2 + \frac{2\epsilon\lambda(t)}{K} \sum_{i=1}^N \gamma_i \bar{v}_i h_i, \end{aligned}$$

which implies that

$$(7.24) \quad \begin{aligned} \frac{d\mathcal{E}(h)}{dt} &\leq - \sum_{i,j=1}^N \mu_{ij} \bar{v}_i \bar{v}_j \left( \frac{h_j}{\bar{v}_j} - \frac{h_j}{v_j} \right)^2 + \frac{2}{K} (\alpha(\bar{v}) + 2\epsilon\|\psi\|_\infty - \alpha(v)) \mathcal{E}(h) \\ &\quad + \frac{2\epsilon\lambda(t)}{K} \sum_{i=1}^N \gamma_i \bar{v}_i h_i. \end{aligned}$$

By (7.23), using the definition of  $\Psi_i$  we also have

$$\begin{aligned} \frac{d}{dt} \sum_{i=1}^N \bar{v}_i v_i &= \frac{1}{K} (\alpha(\bar{v}) - \alpha(v)) \sum_{i=1}^N \bar{v}_i v_i + \frac{\epsilon}{K} \sum_{i=1}^N \gamma_i \bar{v}_i v_i, \\ &\geq \frac{1}{K} (\alpha(\bar{v}) - 2\epsilon\|\psi\|_\infty - \alpha(v)) \sum_{i=1}^N \bar{v}_i v_i. \end{aligned}$$

By Lemma 7.2, for all  $\epsilon \leq \epsilon_1$ ,  $\bar{c}_2 \leq \beta(v) = \sum_{i=1}^N \bar{v}_i v_i \leq \bar{C}_2$  with  $\bar{c}_2$  and  $\bar{C}_2$  independent of  $\epsilon$ . So we have

$$\frac{d}{dt} \log(\beta(v(t))) \geq \frac{1}{K} (\alpha(\bar{v}) - 2\epsilon \|\psi\|_\infty - \alpha(v)),$$

which combined with (7.24) leads to

$$\frac{d\mathcal{E}(h)}{dt} \leq - \sum_{i,j=1}^N \mu_{ij} \bar{v}_i \bar{v}_j \left( \frac{h_j}{\bar{v}_j} - \frac{h_i}{\bar{v}_i} \right)^2 + \frac{d}{dt} \log(\beta^2(v(t))) \mathcal{E}(h) + \frac{8\epsilon \|\psi\|_\infty}{K} \mathcal{E}(h) + \frac{2\epsilon \lambda(t)}{K} \sum_{i=1}^N \gamma_i \bar{v}_i h_i.$$

By using the functional inequality, Lemma 5.3, and rearranging the terms in the above inequality we get

$$(7.25) \quad \frac{d\mathcal{E}(h)}{dt} - \mathcal{E}(h) \frac{d}{dt} \log(\beta^2(v(t))) \leq \left( -C_1(\bar{v}_\epsilon) + \frac{8\epsilon \|\psi\|_\infty}{K} \right) \mathcal{E}(h) + \frac{2\epsilon \lambda(t)}{K} \sum_{i=1}^N \gamma_i \bar{v}_i h_i,$$

where  $C_1(\epsilon)$  is the second largest eigenvalue of some associated linear problem.

Now, we estimate the last term of the above inequality. Recall that by (7.19) we have

$$\sum_{i=1}^N |\gamma_i \bar{v}_i h_i| \leq N \bar{\kappa} \sqrt{\mathcal{E}(h) \mathcal{E}(\bar{v})} \left[ |1 - \lambda(t)| \sqrt{\mathcal{E}(\bar{v})} + \sqrt{\mathcal{E}(h)} \right].$$

By combining the above estimate and (7.25), we achieve

$$(7.26) \quad \frac{d\mathcal{E}(h)}{dt} - \mathcal{E}(h) \frac{d}{dt} \log(\beta^2(v(t))) \leq (-C_1(\bar{v}_\epsilon) + \epsilon C_5) \mathcal{E}(h) + \epsilon C_4 |1 - \lambda(t)| \sqrt{\mathcal{E}(h)},$$

where  $C_5 := \frac{2}{K} \left( 4\|\psi\|_\infty + \frac{N^2 \bar{\kappa} \bar{C}_2}{\bar{c}_1} \right)$  and  $C_4 := \frac{2\bar{C}_2 N \bar{\kappa}}{K}$ .

Recall that  $C_1(\bar{v})$  is an eigenvalue, therefore for  $\epsilon$  small enough, say  $\epsilon \leq \epsilon_4$ , since  $\bar{\omega}^- \leq \bar{v} \leq \bar{\omega}^+$  and  $\|\omega^+ - \omega^-\| \rightarrow 0$  as  $\epsilon \rightarrow 0$  one has  $C_1(\bar{v}) \geq \frac{C_1(\bar{\omega}^+)}{2}$ .

With the latter estimate and by choosing  $\epsilon \leq \min(\epsilon_1, \epsilon_4)$  smaller if necessary, we get

$$(7.27) \quad \frac{d\mathcal{E}(h)}{dt} - \mathcal{E}(h) \frac{d}{dt} \log(\beta^2(v(t))) \leq -\frac{C_1(\bar{\omega}^+)}{4} \mathcal{E}(h) + \epsilon C_4 |1 - \lambda(t)| \sqrt{\mathcal{E}(h)}.$$

The proof now will follow several steps:

*Step One:* Since by (7.13) we have  $|1 - \lambda(t)| \leq \left[ 1 + \frac{N\bar{C}_2}{\bar{c}_1^2} \right]$  for all  $t$ , we claim that

**Claim 7.6.** *For all  $\epsilon \leq \min(\epsilon_1, \epsilon_4)$ , there exists  $t_0$  so that for all  $t \geq t_0$  we have*

$$\sqrt{\mathcal{E}(h)} \leq 2\epsilon \left( \frac{\bar{C}_2}{\bar{c}_2} \right) \frac{4C_4 \left[ 1 + \frac{N\bar{C}_2}{\bar{c}_1^2} \right]}{C_1(\bar{\omega}^+)}$$

Indeed, by (7.11), and Lemma 7.2 we have

$$(7.28) \quad \frac{d\mathcal{E}(h)}{dt} - \mathcal{E}(h) \frac{d}{dt} \log(\beta^2(v(t))) \leq -\frac{C_1(\bar{\omega}^+)}{4} \mathcal{E}(h) + \epsilon C_4 \left[ 1 + \frac{N\bar{C}_2}{\bar{c}_1^2} \right] \sqrt{\mathcal{E}(h)}$$

and we can check that there exists  $t_0 > 0$  so that  $\sqrt{\mathcal{E}(h(t_0))} \leq 2 \frac{\epsilon C_4 \left[ 1 + \frac{N\bar{C}_2}{\bar{c}_1^2} \right]}{C_1(\bar{\omega}^+)}$ . If not, then for all  $t > 0$   $\sqrt{\mathcal{E}(h(t))} > 2 \frac{\epsilon C_4 \left[ 1 + \frac{N\bar{C}_2}{\bar{c}_1^2} \right]}{C_1(\bar{\omega}^+)}$  and by dividing (7.28) by  $\sqrt{\mathcal{E}(h)}$  and rearranging the terms, we get that

$$(7.29) \quad \sqrt{\mathcal{E}(h)} \frac{d}{dt} \log \left( \frac{\mathcal{E}(h)}{\beta^2(v(t))} \right) \leq -\frac{C_1(\bar{\omega}^+)}{4} \sqrt{\mathcal{E}(h)} + \epsilon C_4 \left[ 1 + \frac{N\bar{C}_2}{\bar{c}_1^2} \right] < -\epsilon C_4 \left[ 1 + \frac{N\bar{C}_2}{\bar{c}_1^2} \right].$$

Thus  $F(t) := \log\left(\frac{\mathcal{E}(h)}{\beta^2(v(t))}\right)$  is a decreasing function which by Lemma 7.2, is bounded from below.

Moreover, we have for all  $t > 0$   $\sqrt{\mathcal{E}(h(t))} > 2\frac{\epsilon 4C_4 \left[1 + \frac{N\bar{C}_2}{\bar{c}_1^2}\right]}{C_1(\bar{\omega}^+)}$ . Therefore  $F$  converges as  $t$  tends to  $+\infty$  and  $\frac{dF}{dt} \rightarrow 0$ . Thus for  $t$  large enough, we get the contradiction

$$-\frac{1}{2}\epsilon C_4 \left[1 + \frac{N\bar{C}_2}{\bar{c}_1^2}\right] \leq \sqrt{\mathcal{E}(h)} \frac{d}{dt} \log\left(\frac{\mathcal{E}(h)}{\beta^2(v(t))}\right) \leq \epsilon C_4 \left[1 + \frac{N\bar{C}_2}{\bar{c}_1^2}\right].$$

Let  $\Sigma$  be the set  $\Sigma := \left\{t > t_0 \mid \sqrt{\mathcal{E}(h(t))} > 2\frac{\epsilon 4C_4 \left[1 + \frac{N\bar{C}_2}{\bar{c}_1^2}\right]}{C_1(\bar{\omega}^+)}\right\}$ . Assume that  $\Sigma$  is non empty

otherwise the claim is proved since  $\frac{\bar{C}_2}{\bar{c}_2} > 1$ . Let us denote  $t^* := \inf \Sigma$ . So at  $t^*$  we have  $\sqrt{\mathcal{E}(h(t^*))} = 2\frac{\epsilon 4C_4 \left[1 + \frac{N\bar{C}_2}{\bar{c}_1^2}\right]}{C_1(\bar{\omega}^+)}$ . Again, by dividing(7.28) by  $\sqrt{\mathcal{E}(h)}$  and rearranging the terms, on the set  $\Sigma$  we have

$$(7.30) \quad \sqrt{\mathcal{E}(h)} \frac{d}{dt} \log\left(\frac{\mathcal{E}(h)}{\beta^2(v(t))}\right) \leq -\frac{C_1(\bar{\omega}^+)}{4} \sqrt{\mathcal{E}(h)} + \epsilon C_4 \left[1 + \frac{N\bar{C}_2}{\bar{c}_1^2}\right] \leq 0.$$

Thus  $\log\left(\frac{\mathcal{E}(h)}{\beta^2(v(t))}\right)$  is a decreasing function of  $t$  for all  $t \in \Sigma$ . By arguing on each connected component of  $\Sigma$ , we can check that for all  $t \in \Sigma$

$$\sqrt{\mathcal{E}(h(t))} \leq \frac{\beta(v(t))}{\bar{c}_2} \frac{\epsilon 4C_4 \left[1 + \frac{N\bar{C}_2}{\bar{c}_1^2}\right]}{C_1(\bar{\omega}^+)}.$$

Therefore by using Lemma 7.2 for  $t \geq t^*$  we have

$$\sqrt{\mathcal{E}(h)} \leq 2\frac{\bar{C}_2}{\bar{c}_2} \frac{\epsilon 4C_4 \left[1 + \frac{N\bar{C}_2}{\bar{c}_1^2}\right]}{C_1(\bar{\omega}^+)}.$$

Hence, since  $\frac{\bar{C}_2}{\bar{c}_2} > 1$  we get for all  $t \geq t_0$ ,

$$\sqrt{\mathcal{E}(h)} \leq 2\frac{\bar{C}_2}{\bar{c}_2} \frac{\epsilon 4C_4 \left[1 + \frac{N\bar{C}_2}{\bar{c}_1^2}\right]}{C_1(\bar{\omega}^+)}.$$

□

*Step Two:* First, we define some constant quantities:

$$(7.31) \quad \delta_0 := 2\frac{\bar{C}_2}{\bar{c}_2} \frac{4C_4 \left[1 + \frac{N\bar{C}_2}{\bar{c}_1^2}\right]}{C_1(\bar{\omega}^+)},$$

$$(7.32) \quad d := \left[1 + \frac{N\bar{C}_2}{\bar{c}_1^2}\right],$$

$$(7.33) \quad \epsilon^* := \min \left\{ \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \frac{d}{4kC\delta_0}, \frac{\bar{c}_3}{4C\delta_0} \right\},$$

where the constants  $C, k$  are respectively defined in (7.16),(7.17) and in the Lemma 7.4.

By the previous step, we have that for all  $\epsilon \leq \epsilon^*$ , all  $t \geq t_0$

$$\sqrt{\mathcal{E}(h)} \leq \epsilon \delta_0.$$

We claim that

**Claim 7.7.** For all  $\epsilon \leq \epsilon^*$  there exists  $t_{\epsilon\delta_0}$  such that for all  $t \geq t_{\epsilon\delta_0}$

$$\sqrt{\mathcal{E}(h(t))} \leq \frac{\epsilon\delta_0}{2}.$$



**Proof:**

First, we can check that for  $\epsilon \leq \epsilon^*$  there exists  $t^*$  so that for all  $t \geq t^*$

$$|1 - \lambda(t)| \leq 2kC\epsilon\delta_0.$$

Indeed, by (7.22) and (7.21), we have

$$(7.34) \quad \lambda'(t) = \frac{\lambda(t)}{K\mathcal{E}(\bar{v})}(\theta + o(1) - \tilde{\Psi}(\lambda(t))),$$

with  $o(1) \leq C\epsilon\delta_0$  for  $t \geq t_0$ . Let  $\lambda_{\pm C\epsilon\delta_0}$  be the solution of the ODE

$$(7.35) \quad \lambda'_{\pm C\epsilon\delta_0}(t) = \frac{\lambda_{\pm C\epsilon\delta_0}(t)}{K\mathcal{E}(\bar{v})}(\tilde{\Psi}_{\bar{v}_\epsilon}(1) \pm C\epsilon\delta_0 - \tilde{\Psi}_{\bar{v}_\epsilon}(\lambda_{\pm C\epsilon\delta_0}(t))).$$

Since  $\epsilon \leq \frac{\bar{c}_3}{4C\delta_0}$ , for  $t \geq t_0$  we have

$$\frac{3\bar{c}_3}{4} \leq \tilde{\Psi}_{\bar{v}_\epsilon}(1) \pm C\epsilon\delta_0 \leq \frac{5\bar{C}_3}{4}.$$

Therefore  $\lambda_{\pm C\epsilon\delta_0} \rightarrow \bar{\lambda}_{\pm C\epsilon\delta_0}$  where  $\bar{\lambda}_{\pm C\epsilon\delta_0}$  are the unique positive solutions of  $\tilde{\Psi}_{\bar{v}_\epsilon}(\bar{\lambda}_{\pm C\epsilon\delta_0}) = \tilde{\Psi}_{\bar{v}_\epsilon}(1) \pm C\epsilon\delta_0$ .

Thanks to the strict monotonicity of  $\tilde{\Psi}_{\bar{v}_\epsilon}$ , we also have

$$\bar{\lambda}_{-2C\epsilon\delta_0} < \bar{\lambda}_{-C\epsilon\delta_0} < \bar{\lambda}_{C\epsilon\delta_0} < \bar{\lambda}_{2C\epsilon\delta_0},$$

where  $\bar{\lambda}_{\pm 2C\epsilon\delta_0}$  are the unique positive solutions of  $\tilde{\Psi}_{\bar{v}_\epsilon}(\bar{\lambda}_{\pm 2C\epsilon\delta_0}) = \tilde{\Psi}_{\bar{v}_\epsilon}(1) \pm 2C\epsilon\delta_0$ .

Since  $\epsilon \leq \epsilon^* \leq \frac{\bar{c}_3}{4C\delta_0}$ , it follows that

$$\frac{\bar{c}_3}{2} \leq \tilde{\Psi}_{\bar{v}_\epsilon}(1) \pm 2C\epsilon\delta_0 \leq \frac{3\bar{C}_3}{2}$$

and therefore, by Lemma (7.4), we have

$$(7.36) \quad 0 < \bar{\lambda}_{\pm 2C\epsilon\delta_0} < 1 + \tau_0.$$

Now, recall that for  $t \geq t_0$ ,  $\lambda(t)$  satisfies

$$\lambda'(t) \leq \frac{\lambda(t)}{K\mathcal{E}(\bar{v})}(\tilde{\Psi}_{\bar{v}_\epsilon}(1) + C\epsilon\delta_0 - \tilde{\Psi}_{\bar{v}_\epsilon}(\lambda(t))),$$

$$\lambda'(t) \geq \frac{\lambda(t)}{K\mathcal{E}(\bar{v})}(\tilde{\Psi}_{\bar{v}_\epsilon}(1) - C\epsilon\delta_0 - \tilde{\Psi}_{\bar{v}_\epsilon}(\lambda(t))).$$

Thus, by a standard argumentation, we can show that for  $t \geq t^*$  we have

$$\bar{\lambda}_{-2C\epsilon\delta_0} \leq \lambda(t) \leq \bar{\lambda}_{+2C\epsilon\delta_0}.$$

Therefore, for  $t \geq t^*$  we have

$$|1 - \lambda(t)| \leq \sup\{|1 - \bar{\lambda}_{-2C\epsilon\delta_0}|, |1 - \bar{\lambda}_{+2C\epsilon\delta_0}|\}.$$

By (7.36) and Lemma 7.4, since  $\epsilon \leq \epsilon^* \leq \epsilon_3$ , we deduce that for  $t \geq t^*$ ,

$$|1 - \lambda(t)| \leq k \sup\{|\tilde{\Psi}_{\bar{v}_\epsilon}(1) - \tilde{\Psi}_{\bar{v}_\epsilon}(\bar{\lambda}_{-2C\epsilon\delta_0})|, |\tilde{\Psi}_{\bar{v}_\epsilon}(1) - \tilde{\Psi}_{\bar{v}_\epsilon}(\bar{\lambda}_{+2C\epsilon\delta_0})|\} \leq 2kC\epsilon\delta_0.$$

From the latter estimate, by using (7.27), we see that for  $t \geq t^*$

$$(7.37) \quad \frac{d\mathcal{E}(h)}{dt} - \mathcal{E}(h) \frac{d}{dt} \log(\beta^2(v(t))) \leq -\frac{C_1(\bar{\omega}^+)}{4} \mathcal{E}(h) + 2\epsilon^2 C_4 k C \delta_0 \sqrt{\mathcal{E}(h)}.$$

By following the argumentation of Step one, we can show that there exists  $t_{\epsilon\delta_0}$  such that for  $t \geq t_{\epsilon\delta_0}$  we have

$$\sqrt{\mathcal{E}(h)} \leq \frac{2kC\epsilon^2\delta_0^2}{d}.$$

Hence, we have for all  $t \geq t_{\epsilon\delta_0}$

$$\sqrt{\mathcal{E}(h)} \leq \frac{\epsilon\delta_0}{2},$$

since  $\epsilon\delta_0 < \frac{d}{4kC}$ . □

*Step Three:* Since for all  $t \geq t_{\epsilon\delta_0}$ ,

$$\sqrt{\mathcal{E}(h(t))} \leq \frac{\epsilon\delta_0}{2},$$

by arguing as in the proof of Claim 7.7, we show that for all  $\epsilon \leq \epsilon^*$  there exists  $t_{\frac{\epsilon\delta_0}{2}}$  so that for all  $t \geq t_{\frac{\epsilon\delta_0}{2}}$

$$\sqrt{\mathcal{E}(h)} \leq \frac{\epsilon\delta_0}{4}.$$

By reproducing inductively the above argumentation, for all  $\epsilon \leq \epsilon^*$  we can construct a sequence  $(t_n)_{n \in \mathbb{N}}$  so that for all  $t \geq t_n$  we have

$$\sqrt{\mathcal{E}(h(t))} \leq \frac{\epsilon\delta_0}{2^n}.$$

Hence, for all  $\epsilon \leq \epsilon^*$  we deduce that

$$\lim_{t \rightarrow \infty} \mathcal{E}(h(t)) \rightarrow 0. \quad \square$$

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