Modulus of continuity of some conditionally sub-Gaussian fields, application to stable random fields

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Abstract

In this paper we study modulus of continuity and rate of convergence of series of conditionally sub-Gaussian random fields. This framework includes both classical series representations of Gaussian fields and LePage series representations of stable fields. We enlighten their anisotropic properties by using an adapted quasi-metric instead of the classical Euclidean norm. We specify our assumptions in the case of shot noise series where arrival times of a Poisson process are involved. This allows us to state unified results for harmonizable (multi)operator scaling stable random fields through their LePage series representation, as well as to study sample path properties of their multistable analogous.

Keywords: sub-Gaussian, stable and multistable random fields, operator scaling property, Hölder regularity.

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1 Introduction

In recent years, lots of new random fields and processes have been defined to propose new models for rough real data. To cite a few of them let us mention the (multi)fractional Brownian fields (see [6] for instance) or their stable analogous the linear (multi)fractional stable process [33] and the harmonizable (multi)fractional stable process [12] and some anisotropic generalizations the (multi)fractional Brownian and stable sheets [3, 4] and the (multi)operator scaling Gaussian and stable fields [9, 8]. In the Gaussian setting, sample path regularity relies on mean square regularity. Many studies have investigated finer properties such as modulus of continuity. A powerful technique consists in considering

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a representation of the field as a series of random fields, using for instance Karhunen Loeve decomposition (see [1] Chapter 3), Fourier or wavelet series (as in [16] or [5]). This also allows generalizations to non-Gaussian framework using for instance LePage series [24, 23] for stable distributions (see [19] for instance). Actually, following previous works of LePage [23] and Marcus and Pisier [26], Kono and Maejima proved in [20] that, for $\alpha \in (0,2)$, a symmetric complex α-stable random variable may be represented as a convergent shot noise series of the form

$$\sum_{n=1}^{+\infty} T_n^{-1/\alpha} X_n, \tag{1}$$

with $(T_n)_{n\geq 1}$ the sequence of arrival times of a Poisson process of intensity 1, and $(X_n)_{n\geq 1}$ a sequence of independent identically distributed (i.i.d.) isotropic complex-valued random variables, which is assumed to be independent of $(T_n)_{n\geq 1}$ and such that $\mathbb{E}(|X_1|^\alpha) < +\infty$. When $X_n = V_n g_n$ with $(g_n)_{n\geq 1}$ a sequence of i.i.d. isotropic complex-valued Gaussian random variables independent of $(V_n, T_n)_{n\geq 1}$, the series may be considered as a conditional Gaussian series. Another classical representation consists in choosing $X_n = V_n \varepsilon_n$ with $(\varepsilon_n)_{n\geq 1}$ a sequence of i.i.d. Rademacher random variables i.e. such that $\mathbb{P}(\varepsilon_n = 1) = \mathbb{P}(\varepsilon_n = -1) = 1/2$. Both g_n and ε_n are sub-Gaussian random variables. Sub-Gaussian random variables have been introduced in [16] for the study of random Fourier series. Their main property is that their tail distributions behave like the Gaussian ones so that sample path properties of sub-Gaussian fields also rely on their mean square regularity.

In this paper we study the sample path regularity of the complex-valued series of conditionally sub-Gaussian fields defined as

$$S(x) = \sum_{n=1}^{+\infty} W_n(x) g_n, \text{ for } x \in K_d \subset \mathbb{R}^d, \tag{2}$$

with $(g_n)_{n\geq 1}$ a sequence of independent symmetric sub-Gaussian complex random variables and $(W_n)_{n\geq 1}$ a sequence of complex random fields defined on K_d, a compact d-dimensional interval, independent of $(g_n)_{n\geq 1}$. In this setting we give sufficient assumptions on the sequence $(W_n)_{n\geq 1}$ to get an upper bound of the modulus of continuity of S as well as a uniform rate of convergence, using Lévy inequalities (see [22] for instance) available for series of independent symmetric variables. Then, we focus on shot noises series

$$S(\alpha, u) = \sum_{n=1}^{+\infty} T_n^{-1/\alpha} V_n(\alpha, u) g_n, \ x = (\alpha, u) \in K_{d+1} \subset (0,2) \times \mathbb{R}^d, \tag{3}$$

where $(T_n)_{n\geq 1}$ is as previously the sequence of arrival times of a Poisson process of intensity 1. Assuming the independence of $(T_n)_{n\geq 1}$, $(V_n)_{n\geq 1}$ and $(g_n)_{n\geq 1}$, we state some more convenient conditions based on moments of V_n to ensure that the main assumptions of this paper are fulfilled. In particular when $V_n(\alpha, u) := X_n$ is a symmetric random variable, one of our main result gives a uniform rate of convergence of the shot noise series (1) in α on any compact $K_1 = [a, b] \subset (0, 2)$, which improves the results obtained in [11] on the convergence of such series.

Moreover, we also investigate the case of LePage random series, for which the random fields $(V_n, g_n)_{n\geq 1}$ are chosen to ensure that for all fixed α the series $S(\alpha, \cdot)$ have the same finite dimensional distributions as an isotropic α-stable random field defined through a stochastic integral with

respect to an isotropic α-stable random measure. On the one hand, it allows to include in a general setting some sample path regularity results already obtained in [7, 8] for harmonizable (multi)-operator scaling stable random fields. On the other hand, considering α as a function of $u \in \mathbb{R}^d$, we also investigate sample path properties of multistable random fields that have been introduced in [14]. To illustrate our results, we focus on harmonizable random fields.

The paper falls into the following parts. In Section 2 we recall definition and properties of sub-Gaussian random variables and state our first assumption needed to ensure that the random field S is well-defined. We also introduce a notion of anisotropic local regularity, which is obtained by replacing the isotropic Euclidean norm of $\mathbb{R}^d$ by a quasi-metric that can reveal the anisotropy of the random fields. Section 3 is devoted to our main results concerning both local modulus of continuity of the random field S defined by the series (2) and rate of convergence of this series. Section 4 deals with the particular setting of shot noise series, that is of random fields given by (3). In Section 5 we investigate the case of LePage series and apply our results to study the sample path regularity of stable or even multistable random fields. Technical proofs are postponed to Appendix for reader convenience.

2 Preliminaries

2.1 Sub-Gaussian random variables

Real-valued sub-Gaussian random variables have been defined by [16] and [10] has studied the structure of the class of these random variables and proposed some conditions for continuity of real-valued sub-Gaussian random fields. In this paper we focus on conditionally complex-valued sub-Gaussian random fields, where a complex sub-Gaussian random variable is defined as follows.

Definition 2.1. *A complex-valued random variable Z is sub-Gaussian if there exists $s \in [0, +\infty)$ such that*

$$\forall z \in \mathbb{C}, \mathbb{E}\Big(\mathrm{e}^{\Re(\bar{z}Z)}\Big) \leq \mathrm{e}^{\frac{s^2|z|^2}{2}}. \tag{4}$$

Remark 2.1. *For a real-valued random variable Z, this definition coincides with the definition in [16]. The parameter s is not unique. Kahane [16] called the smallest s such that* (4) *holds the Gaussian shift of the sub-Gaussian variable Z. In this paper, if* (4) *is fulfilled, we say that Z is sub-Gaussian with parameter s.*

Remark 2.2. *A complex-valued random variable Z is sub-Gaussian if and only if $\Re(Z)$ and $\Im(Z)$ are real sub-Gaussian random variables. Note that if Z is sub-Gaussian with parameter s then $\mathbb{E}(\Re(Z)) = \mathbb{E}(\Im(Z)) = 0$ and $\mathbb{E}(\Re(Z)^2) \leq s^2$ as well as $\mathbb{E}(\Im(Z)^2) \leq s^2$. Moreover, our definition includes also complex sub-Gaussian random variables as defined in [15] in the more general setting of random variables with values in a Banach space.*

The main property of sub-Gaussian random variables is that their tail distributions decrease exponentially as the Gaussian ones (see Lemma A.1). Moreover, considering convergent series of independent symmetric sub-Gaussian random variables, a uniform rate of decrease is available and the limit

remains a sub-Gaussian random variable. This result, stated below, is one of the main tool we use to study sample path properties of conditionally sub-Gaussian random fields.

Proposition 2.1. *Let $(g_n)_{n\geq 1}$ be a sequence of independent symmetric sub-Gaussian random variables with parameter $s = 1$. Let us consider a complex-valued sequence $a = (a_n)_{n\geq 1}$ such that*

$$\|a\|_{\ell^2}^2 = \sum_{n=1}^{+\infty} |a_n|^2 < +\infty.$$

1. *Then for any $t \in (0, +\infty)$*

$$\mathbb{P}\left(\sup_{N\in\mathbb{N}\setminus\{0\}} \left| \sum_{n=1}^{N} a_n g_n \right| > t\|a\|_{\ell^2} \right) \leq 8\, \mathrm{e}^{-\frac{t^2}{8}}.$$

2. *Moreover, the series $\sum a_n g_n$ converges almost surely, and the limit $\sum_{n=1}^{+\infty} a_n g_n$ is a sub-Gaussian random variable with parameter $\|a\|_{\ell^2}$.*

Proof. See Appendix, Section A. □

Remark 2.3. *In the previous proposition, assuming that the parameter $s = 1$ is not restrictive since one can replace a_n by $a_n s_n$ and g_n by g_n/s_n when g_n is sub-Gaussian with parameter $s_n > 0$.*

2.2 Conditionally sub-Gaussian series

In the whole paper, for $d \geq 1$, we note $K_d = \prod_{j=1}^{d} [a_j, b_j]$ with $a_j \leq b_j$ a compact d-dimensional interval of $\mathbb{R}^d$. We consider the sequence $(S_N)_{N\in\mathbb{N}}$ defined on $K_d \subset \mathbb{R}^d$ by for all $N \in \mathbb{N}$ and $x \in K_d$,

$$S_N(x) = \sum_{n=1}^{N} W_n(x) g_n. \tag{5}$$

Here and in the sequel we use the convention that $\sum_{n=1}^{0} = 0$. Throughout the paper, we assume that the sequence $(W_n, g_n)_{n\geq 1}$ satisfies the following assumption.

Assumption 1. *Let $(g_n)_{n\geq 1}$ and $(W_n)_{n\geq 1}$ be independent sequences of random variables.*

- *$(g_n)_{n\geq 1}$ is a sequence of independent symmetric complex-valued sub-Gaussian random variables with parameter $s = 1$.*
- *$(W_n)_{n\geq 1}$ is a sequence of complex-valued continuous random fields defined on K_d and such that*

$$\forall x \in K_d,\ \sum_{n=1}^{+\infty} |W_n(x)|^2 < +\infty \quad \text{almost surely.}$$

Under Assumption 1, conditionally to $(W_n)_{n\geq 1}$, each S_N is a sub-Gaussian random field defined on K_d. Moreover, for each x, Proposition 2.1 and Fubini Theorem lead to the almost sure convergence of the series (5) as $N \to +\infty$. The almost sure limit field S defined by

$$S(x) = \sum_{n=1}^{+\infty} W_n(x) g_n, \quad x \in K_d \subset \mathbb{R}^d, \tag{6}$$

is then a conditionally sub-Gaussian random field. In the sequel, we study almost sure uniform convergence and rate of uniform convergence of $(S_N)_{N\in\mathbb{N}}$ as well as the sample path properties of the limit field S.

Assume first that each g_n is a Gaussian random variable and that each W_n is a deterministic random field, which implies that S is a Gaussian centered random field. Then, it is well-known that its sample path properties are given by the behavior of

$$s(x,y) := \left(\sum_{n=1}^{+\infty} |W_n(x) - W_n(y)|^2 \right)^{1/2}, \quad x, y \in K_d, \tag{7}$$

which is proportional to the square root of the variogram of S. In the following, we see that under Assumption 1, the behavior of S is still linked with the behavior of the parameter s. In this more general framework, a key tool is to remark that conditionally to $(W_n)_{n\geq 1}$, S is a sub-Gaussian random field and the random variable $S(x) - S(y)$ is sub-Gaussian with parameter $s(x,y)$ defined by (7).

In the following, the random field S, and then the parameter s, is a priori anisotropic. Therefore, next section deals with an anisotropic generalization of the classical Hölder regularity. In other words, it introduces a notion of regularity which takes into account the anisotropy of the random fields under study.

2.3 Anisotropic local regularity

Let us first recall the notion of quasi-metric (see [29] for instance), which is more adapted to our framework.

Definition 2.2. *A continuous function $\rho : \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty)$ is called a quasi-metric on $\mathbb{R}^d$ if*

1. *ρ is faithful i.e. $\rho(x,y) = 0$ iff $x = y$;*
2. *ρ is symmetric i.e. $\rho(x,y) = \rho(y,x)$;*
3. *ρ satisfies a quasi-triangle inequality: there exists a constant $\kappa \geq 1$ such that*

$$\forall x, y, z \in \mathbb{R}^d,\ \rho(x,z) \leq \kappa \left(\rho(x,y) + \rho(y,z) \right).$$

Observe that a continuous function ρ is a metric on $\mathbb{R}^d$ if and only if ρ is a quasi-metric on $\mathbb{R}^d$ which satisfies Assertion *3.* with $\kappa = 1$. In particular, the Euclidean distance, defined by

$$\rho(x,y) = \|x - y\| = \left(\sum_{i=1}^{d} |x_i - y_i|^2 \right)^{1/2}, \quad \text{for } x = (x_i)_{1\leq i\leq d},\ y = (y_i)_{1\leq i\leq d} \in \mathbb{R}^d,$$

is an isotropic quasi-metric and its following anisotropic generalization

$$(x,y) \mapsto \rho(x,y) := \left(\sum_{i=1}^{d} |x_i - y_i|^{p/a_i} \right)^{1/p} \quad \text{where } p > 0 \text{ and } a_1, \ldots, a_d > 0,$$

is also a quasi-metric. Such quasi-metrics are particular cases of the following general example.

Example 2.1. *Let us consider E a real $d \times d$ matrix whose eigenvalues have positive real parts and $\tau_{_E} : \mathbb{R}^d \to \mathbb{R}^+$ a continuous even function such that*

i) $\tau_{_E}(x) > 0$ for all $x \neq 0$;

ii) $\tau_{_E}(r^E x) = r\tau_{_E}(x)$ for all $r > 0$ and $x \in \mathbb{R}^d$.

Then let us consider the continuous function $\rho_{_E}$, defined on $\mathbb{R}^d \times \mathbb{R}^d$ by

$$\rho_{_E}(x,y) = \tau_{_E}(x-y).$$

Then, by definition of $\tau_{_E}$, $\rho_{_E}$ is faithful and symmetric. Moreover, by Lemma 2.2 of [9], $\rho_{_E}$ also satisfies a quasi-triangle inequality. Hence, $\rho_{_E}$ is a quasi-metric on $\mathbb{R}^d$ and it is adapted to study operator scaling random fields (see [9, 7] for example).

Let us remark that we may define a quasi-metric $\rho_{_E}$ for any matrix E whose eigenvalues have positive real parts. However since $\rho_{_E}^{\beta}$ will define a quasi-metric for E/β whatever $\beta > 0$ is, we may restrict our study to matrix E whose eigenvalues have real parts greater than one. Then, by Proposition 3.5 of [8], there exist $0 < \underline{H} \leq \overline{H} \leq 1$ and two positive finite constants $c_{_{2,1}}, c_{_{2,2}} > 0$ such that for all $x, y \in \mathbb{R}^d$,

$$c_{_{2,1}} \min(\|x-y\|^{\overline{H}}, \|x-y\|^{\underline{H}}) \leq \rho_{_E}(x,y) \leq c_{_{2,2}} \max(\|x-y\|^{\overline{H}}, \|x-y\|^{\underline{H}}).$$

In [8, 7], this comparison is one of the main tool in the study of the regularity of some stable anisotropic random fields. Therefore, throughout the paper, we consider a quasi-metric ρ such that there exist $0 < \underline{H} \leq \overline{H} \leq 1$ and two positive finite constants $c_{_{2,1}}, c_{_{2,2}} > 0$ such that for all $x, y \in \mathbb{R}^d$, with $\|x-y\| \leq 1$,

$$c_{_{2,1}} \|x-y\|^{\overline{H}} \leq \rho(x,y) \leq c_{_{2,2}} \|x-y\|^{\underline{H}}. \tag{8}$$

Before we introduce the anisotropic regularity used in the following, let us briefly comment this assumption done on ρ.

Remark 2.4.

1. *The upper bound is needed in the sequel to construct a particular 2^{-k} net for ρ, whose cardinality can be estimated using the lower bound.*

2. *Let us quote that using the quasi-triangle inequality this implies that for all $r \geq 1$ there exist two finite constants $c_{_{2,1}}(r), c_{_{2,2}}(r) > 0$ such that for all $x, y \in \mathbb{R}^d$, with $\|x-y\| \leq r$,*

$$c_{_{2,1}}(r) \|x-y\|^{\overline{H}} \leq \rho(x,y) \leq c_{_{2,2}}(r) \|x-y\|^{\underline{H}}. \tag{9}$$

3. *It is not restrictive to assume that* $\overline{H} \leq 1$ *since for any* $c > 0$, ρ^c *is also a quasi-metric.*

We will consider the following anisotropic local regularity property.

Definition 2.3. *Let* $\beta \in (0,1]$ *and* $\eta \in \mathbb{R}$. *Let* $x_0 \in K_d$ *with* $K_d \subset \mathbb{R}^d$. *A real-valued function* f *defined on* K_d *belongs to* $\mathcal{H}_{\rho,K_d}(x_0,\beta,\eta)$ *if there exist* $\gamma \in (0,1)$ *and* $C \in (0,+\infty)$ *such that*

$$|f(x)-f(y)| \leq C\,\rho(x,y)^{\beta}|\log(\rho(x,y))|^{\eta}$$

for all $x,y \in B(x_0,\gamma)\cap K_d := \{z \in T; \|z-x_0\| \leq \gamma\}$. *Moreover* f *belongs to* $\mathcal{H}_{\rho}(K_d,\beta,\eta)$ *if there exists* $C \in (0,+\infty)$ *such that*

$$\forall x,y \in K_d,\ |f(x)-f(y)| \leq C\,\rho(x,y)^{\beta}|\log(\min(\rho(x,y),1/2))|^{\eta}$$

Remark 2.5.

1. *A function in* $\mathcal{H}_{\rho}(K_d,\beta,0)$ *may be view as a Lipschitz function on an homogeneous space [25]. Note also that when* ρ *is the Euclidean distance, for any* $\beta \leq 1$ *and* $\eta \leq 0$, *the set* $\mathcal{H}_{\rho}(K_d,\beta,\eta)$ *(respectively* $\mathcal{H}_{\rho,K_d}(x_0,\beta,\eta)$*) is included in the set of Hölder functions of order* β *on* K_d *(respectively around* x_0*).*

2. *Assuming* $\beta \leq 1$ *is not restrictive since, for any* $c > 0$, ρ^c *is also a quasi-metric, as already noticed.*

3. *When* f *belongs to* $\mathcal{H}_{\rho}(K_d,\beta,\eta)$ *it follows that* $f \in \mathcal{H}_{\rho,K_d}(x_0,\beta,\eta)$ *for all* $x_0 \in K_d$. *The converse is also true since* K_d *is a compact set, using a finite covering of* K_d, *Equation* (8) *and the fact that there exists* $h_0 > 0$ *such that* $h \mapsto h^{\beta}|\log(h)|^{\eta}$ *is increasing on* $[0,h_0)$.

The introduction of the logarithmic term appears naturally when considering Gaussian random fields. Actually, [6] proves that for all $\beta \in (0,1]$, a large class of elliptic Gaussian random fields X_β, including the famous fractional Brownian fields, belongs a.s. to $\mathcal{H}_{\rho,K_d}(x_0,\beta,1/2)$ with ρ the Euclidean distance (see Theorem 1.3 in [6]). Moreover, Yimin Xiao also gives some anisotropic examples of Gaussian fields belonging a.s. to $\mathcal{H}_{\rho,K_d}(x_0,1,1/2)$ for some anisotropic quasi-distance $\rho = \rho_E$ associated with E a diagonal matrix (see Theorem 4.2 of [34]). Finally, in [8], we construct stable and Gaussian random fields belonging a.s. to $\mathcal{H}_{\rho_{x_0},K_d}(x_0,1-\varepsilon,0)$ for some convenient ρ_{x_0} (see Theorem 4.6 in [8]).

3 Main results on conditionally sub-Gaussian series

3.1 Local modulus of continuity

In this section, we first give an upper bound of the local modulus of continuity of S defined by (5) under the following local assumption on the conditional parameter (7).

Assumption 2. *Let $x_0 \in K_d$ with $K_d = \prod_{j=1}^d [a_j, b_j] \subset \mathbb{R}^d$. Let us consider ρ a quasi-metric on $\mathbb{R}^d$ satisfying Equation (8). Assume also that almost surely there exist $\gamma > 0$, $\beta \in (0, 1]$, $\eta \in \mathbb{R}$ and $C \in (0, +\infty)$ such that*

$$\forall x, y \in B(x_0, \gamma) \cap K_d,\ s(x, y) \leq C\, \rho(x, y)^\beta |\log(\rho(x, y))|^\eta,$$

where we recall that the conditional parameter s is given by (7).

Let us first comment this assumption.

Remark 3.1. *Note that γ, β, η and C may be random variables. Let us also emphasize that γ, β, η, C and the quasi-metric ρ may depend on x_0.*

Let us now state the main result of this section on the modulus of continuity. The main difference with [20, 7, 8] is that we do not only consider the limit random field S but obtain a uniform upper bound in N for the modulus of continuity of S_N.

Theorem 3.1. *Assume that Assumptions 1 and 2 are fulfilled. Then, almost surely, there exist $\gamma^* \in (0, \gamma)$ and $C \in (0, +\infty)$ such that for all $x, y \in B(x_0, \gamma^*) \cap K_d$,*

$$\sup_{N \in \mathbb{N}} |S_N(x) - S_N(y)| \leq C \rho(x, y)^\beta |\log \rho(x, y)|^{\eta + 1/2}.$$

Moreover, almost surely $(S_N)_{N \in \mathbb{N}}$ converges uniformly on $B(x_0, \gamma^*) \cap K_d$ to S and the limit S belongs to $\mathcal{H}_{\rho, K_d}(x_0, \beta, \eta + 1/2)$.

Proof. See Appendix, Section B.1. □

Remark 3.2. *Under Assumptions 1 and 2, according to the previous theorem, S is almost surely continuous at x_0.*

Remark 3.3. *If Assertion 2 holds for any $x_0 \in K_d$ (with the same quasi-metric ρ), then Theorem 3.1 holds replacing $B(x_0, \gamma^*) \cap K_d$ by all the set K_d. This is simply deduced by covering the compact set K_d. Moreover, if in Assertion 2, γ is deterministic, then Theorem 3.1 holds with $\gamma^* = \gamma$ and S belongs to $\mathcal{H}_\rho(B(x_0, \gamma) \cap K_d, \beta, \eta + 1/2)$.*

3.2 Rate of almost sure uniform convergence

This section is concerned with the rate of uniform convergence of the series $(S_N)_{N \in \mathbb{N}}$. Let us first recall that the partial sum S_N is defined by (5). Under Assumption 1, this series converges to S and, for any integer N, we may consider the rest

$$R_N(x) = S(x) - S_N(x) = \sum_{n=N+1}^{+\infty} W_n(x) g_n, \quad x \in K_d \subset \mathbb{R}^d.$$

Then, conditionally to $(W_n)_{n\geq 1}$, $R_N(x) - R_N(y)$ is a sub-Gaussian random variable with parameter

$$r_N(x,y) = \left(\sum_{n=N+1}^{+\infty} |W_n(x) - W_n(y)|^2 \right)^{1/2}, \quad x, y \in K_d. \tag{10}$$

Observe that $R_0 = S$ and of course that $r_0(x,y) = s(x,y)$. To obtain a rate of uniform convergence for the sequence $(S_N)_{N\in\mathbb{N}}$, the general assumption relies on a rate of convergence for the sequence of parameters $(r_N)_{N\in\mathbb{N}}$.

Assumption 3. *Let $x_0 \in K_d$ with $K_d = \prod_{j=1}^d [a_j, b_j] \subset \mathbb{R}^d \subset \mathbb{R}^d$ and let ρ be a quasi-metric on $\mathbb{R}^d$ satisfying (8). Assume that almost surely there exist $\gamma > 0$, $\beta \in (0,1]$, $\eta \in \mathbb{R}$ and $(b(N))_{N\in\mathbb{N}}$ a positive sequence such that*

$$r_N(x,y) \leq b(N)\rho(x,y)^\beta |\log(\rho(x,y))|^\eta \tag{11}$$

for any $N \in \mathbb{N}$ and any $x, y \in B(x_0,\gamma) \cap K_d$.

Note that Assumption 3 implies Assumption 2. Then, according to Theorem 3.1, almost surely, there exists $\gamma^* \in (0,\gamma)$ such that $R_N = S - S_N$ is continuous on $B(x_0,\gamma^*)$. The following theorem precises the modulus of continuity of R_N with respect to N and a rate of uniform convergence.

Theorem 3.2. *Assume that Assumptions 1 and 3 are fulfilled.*

1. *Then, almost surely, there exists $\gamma^* \in (0,\gamma)$ and $C \in (0,+\infty)$ such that*

$$|R_N(x) - R_N(y)| \leq Cb(N)\sqrt{\log(N+2)}\rho(x,y)^\beta |\log \rho(x,y)|^{\eta+1/2}$$

   for all $N \in \mathbb{N}$ and all $x, y \in B(x_0,\gamma^*) \cap K_d$.

2. *Moreover, if almost surely, for all $N \in \mathbb{N}$,*

$$|R_N(x_0)| \leq b(N)\sqrt{\log(N+2)}, \tag{12}$$

   then, almost surely, there exists $\gamma^* \in (0,\gamma)$ and $C \in (0,+\infty)$ such that

$$|R_N(x)| \leq Cb(N)\sqrt{\log(N+2)}$$

   for all $N \in \mathbb{N}$ and all $x \in B(x_0,\gamma^*) \cap K_d$.

Proof. See Appendix, Section B.2. □

Remark 3.4. *An analoguous of Remark 3.3 holds for strengthening the previous local theorem to get uniform results on K_d or on $B(x_0,\gamma) \cap K_d$ when γ is deterministic.*

4 Shot noise series

4.1 Preliminary results

In this section, we consider the sequence of shot noise series defined by

$$\forall N \in \mathbb{N},\, \forall \alpha \in K_1 = [a,b] \subset (0,2),\, S_N^*(\alpha) = \sum_{n=1}^{N} T_n^{-1/\alpha} X_n,$$

where for all $n \geq 1$, the random variable T_n is the nth arrival time of a Poisson process with intensity 1 and $(X_n)_{n\geq 1}$ is a sequence of i.i.d. symmetric random variables, which is assumed independent of $(T_n)_{n\geq 1}$. Let us first recall that $S_N^*(\alpha)$ converges almost surely to $S^*(\alpha)$ an α-stable random variable as soon as $X_n \in L^\alpha$ (see [32] for instance). Under a strengthened assumption on the integrability of X_n, rate of pointwise almost sure convergence and rate of absolute convergence have also been given in Theorems 2.1 and 2.2 of [11].

Since $(X_n)_{n\geq 1}$ may not be a sequence of sub-Gaussian random variables, we cannot apply the previous section to the sequence $(S_N^*)_{N\in\mathbb{N}}$. However, due to symmetry of $(X_n)_{n\geq 1}$,

$$(X_n)_{n\geq 1} \overset{(d)}{=} (X_n g_n)_{n\geq 1}$$

with $(g_n)_{n\geq 1}$ a Rademacher sequence independent of $(X_n, T_n)_{n\geq 1}$ and we can apply our results to

$$S_N(\alpha) = \sum_{n=1}^{N} W_n(\alpha) g_n \quad \text{with } W_n(\alpha) := T_n^{-1/\alpha} X_n.$$

Moreover, looking at the proof of Theorems 3.1 and 3.2, we see that in these theorems S_N (respectively $R_N = S - S_N$) can replaced by S_N^* (respectively $R_N^* = S^* - S_N^*$), see the proof of next theorem for details. Then, assuming that X_n is sufficiently integrable, we obtain the uniform convergence of S_N^* on a deterministic compact interval $K_1 = [a,b] \subset (0,2)$ and a rate of uniform convergence. These results, stated in the following theorem, strengthen Theorem 2.1 of [11] which deals with the pointwise rate of convergence.

Theorem 4.1. *For any integer $n \geq 1$, let T_n be the nth arrival time of a Poisson process with intensity 1. Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. symmetric random variables, which is assumed independent of $(T_n)_{n\geq 1}$. Furthermore assume that $\mathbb{E}(|X_1|^{2p}) < +\infty$ for some $p > 0$.*

1. *Then, almost surely, for all $b \in (0, \min(2, 2p))$ and for all $a \in (0, b]$, the sequence of partial sums $(S_N^*)_{N\in\mathbb{N}}$ converges uniformly on $[a,b]$.*
2. *Then, almost surely, for all $b \in (0, \min(2, 2p))$ and for all $a \in (0, b)$, for all $p' > 0$ with $1/p' \in (0, 1/b - 1/\min(2p, 2))$,*

$$\sup_{N\in\mathbb{N}} \sup_{\alpha\in[a,b]} N^{1/p'} \left| \sum_{n=N+1}^{+\infty} T_n^{-1/\alpha} X_n \right| < +\infty.$$

Proof. See Appendix, Section C.1. □

4.2 Modulus of continuity and rate of convergence of shot noise series

In this section, we focus on some general shot noise series, which are particular examples of conditionally sub-Gaussian series. For this purpose we assume that the following assumption is fulfilled in all this Section 4.2.

Assumption 4. *We consider independent sequences $(T_n)_{n\geq 1}$, $(V_n)_{n\geq 1}$ and $(g_n)_{n\geq 1}$ that satisfy the following conditions.*

(1). $(g_n)_{n\geq 1}$ is a sequence of independent complex-valued symmetric sub-Gaussian random variables with parameter $s = 1$.

(2). T_n is the nth arrival time of a Poisson process with intensity 1.

(3). $(V_n)_{n\geq 1}$ is a sequence of i.i.d. complex-valued random fields defined on $K_{d+1} \subset (0,2) \times \mathbb{R}^d$.

(4). For any $(\alpha, u) \in K_{d+1} \subset (0,2) \times \mathbb{R}^d$, $V_n(\alpha, u) \in L^\alpha$.

For any integer $n \geq 1$, we consider the complex-valued random field W_n defined by

$$W_n(\alpha, u) := T_n^{-1/\alpha} V_n(\alpha, u), \quad (\alpha, u) \in K_{d+1} \subset (0,2) \times \mathbb{R}^d.$$

Observe that the sequences $(W_n)_{n\geq 1}$ and $(g_n)_{n\geq 1}$ are independent. Moreover, since $|V_n(\alpha, u)|^2 \in L^{\alpha/2}$ and $\alpha/2 \in (0,1)$, according to Theorem 1.4.5 of [32],

$$\sum_{n=1}^{+\infty} |W_n(\alpha, u)|^2 = \sum_{n=1}^{+\infty} T_n^{-2/\alpha} |V_n(\alpha, u)|^2 < +\infty \quad \text{almost surely.}$$

Therefore the sequences $(W_n)_{n\geq 1}$ and $(g_n)_{n\geq 1}$ satisfy Assumption 1 and we can apply Section 3 to the shot noise series S defined on $K_{d+1} \subset (0,2) \times \mathbb{R}^d \subset \mathbb{R}^{d+1}$ by

$$S(\alpha, u) = \sum_{n=1}^{+\infty} W_n(\alpha, u) g_n = \sum_{n=1}^{+\infty} T_n^{-1/\alpha} V_n(\alpha, u) g_n. \tag{13}$$

Keeping the notation of the previous sections, for all $N \in \mathbb{N}$,

$$S_N(\alpha, u) = \sum_{n=1}^{N} T_n^{-1/\alpha} V_n(\alpha, u) g_n \quad \text{and} \quad R_N(\alpha, u) = \sum_{n=N+1}^{+\infty} T_n^{-1/\alpha} V_n(\alpha, u) g_n. \tag{14}$$

Before we study, the modulus of continuity of S and rate of convergence of $(S_N)_{N\in\mathbb{N}}$, let us state some remarks.

Remark 4.1. *Assume that Conditions (1)–(3) of Assumption 4 are fulfilled and that $(g_n)_{n\geq 1}$ is a sequence of i.i.d. random variables. Then Remark 2.6 of [31] proves that Condition (4) is a necessary and sufficient condition for the almost sure convergence of $(S_N(\alpha, u))_{N\in\mathbb{N}}$ for each $(\alpha, u) \in K_{d+1}$. Note that by Itô-Nisio Theorem (see for example Theorem 6.1 of [22]), it is also a necessary and sufficient condition for the convergence in distribution of the sequence $(S_N(\alpha, u))_{N\in\mathbb{N}}$. Then, Condition (4) is not a strong assumption and is clearly essential to ensure that $S(\alpha, u)$ is well-defined by* (13).

Remark 4.2. *Assume that Assumption 4 is fulfilled with $(g_n)_{n\geq 1}$ a sequence of i.i.d. random variables. Then, it is well-known that for each $\alpha \in (0,2)$, $S(\alpha,\cdot)$ is an α-stable symmetric random field, as field in variable u. In Section 5, we will focus on α-stable random fields defined through a stochastic integral and see that, up to a multiplicative constant, such a random field X_α has the same finite distributions as $S(\alpha,\cdot)$ for a suitable choice of $(g_n, V_n)_{n\geq 1}$. The sample path regularity of S in its variable α is not needed to obtain an upper bound of the modulus of continuity of X_α. Nevertheless, this regularity is useful to deal with multistable random fields (see Section 5.3).*

The sequel of this section is devoted to simple criteria, based on some moments of V_n, which ensure that Assumption 3 (and then Assumption 2) is fulfilled. More precisely, the results given below help us to give simple conditions in order to get Assumption 3 and (12) satisfied with a sequence $(b(N))_{N\in\mathbb{N}}$ of the form $b(N) = (N+1)^{-1/p'}$ for some convenient $p' > 0$. Then, under Assumption 4, all the results of Section 3 hold.

Theorem 4.2. *Let $x_0 = (\alpha_0, u_0) \in K_{d+1}$ with $K_{d+1} = [a,b] \times \prod_{j=1}^d [a_j, b_j] \subset (0,2) \times \mathbb{R}^d$ and let ρ be a quasi-metric on $\mathbb{R}^{d+1}$ satisfying Equation (8). Assume that Assumption 4 is fulfilled and that there exist $r \in (0,+\infty)$, $\beta \in (0,1]$, $\eta \in \mathbb{R}$ and $p \in (b/2, +\infty)$ such that $\mathbb{E}\Big(|V_1(x_0)|^{2p}\Big) < +\infty$ and*

$$\mathbb{E}\left(\left[\sup_{\substack{x,y\in K_{d+1}\\ 0<\|x-y\|\leq r}} \frac{|V_1(x)-V_1(y)|}{\rho(x,y)^\beta |\log \rho(x,y)|^\eta}\right]^{2p}\right) < \infty. \tag{15}$$

Let us now set consider the random fields S and S_N given by (13) and (14).

1. *Then, almost surely $(S_N)_{N\in\mathbb{N}}$ converges uniformly on K_{d+1} and its limit S belongs almost surely to $\mathcal{H}_\rho(K_{d+1}, \beta, \max(\eta, 0) + 1/2)$.*
2. *Moreover, when $p' > 0$ is such that $1/p' \in (0, 1/b - 1/\min(2p, 2))$, almost surely*

$$\sup_{N\geq 1} N^{1/p'} \sup_{x\in K_{d+1}} |S(x) - S_N(x)| < +\infty.$$

Proof. See Appendix, Section C.2. □

Example 4.1. *One can consider for V_1 a fractional Brownian field on $\mathbb{R}^d$ with Hurst parameter H so that (15) is satisfied for all $p > 0$ with $\rho(x,y) = \|x-y\|$ for $\|\cdot\|$ the Euclidean norm on $\mathbb{R}^d$, $\beta = H$ and $\eta = 1/2$ (see Theorem 1.3 of [6] for instance).*

Let us now present a method (similar to those used in [19, 7, 8] to bound some conditional variance) to establish (15).

Proposition 4.3. *Let $x_0 = (\alpha_0, u_0) \in K_{d+1}$ with $K_{d+1} = [a,b] \times \prod_{j=1}^d [a_j, b_j] \subset (0,2) \times \mathbb{R}^d$. Let V_1 be a complex-valued random field defined on K_{d+1}. Assume that there exists a random field $(\mathcal{G}(h))_{h\in[0,+\infty)}$ with values in $[0,\infty)$ and such that*

(i). there exists ρ a quasi-metric on $\mathbb{R}^{d+1}$ satisfying Equation (8) such that almost surely, for $x, y \in K_{d+1}$

$$|V_1(x) - V_1(y)| \leq \mathcal{G}(\rho(x,y));$$

(ii). there exists $h_0 \in (0,1]$ such that almost surely, the function $h \mapsto \mathcal{G}(h)$ is monotonic on $[0, h_0]$;

(iii). there exists $p > b/2$ and some constants $\beta \in (0,1]$, $\eta \in \mathbb{R}$ and $C \in (0,\infty)$ such that for some $\varepsilon > 0$ and for $h > 0$ small enough,

$$I(h) := \mathbb{E}\big(\mathcal{G}(h)^{2p}\big) \leq C h^{2p\beta} |\log h|^{2p(\eta - 1/2p - \varepsilon)}$$

Then, Equation (15) holds for $r > 0$ small enough.

Proof. See Appendix, Section C.2. □

Let us conclude this section by the following remark.

Remark 4.3. *If $(X_n)_{n\geq 1}$ is a sequence of independent symmetric random variables, one can replace in all the results of this section $V_n g_n$ by X_n, that is $S_N(\alpha, u)$ (respectively $S(\alpha, u)$) by*

$$S_N^*(\alpha,u) = \sum_{n=1}^{N} T_n^{-1/\alpha} X_n(\alpha,u) \quad \textit{(respectively by } S^*(\alpha,u) = \sum_{n=1}^{+\infty} T_n^{-1/\alpha} X_n(\alpha,u)\textit{)}.$$

In particular, following Example 4.1, let us consider $(X_n)_{n\geq 1}$ a sequence of i.i.d. fractional Brownian fields on $\mathbb{R}^d$ with Hurst parameter H. This sequence is assumed to be independent from the sequence $(T_n)_{n\geq 1}$. Then, Assumptions of Theorem 4.2 are fulfilled with ρ_{d+1} the Euclidean distance on $\mathbb{R}^{d+1}$, $\beta = H$ and $\eta = 1/2$, on any compact $(d+1)$-dimensional interval K_{d+1}. Especially, this leads to an upper bound of the modulus of continuity of

$$S^*(\alpha,u) = \sum_{n=1}^{+\infty} T_n^{-1/\alpha} X_n(u), \quad (\alpha,u) \in (0,2) \times \mathbb{R}^d.$$

on any compact $(d+1)$-dimensional interval K_{d+1}. Then for any fixed $\alpha_0 \in (0,2)$, we also obtain an upper bound for the modulus of continuity of the α_0-stable random field $(S^*(\alpha_0,u))_{u\in\mathbb{R}^d}$. More precisely, $(S^*(\alpha_0,u))_{u\in\mathbb{R}^d}$ is in $\mathcal{H}_{\rho_d}(K_d, H, 1)$ for ρ_d the Euclidean distance on $\mathbb{R}^d$ and for any compact set $K_d \subset \mathbb{R}^d$.

5 Applications to LePage random series

5.1 LePage series representation

Representations in random series of infinitely divisible laws have been studied in [24, 23]. In particular, symmetric α_0-stable laws may be represented using LePage random series. Moreover, such representations have been successfully used to study sample path properties of some symmetric α_0-stable random processes ($d = 1$) and fields (see e.g. [19, 7, 8, 12]).

Let us be more precise on the assumptions on the LePage series under study.

Assumption 5. *Let ν be a σ-finite measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.*

1. *Let $(\xi_n)_{n\geq 1}$ be a sequence of i.i.d. random variables with common law*

$$\mu(d\xi) = m(\xi)\nu(d\xi)$$

equivalent to ν (that is such that $m(\xi) > 0$ for ν-almost every ξ).

2. *$(g_n)_{n\geq 1}$ is a sequence of i.i.d. complex-valued symmetric sub-Gaussian random variables with parameter $s = 1$.*

3. *T_n is the nth arrival time of a Poisson process with intensity 1.*

4. *The sequences $(\xi_n)_{n\geq 1}$, $(g_n)_{n\geq 1}$ and $(T_n)_n$ are independent.*

5. *For any $\alpha \in K_1 \subset (0, 2)$, $f_\alpha : K_d \times \mathbb{R}^d \to \mathbb{C}$ is a deterministic function such that*

$$\forall u \in K_d \subset \mathbb{R}^d, \int_{\mathbb{R}^d} |f_\alpha(u, \xi)|^\alpha \nu(d\xi) < +\infty.$$

Then, setting

$$V_n(\alpha, u) := f_\alpha(u, \xi_n)m(\xi_n)^{-1/\alpha},$$

Assumption 4 is fulfilled so that we can define the sequence of random fields

$$S_{m,N}(\alpha, u) = \sum_{n=1}^{N} T_n^{-1/\alpha} f_\alpha(u, \xi_n)m(\xi_n)^{-1/\alpha} g_n, \ N \in \mathbb{N},$$

with $(\alpha, u) \in K_{d+1} := K_1 \times K_d \subset (0, 2) \times \mathbb{R}^d$ and its almost sure limit S_m, given by

$$S_m(\alpha, u) = \sum_{n=1}^{+\infty} T_n^{-1/\alpha} f_\alpha(u, \xi_n)m(\xi_n)^{-1/\alpha} g_n. \tag{16}$$

Under appropriate assumptions on f_α and m, the previous sections state the uniform convergence of $(S_{m,N})_{N\in\mathbb{N}}$, give a rate of convergence and some results on regularity for its almost sure limit S_m. Precise results on regularity of S_m may be obtained using the following proposition, which states that the finite distributions of S_m does not depend on the choice of the ν-density m.

Proposition 5.1. *Assume that Assumption 5 is fulfilled and let S_m be defined by* (16)*. Let $\left(\tilde{\xi}_n\right)_{n\geq 1}$ be a sequence of i.i.d. random variables with common law*

$$\tilde{\mu}(d\xi) = \tilde{m}(\xi)\nu(d\xi)$$

equivalent to ν. Assume that the sequences $\left(\tilde{\xi}_n\right)_{n\geq 1}$, $(g_n)_{n\geq 1}$ and $(T_n)_{n\geq 1}$ are independent.

1. *Then,*

$$(S_m(\alpha, u))_{(\alpha,u)\in K_{d+1}} \overset{fdd}{=} \left(\sum_{n=1}^{+\infty} T_n^{-1/\alpha} f_\alpha\left(u, \tilde{\xi}_n\right)\tilde{m}\left(\tilde{\xi}_n\right)^{-1/\alpha} g_n\right)_{(\alpha,u)\in K_{d+1}},$$

where $\overset{fdd}{=}$ means equality of finite distributions. In other words, $S_m \overset{fdd}{=} S_{\tilde{m}}$.

2. *Assume moreover that for ν-almost every $\xi \in \mathbb{R}^d$, the map $(\alpha, u) \mapsto f_\alpha(u, \xi)$ is continuous on the compact set $K_{d+1} \subset (0,2) \times \mathbb{R}^d$. Let us consider ρ a quasi-metric on $\mathbb{R}^{d+1}$, $\beta \in (0,1]$ and $\eta \in \mathbb{R}$. Then, S_m belongs almost surely in $\mathcal{H}_\rho(K_{d+1}, \beta, \eta)$ if and only if $S_{\tilde{m}}$ does.*

Proof. See Appendix, Section D.1. □

In particular, when studying the sample path properties of S_m, this result allows us to replace m by an other function $\tilde{m}$ so that the regularity of S_m may be deduced from the regularity of $S_{\tilde{m}}$. For example, replacing m by m_{x_0} depending on x_0 this may lead to a more precised bound for the modulus of continuity of S_m around x_0 (see e.g. Example 5.3).

5.2 α-stable random fields

In this section, let us fix $\alpha = \alpha_0 \in (0,2)$. Then, under Assumption 4, the random field $S_m(\alpha_0, \cdot) = (S_m(\alpha_0, u))_{u \in K_d}$ is an isotropic α_0-stable random field. Moreover, the proof of Proposition 5.1 (see Section D.1) allows to compute its characteristic function and to give a stochastic integral representation of S_m. Actually,

$$S_m(\alpha_0, \cdot) \overset{fdd}{=} d_{\alpha_0}^{-1} \left(\int_{\mathbb{R}^d} f_{\alpha_0}(u, \xi) M_{\alpha_0}(d\xi) \right)_{u \in K_d},$$

with M_{α_0} a complex isotropic α_0-stable random measure on $\mathbb{R}^d$ with control measure ν and

$$d_{\alpha_0} = \mathbb{E}(|\Re(g_1)|^{\alpha_0})^{1/\alpha_0} \left(\frac{1}{2\pi} \int_0^{2\pi} |\cos(\theta)|^{\alpha_0} d\theta \right)^{-1/\alpha_0} \left(\int_0^{+\infty} \frac{\sin(\theta)}{\theta^{\alpha_0}} d\theta \right)^{1/\alpha_0}. \tag{17}$$

When ν is a finite measure (respectively the Lebesgue measure), this stochastic integral representation of $S_m(\alpha_0, \cdot)$ has been provided in [32, 26] (respectively [19, 7]).

Let us note that assumptions of Theorem 4.2 and Proposition 4.3 can be stated in term of the deterministic kernel f_{α_0} to obtain an upper bound of the modulus of continuity of S_m. In general, well-choosing m_{u_0} and applying Proposition 5.1, we obtain a more precise upper bound of the modulus of continuity of $S_m(\alpha_0, \cdot)$ around u_0, which also holds for the random field

$$X_{\alpha_0} = \left(\int_{\mathbb{R}^d} f_{\alpha_0}(u, \xi) M_{\alpha_0}(d\xi) \right)_{u \in K_d}, \tag{18}$$

up to consider one of its modifications. To illustrate how the previous sections can be applied to study the field X_{α_0}, which is defined through a stochastic integral and not a series, let us focus on the case of harmonizable stable random fields. More precisely, we consider

$$f_{\alpha_0}(u, \xi) = \left(e^{i\langle u, \xi \rangle} - 1 \right) \psi_{\alpha_0}(\xi), \quad \forall (u, \xi) \in \mathbb{R}^d \times \mathbb{R}^d, \tag{19}$$

with $\psi_{\alpha_0} : \mathbb{R}^d \to \mathbb{C}$ a borelian function such that

$$\int_{\mathbb{R}^d} \min(1, \|\xi\|^{\alpha_0}) |\psi_{\alpha_0}(\xi)|^{\alpha_0} \nu(d\xi) < +\infty.$$

Note that, since this assumption does not depend on u, the random field X_{α_0} may be defined on the whole space $\mathbb{R}^d$. For the sake of simplicity, in the sequel, we consider the case where ν is the Lebesgue measure and first focus on a random field X_{α_0} which behaves as operator scaling random fields studied in [9].

Proposition 5.2. *Let $\alpha_0 \in (0,2)$ and let X_{α_0} be defined by (18) with ν the Lebesgue measure on $\mathbb{R}^d$. Let E be a real matrix of size $d \times d$ whose eigenvalues have positive real parts. Let $\tau_{_E}$ and $\tau_{_{E^t}}$ be functions as introduced in Example 2.1. Let*

$$q(E) = \mathrm{trace}(E) \quad \text{and} \quad a_1 = \min_{\lambda \in \mathrm{Sp}(E)} \Re(\lambda)$$

with $\mathrm{Sp}(E)$ the spectrum of E, that is the set of the eigenvalues of E. Assume that there exist finite positive constants $c_{_{5,1}}$ and A and $\beta \in (0, a_1)$ such that

$$|\psi_{\alpha_0}(\xi)| \leq c_{_{5,1}} \tau_{_{E^t}}(\xi)^{-\beta - q(E)/\alpha_0}, \quad \text{for almost every } \|\xi\| > A. \tag{20}$$

Then, there exists a modification $X^*_{\alpha_0}$ of X_{α_0} such that almost surely, for any $\varepsilon > 0$, for any compact set $K_d \subset \mathbb{R}^d$,

$$\sup_{\substack{u,v \in K_d \\ u \neq v}} \frac{\left|X^*_{\alpha_0}(u) - X^*_{\alpha_0}(v)\right|}{\tau_{_E}(u-v)^{\beta} |\log \tau_{_E}(u-v)|^{\varepsilon + 1/2 + 1/\alpha_0}} < +\infty.$$

Remark 5.1. *Let us note that the quasi-metric $(x,y) \mapsto \tau_{_E}(x-y)$ may not fulfill Equation (8) since the eigenvalues of E may not be greater than 1. Nevertheless, the quasi-metric $(x,y) \mapsto \tau_{_{E/a_1}}(x-y)$ does and the conclusion with $\tau_{_E}$ in the previous proposition then follows from the comparison*

$$\forall \xi \in \mathbb{R}^d,\ c_1 \tau_{_E}(\xi)^{a_1} \leq \tau_{_{E/a_1}}(\xi) \leq c_2 \tau_{_E}(\xi)^{a_1}$$

with c_1, c_2 two finite positive constants.

Proof. See Appendix, Section D.2. □

An upper bound for the modulus of continuity of such harmonizable random fields is also obtained in [35]. This upper bound is given in term of the Euclidean norm and then does not take into account the anisotropic behavior of X_{α_0}. Even when $\tau_{_E}$ is the Euclidean norm, our result is a little more precise that the one of [35]. The difference is only in the power of the logarithm term.

Let us now give some examples. We keep the notation of the previous proposition and the eigenvalues of the matrix E have always positive real parts.

Example 5.1. (Operator scaling random fields [9]) *Let $\psi : \mathbb{R}^d \to [0, \infty)$ be an E^t-homogenous function, which means that*

$$\forall c \in (0, +\infty),\ \forall \xi \in \mathbb{R}^d,\ \psi\Big(c^{E^t} \xi\Big) = c\psi(\xi)$$

where $c^{E^t} = \exp\big(E^t \log c\big)$. Let us assume that ψ is a continuous function such that $\psi(\xi) \neq 0$ for $\xi \neq 0$. Then we consider the function $\psi_{\alpha_0} : \mathbb{R}^d \to [0, +\infty]$ defined by

$$\psi_{\alpha_0}(\xi) = \psi(\xi)^{-H - q(E)/\alpha_0}.$$

The random field X_{α_0} associated with ψ_{α_0} by (18) is well-defined and is stochastically continuous if and only if $H \in (0, a_1)$.

Let us fix $H \in (0, a_1)$ and recall that X_{α_0} is operator scaling in the sense that

$$\big(X_{\alpha_0}(c^E u)\big)_{u\in\mathbb{R}^d} \overset{fdd}{=} c^H (X_{\alpha_0}(u))_{u\in\mathbb{R}^d},\ \forall c \in [0, +\infty).$$

Since ψ_{α_0} is E^t-homogenous, one easily checks that there exists $C \in (0, +\infty)$ such that

$$\forall \xi \in \mathbb{R}^d,\ \psi_{\alpha_0}(\xi) \leq C \tau_{E^t}(\xi)^{-H-q(E)/\alpha_0}.$$

Then, the assumptions of Proposition 5.2 are fulfilled with $\beta = H$. The corresponding conclusion was stated in Theorem 5.1 of [7] when $H = 1$ and $a_1 > 1$, which is enough to cover the general case using Remark 2.1 of [7].

Example 5.2. (Anisotropic Riesz-Bessel α-stable random fields) *Let us consider*

$$\psi_{\alpha_0}(\xi) = \frac{1}{\tau_{E^t}(\xi)^{2\beta_1/\alpha_0}\big(1+\tau_{E^t}(\xi)^2\big)^{\beta_2/\alpha_0}},\ \xi \in \mathbb{R}^d\backslash\{0\}$$

with two real numbers β_1 and β_2. Assuming that

$$\frac{q(E)}{2} < \beta_1 + \beta_2 \quad \text{and} \quad \beta_1 < \frac{q(E)}{2} + \frac{\alpha_0 a_1}{2},$$

the random field X_{α_0} is well-defined by (18)*. When τ_{E^t} is the Euclidean norm, this random field has been introduced in [35]. Note that it generalizes the Gaussian fractional Riesz-Bessel motion [2] to the stable framework.*

We distinguish two cases. If $\beta_1 + \beta_2 < \frac{q(E)}{2} + \frac{\alpha_0 a_1}{2}$, Proposition 5.2 can be applied with $\beta = \frac{2(\beta_1+\beta_2)-q(E)}{\alpha_0}$. Otherwise, Proposition 5.2 can be applied for any $\beta \in (0, a_1)$.

Random fields defined by (18) have stationary increments so that their regularity on K_d does not depend on the compact set K_d. To avoid this feature one can consider non-stationary generalizations by substituting ψ_{α_0} by a function that also depends on $u \in K_d$. More precisely, we can consider

$$X_{\alpha_0} = \left(\int_{\mathbb{R}^d} \Big(\mathrm{e}^{i\langle u,\xi\rangle} - 1\Big)\psi_{\alpha_0}(u,\xi) M_{\alpha_0}(d\xi)\right)_{u\in K_d} \tag{21}$$

with M_{α_0} a complex isotropic α_0-stable random measure with Lebesgue control measure and ψ_{α_0} a borelian function such that, for all $u \in K_d$,

$$\int_{\mathbb{R}^d} \left|\mathrm{e}^{i\langle u,\xi\rangle} - 1\right|^{\alpha_0} |\psi_{\alpha_0}(u,\xi)|^{\alpha_0} d\xi < +\infty.$$

Under some conditions on ψ_{α_0}, when considering the local behavior of X_{α_0} around a point u_0 one can conveniently choose a Lebesgue density m_{u_0} to obtain an upper bound of the modulus of continuity of the shot noise series $S_{m_{u_0}}(\alpha_0, \cdot)$ given by (16) with

$$f_{\alpha_0}(u,\xi) = \Big(\mathrm{e}^{i\langle u,\xi\rangle} - 1\Big)\psi_{\alpha_0}(u,\xi).$$

For the sake of conciseness, let us illustrate this with multi-operator random fields, which have already been studied in [8].

Example 5.3. (Multi-operator scaling α-stable random fields) *In [8], we consider E a function defined on $\mathbb{R}^d$ with values in the set of real matrix of size $d \times d$ whose eigenvalues have real parts greater than 1 and $\psi : \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty)$ a continuous function such that for any $u \in \mathbb{R}^d$, $\psi(u, \cdot)$ is homogeneous with respect to $E(u)^t$ i.e.*

$$\psi(u, c^{E(u)^t}\xi) = c\psi(u, \xi), \forall \xi \in \mathbb{R}^d, \forall c > 0.$$

Under convenient regularity assumptions on ψ and E, one can define the α_0-stable random field X_{α_0} by (21) with

$$\psi_{\alpha_0}(u, \xi) = \psi(u, \xi)^{-1-q(E(u))/\alpha_0} \text{ and } q(E(u)) = trace(E(u)).$$

Let $K_d = \prod_{j=1}^d [a_j, b_j] \subset \mathbb{R}^d$ and $u_0 \in K_d$. Let us set $K_{d+1} = \{\alpha_0\} \times K_d$ and consider the quasi-metric ρ defined on $\mathbb{R}^{d+1}$ by

$$\rho((\alpha, u), (\alpha', v)) = |\alpha - \alpha'| + \tau_{E(u_0)}(u - v)$$

for all $(\alpha, u), (\alpha', v) \in \mathbb{R} \times \mathbb{R}^d$, which clearly satisfies Equation (8). Then, under assumptions of [8], there exists a Lebesgue density $m_{u_0} > 0$ a.e. such that Assumption 2 holds for $S_{m_{u_0}}$ on K_{d+1} with $\eta = 0$ and all $\beta \in (0, 1)$, adapting similar arguments as in Proposition 5.2 (see Lemma 4.7 of [8]). Therefore, following a part of the proof of Proposition 5.2, there exists a modification $X^*_{\alpha_0}$ of X_{α_0} such that almost surely,

$$\lim_{r \downarrow 0} \sup_{\substack{u,v \in B(u_0,r) \cap K_d \\ u \neq v}} \frac{\left|X^*_{\alpha_0}(u) - X^*_{\alpha_0}(v)\right|}{\tau_{E(u_0)}(u - v)^{1-\varepsilon}} < +\infty$$

for any $\varepsilon \in (0, 1)$. This is Theorem 4.6 of [8].

For the sake of conciseness, we do not develop other examples. Nevertheless, let us mention that our results can also be applied to harmonizable fractional α-stable sheets or even to operator stable sheets. In particular, this improves the result stated in [27] for fractional α-stable sheets. Note that we can also deal with real symmetric measure W_α.

5.3 Multistable random fields

Multistable random fields have first been introduced in [14] and then studied in [13]. Each marginal $X(u)$ of such a random field is a stable random variable but its stability index is allowed to depend on the position u.

Generalizing the class of multistable random fields introduced in [21], we consider a multistable random field defined by a LePage series. More precisely, under Assumption 5, we consider

$$\tilde{S}_m(u) = \sum_{n=1}^{+\infty} T_n^{-1/\alpha(u)} f_{\alpha(u)}(u, \xi_n) m(\xi_n)^{-1/\alpha(u)} g_n, \quad u \in K_d$$

where $\alpha : K_d \to (0, 2)$ is a function. Therefore, using Section 4, we can bound the modulus of continuity of the shot noise series S_m defined by (16). Since

$$\tilde{S}_m(u) = S_m(\alpha(u), u),$$

we then obtain an upper bound for the modulus of continuity of $\tilde{S}$. In particular, assuming that α is smooth enough, we obtain the following theorem.

Proposition 5.3. *Let $K_d = \prod_{j=1}^{d}[a_j, b_j] \subset \mathbb{R}^d$. Let us choose $u_0 \in K_d$. Let $\tilde{\rho}$ be a quasi-metric on $\mathbb{R}^d$ satisfying Equation (8) and let $\alpha : K_d \to (0,2)$ belong to $\mathcal{H}_{\tilde{\rho}}(K_d, 1, 0)$. Let us set*

$$a = \min_{K_d} \alpha, \;\; b = \max_{K_d} \alpha \;\; \text{and} \;\; K_1 = [a, b] \subset (0,2)$$

and consider the quasi-metric ρ defined on $\mathbb{R} \times \mathbb{R}^d$ by

$$\rho\big((\alpha, u), (\alpha', v)\big) = \big|\alpha - \alpha'\big| + \tilde{\rho}(u, v).$$

Assume that Assumption 5 is fulfilled and that Equation (15) holds on $K_{d+1} = [a, b] \times K_d$ for some $p > b/2$, $\beta \in (0, 1]$ and $\eta \in \mathbb{R}$. Assume also that

$$\mathbb{E}\Big(|V_1(\alpha(u_0), u_0)|^{2p}\Big) = \int_{\mathbb{R}^d} \Big|f_{\alpha(u_0)}(u_0, \xi)\Big|^{2p} m(\xi)^{1-2p/\alpha(u_0)} d\xi < +\infty.$$

1. *Then, almost surely, $(\tilde{S}_{m,N})_{N\in\mathbb{N}}$ converges uniformly on K_d to $\tilde{S}_m$ and the limit $\tilde{S}_m$ belongs to $\mathcal{H}_{\tilde{\rho}}(K_d, \beta, \max(\eta, 0) + 1/2)$.*

2. *Moreover, for all $p' > 0$ such that $1/p' \in (0, 1/b - 1/\min(2p, 2))$,*

$$\sup_{N\geq 1} N^{1/p'} \sup_{u\in K_d} \Big|\tilde{S}_m(u) - \tilde{S}_{m,N}(u)\Big| < +\infty.$$

Proof. See Appendix, Section D.3. □

Remark 5.2. *Let us recall that $\tilde{S}_m \in \mathcal{H}_{\tilde{\rho}}(K_d, \beta, \max(\eta, 0)+1/2)$ if and only if $\tilde{S}_{\tilde{m}} \in \mathcal{H}_{\tilde{\rho}}(K_d, \beta, \max(\eta, 0)+1/2)$, with $\tilde{m}$ an other ν-density equivalent to ν, by Proposition 5.1.*

Note that other assumptions under which the conclusion still holds may be given to apply the results of the paper. Moreover, the study of $\tilde{S}$ may also be done by studying directly the conditional parameter associated with $\tilde{S}$. To finish, note also that more general quasi-metrics ρ can be considered.

To illustrate the previous proposition, we only focus on multistable random fields obtained replacing in a LePage series representation of an harmonizable operator scaling stable random field the index α by a function. Many other examples can be given, such as multistable anisotropic Riesz-Bessel random fields or the class of linear multistable random fields defined in [13].

Corollary 5.4 (Multistable versions of harmonizable operator scaling random fields). *Let E be a real matrix of size $d \times d$ such that $\min_{\lambda\in \mathrm{Sp}E} \Re(\lambda) > 1$. Let us consider $\rho_{_E}$ and $\tau_{_E}$ as defined in Example 2.1. Let us also consider $\psi : \mathbb{R}^d \to [0, \infty)$ a continuous, E^t-homogeneous function such that $\psi(\xi) \neq 0$ for $\xi \neq 0$. Then we set*

$$f_{_\alpha}(u, \xi) = \Big(\mathrm{e}^{i\langle u,\xi\rangle} - 1\Big)\psi(\xi)^{-1-q(E)/\alpha}$$

with $q(E) = \mathrm{trace}(E)$. *Let* m *be a Lebesgue density a.e. positive on* $\mathbb{R}^d$, $(\xi_n, T_n, g_n)_{n\geq 1}$ *be as in Assumption 5 with* ν *the Lebesgue measure and consider a function* $\alpha : \mathbb{R}^d \to (0,2)$. *Therefore, the multistable random field*

$$\tilde{S}_m(u) = \sum_{n=1}^{+\infty} T_n^{-1/\alpha(u)} f_{\alpha(u)}(u, \xi_n) m(\xi_n)^{-1/\alpha(u)} g_n$$

is well-defined on $\mathbb{R}^d$. *Moreover if* $\alpha \in \mathcal{H}_{\rho_E}\big(\mathbb{R}^d, 1, 0\big)$, *then for any* $u_0 \in \mathbb{R}^d$ *and* $\varepsilon > 0$, *there exists* $r \in (0,1]$ *such that almost surely*

$$\sup_{\substack{u,v \in B(u_0,r) \\ u \neq v}} \frac{\left|\tilde{S}_m(u) - \tilde{S}_m(v)\right|}{\tau_E(u-v)|\log \tau_E(u-v)|^{1/\alpha(u_0)+1/2+\varepsilon}} < +\infty.$$

Proof. See Appendix, Section D.3. □

Remark 5.3. *In particular, when* $E = \mathrm{Id}$, τ_E *is the Euclidean norm and we obtain an upper bound of the modulus of continuity of multistable versions of fractional harmonizable stable fields.*

A Proof of Proposition 2.1

The proof of Proposition 2.1 is based on the following lemma.

Lemma A.1. *If* Z *is a complex-valued sub-Gaussian random variable with parameter* $s \in (0, +\infty)$, *then*

$$\forall t \in (0,+\infty),\ \mathbb{P}(|Z| > t) \leq 4\mathrm{e}^{-\frac{t^2}{8s^2}}.$$

Proof. Let $t \in (0,+\infty)$. Then,

$$\mathbb{P}(|Z| > t) \leq \mathbb{P}\bigg(|\Re(Z)| > \frac{t}{2}\bigg) + \mathbb{P}\bigg(|\Im(Z)| > \frac{t}{2}\bigg).$$

Since Z is sub-Gaussian with parameter s, $\Re(Z)$ and $\Im(Z)$ are real-valued sub-Gaussian random variables with parameter s. Then applying [16],

$$\max\left(\mathbb{P}\bigg(|\Re(Z)| > \frac{t}{2}\bigg), \mathbb{P}\bigg(|\Im(Z)| > \frac{t}{2}\bigg)\right) \leq 2\exp\left(-\frac{t^2}{8s^2}\right),$$

which leads to the conclusion. □

Let us now prove Proposition 2.1.

Proof of Proposition 2.1. Let $t \in (0,+\infty)$. Proposition 2.1 is straightforward if $\|a\|_{\ell^2} = 0$, that is if $a = 0$. Then, in the following we assume that $a \neq 0$. Since the sequence $(g_n)_{n\geq 1}$ is symmetric, by the Lévy inequalities (see Proposition 2.3 in [22]), for any $M \in \mathbb{N}\backslash\{0\}$,

$$\mathbb{P}\left(\sup_{1\leq P\leq M}\left|\sum_{n=1}^{P} a_n g_n\right| > t\|a\|_{\ell^2}\right) \leq 2\mathbb{P}\left(\left|\sum_{n=1}^{M} a_n g_n\right| > t\|a\|_{\ell^2}\right).$$

We now prove that $\sum_{n=1}^{M} a_n g_n$ is sub-Gaussian. By independence of the random variables g_n,

$$\forall z \in \mathbb{C},\ \mathbb{E}\Big(\mathrm{e}^{\Re\left(\overline{z}\sum_{n=1}^{M} a_n g_n\right)}\Big) = \prod_{n=1}^{M} \mathbb{E}\Big(\mathrm{e}^{\Re(\overline{z} a_n g_n)}\Big).$$

Then since each g_n is sub-Gaussian with parameter $s = 1$,

$$\forall z \in \mathbb{C},\ \mathbb{E}\Big(\mathrm{e}^{\Re\left(\overline{z}\sum_{n=1}^{M} a_n g_n\right)}\Big) \leq \prod_{n=1}^{M} \mathrm{e}^{|z|^2|a_n|^2} = \mathrm{e}^{|z|^2\sum_{n=1}^{M}|a_n|^2}. \tag{22}$$

Therefore, for any integer M, $\sum_{n=1}^{M} a_n g_n$ is sub-Gaussian with parameter $s_M = \left(\sum_{n=1}^{M}|a_n|^2\right)^{1/2}$. Since $a \neq 0$, for M large enough, $s_M \neq 0$ and then applying Lemma A.1,

$$\forall t > 0,\ \mathbb{P}\left(\sup_{1\leq P\leq M}\left|\sum_{n=1}^{P} a_n g_n\right| > t\|a\|_{\ell^2}\right) \leq 8\exp\left(-\frac{t^2\|a\|_{\ell^2}^2}{8s_M^2}\right).$$

Then, for M large enough,

$$\forall t > 0,\ \mathbb{P}\left(\sup_{1\leq P\leq M}\left|\sum_{n=1}^{P} a_n g_n\right| > t\|a\|_{\ell^2}\right) \leq 8\mathrm{e}^{-\frac{t^2}{8}}$$

since $0 < s_M^2 \leq \|a\|_{\ell^2}^2$. Assertion *1.* follows letting $M \to +\infty$.

Let us now prove Assertion *2.*. Let us first assume that there exists $N \in \mathbb{N}\backslash\{0\}$, such that

$$\forall n \geq N,\ a_n = 0.$$

Then, according to the previous lines, $\sum_{n=1}^{+\infty} a_n g_n = \sum_{n=1}^{N} a_n g_n$ is a sub-Gaussian random variable with parameter

$$s_N = \left(\sum_{n=1}^{N}|a_n|^2\right)^{1/2} = \left(\sum_{n=1}^{+\infty}|a_n|^2\right)^{1/2} = \|a\|_{\ell^2}.$$

Therefore to prove Assertion *2.*, we now assume that

$$\forall N \in \mathbb{N}\backslash\{0\}, \exists n \geq N,\ a_n \neq 0,$$

so that $\sum_{n=N}^{+\infty}|a_n|^2 \neq 0$ for any integer $N \geq 1$. Then, applying Assertion *1.* replacing a_n by $a_n\mathbf{1}_{n\geq N}$, we have

$$\forall \varepsilon > 0,\ \forall N \in \mathbb{N}\backslash\{0\},\ \mathbb{P}\left(\sup_{P\geq N}\left|\sum_{n=N}^{P} a_n g_n\right| > \varepsilon\right) \leq 8\mathrm{e}^{-\frac{\varepsilon^2}{8\sum_{n=N}^{+\infty}|a_n|^2}}.$$

Since $\|a\|_{\ell^2}^2 = \sum_{n=1}^{+\infty}|a_n|^2 < +\infty$, this implies that $\left(\sum_{n=1}^{N} a_n g_n\right)_N$ is a Cauchy sequence in probability. Then, by Lemma 3.6 in [17], the series $\sum_{n=1}^{+\infty} a_n g_n$ converges in probability. By Itô-Nisio Theorem (see [22] for instance), this series also converges almost surely, since the random variables g_n, $n \geq 1$, are independent. Moreover, letting $M \to +\infty$ in (22) and applying Lebesgue monotone convergence Theorem, we obtain that $\sum_{n=1}^{+\infty} a_n g_n$ is sub-Gaussian with parameter $\|a\|_{\ell^2}$. We conclude the proof noting that

$$\forall t > 0,\ \mathbb{P}\left(\left|\sum_{n=1}^{+\infty} a_n g_n\right| > t\|a\|_{\ell^2}\right) \leq \mathbb{P}\left(\sup_{P\geq 1}\left|\sum_{n=1}^{P} a_n g_n\right| > t\|a\|_{\ell^2}\right) \leq 8\mathrm{e}^{-\frac{t^2}{8}}.$$

□

B Main results on conditionally sub-Gaussian series

B.1 Proof of Theorem 3.1

Let us recall that $x_0 \in K_d = \prod_{j=1}^d [a_j, b_j] \subset \mathbb{R}^d$. We assume, without loss of generality, that

$$\forall 1 \leq j \leq d,\, a_j < b_j.$$

Actually, if some $a_j = b_j$, studying $(S_N)_{N\geq 1}$ and its limits S on K_d, we identify them as random fields defined on $K_{d'} \subset \mathbb{R}^{d'}$ for $d' < d$. Note that if $a_j = b_j$ for all $1 \leq j \leq d$, there is nothing to prove.

We also assume that $\gamma(\omega) \in (0,1)$, which is not restrictive and allows us to apply Equation (8) as soon as $\|x - y\| \leq \gamma(\omega)$ (with $c_{2,1}$ and $c_{2,2}$ which do not depend on γ).

First Step We first introduce a convenient sequence $(\mathcal{D}_{\nu_k})_{k\geq 1}$ of countable sets included on dyadics, which is linked to the quasi-metric ρ. It allows to follow some arguments of the proof of the Kolmogorov's Lemma to obtain an upper bound for the modulus of continuity of S.

For any $k \in \mathbb{N}\backslash\{0\}$ and $j = (j_1, \ldots, j_d) \in \mathbb{Z}^d$, we set

$$x_{k,j} = \frac{j}{2^k} \quad \text{and} \quad \mathcal{D}_k = \Big\{ x_{k,j} : j \in \mathbb{Z}^d \Big\}.$$

For any $k \in \mathbb{N}\backslash\{0\}$, we then consider

$$\nu_k = \inf \Big\{ n \in \mathbb{N}\backslash\{0\} : c_{2,2} d^{\underline{H}/2} 2^{-n\underline{H}} \leq 2^{-k} \Big\}$$

where the constant $c_{2,2}$ is given by Equation (8). Choosing $c_{2,2}$ large enough (which is not restrictive) and following the ideas of Step 1 of the proof of Theorem 5.1 of [7], one easily checks that $(\nu_k)_{k\geq 1}$ is an increasing sequence such that

$$\lim_{k\to+\infty} k/\nu_k = \underline{H}.$$

In particular, the sequence $(\mathcal{D}_{\nu_k})_{k\geq 1}$ is increasing and $\mathcal{D} = \bigcup_{k=1}^{+\infty} \mathcal{D}_k = \bigcup_{k=1}^{+\infty} \mathcal{D}_{\nu_k}$. Let us note that since $a_j < b_j$ for any $1 \leq j \leq d$, $\mathcal{D} \cap K_d$ is dense in K_d.
Moreover one also checks that for k large enough, $\mathcal{D}_{\nu_k} \cap K_d$ is a 2^{-k} net of K_d for ρ, which means that for any $x \in K_d$, there exists $j \in \mathbb{Z}^d$ such that $\rho(x, x_{\nu_k,j}) \leq 2^{-k}$, with $x_{\nu_k,j} = j/2^{\nu_k} \in K_d$.

Second Step This step is inspired from Step 2 of [7, 8]. The main difference is that we use Proposition 2.1 to obtain a uniform control in N.

For $k \in \mathbb{N}\backslash\{0\}$ and $(i,j) \in \mathbb{Z}^d$, we consider

$$E_{i,j}^k = \left\{ \omega : \sup_{N\geq 1} |S_N(x_{\nu_k,i}) - S_N(x_{\nu_k,j})| > s(x_{\nu_k,i}, x_{\nu_k,j})\, \varphi(\rho(x_{\nu_k,i}, x_{\nu_k,j})) \right\}$$

with, following [18],

$$\varphi(t) = \sqrt{8Ad \log \frac{1}{t}}, \quad t > 0, \tag{23}$$

for $A > 0$ conveniently chosen later. We choose $\delta \in (0,1)$ and set for $k \in \mathbb{N}\backslash\{0\}$,

$$\delta_k = 2^{-(1-\delta)k}, \quad I_k = \left\{ (i,j) \in \left(\mathbb{Z}^d \cap 2^{\nu_k} K_d \right)^2 : \rho\left(x_{\nu_k,i}, x_{\nu_k,j}\right) \leq \delta_k \right\} \tag{24}$$

and

$$E_k = \bigcup_{(i,j)\in I_k} E_{i,j}^k.$$

Since φ is a decreasing function and $s \geq 0$, for any $(i,j) \in I_k$

$$\mathbb{P}\Big(E_{i,j}^k\Big) \leq \mathbb{P}\left(\sup_{N\geq 1} |S_N(x_{\nu_k,i}) - S_N(x_{\nu_k,j})| > s(x_{\nu_k,i}, x_{\nu_k,j})\, \varphi(\delta_k) \right).$$

Since $(g_n)_{n\geq 1}$ is a sequence of symmetric independent sub-Gaussian random variables with parameter $s = 1$, conditionning to $(W_n)_{n\geq 1}$ and applying Assertion 1. of Proposition 2.1, one has

$$\forall x, y \in K_d,\, \mathbb{P}\left(\sup_{N\geq 1} |S_N(x) - S_N(y)| > t s(x,y) \right) \leq 8\, \mathrm{e}^{-\frac{t^2}{8}}$$

by definition of s and S_N. Then,

$$\mathbb{P}\Big(E_{i,j}^k\Big) \leq 8\, \mathrm{e}^{-\frac{\varphi(\delta_k)^2}{8}} \leq 8\mathrm{e}^{-Ad(1-\delta)k\log 2}.$$

Let us note that there exists a finite positive constant $c_1 \in (0, +\infty)$ such that

$$\forall k \in \mathbb{N}\backslash\{0\},\, \mathrm{card}\, I_k \leq c_1 2^{2\nu_k d} \delta_k^{\frac{d}{\underline{H}}}$$

since $K_d \subset \mathbb{R}^d$ is a compact set and since ρ satisfies Equation (9). Moreover, according to the first step,

$$\lim_{k\to+\infty} \frac{k}{\nu_k} = \underline{H}.$$

Hence, there exists a finite positive constant c_2 such that

$$\sum_{k=1}^{+\infty} \mathbb{P}(E_k) \leq \sum_{k=1}^{+\infty} \sum_{(i,j)\in I_k} \mathbb{P}\left(E_{i,j}^k\right) \leq c_2 \sum_{k=1}^{+\infty} \mathrm{e}^{-\left(A(1-\delta) - \frac{2}{\underline{H}} + \frac{1-\delta}{H}\right) k d \log 2}.$$

Therefore, choosing $A > \frac{2}{\underline{H}} - \frac{1}{H}$ and δ small enough, $\displaystyle\sum_{k=1}^{+\infty} \mathbb{P}(E_k) < +\infty$, such that by the Borel-Cantelli Lemma,

$$\mathbb{P}\left(\bigcup_{k=1}^{+\infty} \bigcap_{\ell=k}^{+\infty} E_\ell^c \right) = 1.$$

Then, by definition of E_k and φ and by Assumption 2, there exists an event $\tilde{\Omega}_{x_0}$ with $\mathbb{P}(\tilde{\Omega}_{x_0}) = 1$ such that, for any $\omega \in \tilde{\Omega}_{x_0}$ there exists $k^*(\omega)$ such that for every $k \geq k^*(\omega)$ and for all $x, y \in \mathcal{D}_{\nu_k}$ with $x, y \in B(x_0, \gamma(\omega)) \cap K_d$ and $\rho(x,y) \leq \delta_k = 2^{-(1-\delta)k}$,

$$\sup_{N\geq 1} |S_N(x) - S_N(y)| \leq C \rho(x,y)^\beta |\log(\rho(x,y))|^{\eta+1/2}. \tag{25}$$

Third Step: In this step we prove that (25) holds, up to a multiplicative constant, for any $x, y \in \mathcal{D}$ closed enough to x_0. This step is adapted from Step 4 of the proof of Theorem 5.1 in [7], taking care that (25) only holds for some $x, y \in \mathcal{D}_{\nu_k} \cap K_d$ randomly closed enough of x_0. Let us mention that this step has been omitted in the proof of the main result of [8] but is not trivial. We then decide to provide a proof here for the sake of completeness and clearness.

Let us now fix $\omega \in \tilde{\Omega}_{x_0}$ and denote by $\kappa \geq 1$ the constant appearing in the quasi-triangle inequality satisfied by ρ. We also consider the function F defined on $(0, +\infty)$ by

$$F(h) := h^{\beta} |\log(h)|^{\eta + 1/2}.$$

Observe that F is a random function since β and η are random variables. Then, we choose $k_0 = k_0(\omega) \in \mathbb{N}$ such that the three following assertions are fulfilled:

(a) F is increasing on $(0, \delta_{k_0}]$, where δ_k is given by (24),

(b) for all $k \geq k_0(\omega)$ $\mathcal{D}_{\nu_k} \cap K_d$ is a 2^{-k} net of K_d for ρ

(c) $2^{k_0} \delta_{k_0+1} > 3\kappa^2$.

Up to change $k^*(\omega)$, we can assume that $k^*(\omega) \geq k_0$ and that

$$\gamma(\omega) \geq \left(\frac{\delta_{k^*(\omega)}}{3\kappa^2 c_{2,2}} \right)^{1/\underline{H}} := r^*(\omega). \tag{26}$$

where $\underline{H}$ and $c_{2,2}$ are defined in Equation (8).
Let us now consider $x, y \in \mathcal{D} \cap K_d$ such that $x \neq y$ and

$$\max\left(\|x - x_0\|, \|y - x_0\| \right) \leq \frac{r^*(\omega)}{2} := \gamma^*(\omega).$$

Then, by definition of $r^*(\omega)$, $x, y \in B(x_0, \gamma^*(\omega)) \cap K_d \subset B(x_0, \gamma(\omega)) \cap K_d$ and

$$3\kappa^2 c_{2,2} \|x - y\|^{\underline{H}} \leq \delta_{k^*(\omega)}.$$

Note also that $\|x - y\| \leq 2\gamma^*(\omega) = r^*(\omega) \leq \gamma(\omega) \leq 1$. Hence, the upper bound of Equation (8) leads to

$$3\kappa^2 \rho(x, y) \leq \delta_{k^*(\omega)}.$$

Then, there exists a unique $k \geq k^*(\omega)$ such that

$$\delta_{k+1} < 3\kappa^2 \rho(x, y) \leq \delta_k. \tag{27}$$

Furthermore, since $x, y \in \mathcal{D} \cap K_d$, there exists $n \geq k+1$ such that $x, y \in \mathcal{D}_{\nu_n} \cap K_d$ and for $j = k, \ldots, n-1$, there exist $x^{(j)} \in \mathcal{D}_{\nu_j} \cap K_d$ and $y^{(j)} \in \mathcal{D}_{\nu_j} \cap K_d$ such that

$$\rho\Big(x, x^{(j)}\Big) \leq 2^{-j} \text{ and } \rho\Big(y, y^{(j)}\Big) \leq 2^{-j}. \tag{28}$$

Let us now fix $N \geq 1$ and focus on $S_N(x) - S_N(y)$. Then, setting $x^{(n)} = x$ and $y^{(n)} = y$,

$$S_N(x) - S_N(y) = \left(S_N\left(x^{(k)}\right) - S_N\left(y^{(k)}\right)\right) + \sum_{j=k}^{n-1}\left(S_N\left(x^{(j+1)}\right) - S_N\left(x^{(j)}\right)\right) + \sum_{j=k}^{n-1}\left(S_N\left(y^{(j+1)}\right) - S_N\left(y^{(j)}\right)\right). \tag{29}$$

The following lemma, whose proof is postponed to the end of the section for the sake of clearness, allows to apply (25) for each term of the right hand side of the last inequality.

Lemma B.1. *Choosing $k^*(\omega)$ large enough, the sequences $\left(x^{(j)}\right)_{k\leq j\leq n}$ and $\left(y^{(j)}\right)_{k\leq j\leq n}$ satisfy the three following assertions.*

1. *$x^{(j)}, y^{(j)} \in B(x_0, \gamma(\omega))$ for any $j = k, \ldots, n$,*
2. *for any $j = k, \ldots, n-1$, $\max(\rho(x^{(j+1)}, x^{(j)}), \rho(y^{(j+1)}, y^{(j)})) \leq \delta_{j+1}$,*
3. *$\rho(x^{(k)}, y^{(k)}) \leq \delta_k$.*

Therefore, by Assertion *1.*, up to change $k^*(\omega)$, Equations (25) and (29) lead to

$$|S_N(x) - S_N(y)| \leq C\left(F\left(\rho(x^{(k)}, y^{(k)})\right) + \sum_{j=k}^{n-1}\left[F\left(\rho\left(x^{(j+1)}, x^{(j)}\right)\right) + F\left(\rho\left(y^{(j+1)}, y^{(j)}\right)\right)\right]\right).$$

Since F is increasing on $(0, \delta_{k_0}]$ and since $j \geq k_0$, by Assertions *2.* and *3.* of Lemma B.1,

$$|S_N(x) - S_N(y)| \leq C\left(F\left(\rho(x^{(k)}, y^{(k)})\right) + 2\sum_{j=k}^{n-1} F(\delta_{j+1})\right),$$

which implies, by definition of F that

$$|S_N(x) - S_N(y)| \leq C\Big(F\left(\rho(x^{(k)}, y^{(k)})\right) + 2\tilde{C}F(\delta_{k+1})\Big),$$

where

$$\tilde{C}(\omega) = 2\sum_{j=0}^{+\infty} \delta_j^{\beta(\omega)}(j+1)^{\max(\eta(\omega)+1/2,0)} < +\infty$$

since $\beta > 0$ and $\delta_j = 2^{-(1-\delta)j}$ with $\delta < 1$. Then, since F is increasing on $(0, \delta_0)$, by Assertion *3.* of Lemma B.1 and Equation (27), we get

$$|S_N(x) - S_N(y)| \leq C(1 + 2\tilde{C})F(3\kappa^2\rho(x, y)),$$

for every $N \geq 1$ and $x, y \in \mathcal{D} \cap B(x_0, \gamma^*(\omega)) \cap K_d$.

Therefore, by continuity of ρ and each S_N and by density of $\mathcal{D} \cap K_d$ in K_d

$$|S_N(x) - S_N(y)| \leq C(1 + 2\tilde{C})F(3\kappa^2\rho(x, y)), \tag{30}$$

for every $N \geq 1$ and $x, y \in B(x_0, \gamma^*(\omega)) \cap K_d$.

Fourth Step: Uniform convergence of S_N. Let us now consider

$$\Omega'_{x_0} = \bigcap_{u \in \mathcal{D}} \left\{ \lim_{N \to +\infty} S_N(u) = S(u) \right\} \cap \tilde{\Omega}_{x_0}.$$

Observe that $\mathbb{P}\Big(\Omega'_{x_0}\Big) = 1$. Let us now fix $\omega \in \Omega'_{x_0}$. Hence, by Equation (30), the sequence $(S_N(\cdot)(\omega))_{N \in \mathbb{N}}$, which converges pointwise on $\mathcal{D} \cap B(x_0, \gamma^*(\omega)) \cap K_d$ is uniformly equicontinuous on $B(x_0, \gamma^*(\omega))$. Since $\mathcal{D} \cap B(x_0, \gamma^*(\omega)) \cap K_d$ is dense in $B(x_0, \gamma^*(\omega)) \cap K_d$, by Theorem I.26 and adapting Theorem I.27 in [30], $(S_N(\cdot)(\omega))_{N \in \mathbb{N}}$ converges uniformly on $B(x_0, \gamma^*(\omega)) \cap K_d$. Therefore, its limit S is continuous on $B(x_0, \gamma^*(\omega)) \cap K_d$. Moreover, letting $N \to +\infty$ in (30) (which holds since $\omega \in \tilde{\Omega}_{x_0}$), we get

$$|S(x) - S(y)| \leq C(1 + 2\tilde{C}) F(3\kappa^2 \rho(x, y)), \tag{31}$$

for every $x, y \in B(x_0, \gamma^*(\omega)) \cap K_d$, which concludes the proof. □

To conclude this section, let us now prove Lemma B.1.

Proof of Lemma B.1. Let us first observe that $x^{(n)} = x \in B(x_0, \gamma(\omega)) \cap K_d$ and $y^{(n)} = y \in B(x_0, \gamma(\omega)) \cap K_d$. Let us now fix $j \in \{k, \ldots, n-1\}$. The lower bound of Equation (8) leads to

$$\left\| x^{(j)} - x_0 \right\| \leq \left\| x^{(j)} - x \right\| + \|x - x_0\| \leq \frac{\rho\big(x^{(j)}, x\big)^{1/\overline{H}}}{c_{2,1}^{1/\overline{H}}} + \|x - x_0\|.$$

Since $x_0 \in B(x_0, r^*(\omega)/2)$ with r^* satisfying Equation (26) and since $\rho\big(x^{(j)}, x\big) \leq 2^{-j}$ with $j \geq k \geq k^*(\omega)$, we have

$$\left\| x^{(j)} - x_0 \right\| \leq \frac{2^{-k^*(\omega)/\overline{H}}}{c_{2,1}^{1/\overline{H}}} + \frac{\gamma(\omega)}{2}.$$

Then, choosing $k^*(\omega)$ large enough, $x^{(j)} \in B(x_0, \gamma(\omega))$ for $j = k, \ldots, n-1$. The same holds for $y^{(j)}$. Assertion *1.* is then proved.

Let us now observe that since $j \geq k_0$ and since $\kappa \geq 1$,

$$2^j \delta_{j+1} \geq 2^{k_0} \delta_{k_0+1} > 3\kappa^2 \geq 3\kappa \tag{32}$$

by definition of k_0 (see the third step of the proof of Theorem 3.1). Then, using the quasi-triangle inequality fulfilled by ρ and (28), we obtain that

$$\rho(x^{(j+1)}, x^{(j)}) \leq 3\kappa 2^{-(j+1)} \leq \frac{\delta_{j+1}}{2} \leq \delta_{j+1}.$$

Since the same holds for $\rho(y^{(j+1)}, y^{(j)})$, Assertion *2.* is fulfilled. Moreover, applying twice the quasi-triangle inequality fulfilled by ρ and Equations (27), (28) and (32) (with $j = k$), we obtain

$$\rho(x^{(k)}, y^{(k)}) \leq \kappa^2 (2^{1-k} + \rho(x, y)) \leq 3\kappa^2 \rho(x, y) \leq \delta_k,$$

which is Assertion *3.* □

B.2 Proof of Theorem 3.2

Let us first observe that Theorem 3.1 holds. Then, for almost ω, up to choose γ smaller, the sequence of continuous functions $(S_N(\cdot)(\omega))_{N\in\mathbb{N}}$ converges uniformly on $B(x_0,\gamma(\omega))\cap K_d$, which implies that each $R_N(\cdot)(\omega)$ is continuous on $B(x_0,\gamma(\omega))\cap K_d$. As in the proof of Theorem 3.2, we assume without loss of generality that $K_d=\prod_{j=1}^d[a_j,b_j]$ with $a_j<b_j$.

The proof of Assertion *1.* is quite similar to the proof of Equation (30), see Theorem 3.1. Therefore, we only sketch it.

Let $(\nu_k)_{k\geq 1}$ be the sequence introduced in Step 1 of the proof of Theorem 3.1. For $k\in\mathbb{N}\backslash\{0\}$, $N\in\mathbb{N}$ and $(i,j)\in\mathbb{Z}^d$, we consider

$$E_{i,j}^{k,N}=\Big\{\omega : |R_N(x_{\nu_k,i})-R_N(x_{\nu_k,j})|>\sqrt{\log(N+2)}r_N(x_{\nu_k,i},x_{\nu_k,j})\,\varphi(\rho(x_{\nu_k,i},x_{\nu_k,j}))\Big\}$$

with r_N defined by (10) and φ by (23). Then, we replace in Step 1 of the proof of Theorem 3.1 the set E_k by

$$E'_k=\bigcup_{N=0}^{+\infty}\bigcup_{(i,j)\in I_k}E_{i,j}^{k,N},$$

with I_k and δ_k defined by (24). To bound $\mathbb{P}\Big(E_{i,j}^{k,N}\Big)$ we proceed as in the proof of Theorem 3.1 except that we apply Assertion *2.* of Proposition 2.1 instead of Assertion *1.* Then, choosing the constant A, which appears in the definition of φ, and $\delta\in(0,1)$ such that

$$A(1-\delta)-\frac{2}{\underline{H}}+\frac{1-\delta}{\overline{H}}>0\quad\text{ and }\quad A(1-\delta)\log 2>1$$

and following Step 1 of the proof of Theorem 3.1, we obtain that

$$\sum_{k=1}^{+\infty}\mathbb{P}\big(E'_k\big)\leq c_2\sum_{N=2}^{+\infty}2^{-A(1-\delta)\log N}=c_2\sum_{N=2}^{+\infty}N^{-A(1-\delta)\log 2}<+\infty.$$

with c_2 a finite positive constant. Then, by Borel-Cantelli Lemma, the definition of φ and Assumption 3, almost surely there exists an integer $k^*(\omega)$ such that for every $k\geq k^*(\omega)$, for all $N\in\mathbb{N}$, and for all $x,y\in\mathcal{D}_{\nu_k}$ with $x,y\in B(x_0,\gamma(\omega))\cap K_d$ and $\rho(x,y)\leq\delta_k=2^{-(1-\delta)k}$

$$|R_N(x)-R_N(y)|\leq Cb(N)\sqrt{\log(N+2)}\rho(x,y)^\beta|\log(\rho(x,y))|^{\eta+1/2}.$$

In addition, replacing in Step 2 of the proof of Theorem 3.1, S_N by R_N (which still be, for almost ω, continuous on $B(x_0,\gamma(\omega))\cap K_d$), we obtain that for almost ω, there exists $\gamma^*\in(0,\gamma)$, such that

$$|R_N(x)-R_N(y)|\leq Cb(N)\sqrt{\log(N+2)}\rho(x,y)^\beta|\log(\rho(x,y))|^{\eta+1/2}. \tag{33}$$

for every $N\in\mathbb{N}$ and $x,y\in B(x_0,\gamma^*(\omega))\cap K_d$. This establishes Assertion *1.*

Let us now assume that Equation (12) holds. Since

$$|R_N(x)|\leq|R_N(x)-R_N(x_0)|+|R_N(x_0)|,$$

Assertion *2.* of Theorem 3.2 follows from Equations (33) and (12) and the continuity of ρ on the compact set $B(x_0,\gamma(\omega))\cap K_d$. The proof of Theorem 3.2 is then complete. □

C Shot noise series

C.1 Proof of Theorem 4.1

Let $(g_n)_{n\geq 1}$ be a Rademacher sequence, that is a sequence of i.i.d. random variables with symmetric Bernoulli distribution. This Rademacher sequence is assumed to be independent of $(T_n, X_n)_{n\geq 1}$. Then, by independence and also by symmetry of the sequence $(X_n)_{n\geq 1}$, $(X_n g_n)_{n\geq 1}$ has the same distribution as $(X_n)_{n\geq 1}$ and is independent of the sequence $(T_n)_{n\geq 1}$.

Let us now set

$$W_n(\alpha) = T_n^{-1/\alpha} X_n \quad \text{and} \quad S_N(\alpha) = \sum_{n=1}^{N} W_n(\alpha) g_n,$$

so that $\{S_N^*(\alpha), N \geq 1\}$ has the same finite distribution as $\{S_N(\alpha), N \geq 1\}$. Moreover, since

$$\sum_{n=1}^{N} |W_n(\alpha)|^2 = \sum_{n=1}^{N} T_n^{-2/\alpha} |X_n|^2,$$

with $X_n \in L^{2p}$ (with $p > 0$), Assumption 1 is fulfilled on any $K_1 = [a, b] \subset (0, \min(2, 2p))$ (see [32] for instance).

Let us now fix $a, b \in (0, \min(2, 2p))$ such that $a < b$ and remark that, for any $\alpha, \alpha' \in [a, b]$, by the Mean Value Theorem, almost surely, there exists $c_n \in (a, b)$ such that

$$\left(T_n^{-1/\alpha} - T_n^{-1/\alpha'}\right) = \frac{1}{c_n^2} T_n^{-1/c_n} (\ln T_n) (\alpha - \alpha').$$

Let us fix $a' \in (0, a)$ and $b' \in (b, \min(2, 2p))$. Then, there exists a finite positive constant c_1 such that almost surely, for all $c \in [a, b]$ and $n \geq 1$,

$$\frac{1}{c^2} T_n^{-1/c} |\ln T_n| \leq c_1 \max\left(T_n^{-1/b'}, T_n^{-1/a'}\right). \tag{34}$$

It follows that, almost surely, for all $\alpha, \alpha' \in [a, b]$,

$$s(\alpha, \alpha') := \left(\sum_{n=1}^{+\infty} |W_n(\alpha) - W_n(\alpha')|^2\right)^{1/2} \leq C_2 |\alpha - \alpha'|, \tag{35}$$

with $C_2 = c_1 \left(\sum_{n=1}^{+\infty} T_n^{-2/b'} |X_n|^2 + \sum_{n=1}^{+\infty} T_n^{-2/a'} |X_n|^2\right)^{1/2} < +\infty$ since $|X_n|^2 \in L^p$ with $2p > b' \geq a'$ and $a', b' \in (0, 2)$.

Let us now remark that for all $\alpha, \alpha' \in [a, b]$,

$$\{(S_N^*(\alpha) - S_N^*(\alpha'), s(\alpha, \alpha')); N \geq 1\} \overset{fdd}{=} \{(S_N(\alpha) - S_N(\alpha'), s(\alpha, \alpha')); N \geq 1\}.$$

This allows us to replace S_N by S_N^* in the Second Step of the proof of Theorem 3.1. Then, the third and the fourth step of this proof still hold replacing S_N by S_N^* and the limit S by the limit S^* since each S_N^* is continuous (as S_N is) and since S_N^* converges pointwise to S^*. It follows that, for any $x_0 = \alpha_0 \in [a, b]$, almost surely, there exists $\gamma^* \in (0, 1)$ such that $(S_N^*(\cdot)(\omega))_{N\in\mathbb{N}}$ converges uniformly on $B(\alpha_0, \gamma^*(\omega)) \cap [a, b]$ to $S^*(\cdot)(\omega)$. Then, by covering the compact set $[a, b]$, we deduce that almost

surely, $(S_N^*)_{N\in\mathbb{N}}$ converges uniformly on $[a,b]$ to S^*. Since this holds for any $0<a<b<\min(2,2p)$, Assertion 1. of Theorem 4.1 is established.

Let us now focus on the rate of convergence $(S_N^*)_{N\in\mathbb{N}}$. Since almost surely the sequence of continuous random fields $(S_N^*)_{N\in\mathbb{N}}$ converges uniformly on $[a,b]$, for all $N\in\mathbb{N}$ the rest R_N^*, defined by

$$R_N^*(\alpha) := \sum_{n=N+1}^{+\infty} T_n^{-1/\alpha} X_n,$$

is also continuous on $[a,b]$. Remark also that we have, for all $\alpha,\alpha'\in[a,b]$ and $N\in\mathbb{N}$,

$$(R_N^*(\alpha)-R_N^*(\alpha'), r_N(\alpha,\alpha')) \stackrel{d}{=} (R_N(\alpha)-R_N(\alpha'), r_N(\alpha,\alpha')),$$

where $R_N(\alpha)=\sum_{n=N+1}^{+\infty} T_n^{-1/\alpha} X_n g_n = S(\alpha)-S_N(\alpha)$ and

$$r_N\big(\alpha,\alpha'\big) = \left(\sum_{n=N+1}^{+\infty} |X_n|^2 \Big| T_n^{-1/\alpha} - T_n^{-1/\alpha'}\Big|^2\right)^{1/2}.$$

As done for S_N, the previous lines allow to replace R_N by R_N^* in the proof of Theorem 3.2. Moreover, applying as previously the Mean Value Theorem and Equation (34), we obtain that almost surely, for all $N\in\mathbb{N}$, and $\alpha,\alpha'\in[a,b]$,

$$r_N\big(\alpha,\alpha'\big) \le c_1|\alpha-\alpha'|\left(\sum_{n=N+1}^{+\infty} T_n^{-2/b'}|X_n|^2 + \sum_{n=1}^{+\infty} T_n^{-2/a'}|X_n|^2\right)^{1/2} \tag{36}$$

where $b'\in(b,\min(2,2p))$ and $a'\in(0,a)$. Let us now fix $p'>0$ such that $1/p'\in(0,1/b-1/\min(2p,2))$. Up to change b', we assume without loss of generality that $1/p'\in(0,1/b'-1/\min(2p,2))$. Then, by Theorem 2.2 in [11], almost surely

$$\sup_{N\ge 0} N^{2/p'} \sum_{n=N+1}^{+\infty} T_n^{-2/b'}|X_n|^2 < +\infty$$

since $X_n^2\in L^p$ with $p>b'/2$ and $b'/2\in(0,1)$. Moreover, since $a'\ge b$, $1/p'\in(0,1/a'-1/\min(2p,2))$ and then by Theorem 2.2 in [11], almost surely

$$\sup_{N\ge 0} N^{2/p'} \sum_{n=N+1}^{+\infty} T_n^{-2/a'}|X_n|^2 < +\infty$$

Therefore, Assumption 3 is fulfilled for all $x_0=\alpha_0\in[a,b]$ with $b(N)=(N+1)^{-1/p'}$. Note also that by Theorem 2.1 in [11], for all $x_0=\alpha_0\in[a,b]$, almost surely

$$\sup_{N\ge 0} N^{1/p'}\left|\sum_{n=N+1}^{+\infty} T_n^{-1/\alpha_0} X_n\right| < +\infty.$$

Therefore, applying Theorem 3.2 (substituting R_N by R_N^*), for any $x_0=\alpha_0\in[a,b]$, almost surely there exists $\gamma^*\in(0,1)$ such that

$$\sup_{N\in\mathbb{N}} \sup_{\alpha\in B(\alpha_0,\gamma^*)\cap[a,b]} N^{1/p'}|R_N^*(\alpha)| < +\infty$$

Then, by covering the compact set $[a,b]$, we deduce that almost surely,

$$\sup_{N\in\mathbb{N}}\sup_{\alpha\in[a,b]} N^{1/p'}|R_N^*(\alpha)| < +\infty,$$

which concludes the proof. □

C.2 Modulus of continuity and rate of convergence

This section is devoted to the proofs of the results stated in Section 4.2. Let us first establish Theorem 4.2.

Proof of Theorem 4.2. Let $x_0 = (\alpha_0, u_0) \in K_{d+1} = [a,b] \times \prod_{j=1}^d [a_j, b_j] \subset (0,2) \times \mathbb{R}^d$.

Let us assume that $p > b/2$ and consider s the conditional parameter defined by (7). Then, for any $x = (\alpha, u) \in K_{d+1}$ and $y = (\alpha', v) \in K_{d+1}$,

$$s(x,y) \le s_1(x,y) + s_2(x,y), \tag{37}$$

where

$$s_1(x,y) = \left(\sum_{n=1}^{+\infty} T_n^{-2/\alpha}|V_n(x) - V_n(y)|^2\right)^{1/2}$$

and

$$s_2(x,y) = \left(\sum_{n=1}^{+\infty}\left(T_n^{-1/\alpha} - T_n^{-1/\alpha'}\right)^2|V_n(y)|^2\right)^{1/2}.$$

Let us now remark that almost surely, there exists a finite positive constant c_1 such that for all $n \ge 1$,

$$\max_{\gamma\in[a,b]} T_n^{-1/\gamma} \le c_1 \max\left(T_n^{-1/b}, T_n^{-1/a}\right). \tag{38}$$

Hence, for any $x, y \in K_{d+1}$,

$$s_1(x,y) \le c_1\rho(x,y)^\beta|\log\min\left(\rho(x,y),1/2\right)|^\eta\left(\sum_{n=1}^{+\infty} T_n^{-2/b}|Y_n|^2 + \sum_{n=1}^{+\infty} T_n^{-2/a}|Y_n|^2\right)^{1/2}$$

with

$$Y_n = \sup_{\substack{x,y\in K_{d+1}\\ x\neq y}} \frac{|V_n(x) - V_n(y)|}{\rho(x,y)^\beta|\min\left(\rho(x,y),1/2\right)|^\eta}.$$

Since the map $h \mapsto h^\beta|\log h|^\eta$ is increasing around 0, Equation (15) implies that

$$\mathbb{E}\big(Y_n^{2p}\big) = \mathbb{E}\left(\left[\sup_{\substack{x,y\in K_{d+1}\\ x\neq y}} \frac{|V_n(x) - V_n(y)|}{\rho(x,y)^\beta|\log\min\left(\rho(x,y),1/2\right)|^\eta}\right]^{2p}\right) < +\infty.$$

This follows from a covering argument of the compact set K_{d+1} and Equation (8). Then, since $2p > b \ge a$ and since the random variables Y_n, $n \ge 1$, are i.i.d, Theorem 1.4.5 of [32] ensures that $\sum_{n=1}^{+\infty} T_n^{-2/b}|Y_n|^2 < +\infty$ and $\sum_{n=1}^{+\infty} T_n^{-2/a}|Y_n|^2 < +\infty$ almost surely. Therefore there exists a finite positive random variable C_2 such that

$$s_1(x,y) \le C_2\rho(x,y)^\beta|\log\min\left(\rho(x,y),1/2\right)|^\eta$$

almost surely for any $x, y \in K_{d+1}$.

Let us now focus on s_2. Observe that $|V_n(y)| \leq X_n$, with

$$X_n = |V_n(x_0)| + c_2 Y_n$$

for $c_2 = \sup_{z \in K_{d+1}} \rho(x_0, z)^\beta |\log \min(\rho(x_0, z)), 1/2|^\eta$. Let us remark that by continuity of ρ on the compact set $\{x_0\} \times K_{d+1}$, $c_2 < +\infty$. Moreover, since $V_n(x_0) \in L^{2p}$ by assumption, $(X_n)_{n \geq 1}$ still be a sequence of i.i.d. variables in L^{2p} and following the same lines as for Equation (35), we obtain that, almost surely, for any $x, y \in K_{d+1}$,

$$s_2(x, y) \leq C_2 |\alpha - \alpha'|,$$

up to change the random variable C_2. Let us also note that by Equation (9), there exist finite positive constants c_3 and c_4 such that for any $x = (\alpha, u) \in K_{d+1}$ and any $y = (\alpha', v) \in K_{d+1}$,

$$\left|\alpha - \alpha'\right| \leq c_3 \rho(x, y)^{1/\overline{H}} \leq c_4 \rho(x, y)$$

since $\overline{H} \leq 1$. Hence, since $\beta \in (0, 1]$, up to change C_2, almost surely, for any $x, y \in K_{d+1}$,

$$s(x, y) \leq C_2 \rho(x, y)^\beta |\log \min(\rho(x, y), 1/2)|^{\max(\eta, 0)}.$$

This, combined with Remarks 3.3, leads to Assertion 1. of Theorem 4.2.

Let us now prove Assertion 2. of Theorem 4.2. Let us choose $p' > 0$ such that $1/p' \in (0, 1/b - 1/\min(2, 2p))$. Then, replacing in the previous lines s by the parameter r_N and Theorem 1.4.5 of [32] by Theorem 2.2 of [11], we obtain: there exists C_3 a finite positive random variable such that almost surely, for any $x, y \in K_{d+1}$, and for any $N \in \mathbb{N}$,

$$r_N(x, y) \leq C_3 (N+1)^{-1/p'} \rho(x, y)^\beta |\log \rho(x, y)|^{\max(\eta, 0)}.$$

In other words, Assumption 3 holds with $\gamma^* = 1$ deterministic and $b(N) = (N+1)^{-1/p'}$. Note also that by Theorem 2.1 in [11], almost surely

$$\sup_{N \geq 0} N^{1/p'} |R_N(x_0)| < +\infty.$$

Therefore, by Theorem 3.2 and a covering argument, almost surely,

$$\sup_{N \in \mathbb{N}} N^{1/p'} \sup_{x \in K_{d+1}} |R_N(x)| < +\infty,$$

which concludes the proof. □

Let us now prove Proposition 4.3.

Proof of Proposition 4.3. Since Equation (8) is fulfilled, there exists $r \in (0, 1)$ such that $\rho(x, y) \leq h_0$ for all $x, y \in K_{d+1}$ with $\|x - y\| \leq r$. Then, the assumptions done imply that

$$X_1 := \sup_{\substack{x, y \in K_{d+1} \\ 0 < \|x - y\| \leq r}} \frac{|V_1(x) - V_1(y)|}{\rho(x, y)^\beta |\log \rho(x, y)|^\eta} \leq \sup_{h \in (0, h_0]} \frac{\mathcal{G}(h)}{h^\beta |\log h|^\eta} := G.$$

Note that assuming $h_0 = 2^{-k_0}$ with an integer $k_0 \geq 1$ (which is not restrictive),

$$G = \sup_{k \geq k_0} \sup_{h \in (2^{-k-1}, 2^{-k}]} \frac{\mathcal{G}(h)}{F(h)}$$

where the function F is defined by

$$F(h) := h^{\beta} |\log h|^{\eta}.$$

Therefore, up to choose h_0 small enough, using that F is increasing on $(0, h_0]$ and the monotonicity of $\mathcal{G}$, we obtain:

$$G^{2p} \leq \sum_{k=k_0}^{+\infty} \sup_{h \in (2^{-k-1}, 2^{-k}]} \left(\frac{\mathcal{G}(h)}{F(h)} \right)^{2p} \leq \max(2^{\eta}, 1)^{2p} \sum_{k=k_0}^{+\infty} \left(\frac{\mathcal{G}(2^{-k})}{F(2^{-k})} \right)^{2p}.$$

Moreover, up to choose h_0 small enough (and then r smaller), by assumption,

$$\forall h \in (0, h_0], \, \mathbb{E}\big(\mathcal{G}(h)^{2p}\big) \leq F(h)^{2p} |\log h|^{-1-2p\varepsilon}.$$

Therefore, we have

$$\mathbb{E}(X_1^{2p}) \leq \mathbb{E}(G^{2p}) \leq \max(2^{\eta}, 1)^{2p} \sum_{k=k_0}^{+\infty} |k \log 2|^{-1-2p\varepsilon} < +\infty,$$

which concludes the proof. □

D Application to LePage random series

D.1 Proof of Proposition 5.1

Let $K_{d+1} = [a, b] \times \prod_{j=1}^{d} [a_j, b_j] \subset (0, 2) \times \mathbb{R}^d$. Let us fix an integer $p \geq 1$ and consider $x^{(j)} = \big(\alpha_j, u^{(j)}\big) \in K_{d+1}$ for each integer $1 \leq j \leq p$. Then, we set $\vec{x} = \big(x^{(1)}, \ldots, x^{(p)}\big)$. Choosing

$$\mathcal{S} = \Big\{ \xi \in \mathbb{R}^d ; m(\xi) > 0 \Big\} \times \mathbb{C},$$

we can define $H_{\vec{x}} : (0, +\infty) \times \mathcal{S} \to \mathbb{C}^p$ by

$$H_{\vec{x}}(r, (\xi, g)) = \Big(r^{-1/\alpha_1} f_{\alpha_1}(u^{(1)}, \xi)\, m(\xi)^{-1/\alpha_1} g, \ldots, r^{-1/\alpha_p} f_{\alpha_p}(u^{(p)}, \xi)\, m(\xi)^{-1/\alpha_p} g \Big). \tag{39}$$

Let us note that,

$$\sum_{n=1}^{N} H_{\vec{x}}(T_n, (\xi_n, g_n)) = \Big(S_{m,N}\Big(x^{(1)}\Big), \ldots, S_m\Big(x^{(p)}\Big) \Big) \text{ a.s.}$$

and then converges almost surely to $\big(S_m\big(x^{(1)}\big), \ldots, S_m\big(x^{(p)}\big)\big)$. Then, since g_1 is symmetric, setting $\mathbb{P}_g$ its distribution, Theorem 2.4 of [31] ensures that

$$\forall \lambda = (\lambda_1, \ldots, \lambda_p) \in \mathbb{C}^p, \, \forall z \in \mathbb{C}, \, \mathbb{E}\Big(\mathrm{e}^{i\Re\left(\overline{z} \sum_{j=1}^{p} \lambda_j S_m\left(x^{(j)}\right)\right)} \Big) = \exp\big(I_{\vec{x},\lambda}(z) \big)$$

where

$$I_{\vec{x},\lambda}(z) = \int_{(0,+\infty)\times\mathcal{S}} \Big(\mathrm{e}^{i\Re(\overline{z}\langle \lambda, H_{\vec{x}}(r,(\xi,g))\rangle)} - 1 - i\Re\big(\overline{z}\langle \lambda, H_{\vec{x}}(r,(\xi,g))\rangle \mathbf{1}_{|\Re(\overline{z}\langle \lambda, H_{\vec{x}}(r,(\xi,g))\rangle)| \leq 1}\big) \Big) m(\xi) dr \nu(d\xi) \mathbb{P}_g(dg).$$

Then, by definition of $H_{\vec{x}}$, using the change or variable $t = rm(\xi)$, and the fact that $m(\xi) > 0$ for ν-almost every ξ, we get

$$I_{\vec{x},\lambda}(z) = \int_{(0,+\infty)\times\mathbb{R}^d\times\mathbb{C}} \Big(\mathrm{e}^{i\Re(\overline{z}\langle\lambda, J_{\vec{x}}(t,(\xi,g))\rangle)} - 1 - i\Re\big(\overline{z}\langle\lambda, J_{\vec{x}}(t,(\xi,g))\rangle \mathbf{1}_{|\Re(\overline{z}\langle\lambda, J_{\vec{x}}(t,(\xi,g))\rangle)|\le 1}\big)\Big) dt\nu(d\xi)\mathbb{P}_g(dg)$$

with

$$J_{\vec{x}}(t,(\xi,g)) = \Big(t^{-1/\alpha_1} f_{\alpha_1}(u^{(1)},\xi)\, g, \ldots, t^{-1/\alpha_p} f_{\alpha_p}(u^{(p)},\xi) g \Big).$$

Therefore, $I_{\vec{x},\lambda}$ does not depend on the function m, and then neither does the distribution of the vector $\big(S_m\big(x^{(1)}\big), \ldots, S_m\big(x^{(p)}\big)\big)$. Since this holds for any p and $\vec{x}$, Assertion 1. is established, that is replacing m by $\tilde{m}$ and then ξ_n by $\tilde{\xi_n}$,

$$(S_m(x))_{x\in K_{d+1}} \overset{fdd}{=} (S_{\tilde{m}}(x))_{x\in K_{d+1}}.$$

Let us now consider the space $B = \mathcal{C}(K_{d+1}, \mathbb{C})$ of complex-valued continuous functions defined on the compact set K_{d+1}. This space is endowed with the topology of the uniform convergence, so that it is a Banach space.

Let us assume that $S_{\tilde{m}}$ belongs almost surely to $\mathcal{H}_\rho(K_{d+1}, \beta, \eta) \subset B$. For any $\vec{x} = \big(x^{(1)}, \ldots, x^{(p)}\big) \in K^p$, in view of its characteristic function given above, the vector $\big(S_{\tilde{m}}\big(x^{(1)}\big), \ldots, S_{\tilde{m}}\big(x^{(p)}\big)\big)$ is infinitely divisible and its Lévy measure is given by

$$\begin{aligned} F_{\vec{x}}(A) &= \int_{(0,+\infty)\times\mathbb{R}^d\times\mathbb{C}} \mathbf{1}_{A\setminus\{0\}}(J_{\vec{x}}(t,(\xi,g)))\, dtd\xi\mathbb{P}_g(dg) \\ &= \int_{(0,+\infty)\times\mathcal{S}} \mathbf{1}_{A\setminus\{0\}}(H_{\vec{x}}(r,(\xi,g)))m(\xi)drd\xi\mathbb{P}_g(dg) \end{aligned}$$

for any Borel set $A \in \mathcal{B}(\mathbb{C}^p)$. We first assume that $(\alpha, u) \mapsto f_\alpha(u,\xi)$ belongs to B for all $\xi \in \mathbb{R}^d$ so that we may define

$$\begin{array}{rccl} H: & (0,+\infty)\times\mathcal{S} & \longrightarrow & B \\ & (r,(\xi,g)) & \mapsto & \Big((\alpha,u) \mapsto r^{-1/\alpha} f_\alpha(u,\xi) m(\xi)^{-1/\alpha}\Big). \end{array}$$

Since $H_{\vec{x}}$ is defined by (39), one checks that $(S_{\tilde{m}}(x))_{x\in K_{d+1}}$ is a B-valued infinitely divisible random variable with Lévy measure defined by

$$F(A) = \int_E \mathbf{1}_{A\setminus\{0\}}(H(r,(\xi,g)))m(\xi)drd\xi P_g(dg), \quad A \in \mathcal{B}(E).$$

Then, by Theorem 2.4 of [31],

$$\sum_{n=1}^{N} H(T_n, (\xi_n, g_n))$$

converges almost surely in B as $N \to +\infty$. Then, by definition of H, the sequence $(S_{m,N})_{N\in\mathbb{N}}$ converges in B almost surely. Therefore, its limit S_m is almost surely continuous on K_{d+1}.

Let us now consider $\mathcal{D} \subset K_{d+1}$ a countable dense set in K_{d+1}. Since almost surely $S_{\tilde{m}}$ belongs to $\mathcal{H}_\rho(K_{d+1}, \beta, \eta)$, we get that

$$\sup_{\substack{x,y\in\mathcal{D}\\ x\neq y}} \frac{|S_{\tilde{m}}(x) - S_{\tilde{m}}(y)|}{\rho(x,y)^\beta |\log(\min(\rho(x,y), 1/2))|^\eta} < +\infty$$

almost surely. Therefore, almost surely

$$\sup_{\substack{x,y\in\mathcal{D}\\ x\neq y}} \frac{|S_m(x)-S_m(y)|}{\rho(x,y)^\beta |\log\left(\min\left(\rho(x,y),1/2\right)\right)|^\eta} < +\infty$$

since $\mathcal{D}$ is a countable set and since $S_m \overset{fdd}{=} S_{\tilde{m}}$. Then, by continuity of ρ, by almost sure continuity of S_m and by density of $\mathcal{D}$ on K_{d+1},

$$\sup_{\substack{x,y\in K_{d+1}\\ x\neq y}} \frac{|S_m(x)-S_m(y)|}{\rho(x,y)^\beta |\log\left(\min\left(\rho(x,y),1/2\right)\right)|^\eta} < +\infty$$

almost surely, that is S_m belongs almost surely to $\mathcal{H}_\rho(K_{d+1},\beta,\eta)$. This establishes Assertion 2. when $(\alpha,u)\mapsto f_\alpha(u,\xi)$ is continuous for all $\xi\in\mathbb{R}^d$.

Assume now that $(\alpha,u)\mapsto f_\alpha(u,\xi)$ is continuous for $\xi\in\mathbb{R}^d\backslash\mathcal{N}$ with $\nu(\mathcal{N})=0$ and set

$$g_\alpha(u,\xi) := f_\alpha(u,\xi)\mathbf{1}_{\mathbb{R}^d\backslash\mathcal{N}}(\xi).$$

Then, almost surely, for all $x=(\alpha,u)\in(0,2)\times\mathbb{R}^d$ and all $N\geq 1$,

$$S_{m,N}(x) = \sum_{n=1}^{N} T_n^{-1/\alpha} g_\alpha(u,\xi_n) m(\xi_n)^{-1/\alpha} g_n,$$

and the conclusion follows from the previous lines since $(\alpha,u)\mapsto g_\alpha(u,\xi)$ is continuous on K_{d+1} for all $\xi\in\mathbb{R}^d$. The proof of Proposition 5.1 is then complete. □

D.2 Proof of Proposition 5.2

Let us first note that using Remark 5.1, we can and may assume without loss of generality that $a_1=1$, up to replace E by E/a_1 and $\tau_{_E}$ by $\tau_{_{E/a_1}}^{1/a_1}$.

Let us choose $\zeta>0$ arbitrarily small and consider the Borel function $\tilde{m}$ defined on $\mathbb{R}^d$ by

$$\tilde{m}(\xi) = \|\xi\|^{\alpha_0}\mathbf{1}_{\|\xi\|\leq A} + \tau_{_{E^t}}(\xi)^{-q(E)}\left|\log\tau_{_{E^t}}(\xi)\right|^{-1-\zeta}\mathbf{1}_{\|\xi\|>A}.$$

Observe that $\tilde{m}$ is positive on $\mathbb{R}^d\backslash\{0\}$. Then,

$$0<c=\int_{\mathbb{R}^d}\tilde{m}(\xi)d\xi = c_{_1}+c_{_2}$$

with

$$c_{_1}=\int_{\|\xi\|\leq A}\|\xi\|^{\alpha_0}d\xi \quad \text{and} \quad c_{_2}=\int_{\|\xi\|>A}\tau_{_{E^t}}(\xi)^{-q(E)}\left|\log\tau_{_{E^t}}(\xi)\right|^{-1-\zeta}d\xi.$$

Let us first observe that $c_{_1}<\infty$ since $\alpha_0>0$. To prove that $c_{_2}$ is also a finite constant, we need some tools given in [28, 9]. As in Chapter 6 of [28], let us consider the norm $\|\cdot\|_{_{E^t}}$ defined by

$$\|x\|_{_{E^t}} = \int_0^1 \left\|\theta^{E^t}x\right\|\frac{d\theta}{\theta},\ \ \forall x\in\mathbb{R}^d \tag{40}$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^d$. Then, according to Chapter 6 of [28], $\|\cdot\|_{E^t}$ is a norm on $\mathbb{R}^d$ such that the map

$$\begin{array}{rccc} \Psi_{E^t}: & (0,+\infty)\times S_{E^t} & \longrightarrow & \mathbb{R}^d\setminus\{0\} \\ & (r,\theta) & \longmapsto & r^{E^t}\theta \end{array}$$

is a homeomorphism, where

$$S_{E^t}=\{\xi\in\mathbb{R}^d : \|\xi\|_{E^t}=1\}$$

is the unit sphere for $\|\cdot\|_{E^t}$. According to the change of variables in polar coordinates (see [9]) there exists a finite positive Radon measure σ_{E^t} on S_{E^t} such that for all measurable function φ non-negative or in $L^1(\mathbb{R}^d,d\xi)$,

$$\int_{\mathbb{R}^d}\varphi(\xi)d\xi=\int_0^{+\infty}\int_{S_{E^t}}\varphi(r^{E^t}\theta)\sigma_{E^t}(d\theta)r^{q(E)-1}dr,$$

using the fact that $q(E)=\mathrm{trace}(E^t)$. Applying this change of variables, it follows that $c_2<\infty$ since $\zeta>0$. Hence, $m=\tilde{m}/c$ is well-defined and $\mu(d\xi)=m(\xi)d\xi$ is a probability measure equivalent to the Lebesgue measure. Then we may consider $S_m(\alpha_0,u)$ defined by (16) for $u\in\mathbb{R}^d$ so that

$$X_{\alpha_0}\overset{fdd}{=}d_{\alpha_0}S_m(\alpha_0,\cdot)$$

with d_{α_0} given by (17).

To study the sample path regularity of $S_m(\alpha_0,\cdot)$ on $K_d=\prod_{j=1}^d[a_j,b_j]$, we apply Proposition 4.3 on $K_{d+1}=\{\alpha_0\}\times K_d\subset(0,2)\times\mathbb{R}^d$ for

$$V_1(\alpha_0,u)=f_{\alpha_0}(u,\xi_1)m(\xi_1)^{-1/\alpha_0}$$

with f_{α_0} defined by (19). We recall that here ξ_1 is a random vector of $\mathbb{R}^d$ with density m. Therefore let us now check that assumptions of Proposition 4.3 are fulfilled.

For $h>0$ and $\xi\in\mathbb{R}^d$ we consider

$$g(h,\xi)=\min(c_{E^t}\|h^{E^t}\xi\|_{E^t},1)|\psi_{\alpha_0}(\xi)|,$$

where $c_{E^t}>0$ is chosen such that $|e^{i\langle u,\xi\rangle}-1|\le c_{E^t}\|\tau_E(u)^{E^t}\xi\|_{E^t}$. We consider the quasi-metric defined on $\mathbb{R}^{d+1}$ by

$$\rho((\alpha,u),(\alpha',v))=|\alpha-\alpha'|+\rho_E(u,v),\quad\forall(\alpha,u),(\alpha',v)\in\mathbb{R}\times\mathbb{R}^d,$$

which clearly satisfies Equation (8). By definition of V_1, g and $\|\cdot\|_{E^t}$, the random field $\mathcal{G}=(g(h,\xi_1))_{h\in[0,+\infty)}$ satisfies *(i).* and *(ii).* of Proposition 4.3. It remains to consider assumption *(iii).* Let

$$I(h)=\mathbb{E}(\mathcal{G}(h)^2)=\int_{\mathbb{R}^d}g(h,\xi)^2m(\xi)^{1-2/\alpha_0}d\xi. \tag{41}$$

Since ψ_{α_0} satisfies (20),

$$I(h)=\int_{\mathbb{R}^d}m(\xi)^{1-2/\alpha_0}\min\left(c_{E^t}\left\|h^{E^t}\xi\right\|_{E^t},1\right)^2|\psi_{\alpha_0}(\xi)|^2d\xi\le I_1(h)+I_2(h)$$

with

$$I_1(h)=c^{2/\alpha_0-1}c_{E^t}^2\int_{\|\xi\|\le A}\left\|h^{E^t}\xi\right\|_{E^t}^2\|\xi\|^{\alpha_0-2}|\psi_{\alpha_0}(\xi)|^2\,d\xi,$$

where A is given by the condition (20), and

$$I_2(h) = c^{2/\alpha_0-1}c_{5,1}\int_{\mathbb{R}^d}\min\left(c_{E^t}\left\|h^{E^t}\xi\right\|_{E^t},2\right)^2\tau_{E^t}(\xi)^{-q(E)-2\beta}\left|\log\tau_{E^t}(\xi)\right|^{(1+\zeta)(2/\alpha_0-1)}d\xi.$$

From Lemma 3.2 of [7] there exists $C_1>0$ such that for all $h\in(0,e^{-1}]$

$$I_1(h)\le C_1h^{2a_1}|\log(h)|^{2(d-1)}.$$

Moreover, using again the change of variables in polar coordinates, there exists $C_2>0$ such that for all $h\in(0,e^{-1}]$,

$$I_2(h)\le C_2h^{2\beta}|\log(h)|^{(1+\zeta)(2/\alpha_0-1)}.$$

Since $\beta<a_1$, one can find $C_3>0$ such that

$$I(h)\le C_3h^{2\beta}|\log(h)|^{2(1+\zeta)(1/\alpha_0-1/2)}. \tag{42}$$

Hence, assumption *(iii).* of Proposition 4.3 is also fulfilled and applying this proposition, it follows that (15) is satisfied with β and $\eta=1/\alpha_0+\varepsilon$, for all $\varepsilon>0$. Then, by Theorem 4.2, almost surely $S_m\in\mathcal{H}_\rho(K_{d+1},\beta,1/\alpha_0+1/2+\varepsilon)$. By definition of ρ and K_{d+1}, this means that

$$S_m(\alpha_0,\cdot)\in\mathcal{H}_{\rho_{E}}(K_d,\beta,1/\alpha_0+1/2+\varepsilon).$$

Since $d_{\alpha_0}S(\alpha_0,\cdot)\overset{fdd}{=}X_{\alpha_0}$, it follows that a.s.

$$C:=\sup_{u,v\in\mathcal{D},u\neq v}\frac{|X_{\alpha_0}(u)-X_{\alpha_0}(v)|}{\tau_{E}(u-v)^\beta|\log\tau_{E}(u-v)|^{1/\alpha_0+1/2+\varepsilon}}<+\infty,$$

where $\mathcal{D}\subset K_d$ is a countable dense in $K_d=\prod_{j=1}^d[a_j,b_j]$. So let us write Ω^* this event and let us define a modification of X_{α_0} on K_d.
First, if $\omega\notin\Omega^*$, we set $X^*_{\alpha_0}(u)(\omega)=0$ for all $u\in K_d$. Let us now fix $\omega\in\Omega^*$. Then, we set

$$X^*_{\alpha_0}(u)(\omega)=X_{\alpha_0}(u)(\omega),\forall u\in\mathcal{D}.$$

Let us now consider $u\in K_d$. Then, there exists $u^{(n)}\in\mathcal{D}$ such that $\lim_{n\to+\infty}u^{(n)}=u$. It follows that,

$$\left|X^*_{\alpha_0}\left(u^{(n)}\right)(\omega)-X^*_{\alpha_0}\left(u^{(m)}\right)(\omega)\right|\le C(\omega)\tau_{E}(u^{(n)}-u^{(m)})^\beta|\log\tau_{E}(u^{(n)}-u^{(m)})|^{1/\alpha_0+1/2+\varepsilon},$$

so that $\left(X^*_{\alpha_0}\left(u^{(n)}\right)(\omega)\right)_n$ is a Cauchy sequence and hence converges. We set

$$X^*_{\alpha_0}(u)(\omega)=\lim_{n\to+\infty}X^*_{\alpha_0}\left(u^{(n)}\right)(\omega).$$

Remark that this limit does not depend on the choice of $\left(u^{(n)}\right)_n$. Observe also that $X^*_{\alpha_0}(\cdot)(\omega)$ is then well-defined on K_d.

Moreover, by continuity of τ_{E} we have therefore

$$C(\omega)=\sup_{u,v\in K_d,u\neq v}\frac{|X^*_{\alpha_0}(u)(\omega)-X^*_{\alpha_0}(v)(\omega)|}{\tau_{E}(u-v)^\beta|\log\tau_{E}(u-v)|^{1/\alpha_0+1/2+\varepsilon}}<+\infty$$

for all $\omega \in \Omega$ and $X^*_{\alpha_0}$ is continuous on K_d. To conclude the proof, let us prove that $X^*_{\alpha_0}$ is a modification of X_{α_0}. Since $X^*_{\alpha_0}(u)(\omega) = X_{\alpha_0}(u)(\omega)$ for any $\omega \in \Omega^*$ and any $u \in \mathcal{D}$, it is sufficient to prove that X_{α_0} is stochastically continuous. Since m is positive and since $\mu(d\xi) = m(\xi)d\xi$ is a probability measure, by Hölder Inequality, for $u, v \in \mathbb{R}^d$ we have

$$\begin{aligned}\int_{\mathbb{R}^d} \left|f_{\alpha_0}(u,\xi) - f_{\alpha_0}(v,\xi)\right|^{\alpha_0} d\xi &\leq \left(\int_{\mathbb{R}^d} \left|f_{\alpha_0}(u,\xi) - f_{\alpha_0}(v,\xi)\right|^2 m(\xi)^{1-2/\alpha_0} d\xi\right)^{\alpha_0/2} \\ &\leq I(\tau_E(u-v))^{\alpha_0/2}\end{aligned}$$

where I is defined by (41). Moreover, in view of Equation (42), $I(\tau_E(u-v))^{\alpha_0/2} \to 0$ as u tends to v. Therefore it follows that X_{α_0} is stochastically continuous on $\mathbb{R}^d$, by Proposition 6.2.3 of [32]. This conclude the proof.

D.3 Multistable random fields

This section is devoted to the proofs of the results stated in Section 5.3. Let us first establish Proposition 5.3.

Proof of Proposition 5.3. Since $\tilde{\rho}$ satisfies Equation (8), so does ρ. Then, Assumptions of Theorem 4.2 are fulfilled, which implies that $(S_N)_{N\in\mathbb{N}}$ converges uniforlmy to S on $K_{d+1} = [a,b] \times K_d$. Therefore, $\left(\tilde{S}_N\right)_{N\in\mathbb{N}}$ converges uniformly to $\tilde{S}$ on K_d since $\tilde{S}_N(u) = S_N(\alpha(u), u)$ and $\tilde{S}(u) = S(\alpha(u), u)$ and α is continuous.
Moreover, by Theorem 4.2 and since $\tilde{S}(u) = S(\alpha(u), u)$, there exists a finite positive random variable C such that for any $u, v \in K_d$,

$$\left|\tilde{S}(u) - \tilde{S}(v)\right| \leq C\rho(x(u), x(v))^{\beta} |\log \min\left(\rho(x(u), x(v)), 1/2\right)|^{\max(\eta,0)+1/2}$$

where $x(w) = (\alpha(w), w)$. Moreover, by definition of ρ and since $\alpha \in \mathcal{H}_{\tilde{\rho}}(K_d, 1, 0)$, there exists a finite positive constant c_1 such that

$$\forall u, v \in K_d,\ \rho(x(u), x(v)) \leq c_1 \tilde{\rho}(u, v).$$

Let us now recall that since $\tilde{\rho}$ is continuous on the compact set $K_d \times K_d$,

$$M = \sup_{u,v\in K_d} \tilde{\rho}(u, v) < +\infty.$$

Then, up to change C, for all $u, v \in K_d$,

$$\left|\tilde{S}(u) - \tilde{S}(v)\right| \leq C\tilde{\rho}(u, v)^{\beta} |\log \min\left(\tilde{\rho}(u, v), 1/2\right)|^{\max\left(\eta,0\right)+1/2}$$

since $h \mapsto h^{\beta} |\log \min(h, 1/2)|^{\max\left(\eta,0\right)+1/2}$ is increasing around 0 and bounded on $[0, M]$. Assertion 1. is then proved. Moreover, Assertion 2. is a direct consequence of Assertion 2. of Theorem 4.2. The proof is then complete. □

Let us conclude this paper by the proof of Corollary 5.4.

Proof of Corollary 5.4. Let $K_d = \prod_{j=1}^d [a_j, b_j] \subset \mathbb{R}^d$ and $u_0 \in K_d$. Let us set

$$a = \min_{K_d} \alpha, \ b = \max_{K_d} \alpha \ \text{ and } \ K_{d+1} = [a,b] \times K_d \subset (0,2) \times \mathbb{R}^d$$

Let us first note that Assertion *5.* of Assumption 5 is fulfilled by f_α with $K_1 = [a,b]$ and then $\tilde{S}$ is well-defined. Let us now consider ρ_{E} and τ_{E} as defined in Example 2.1. Then we set

$$\tilde{m}(\xi) = \frac{c_\zeta}{\tau_{E^t}(\xi)^{q(E)} \big|\log \tau_{E^t}(\xi)\big|^{1+\zeta}},$$

with $\zeta > 0$ a parameter chosen arbitrarily small. Therefore, let us consider

$$\tilde{V}_n(\alpha, u) = f_\alpha\Big(u, \tilde{\xi}_n\Big) \tilde{m}(\xi_n)^{-1/\alpha}$$

where $\Big(\tilde{\xi_n}\Big)_{n\geq 1}$ is a sequence of i.i.d. random variables with common distribution $\tilde{\mu}(d\xi) = \tilde{m}(\xi)d\xi$. The sequence $\Big(\tilde{\xi_n}\Big)_{n\geq 1}$ is assumed to be independent from $(T_n, g_n)_{n\geq 1}$. Then, Assumption 4 is fulfilled. Moreover,

$$\Big|\tilde{V}_n(\alpha,u) - \tilde{V}_n\big(\alpha',v\big)\Big| \leq \Big|\tilde{V}_n(\alpha,u) - \tilde{V}_n(\alpha,v)\Big| + \Big|\tilde{V}_n(\alpha,v) - \tilde{V}_n\big(\alpha',v\big)\Big|.$$

Let us set

$$C_1 = \sup_{\substack{u,v\in K_d \\ 0<\|u-v\|\leq r}} \sup_{\alpha\in[a,b]} \frac{\Big|\tilde{V}_1(\alpha,u) - \tilde{V}_1(\alpha,v)\Big|}{\rho_{E}(u,v)|\log \rho_{E}(u,v)|^\eta} \quad \text{and} \quad C_2 = \sup_{\substack{\alpha,\alpha'\in[a,b] \\ \alpha\neq\alpha'}} \sup_{u\in K_d} \frac{\Big|\tilde{V}_1(\alpha,u) - \tilde{V}_1(\alpha',u)\Big|}{|\alpha-\alpha'|}$$

where $r > 0$ and the choice of $\eta \in \mathbb{R}$ is given below. Then, for any $x = (\alpha, u) \in K_{d+1}$ and any $y = (\alpha', v) \in K_{d+1}$ such that $\|x - y\| \leq r$,

$$\begin{aligned}\Big|\tilde{V}_1(x) - \tilde{V}_1(y)\Big| &\leq (C_1 + C_2)\big(\rho_{E}(u,v)|\log \rho_{E}(u,v)|^\eta + \big|\alpha - \alpha'\big|\big) \\ &\leq c_1(C_1+C_2)\rho(x,y)|\log \rho(x,y)|^\eta\end{aligned}$$

where $c_1 \in (0, +\infty)$ is a finite constant and $\rho(x,y) = \rho_{E}(u,v) + |\alpha - \alpha'|$. Then, to apply Assertion 1. of Proposition 5.3 with $\tilde{\rho} = \rho_{E}$ and $\beta = 1$, it suffices to establish that C_1, C_2 and $\tilde{V}_1(\alpha(u_0), u_0) \in L^2$ (since $b < 2$).

Let us first deal with $\tilde{V}_1(\alpha(u_0), u_0)$. By definition of $\tilde{V}_1$ and $\tilde{m}$,

$$\mathbb{E}\bigg(\Big|\tilde{V}_1(\alpha(u_0),u_0)\Big|^2\bigg) = \int_{\mathbb{R}^d} \Big|\mathrm{e}^{i\langle u_0,\xi\rangle} - 1\Big|^2 \psi(\xi)^{-2-2q(E)/\alpha(u_0)} \tau_{E^t}(\xi)^{(2/\alpha(u_0)-1)q(E)} \big|\log \tau_{E^t}(\xi)\big|^{(1+\zeta)(2/\alpha(u_0)-1)} d\xi.$$

Then, using polar coordinates associated with E^t (see [28]),

$$\mathbb{E}\bigg(\Big|\tilde{V}_1(\alpha(u_0),u_0)\Big|^2\bigg) \leq c_2 \int_0^{+\infty} \min\Big(\Big\|t^{E^t}\Big\|, 1\Big)^2 t^{-3} |\log t|^{2(1+\zeta)/\alpha(u_0)-1} dt$$

with c_2 a finite positive constant. Hence, Lemma 2.1 of [9] proves that $V_1(\alpha(u_0), u_0) \in L^2$ for any choice of ζ.

Let us now consider the random variable C_1. By homogeneity and continuity of ψ, there exists a finite positive constant c_3 such that for any $u, v \in K_d$,

$$\sup_{\alpha\in[a,b]} \Big|\tilde{V}_1(\alpha,u) - \tilde{V}_1(\alpha,v)\Big| \leq c_3 \Big|\mathrm{e}^{i\langle u-v,\tilde{\xi}_1\rangle} - 1\Big| Z_1$$

with

$$Z_1 = \tau_{E^t}\left(\tilde{\xi}_1\right)^{-1} \max\left(\left|\log \tau_{E^t}\left(\tilde{\xi}_1\right)\right|^{(1+\zeta)/a}, \left|\log \tau_{E^t}\left(\tilde{\xi}_1\right)\right|^{((1+\zeta)/b}\right).$$

Combining the proofs of Propositions 4.3 and 5.2, we obtain that for any $\varepsilon > 0$, choosing η small enough,

$$\mathbb{E}\left(\left[\sup_{\substack{u,v\in K_d \\ 0<\|u-v\|\leq r}} \frac{\left|\mathrm{e}^{i\langle u-v,\tilde{\xi}_1\rangle} - 1\right| Z_1}{\rho_{_E}(u,v)|\log \rho_{_E}(u,v)|^{1/a+\varepsilon}}\right]^2\right) < +\infty,$$

This implies that for any $\varepsilon > 0$, $C_1 \in L^2$ for $\eta = 1/a + \varepsilon$ and ζ well-chosen.

Let us now study C_2. Since K_d is a compact set, using polar coordinates and the Mean Value Theorem, we have

$$\sup_{v\in K_d} \left|\tilde{V}_1(\alpha, v) - \tilde{V}_1\big(\alpha', v\big)\right| \leq c_4|\alpha - \alpha'|Z_2$$

with $Z_2 = \min\left(\left\|\tau_{E^t}(\xi_n)^{E^t}\right\|, 1\right) Z_1\left|\log \tau_{E^t}(\xi_n) + c_5\right|$ and c_4 and c_5 two finite positive constants. Using polar coordinates, one checks that $Z_2 \in L^2$, which implies that $C_2 \in L^2$.

Therefore, for any $\varepsilon > 0$, Assumptions of Assertion 1. of Proposition 5.3 are fulfilled for a well-chosen ζ. This implies that almost surely, for any $\varepsilon > 0$, $\tilde{S}_m \in \mathcal{H}_{\rho_{_E}}(K_d, 1, 1/a + 1/\varepsilon)$ with $a = \min_{K_d} \alpha$. Hence, for any $\varepsilon > 0$, $\tilde{S}_m \in \mathcal{H}_{\rho_{_E}, B(u_0,r)}(u_0, 1, 1/\alpha(u_0) + 1/2 + 1/\varepsilon)$ for r small enough. This concludes the proof.

□

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