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# RANDOM WALKS IN THE QUARTER PLANE, HARMONIC FUNCTIONS AND CONFORMAL MAPPINGS

KILIAN RASCHEL

ABSTRACT. We propose here a new approach for finding harmonic functions of killed random walks with small steps in the quarter plane. It is based on solving a certain functional equation that satisfies the generating function of the values taken by the harmonic functions. As a first application of our results, we obtain a simple expression for the harmonic function that governs the asymptotic tail distribution of the first exit time from the quarter plane. As another corollary, we prove, in the zero drift case, the uniqueness of the harmonic function.

## 1. INTRODUCTION

**Context.** Random processes conditioned on staying in cones of  $\mathbf{Z}^d$  arouse a great interest in the mathematical community, as they appear in several distinct domains: quantum random walks [1, 2], random matrices [8], non-colliding random walks [5, 6, 9], etc. A usual way to realize this conditioning consists in using Doob  $h$ -transforms, thanks to functions which are harmonic for the process, positive within the cone and equal to zero elsewhere—or equivalently harmonic and positive for the underlying killed process. It is therefore natural to be interested in finding some (all, if possible) positive harmonic functions for processes in cones of  $\mathbf{Z}^d$  killed at the first exit time from the cone.

In the literature, results on harmonic functions for killed processes in cones are broken up, and most of them concern very particular cones, as half spaces  $\mathbf{Z}_+ \times \mathbf{Z}^{d-1}$  and orthants  $\mathbf{Z}_+^d$ . Regarding random walks with *non-zero drift*, general results are obtained in [16, 18] for the above domains when  $d = 2$ : the Martin boundary is found, and happens to be composed of infinitely many harmonic functions. For random walks with *zero drift*, the results are rare, and typically require a strong underlying structure: the random walks are cartesian products in [20]; they are associated with Lie algebras in [1, 2]; certain reflexion groups are supposed to be finite in [22]. This is problematic, as these harmonic functions are useful to construct other important processes (see above). Last but not least, knowing the harmonic functions for zero drift random walks in  $\mathbf{Z}_+^{d-1}$  is necessary for building the harmonic functions of walks with drift in  $\mathbf{Z}_+^d$ , see [14].

In this article, we prove that for the whole class of walks with small steps and zero drift killed at the boundary of the quadrant  $\mathbf{Z}_+^2$ , there exists exactly one harmonic function. Further, the unique harmonic function is expressed in terms of sine and arcsine functions.

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Our approach<sup>1</sup> is new—to the best of our knowledge—and is based on functional equations that satisfy the generating functions of the values taken by the harmonic functions.

Before presenting it (in point 5 below), we notice that in the literature, there are several ways to obtain harmonic functions:

1. The most elementary approach just uses the definition of a harmonic function: if  $P$  denotes the generator of the Markov chain under consideration, we must have  $Pf = f$  inside of the cone, and  $f = 0$  outside. Unfortunately, except for a few particular cases (see, e.g., [26]), solving directly this discrete Dirichlet problem is out of range.
2. A second one, based on Martin boundary theory, is used in [1, 14, 15, 16, 18, 22]. Briefly said, given a (transient) Markov process, this theory provides explicit expressions (as integral formulations) for all its (super)harmonic functions. To find the Martin boundary, it suffices [7] to compute the asymptotic behavior of quotients of Green functions. More details are given in Section 5.1. The main and profound difficulty of this approach lies in finding the (asymptotic of quotients of) Green functions.
3. A third one, which is studied in [5, 6, 9, 17], is based on the fact that in many examples, the asymptotic tail distribution of the exit time  $\tau_x$  from the cone of the random walk started at  $x$  is governed by a certain positive harmonic function  $V(x)$ : as  $n \rightarrow \infty$ ,

$$(1.1) \quad \mathbf{P}[\tau_x > n] \sim \varkappa V(x)F(n),$$

where  $\varkappa > 0$ . In comparison with point 2, this way presents the drawback of giving at most one harmonic function.

4. Another possibility consists in comparing the solutions to the discrete and the classical (continuous) Dirichlet problems, and to use the numerous results which exist concerning the latter. Unfortunately, this comparison only exists for bounded domains (among which truncated cones), see [25].
5. A last and new approach consists in showing that the generating function of the values taken by the harmonic functions satisfies some functional equation, and in solving the latter. While the functional equation is easily obtained (it turns out to be a rewriting of the Dirichlet problem of point 1), its resolution will be more arduous!

We now briefly present approach 5 in the one-dimensional case. Let  $X = (X(n))_{n \in \mathbf{Z}_+}$  be a random walk on  $\mathbf{Z}_+$ , with jumps  $\{p_{-1}, p_1\}$  to the two nearest neighbors, and killed at the boundary of  $\mathbf{Z}_+$  (the boundary is here simply the origin  $\{0\}$ ). Let  $f$  be a harmonic function for this process. We have  $f(0) = 0$  and, for any  $i_0 \geq 1$ ,

$$(1.2) \quad f(i_0) = p_1 f(i_0 + 1) + p_{-1} f(i_0 - 1).$$

Let  $H(x) = \sum_{i_0 \geq 1} f(i_0) x^{i_0 - 1}$ . Multiplying (1.2) by  $x^{i_0 - 1}$  and summing w.r.t.  $i_0 \geq 1$  yields

$$(x^2 p_{-1} - x + p_1) H(x) = p_1 f(1).$$

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<sup>1</sup>We wish to thank D. Denisov and V. Wachtel for suggesting us this new approach, during a workshop at Eindhoven University in February 2012.

It is then obvious to deduce an expression of  $H(x)$ , and also, for any  $i_0 \geq 1$ , of  $f(i_0)$ , via Cauchy's formulæ.

**Results.** This work aims at presenting the approach summarized in point 5 (i.e., finding harmonic functions by solving functional equations) for the walks with small steps (i.e., with jumps only to the eight nearest neighbors) in the quarter plane  $\mathbf{Z}_+^2$ , see Figure 1. Our motivations to restrict ourselves to this class are the following: First, it is of special

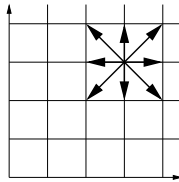


FIGURE 1. Walks with small steps in the quarter plane

interest in probability [1, 4, 10, 11, 12, 16, 18, 22] as in combinatorics [3, 21]. Second, our way to solve the functional equation we shall obtain for (the generating function of) the harmonic functions needs complex analysis, and thus dimension 2. The hypothesis on the small steps is less crucial, and will be commented in Section 6. Further, we shall mostly assume that the increments of the random walk have zero mean, as it is the most interesting case (see above). This article is hinged on the following main results:

1. We first state a functional equation satisfied by the generating function of the harmonic functions (Section 2). We solve it (Section 4), and we show that the solutions are closely related to a certain conformal mapping (introduced in Section 3).
2. We then prove that for the class of zero-mean random walks with small steps in the quarter plane, there is exactly one harmonic function (Section 5).
3. We also obtain the following two important corollaries (Section 5): in [6], it is proved that for the zero-mean random walks in the quarter plane, Equation (1.1) holds, where  $V(x)$  is shown to be harmonic, but is not made explicit. Here we find an expression for  $V(x)$ , as a Taylor coefficient of a (relatively) simple function involving sine and arcsine functions. The second consequence is about lattice path enumeration, and more precisely about the counting of walks with small steps in the quarter plane. Recently, in [6, Section 1.5], the asymptotics of the number of excursions (walks starting and ending at given points) was obtained, up to some multiplicative constant involving two harmonic functions. The analysis we lead here allows us to obtain explicit expressions for both these functions.
4. Finally, we present possible extensions of our approach and results (Section 6). Firstly, we show that our methods do work for random walks with non-zero mean increments. In particular, we obtain expressions for the harmonic functions in that case (this improves some results in [16, 18]). Regarding the non-zero drift case, we also prove that all harmonic functions converge to the unique harmonic function, as the drift goes to 0. Further, we see

that our approach also gives expressions for  $t$ -harmonic functions (i.e., functions  $f$  such that  $Pf = tf$  within the cone,  $P$  being the generator of the random walk and  $t > 0$ ).

## 2. FUNCTIONAL EQUATIONS FOR HARMONIC FUNCTIONS

**2.1. Harmonic functions for walks with small steps in the quarter plane.** Denote by  $(X, Y) = (X(n), Y(n))_{n \in \mathbf{Z}_+}$  a random walk in the quarter plane  $\mathbf{Z}_+^2$ , and let  $\mathbf{P}_{(i_0, j_0)}[\mathcal{E}]$  be the probability of event  $\mathcal{E}$  conditional on  $(X(0), Y(0)) = (i_0, j_0)$ . Throughout we shall make the following assumptions:

- (H1) The walk is homogeneous inside of the quarter plane, with transition probabilities  $\{p_{i,j}\}_{-1 \leq i,j \leq 1}$  to the eight nearest neighbors ( $p_{0,0} = 0$ );
- (H2) In the list  $p_{1,1}, p_{1,0}, p_{1,-1}, p_{0,-1}, p_{-1,-1}, p_{-1,0}, p_{-1,1}, p_{0,1}$ , there are no three consecutive zeros;
- (H3) The drifts are zero:  $\sum_{-1 \leq i,j \leq 1} ip_{i,j} = 0$  and  $\sum_{-1 \leq i,j \leq 1} jp_{i,j} = 0$ .

With assumption (H1) we can use the general framework for walks in the quarter plane developed by Fayolle, Iasnogorodski and Malyshev [11].

In this note we are interested in functions  $f$  which are harmonic for these random walks, i.e., in functions  $f = (f(i_0, j_0))_{(i_0, j_0) \in \mathbf{Z}_+^2}$  such that:

- (P1) For any  $i_0, j_0 \geq 1$ ,  $f(i_0, j_0) = \sum_{-1 \leq i,j \leq 1} p_{i,j} f(i_0 + i, j_0 + j)$ .

More specifically, we restrict ourselves to the harmonic functions which are

- (P2) zero on the boundary: if  $i_0 j_0 = 0$  then  $f(i_0, j_0) = 0$ ;
- (P3) positive inside of the quadrant: if  $i_0, j_0 > 0$  then  $f(i_0, j_0) > 0$ .

A harmonic function  $f$  satisfying (P2) and (P3) is positive harmonic for the random walk killed at the boundary of the quarter plane.

**2.2. Structure of the remaining of Section 2.** We fix a harmonic function  $f$  satisfying (P1), (P2) and (P3). We first obtain, in Section 2.3, a functional equation for the generating function

$$(2.1) \quad H(x, y) = \sum_{i_0, j_0 \geq 1} f(i_0, j_0) x^{i_0-1} y^{j_0-1}.$$

This functional equation is stated in (2.4), and involves a certain bivariate polynomial  $L(x, y)$ , see (2.3), that we study in Section 2.4. We then prove, in Section 2.5, that both generating functions

$$(2.2) \quad H(x, 0) = \sum_{i_0 \geq 1} f(i_0, 1) x^{i_0-1}, \quad H(0, y) = \sum_{j_0 \geq 1} f(1, j_0) y^{j_0-1},$$

satisfy certain (rather simple) boundary value problems. Finally, starting from the latter, we obtain explicit expressions of  $H(x, 0)$  and  $H(0, y)$ , then of  $H(x, y)$  via (2.4), and finally of  $f(i_0, j_0)$ , for any  $i_0, j_0 \geq 1$ , via the classical Cauchy's formulæ.

**2.3. A first functional equation.** The following polynomial—also called the kernel of the random walk—will be of the highest importance:

$$(2.3) \quad L(x, y) = xy[\sum_{-1 \leq i,j \leq 1} p_{i,j} x^{-i} y^{-j} - 1].$$

**Lemma 1.** *For any random walk with property (H1) (but not necessarily (H2) and (H3)), the generating function  $H(x, y)$  defined in (2.1) satisfies the functional equation*

$$(2.4) \quad L(x, y)H(x, y) = L(x, 0)H(x, 0) + L(0, y)H(0, y) - L(0, 0)H(0, 0).$$

*Proof.* The proof of (2.4) is a simple consequence of (P1) and (P2).  $\square$

We now make a series of remarks on this functional equation.

*Example.* Let us verify (2.4) for the simple random walk, i.e., for the model with transition probabilities  $p_{0,1} = p_{1,0} = p_{0,-1} = p_{-1,0} = 1/4$ . In that case, it is well known (see, e.g., [20]) that there exists a unique positive harmonic function (up to the multiplicative constants), with the product form  $f(i_0, j_0) = i_0 j_0$ . With (2.1) and (2.2), we then obtain

$$(2.5) \quad H(x, 0) = \frac{1}{(1-x)^2}, \quad H(0, y) = \frac{1}{(1-y)^2}, \quad H(x, y) = \frac{1}{(1-x)^2} \frac{1}{(1-y)^2}.$$

Furthermore,  $L(x, y) = (y/4)(x-1)^2 + (x/4)(y-1)^2$ ,  $L(x, 0) = x/4$ ,  $L(0, y) = y/4$  and  $L(0, 0) = 0$ , see (2.3), and it becomes straightforward to verify that (2.4) does hold.

*Around the functional equation (2.4).* In the literature, there actually exist many papers on problems that can be reduced to solving functional equations closed to (2.4). This typically arises in probability theory [10, 11, 22], in queueing theory [4], and in combinatorics as well [3, 21]. In our case, contrary to most of the functional equations present in the above-cited works:

- The coefficient  $L(x, 0)$  (resp.  $L(0, y)$ ) in front of unknown  $H(x, 0)$  (resp.  $H(0, y)$ ) depends only on one of the two variables  $x$  and  $y$ ;
- In the right-hand side of (2.4), there are no non-homogeneous terms depending on  $x$  and  $y$ .

For these two reasons, the analytic description of the solutions to the functional equation (2.4) we shall propose here will be more precise than in other works.

*Domain of validity of (2.4).* Equation (2.4) holds (at least) on the domain  $\{(x, y) \in \mathbf{C}^2 : |x| \leq 1, |y| \leq 1\}$ . Indeed, we have:

**Lemma 2.** *For a random walk satisfying to (H1), (H2) and (H3), the radius of convergence of  $H(x, 0)$  and  $H(0, y)$  is one. Further, for any non-zero harmonic function,  $H(1, 0) = H(0, 1) = \infty$ .*

The proof of Lemma 2 we shall give here<sup>2</sup> uses large deviation theory. It is worth noting that it is the only place in this article where a result is not derived as a consequence of the functional equation (2.4) (and it is an open problem to decide whether it is possible to do so). It is postponed to Appendix A.

It is worth remarking that for a random walk (satisfying to (H1) and (H2) but) having a non-zero drift, Lemma 2 does not hold anymore. This is precisely this phenomenon which eventually will entail that the Martin boundary of a random walk with a non-zero drift is composed of infinitely many harmonic functions [16, 18]; see Section 6.1 for more details.

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<sup>2</sup>We are grateful to I. Ignatiuk-Robert for suggesting us the ideas of this proof.

**2.4. Basic properties of the kernel.** This part is purely technical, and presents some properties of the kernel  $L(x, y)$  introduced in (2.3), that are crucial for the sequel.

*Branch points.* The kernel  $L(x, y)$  can also be written as<sup>3</sup>

$$(2.6) \quad L(x, y) = \alpha(x)y^2 + \beta(x)y + \gamma(x) = \tilde{\alpha}(y)x^2 + \tilde{\beta}(y)x + \tilde{\gamma}(y),$$

where

$$(2.7) \quad \begin{aligned} \alpha(x) &= p_{-1,-1}x^2 + p_{0,-1}x + p_{1,-1}, & \beta(x) &= p_{-1,0}x^2 - x + p_{1,0}, & \gamma(x) &= p_{-1,1}x^2 + p_{0,1}x + p_{1,1}, \\ \tilde{\alpha}(y) &= p_{-1,-1}y^2 + p_{-1,0}y + p_{-1,1}, & \tilde{\beta}(y) &= p_{0,-1}y^2 - y + p_{0,1}, & \tilde{\gamma}(y) &= p_{1,-1}y^2 + p_{1,0}y + p_{1,1}. \end{aligned}$$

We also define

$$(2.8) \quad \delta(x) = \beta(x)^2 - 4\alpha(x)\gamma(x), \quad \tilde{\delta}(y) = \tilde{\beta}(y)^2 - 4\tilde{\alpha}(y)\tilde{\gamma}(y).$$

The roots of these polynomials are called the branch points of the kernel. The following facts regarding the polynomial  $\delta$  are proved in [11, Chapter 2]: Under (H1), (H2) and (H3),  $\delta$  has degree three or four. Further, 1 is a root of order two of  $\delta$ . We denote its roots by  $\{x_\ell\}_{1 \leq \ell \leq 4}$ , with

$$(2.9) \quad |x_1| \leq 1 \leq |x_4|, \quad x_2 = x_3 = 1,$$

and  $x_4 = \infty$  if  $\delta$  has degree three. We have  $x_1 \in [-1, 1)$  and  $x_4 \in (1, \infty) \cup \{\infty\} \cup (-\infty, -1]$ . Further, on  $\mathbf{R}$ ,  $\delta(x)$  is negative if and only if  $x \in [x_1, x_4] \setminus \{1\}$ . The polynomial  $\tilde{\delta}$  in (2.8) and its roots  $\{y_\ell\}_{1 \leq \ell \leq 4}$  satisfy similar properties.

*Algebraic functions defined by the kernel.* In what follows, we denote by  $X(y)$  and  $Y(x)$  the algebraic functions defined by  $L(X(y), y) = 0$  and  $L(x, Y(x)) = 0$ . With (2.6) and (2.8) we have

$$(2.10) \quad X(y) = \frac{-\tilde{\beta}(y) \pm \tilde{\delta}(y)^{1/2}}{2\tilde{\alpha}(y)}, \quad Y(x) = \frac{-\beta(x) \pm \delta(x)^{1/2}}{2\alpha(x)}.$$

Functions  $X(y)$  and  $Y(x)$  both have two branches, called  $X_0, X_1$  and  $Y_0, Y_1$ , which are meromorphic on  $\mathbf{C} \setminus [x_1, x_4]$  and  $\mathbf{C} \setminus [y_1, y_4]$ , respectively. We fix the notations by requiring that on the whole of  $\mathbf{C}$ ,

$$|X_0(y)| \leq |X_1(y)|, \quad |Y_0(x)| \leq |Y_1(x)|.$$

*On a curve defined by the complex values of function  $X(y)$ .* To conclude Section 2.4, we introduce the curve

$$X([y_1, 1]) = X_0([y_1, 1]) \cup X_1([y_1, 1]),$$

with is symmetrical w.r.t. the real axis (since for  $y \in [y_1, 1]$ ,  $X_0(y)$  and  $X_1(y)$  are complex conjugates), and goes around point  $x_1$  (see [11, Theorem 5.3.3]).

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<sup>3</sup>For readers used to [11], we notice that the notations we take here are not the usual ones, as the kernel  $L(x, y)$  in (2.3) is the “reversed” one of the kernel  $Q(x, y)$  in [11]:  $L(x, y) = x^2y^2Q(1/x, 1/y)$ .

The curve  $X([y_1, 1])$  is smooth except at  $X(1) = 1$ , where it may have a corner point. More specifically, at 1 the angle between the segment  $[x_1, 1]$  and the curve is given by [12]

$$(2.11) \quad \pm \theta = \pm \arccos \left( - \frac{\sum_{-1 \leq i, j \leq 1} ij p_{i,j}}{\sqrt{(\sum_{-1 \leq i, j \leq 1} i^2 p_{i,j}) \cdot (\sum_{-1 \leq i, j \leq 1} j^2 p_{i,j})}} \right).$$

Denote by  $\mathcal{G}_X$  the open set bounded by the curve  $X([y_1, 1])$ , which in addition contains the point  $x_1$  (or equivalently 0). For instance, in the case of the simple random walk (with  $p_{0,1} = p_{1,0} = p_{0,-1} = p_{-1,0} = 1/4$ ),  $X([y_1, 1])$  is the unit circle (see [11, Theorem 5.3.3]), hence  $\mathcal{G}_X$  is the unit disc. In that case,  $X([y_1, 1])$  is smooth at 1, which is coherent with  $\theta = \arccos(0) = \pi/2$  in (2.11).

**2.5. A second functional equation.** Let  $\mathcal{G}_X$  be the set introduced above, and let  $\overline{\mathcal{G}_X}$  be its closure (in  $\mathbf{C} \cup \{\infty\}$ ). This part aims at showing that  $H(x, 0)$  satisfies the following boundary value problem (below and throughout,  $\bar{x}$  stands for the conjugate value of  $x \in \mathbf{C}$ ):

- (i)  $H(x, 0)$  is analytic in  $\mathcal{G}_X$ ;
- (ii)  $H(x, 0)$  is continuous on  $\overline{\mathcal{G}_X} \setminus \{1\}$ ;
- (iii) For all  $x$  on the boundary of  $\mathcal{G}_X$  except at 1,  $L(x, 0)H(x, 0) - L(\bar{x}, 0)H(\bar{x}, 0) = 0$ .

The problem of finding functions satisfying (i), (ii) and (iii) is a particular instance of a boundary value problem with shift (the complex conjugation plays in (iii) the role of the shift), see [19] for an extensive treatment of this topic.

*Proof of (i).* Note that from Lemma 2, we already know that  $H(x, 0)$  is analytic within the open unit disc. Accordingly, if the domain  $\mathcal{G}_X$  is included in the unit disc, there is nothing to prove. Note that like the simple random walk (see Section 2.4), many walks do satisfy this property.

On the other hand, there also exist walks such that part of  $\mathcal{G}_X$  lies outside the unit disc. This is for example the case for the (Gessel's) walk with  $p_{1,1} = p_{1,0} = p_{-1,-1} = p_{-1,0} = 1/4$ , see [21]. In that case, (i) follows from an analytic continuation argument based on (2.4). This argument is exactly the same as that used in [21, Section 4], and we refer to this work for the details.  $\square$

*Proof of (ii).* See [21, Section 4], where a stronger statement (on the analytic continuation of the whole of  $\mathbf{C} \setminus [1, x_4]$ ) is proved for a quite similar functional equation.  $\square$

*Proof of (iii).* Proving (iii) is classical [11, 18, 21]: it suffices to evaluate (2.4) both at  $X_0(y)$  and  $X_1(y)$ , for all  $y \in [y_1, 1]$ . This way, the kernel  $L(x, y)$  vanishes, and the left-hand side of (2.4) too. Finally, taking the difference of the equations corresponding to  $X_0(y)$  and  $X_1(y)$  yields (iii).  $\square$

**2.6. A key lemma.** We conclude Section 2 by stating a lemma (all-present in Section 4) on a certain boundary value problem closed to that of Section 2.5.

**Lemma 3.** *Let  $h(x)$  be a function satisfying (i),  $\tilde{\text{(ii)}}$  and (iii), where*

- $\tilde{\text{(ii)}}$   $h(x)$  is continuous on  $\overline{\mathcal{G}_X}$ .

*Then  $h(x)$  necessarily is a constant function.*



*Proof.* The proof of this result is contained in both [11] and [19].  $\square$

### 3. A PERFECT CONFORMAL MAPPING

In this section we introduce a certain function  $w(x)$  that will be crucial for our analysis. Indeed, we shall see that any function  $H(x, 0)$  solution to (i), (ii) and (iii) can be expressed as  $S(w(x))/L(x, 0)$ , where  $S$  is a rational function and  $L(x, 0)$  has been defined in (2.3)<sup>4</sup>. Unexpectedly, this function  $w(x)$  happens to be a conformal mapping of a certain domain naturally attached to the random walk, namely,  $\mathcal{G}_X$  (introduced in Section 2.4). Luckily, this function  $w(x)$  has been carefully analyzed in [12]; in particular, an expression for  $w(x)$  was obtained. Let

$$(3.1) \quad f(x) = \begin{cases} \frac{d''(x_4)}{6} + \frac{d'(x_4)}{x - x_4} & \text{if } x_4 \neq \infty, \\ \frac{d''(0)}{6} + \frac{d'''(0)x}{6} & \text{if } x_4 = \infty, \end{cases} \quad T(x) = \left[ \frac{1}{3} - \frac{2f(x)}{d''(1)} \right]^{-1/2},$$

and

$$(3.2) \quad w(x) = \sin \left( \frac{\pi}{\theta} \left[ \arcsin(T(x)) - \frac{\pi}{2} \right] \right)^2,$$

with  $\theta$  as in (2.11). It is proved in [12] that  $w(x)$  enjoys the following nice properties:

- (W1)  $w$  is analytic in  $\mathcal{G}_X$ ;
- (W2)  $w$  is continuous on  $\overline{\mathcal{G}_X} \setminus \{1\}$ ;
- (W3)  $w$  establishes a conformal mapping of  $\mathcal{G}_X$  onto the complex plane  $\mathbf{C}$  cut along a segment;
- (W4) There exists  $c \neq 0$  such that for  $x$  in the neighborhood of 1,

$$(3.3) \quad w(x) = \frac{c + o(1)}{(1-x)^{\pi/\theta}};$$

- (W5) For all  $x$  on the boundary of  $\mathcal{G}_X$ ,  $w(x)$  is real and  $w(x) = w(\bar{x})$ .

A priori, the function  $w(x)$  in (3.2) is well defined only for  $|T| = |T(x)| \leq 1$ . For  $|T| \geq 1$ , we have  $\arcsin(T) = \pi/2 \pm i \ln(T + \sqrt{T^2 - 1})$ , and as in [12, Section 2.2] we can write

$$(3.4) \quad w(x) = -\frac{1}{4} \left[ \left( T + \sqrt{T^2 - 1} \right)^{2\pi/\theta} - 2 + \left( T - \sqrt{T^2 - 1} \right)^{2\pi/\theta} \right].$$

**Remark.** It is also shown in [12] that  $w(x)$  is the unique function for which (W1)–(W5) hold (up to additive and multiplicative constants).

*Example.* For the simple random walk, it is easy to compute  $x_4 = 3 + 2\sqrt{2}$ ,  $d'(x_4) = 4 + 3\sqrt{2}$ ,  $d''(x_4) = 11/2 + 3\sqrt{2}$ ,  $d''(1) = -1/2$ ,  $T(x) = [2(2 + \sqrt{2})(x - 1)/(x - 3 - 2\sqrt{2})]^{-1/2}$ . Further,  $\pi/\theta = 2$ , so that with (3.4),

$$(3.5) \quad w(x) = -4T^2(T^2 - 1) = \frac{(x - 3 + 2\sqrt{2})(x - 3 - 2\sqrt{2})}{2(x - 1)^2} = \frac{1}{2} - \frac{2x}{(x - 1)^2}.$$

---

<sup>4</sup>We would like to insist on the fact that the function  $S(w(x))/L(x, 0)$  will not be the generating function associated with a harmonic function for any rational function  $S$ ; however, it will solve the problem (i), (ii) and (iii) for any  $S$ .

## 4. SOLVING THE FUNCTIONAL EQUATIONS

In this part we characterize the harmonic functions satisfying (P1), (P2) and (P3). Precisely, we shall prove the following result: let  $w(x)$  be the function defined in Section 3. Subtracting by  $w(0)$ , we assume that  $w(0) = 0$ . Define

$$(4.1) \quad \nu = -w(X_0(0)), \quad \mu = \begin{cases} \frac{p_{0,1}f(1,1)}{w'(0)} & \text{if } p_{1,1} = 0, \\ -\frac{p_{1,1}f(1,1)}{w(X_0(0))} & \text{if } p_{1,1} \neq 0. \end{cases}$$

**Theorem 4.** *Define  $\mu$  and  $\nu$  as in (4.1). We have*

$$(4.2) \quad H(x, 0) = \mu \frac{w(x) + \nu}{L(x, 0)}.$$

Of course, a similar expression could be written for  $H(0, y)$ , after a suitable change of the parameters (namely,  $p_{i,j} \rightarrow p_{j,i}$ ). As for  $H(0, 0)$ , it is simply equal to  $f(1, 1)$ , see (2.1), and an expression for the complete generating function  $H(x, y)$  then follows from (2.4). Finally, the Taylor coefficients  $f(i_0, j_0)$  of  $H(x, y)$  can be obtained via Cauchy's formulæ.

The starting point for proving Theorem 4 is the boundary value problem that satisfies the function  $H(x, 0)$  (see Section 2.5). In Section 4.1, we show that  $H(x, 0)$  is necessarily of the form  $S(w(x))/L(x, 0)$ , where  $S$  is a rational function. In Section 4.2, we prove that  $S(X)$  is actually a polynomial of the first order, and in Section 4.3 we find the expression of this polynomial (i.e., the constants  $\mu$  and  $\nu$  in (4.2)).

**4.1. Expression of the solutions to the boundary value problem.** For a given non-real  $z_0$  and any non-zero harmonic function  $f(i_0, j_0)$ , consider the function

$$\mathcal{H}(x) = \frac{1}{L(x, 0)H(x, 0) - z_0}.$$

We first notice that for  $x$  on  $X([y_1, 1])$  (the boundary of  $\mathcal{G}_X$ ),  $\mathcal{H}(x) \neq \infty$ . Indeed, remember (Section 2.5) that  $L(x, 0)H(x, 0)$  takes real values on  $X([y_1, 1])$  and that  $z_0$  is non-real. Further, let us denote by  $N_p$  the number of poles that  $\mathcal{H}(x)$  has within  $\mathcal{G}_X$ . First, by the principle of isolated singularities [24, Section 3.2], we obtain that  $N_p < \infty$  (for any non-real  $z_0$ ). Second:

**Lemma 5.** *For any non-real  $z_0$ , we have  $N_p \geq 1$ .*

*Proof.* Assume that  $N_p = 0$ . Then  $\mathcal{H}(x)$  satisfies (i), (ii) and (iii), so that by Lemma 3, we conclude that  $\mathcal{H}(x)$  is a constant function  $C$ . Accordingly,  $H(x, 0) = (z_0 + 1/C)/L(x, 0)$ . Having  $z_0 + 1/C \neq 0$  is impossible, as  $1/L(x, 0)$  is either not a power series (if  $L(0, 0) = 0$ ), or is a power series whose certain coefficients are negative (if  $L(0, 0) \neq 0$ ). It follows that  $z_0 + 1/C = 0$ , and thus that  $H(x, 0)$  is identically zero. Accordingly,  $f(i_0, j_0) = 0$  for any  $i_0, j_0 \geq 0$ , which is a contradiction with our starting assumption.  $\square$

Though this is not crucial for our analysis, it is interesting to note that we can prove, purely analytically, that  $N_p$  is intrinsic, in the sense that it does not depend on  $z_0$ :

**Lemma 6.**  *$N_p$  is independent of the choice of  $z_0 \in \mathbf{C} \setminus \mathbf{R}$ .*

*Proof.* The number of poles  $N_p$  can be calculated as follows (see [24, Section 3.8.2]): if the contour  $X([y_1, 1])$  is traversed in the counter-clockwise sense, and if  $N_z$  (resp.  $N_p$ ) is the number of zeros (resp. poles) of  $\mathcal{H}(x)$  inside  $X([y_1, 1])$  (they are counted as many times as their order indicates), we have<sup>5</sup>

$$(4.3) \quad \frac{1}{2\pi i} \int_{X([y_1, 1])} \frac{\mathcal{H}'(x)}{\mathcal{H}(x)} dx = N_z - N_p = -N_p,$$

since  $N_z = 0$  (indeed,  $H(x, 0) \neq \infty$  for  $x \in \mathcal{G}_X$ ). By a classical reasoning ( $N_p$  takes only integer values and is continuous w.r.t.  $z_0$  in the positive (or negative) half plane), we obtain that  $N_p$  is constant for  $z_0$  in a given half plane. Since the coefficients of  $L(x, 0)H(x, 0)$  are real, we reach the conclusion that  $N_p$  takes the same constant value for  $z_0$  in the two half planes, and the proof of Lemma 6 is completed.  $\square$

We now show the existence of a function  $C(x)$  such that  $\mathcal{H}(x) - C(x)$  satisfies (i), (ii) and (iii). With Lemma 3, it will then be obvious to deduce that  $\mathcal{H}(x) - C(x)$  is a constant function. Let us call  $x_1, \dots, x_q$  the poles of  $\mathcal{H}(x)$  within  $\mathcal{G}_X$ , with respective multiplicities  $n_1, \dots, n_q$ , so that  $n_1 + \dots + n_q = N_p$ . Since the function

$$\frac{1}{w(x) - w(x_\ell)}$$

has a pole of order 1 at  $x_\ell$  (see property (W3)), is analytic in  $\mathcal{G}_X \setminus \{x_\ell\}$  and continuous in  $\overline{\mathcal{G}_X} \setminus \{x_\ell\}$ , we can find a polynomial  $P_\ell$  of order  $n_\ell$  such that  $P_\ell(0) = 0$  and

$$\mathcal{H}(x) - P_\ell \left( \frac{1}{w(x) - w(x_\ell)} \right)$$

is analytic at  $x_\ell$ . The following choice for  $C(x)$  is thus suitable:

$$C(x) = \sum_{\ell=1}^q P_\ell \left( \frac{1}{w(x) - w(x_\ell)} \right).$$

We conclude that the function  $\mathcal{H}(x) - C(x)$  is constant. Since  $w(1) = \infty$  (see (W4)), we have  $C(1) = 0$  (remember that  $P_\ell(0) = 0$ ). Furthermore, we have  $H(1, 0) = \infty$  (Lemma 2), so that  $\mathcal{H}(1) = 0$ , and the constant function  $\mathcal{H}(x) - C(x)$  is zero. Finally, we have

$$H(x, 0) = \frac{1}{L(x, 0)} \left( z_0 + \frac{1}{C(x)} \right).$$

**4.2. Form of the rational function.** To summarize Section 4.1, given some  $z_0 \in \mathbf{C} \setminus \mathbf{R}$ , we have constructed a rational function

$$(4.4) \quad R(X) = \sum_{\ell=1}^q P_\ell \left( \frac{1}{X - w(x_\ell)} \right)$$

such that  $H(x, 0) = (z_0 + 1/R(w(x)))/L(x, 0)$ . We now prove that  $q = 1$  in (4.4), and that the polynomial  $P_1$  has order 1.

**Lemma 7.** *The rational function  $1/R(X)$  must be a polynomial.*

<sup>5</sup>A priori, this Cauchy's theorem cannot be applied as such, as the contour  $X([y_1, 1])$  is not included in a domain where  $\mathcal{H}(x)$  is analytic. However, since  $\mathcal{H}(x)$  can be analytically continued through any point of  $X([y_1, 1]) \setminus \{1\}$  (see [21, Section 4] for a similar statement), and since  $\mathcal{H}(x) \rightarrow \infty$  as  $x \rightarrow 1$ , this result still holds.

Before proving Lemma 7, note its following consequence:

**Corollary 8.** *We have  $q = 1$  in (4.4) and  $P_1(X) = \mu X^{n_1}$ , for some  $\mu \in \mathbf{C}$  and  $n_1 \in \mathbf{Z}_+$ .*

*Proof.* It is elementary to prove that for a rational function  $R(X)$  as in (4.4), a necessary and sufficient condition for  $1/R(X)$  to be a polynomial is that  $q = 1$  and  $P_1(X) = \mu X^{n_1}$  in (4.4).  $\square$

*Proof of Lemma 7.* Assume that  $1/R(X)$  is not a polynomial. Then there exists  $z_1 \in \mathbf{C}$  such that  $1/R(z_1) = \infty$ . Introduce  $t_1 \in \overline{\mathcal{G}_X}$  such that<sup>6</sup>  $w(t_1) = z_1$ . Note that  $t_1 \neq 1$ , as  $w(1) = \infty$ . We conclude that  $H(t_1, 0) = \infty$ , which is an apparent contradiction with (i) or (ii) of Section 2.5.  $\square$

So far, we have shown that

$$(4.5) \quad H(x, 0) = \frac{z_0 + \mu(w(x) - w(x_1))^{n_1}}{L(x, 0)}.$$

**Lemma 9.** *We have  $n_1 = 1$  and  $\mu \in \mathbf{R} \setminus \{0\}$  in (4.5).*

*Proof.* First, we have  $\mu \neq 0$  (we already saw that  $1/L(x, 0)$  cannot be the generating function of positive numbers). As  $x \in (0, 1)$  goes to 1, both  $w(x)$  and  $H(x, 0)$  are real and go to  $\infty$ . For the polynomial  $z_0 + \mu(X - w(x_1))^{n_1}$  to be real, we must have  $\mu \in \mathbf{R}$ . If  $n_1 \geq 2$ , the same argument (together with the binomial formula) gives that  $w(x_1)$  and  $z_0$  have to be real too. But  $z_0 \in \mathbf{C} \setminus \mathbf{R}$ , so that  $n_1 = 1$ .  $\square$

Finally, we have proved (4.2), and it remains to compute the constants  $\mu$  and  $\nu$ .

**4.3. Computation of the constants.** This part aims at computing the values of  $\mu$  and  $\nu$  in (4.2) given in (4.1).

*Case  $L(0, 0) = 0$ .* This is a simple case: taking the limit of (4.2) as  $x \rightarrow 0$  gives that  $\nu = 0$ . The value of  $\mu$  then follows by examining the first term in the expansion of (4.2) at  $x = 0$ .

*Case  $L(0, 0) \neq 0$ .* A similar reasoning as in Section 4.1 and Section 4.2 gives that

$$(4.6) \quad H(0, y) = \tilde{\mu} \frac{\tilde{w}(y) + \tilde{\nu}}{L(0, y)},$$

where  $\tilde{w}(y)$  is the analogue of  $w(x)$ , replacing the domain  $\mathcal{G}_X$  by  $\mathcal{G}_Y$ , the domain bounded by the curve  $Y([x_1, 1])$  and containing  $y_1$ . Similar properties as (W1)–(W5) hold for  $\tilde{w}(y)$ . Further, we fix  $\tilde{w}(y)$  so as  $\tilde{w}(0) = 0$  and (3.3) holds (with exactly the same value of  $c$ ).

We show that for (4.2) and (4.6) to hold, we must have  $\mu = \tilde{\mu}$ . To this aim, evaluate the functional equation (2.4) at  $(X_0(y), y)$ ; we obtain

$$0 = L(X_0(y), 0)H(X_0(y), 0) + L(0, y)H(0, y) - L(0, 0)H(0, 0).$$

Using (4.2) and (4.6), we may write

$$(4.7) \quad 0 = \mu(w(X_0(y)) + \nu) + \tilde{\mu}(\tilde{w}(y) + \tilde{\nu}) - L(0, 0)H(0, 0).$$

<sup>6</sup>If  $t_1$  belongs to the (open) set  $\mathcal{G}_X$ , then  $t_1$  is unique, while if  $t_1$  is on the boundary, two values (namely,  $t_1$  and  $\bar{t}_1$ ) are possible.

In the neighborhood of 1, we have  $X_0(y) = 1 + \exp(i\theta)(y - 1) + o(y - 1)$ , see [12, Equation (2.8)], so that

$$w(X_0(y)) = \frac{c + o(1)}{(1 - X_0(y))^{\pi/\theta}} = -\frac{c + o(1)}{(1 - y)^{\pi/\theta}}.$$

Multiplying Equation (4.7) by  $(1 - y)^{\pi/\theta}$  and letting  $y \rightarrow 1$ , we obtain that  $\mu = \tilde{\mu}$ .

Now we come back to (4.2) and (4.6). Evaluating them at  $x = 0$  and  $y = 0$ , respectively, we obtain that  $\nu = \tilde{\nu} = p_{1,1}f(1, 1)/\mu$ . To find  $\mu$  in the case  $L(0, 0) \neq 0$ , we can use (4.7):

$$\mu = -\frac{p_{1,1}f(1, 1)}{w(X_0(y)) + \tilde{w}(y)}.$$

The function in the denominator above is constant, and can be evaluated at any value of  $y$ , for example 0.

**4.4. Example.** We consider the simple random walk, for which  $\mathcal{G}_X$  is the unit disc, see Section 2.4. With Theorem 4, in order to compute  $H(x, 0)$ , it suffices to calculate  $w(x)$ ,  $L(x, 0)$ ,  $\mu$  and  $\nu$ . First, the function

$$w(x) = \frac{x}{(1 - x)^2}$$

manifestly satisfies (W1)–(W5) (remember that  $\theta = \pi/2$ ); see also (3.5). Furthermore, we have  $L(x, 0) = x/4$ , so that  $L(0, 0) = 0$ , and for this reason  $\nu = 0$ . Finally, with (4.1),  $\mu = f(1, 1)/4$ , and thus

$$H(x, 0) = \frac{f(1, 1)}{(1 - x)^2}.$$

This does match with (2.5).

## 5. APPLICATIONS

In this section we present four applications of our results, which successively concern Martin boundary, (non-)algebraicity of the harmonic function, exit time from cones, and lattice path enumeration.

### 5.1. Martin boundary.

*Brief account on Martin boundary theory* [7]. For a (transient) Markov chain with state space  $E$ , the Martin compactification of  $E$  is the smallest compactification  $\overline{E}$  of  $E$  for which the Martin kernels  $y \mapsto k_y^x = G_y^x/G_y^{x_0}$  extend continuously ( $G_y^x$  is the Green function, and we denote by  $x_0$  any reference state). The set  $\overline{E} \setminus E$  is called the full Martin boundary. For  $\alpha \in \overline{E}$ ,  $x \mapsto k_\alpha^x$  is superharmonic, and  $\partial_m E = \{\alpha \in \overline{E} \setminus E : x \mapsto k_\alpha^x \text{ is minimal}^7 \text{ harmonic}\}$  is the minimal Martin boundary. Then, any superharmonic (resp. harmonic) function  $h$  can be written as  $h(x) = \int_{\overline{E}} k_y^x \mu(dy)$  (resp.  $h(x) = \int_{\partial_m E} k_y^x \mu(dy)$ ), where  $\mu$  is some finite measure, uniquely characterized in the second case above.

<sup>7</sup>A harmonic function  $h$  is minimal if  $0 \leq g \leq h$  with  $g$  harmonic implies  $g = ch$  for some constant  $c$ .

*Our contribution.* In Section 4, provided that the drift of the walk is zero, we have proved the uniqueness of the harmonic function. In other words:

**Theorem 10.** *For all random walks satisfying (H1), (H2) and (H3), the Martin boundary is reduced to one point.*

In particular, all results [1, 2, 20, 22] regarding the uniqueness of the harmonic functions in some particular cases of walks in the quarter plane also follow from Theorem 10.

## 5.2. Nature of the generating function $H(x, y)$ and group of the walk.

*Nature of the generating function  $H(x, y)$ .* Let  $\theta$  be defined in (2.11). We have the following link between the nature (rational, algebraic, non-algebraic) of  $H(x, y)$  and  $\theta$ :

**Proposition 11.** *The generating function  $H(x, y)$  is rational (resp. algebraic non-rational, non-algebraic) if and only if  $\pi/\theta$  belongs to  $\mathbf{Z}$  (resp.  $\mathbf{Q}$ ,  $\mathbf{R} \setminus \mathbf{Q}$ ).*

*Proof.* From the functional equation (2.4) and by symmetry, it suffices to prove the above statement for  $H(x, 0)$ . In fact, thanks to Theorem 4, it is enough to prove it for  $w(x)$ . If  $\pi/\theta$  belongs to  $\mathbf{Z}$  or  $\mathbf{Q}$ , Proposition 11 easily follows from (3.4). If  $\pi/\theta \in \mathbf{R} \setminus \mathbf{Q}$ , then it is a corollary of (3.3) that  $w(x)$  cannot be an algebraic function.  $\square$

A direct consequence of Proposition 11 is the following:

**Corollary 12.** *The harmonic function  $f(i_0, j_0)$  is polynomial in  $i_0$  and  $j_0$  if and only if  $\pi/\theta \in \mathbf{Z}$ .*

However, it is an open problem to decide whether  $f(i_0, j_0)$  is homogeneous<sup>8</sup> or not. For more details about this homogeneity problem, we refer to [22, Section 1.3].

*(Ir)rationality of  $\pi/\theta$  and the group of the walk.* As suggested by Proposition 11,  $\pi/\theta$  is indeed a key-quantity. However, it is not simple to check whether  $\pi/\theta$  is rational or not, and this is why we propose the following alternative way to find the (ir)rationality of  $\pi/\theta$ : define  $Q(x, y) = x^2 y^2 L(1/x, 1/y)$ , with  $L$  as in (2.3), and introduce

$$\xi(x, y) = \left( x, \frac{1 \sum_{-1 \leq i \leq 1} p_{i, -1} x^i}{y \sum_{-1 \leq i \leq 1} p_{i, +1} x^i} \right), \quad \eta(x, y) = \left( \frac{1 \sum_{-1 \leq j \leq 1} p_{-1, j} y^j}{x \sum_{-1 \leq j \leq 1} p_{+1, j} y^j}, y \right).$$

It is quite easy to see that if  $Q(x, y) = 0$ , then  $Q(\xi(x, y)) = Q(\eta(x, y)) = 0$ . In particular,  $\xi$  and  $\eta$  can be interpreted as automorphisms of the surface  $\{(x, y) \in \mathbf{C} \cup \{\infty\} : Q(x, y) = 0\}$ . The group of the walk [11, Section 2.4] is then precisely the automorphism group  $W = \langle \xi, \eta \rangle$  generated by  $\xi$  and  $\eta$ .

Further, it is proved in [12] that under (H1), (H2) and (H3), the group is finite if and only if  $\pi/\theta \in \mathbf{Q}$ . In other words, we can reformulate Proposition 11 by replacing “ $\pi/\theta \in \mathbf{Q}$ ” (resp. “ $\pi/\theta \in \mathbf{R} \setminus \mathbf{Q}$ ”) by “the group  $W$  is finite” (resp. “the group  $W$  is infinite”), knowing that in many examples [3, 11, 18, 22], the order of  $W$  is already computed.

<sup>8</sup>A polynomial  $P(i_0, j_0)$  is homogeneous if there exists  $\alpha \in \mathbf{Z}_+$  such that  $P(\lambda i_0, \lambda j_0) = \lambda^\alpha P(i_0, j_0)$  for any  $i_0, j_0$  and  $\lambda$ .

*Example.* For the simple random walk, we saw that  $\pi/\theta = 2$ ; therefore, the group  $W$  must be finite. Let us verify this fact directly: we have  $\xi(x, y) = (x, 1/y)$  and  $\eta(x, y) = (1/x, y)$ , so that  $W$  has order 4 (besides  $\xi$  and  $\eta$ ,  $W$  includes the identity element, as well as the compound  $\xi \circ \eta = \eta \circ \xi = (1/x, 1/y)$ ). Note that the order of the group  $W$  is twice  $\pi/\theta$ , which is not surprising: in general, the order of  $W$  is  $2 \inf\{p \in \mathbf{Z}_+ \setminus \{0\} : p\theta/\pi \in \mathbf{Z}\}$ .

**5.3. Exit time from the cone.** As written in the introduction, it is proved in [6] that for the zero-mean random walks in the quarter plane, Equation (1.1) holds, where  $V(i_0, j_0)$  is some non-explicit harmonic function. Let  $f(i_0, j_0)$  be the unique harmonic function found in Section 4.

**Corollary 13.** *There exists  $\chi > 0$  such that for any  $(i_0, j_0) \in \mathbf{Z}_+^2$ ,  $V(i_0, j_0) = \chi f(i_0, j_0)$ .*

**Remark.** The conformal mapping  $w(x)$  is related to the first exit time from the cone, as  $V(i_0, j_0)$  governs the asymptotic tail distribution of the latter (Corollary 13). Actually, the link seems to be stronger, as the following holds: let  $F(n)$  and  $\theta$  be introduced in (1.1) and (2.11), respectively. It is proved in [6, Theorem 1 and Example 2] that  $F(n) = n^{-\pi/(2\theta)}$ , and in (3.3) that  $w(x) = (c + o(1))/(1 - x)^{\pi/\theta}$ , with the same angle  $\theta$ .

**5.4. Lattice path enumeration.** A point in the lattice  $\mathbf{Z}_+^2$  has eight neighbors, diagonal neighbors included. Choosing a subset of the set of neighbors and considering it as the set of admissible steps of a walk in the lattice, we obtain one of  $2^8$  possible models of walks in  $\mathbf{Z}_+^2$ . Bousquet-Mélou and Mishna [3] showed that, after eliminating trivial cases, and also those which can be reduced to the walks in a half plane, there remain 79 inherently different models. Recently, in [6, Section 1.5], the asymptotics of the number  $N_n(x, y)$  of excursions (the number of walks starting at  $x$  and ending at  $y$  after  $n$  steps) was obtained: as  $n \rightarrow \infty$ ,

$$N_n(x, y) = CV(x)V'(y)\rho^n n^\alpha(1 + o(1)),$$

where the constants  $C$ ,  $\rho$  and  $\alpha$  are found explicitly, while  $V(x)$  and  $V'(y)$  are not. In fact, with the notations of Section 2,  $V(x)$  is the harmonic function for the random walk  $(X, Y)$ , and  $V'(y)$  is the harmonic function for the reversed process  $-(X, Y)$ . If the drift of the walk (see (H3)) is zero, then the drift of the reversed walk is zero too, and the results of Section 4 provide expressions for  $V(x)$  and  $V'(y)$ . If the drift is non-zero, the harmonic functions have been found in [18]. They could also be obtained from Section 6 below, where we present some extensions (for instance, to the non-zero drift case) of our approach and results.

## 6. EXTENSIONS

Throughout this paper, we mostly assumed that the random walks had zero drift (H3). However, our approach can also be applied to obtain the Martin boundary in the non-zero drift case (Section 6.1), and to compute  $t$ -harmonic functions (Section 6.2); we also say a word on a possible generalization to random walks with more general jumps (Section 6.3).



**6.1. Random walks with non-zero drift.** In this section, we consider random walks satisfying (H1) and (H2) (but not (H3)). We first recall the existing results in the literature on Martin boundary [16, 18]. Then we show how the approach in this paper allows us to retrieve these results. Finally, we present an interesting phenomenon on the convergence of the harmonic functions of the non-zero drift case to the unique harmonic function in the zero drift case. In this section, we focus on ideas, and therefore just sketch the proofs. We illustrate all three paragraphs by the example of the simple random walk.

*Existing results.* For random walks with non-zero drift killed at the boundary of the quarter plane, the minimal Martin boundary (see Section 5 for a brief account on Martin boundary theory) is homeomorphic to  $[0, \pi/2]$ , see [16, 18]. In concrete terms, this means that the set of all minimal harmonic functions is homeomorphic to  $[0, \pi/2]$ , and therefore can be parametrized by  $\gamma \in [0, \pi/2]$ . Further, an expression for the harmonic functions  $f_\gamma$  is given in [18], in terms of solutions to certain boundary value problems. While this expression is typically complex, it may admit nice simplifications (in fact, when a certain group related to the walk is finite [18]).

For instance, for the simple random walk, we have (see [18, Section 5.1])

$$(6.1) \quad f_\gamma(i_0, j_0) = \begin{cases} (s_+(0)^{i_0} - s_-(0)^{i_0})j_0\tilde{s}_+(0)^{j_0} & \text{if } \gamma = 0, \\ (s_+(\gamma)^{i_0} - s_-(\gamma)^{i_0})(\tilde{s}_+(\gamma)^{j_0} - \tilde{s}_-(\gamma)^{j_0}) & \text{if } \gamma \in (0, \pi/2), \\ i_0s_+(\pi/2)^{i_0}(\tilde{s}_+(\pi/2)^{j_0} - \tilde{s}_-(\pi/2)^{j_0}) & \text{if } \gamma = \pi/2, \end{cases}$$

where

$$r(\gamma) = \frac{1 - \sqrt{1 - (1 - \tan^2 \gamma)(1 - 4p_{0,-1}p_{0,1} + 4p_{-1,0}p_{1,0} \tan^2 \gamma)}}{2p_{1,0}(1 - \tan^2 \gamma)},$$

$$\tilde{r}(\gamma) = \frac{1 - \sqrt{1 - (1 - \tan^{-2} \gamma)(1 - 4p_{-1,0}p_{1,0} + 4p_{0,-1}p_{0,1} \tan^{-2} \gamma)}}{2p_{1,0}(1 - \tan^{-2} \gamma)},$$

and

$$s_\pm(\gamma) = r(\gamma) \pm \sqrt{r(\gamma)^2 - p_{-1,0}/p_{1,0}}, \quad \tilde{s}_\pm(\gamma) = \tilde{r}(\gamma) \pm \sqrt{\tilde{r}(\gamma)^2 - p_{0,-1}/p_{0,1}}.$$

*Retrieving the existing results.* In this paragraph, we still consider non-zero drift random walks, and we explain how our methods provide the Martin boundary, and even expressions for harmonic functions. In the zero drift case, the point 1 plays a crucial role, in particular for the boundary value problem of Section 2.5, see (i). This is a consequence of the radius of convergence (equal to 1) of  $H(x, 0)$ —or equivalently of the non-exponential growth of  $f(i_0, 1)$  as  $i_0 \rightarrow \infty$ . For non-zero drift walks, the growth of  $f(i_0, 1)$  is now exponential. In fact, we could prove that this exponential factor necessarily takes the form  $1/p$ , with  $p \in [x_2, X(y_2)]$ <sup>9</sup>, see (6.1) for the case of the simple random walk. We thus have infinitely many boundary value problems, parametrized by  $p$ , as follows:

- (i)  $H(x, 0)$  is analytic in  $\mathcal{G}_X$ , except at  $p \in [x_2, X(y_2)]$ , where it has a pole<sup>10</sup>;
- (ii)  $H(x, 0)$  is continuous on  $\overline{\mathcal{G}_X} \setminus \{p\}$ ;

<sup>9</sup>In the zero drift case, we have  $x_2 = y_2 = X(y_2) = 1$ . In the non-zero drift case, the points  $x_2$  and  $y_2$  are defined as in Section 2.4, by replacing the inequality (2.9) by the following:  $|x_1| \leq x_2 \leq x_3 \leq |x_4|$  and  $|y_1| \leq y_2 \leq y_3 \leq |y_4|$ .

<sup>10</sup>In fact, the pole is simple if  $p \in [x_2, X(y_2))$ , and has order two if  $p = X(y_2)$ .



(iii) For all  $x$  on the boundary of  $\mathcal{G}_X$ ,  $L(x, 0)H(x, 0) - L(\bar{x}, 0)H(\bar{x}, 0) = 0$ .

The resolution of (i), (ii) and (iii) (that we shall not tackle here) would use an analogue of the conformal mapping  $w$ , namely, a function such that:

(W1)  $w$  is analytic in  $\mathcal{G}_X$ , except at  $p \in [x_2, X(y_2)]$ , where it has a pole<sup>10</sup>;

(W2)  $w$  is continuous on  $\overline{\mathcal{G}_X} \setminus \{p\}$ ;

(W3)  $w$  establishes a conformal mapping of  $\mathcal{G}_X$  onto the complex plane cut  $\mathbf{C}$  along a segment;

(W4) For all  $x$  on the boundary of  $\mathcal{G}_X$ ,  $w(x) = w(\bar{x})$ .

At this point, we wish to emphasize the following main difference between the zero drift case and the non-zero drift case: the expression of  $w(x)$  becomes more elaborate (more precisely, in place of sine and arcsine functions, it now involves two  $\wp$ -Weierstrass functions with different periods<sup>11</sup>). Despite this intrinsic complexity, expressions for  $w(x)$  are available in the literature: see [18, Section 3.1] for  $p = x_2$ , and [21, Section 2] for simplifications of  $w(x)$  in some particular cases. Once  $w(x)$  is known, Equation (4.2) holds, and an explicit expression for  $H(x, 0)$  follows.

In some sense, this approach is more efficient than the one used in [18], since we give here an expression for the harmonic functions directly in terms of the function  $w(x)$ , and not in terms of solutions to some other boundary value problems.

For the simple random walk,  $\mathcal{G}_X$  is the circle with center at 0 and radius  $\sqrt{p_{1,0}/p_{-1,0}}$ . The function

$$w(x) = \frac{x}{(x-p)(x-(p_{1,0}/p_{-1,0})/p)}$$

therefore satisfies (W1)–(W4), and with (4.2) we conclude that

$$H(x, 0) = \frac{w(x)}{L(x, 0)} = \frac{1/p_{0,1}}{(x-p)(x-(p_{1,0}/p_{-1,0})/p)} = \frac{(1/p_{0,1})(p_{-1,0}/p_{1,0})}{(1-s_+(\gamma)x)(1-s_-(\gamma)x)},$$

provided that  $\gamma$  is defined by  $1/s_+(\gamma) = p$ . The latter equation has a solution if and only if  $p \in [x_2, X(y_2) = \sqrt{p_{1,0}/p_{-1,0}}]$ . Then, using the functional equation (2.4), this forces the pole of  $H(0, y)$  in  $\mathcal{G}_Y$  to be  $Y_0(p)$ , that is,  $1/\tilde{s}_+(\gamma)$ .

*Convergence of the harmonic functions of the non-zero drift case to the harmonic functions in the zero-drift case.* Consider a random walk without drift, and with jumps  $\{p_{i,j}\}_{-1 \leq i,j \leq 1}$ . Denote by  $f$  its unique harmonic function. Let us introduce any family of random walks with non-zero drift and jumps  $\{q_{i,j}^n\}_{-1 \leq i,j \leq 1}$ , such that for all  $i$  and  $j$ ,  $\lim_{n \rightarrow \infty} q_{i,j}^n = p_{i,j}$ . For any  $n$ , there exist infinitely many harmonic functions  $f_\gamma^n$ , parametrized by  $\gamma \in [0, \pi/2]$ . We normalize all these functions by requiring that  $f(1, 1) = f_\gamma^n(1, 1) = 1$ . The main result in this paragraph is the following convergence result: for any  $i_0, j_0$  and any  $\gamma$ , we have

$$(6.2) \quad \lim_{n \rightarrow \infty} f_\gamma^n(i_0, j_0) = f(i_0, j_0).$$

We now mention two open problems concerning (6.2):

<sup>11</sup>This is due to the fact that a certain Riemann surface—namely,  $\{(x, y) \in (\mathbf{C} \cup \{\infty\})^2 : L(x, y) = 0\}$ —sees its genus passing from 0 to 1.

- The proof that we could write for (6.2) is purely analytical. It is an open problem to decide whether this can be obtained by using only general harmonic analysis.
- The convergence (6.2) to the same harmonic function  $f(i_0, j_0)$  can be proved independently of knowing that in the zero drift case, there exists a unique harmonic function. It is an open problem to prove that the convergence (6.2) implies the uniqueness of the harmonic function in the zero drift case.

Let us now illustrate (6.2) with the simple random walk, for  $\gamma \in (0, \pi/2)$ . With (6.1), the function  $H_\gamma(x, y) = \sum_{i_0, j_0 \geq 1} f_\gamma(i_0, j_0) x^{i_0-1} y^{j_0-1}$  equals

$$\frac{(s_+(\gamma) - s_-(\gamma))(\tilde{s}_+(\gamma) - \tilde{s}_-(\gamma))}{(1 - xs_+(\gamma))(1 - xs_-(\gamma))(1 - y\tilde{s}_+(\gamma))(1 - y\tilde{s}_-(\gamma))}.$$

As the drift goes to 0, the four points  $s_+(\gamma), s_-(\gamma), \tilde{s}_+(\gamma), \tilde{s}_-(\gamma)$  all converge to 1, so that dividing by the constant  $(s_+(\gamma) - s_-(\gamma))(\tilde{s}_+(\gamma) - \tilde{s}_-(\gamma))$ , the generating function  $H_\gamma(x, y)$  goes to  $1/[(1-x)^2(1-y)^2]$ , which indeed is the generating function for the zero-drift case, see Section 2.3. The convergence of the generating functions yields the convergence of its coefficients.

**6.2.  $t$ -harmonic functions.** In this work, we have determined the harmonic functions, that is, the  $t$ -harmonic functions (satisfying  $Pf = tf$ ) for  $t = 1$ . It is also interesting to consider  $t$ -harmonic functions other values of  $t$  (many motivations are given in [26, Chapter 2]). It turns out that our approach can be extended to find the latter: indeed, the key functional equation (2.4) still holds, if now

$$L(x, y) = L(x, y; t) = xy[\sum_{-1 \leq i, j \leq 1} p_{i,j} x^{-i} y^{-j} - t].$$

Accordingly, the methods presented in this article can be used to find  $t$ -harmonic functions, with, nevertheless, the following additional difficulty concerning the conformal mapping  $w(x)$ : mutatis mutandis, as for the non-zero drift case (see Section 6.1), introducing the variable  $t$  would have the effect of involving  $\wp$ -Weierstrass functions instead of sine and arcsin functions in the expression of the crucial conformal mapping  $w(x)$ .

**6.3. More general jumps.** In this part we consider random walks in the quarter plane, that do not satisfy the small step hypothesis (H1). Let us only assume that it has bounded jumps, which means that there exist  $I^\pm$  and  $J^\pm$ , such that  $p_{i,j} = 0$  if  $i > I^+, i < I^-, j > J^+$  or  $j < J^-$ . We are interested in harmonic functions satisfying to (P1), (P2)<sup>12</sup> and (P3). Let us define the sectional generating functions

$$F^{j_0}(x) = \sum_{i_0 \geq 1} f(i_0, j_0) x^{i_0-1}, \quad G^{i_0}(y) = \sum_{j_0 \geq 1} f(i_0, j_0) y^{j_0-1}.$$

The following functional equation then holds:

$$(6.3) \quad \begin{aligned} L(x, y)H(x, y) &= \sum_{1 \leq j_0 \leq J^+} F^{j_0}(x) y^{j_0-1} \sum_{\substack{I^- \leq i \leq I^+ \\ j_0 \leq j \leq J^+}} p_{i,j} x^{-(i-1)} y^{-(j-1)} \\ &+ \sum_{1 \leq i_0 \leq I^+} G^{i_0}(y) x^{i_0-1} \sum_{\substack{i_0 \leq i \leq I^+ \\ J^- \leq j \leq J^+}} p_{i,j} x^{-(i-1)} y^{-(j-1)} \\ &- \sum_{\substack{1 \leq i_0 \leq I^+ \\ 1 \leq j_0 \leq J^+}} f(i_0, j_0) x^{i_0-1} y^{j_0-1} \sum_{\substack{i_0 \leq i \leq I^+ \\ j_0 \leq j \leq J^+}} p_{i,j} x^{-(i-1)} y^{-(j-1)}. \end{aligned}$$

<sup>12</sup>In fact, (P2) has to be replaced by  $f(i_0, j_0) = 0$  if  $i_0 \leq 0$  or  $j_0 \leq 0$ .

This equation is far more complicated than (2.4), because instead of 2 unknowns (namely,  $H(x, 0)$  and  $H(0, y)$ ) in the right part of (2.4), there are now  $I^+ + J^+$  unknowns (namely,  $F^{j_0}(x)$  for  $1 \leq j_0 \leq J^+$  and  $G^{i_0}(y)$  for  $1 \leq i_0 \leq I^+$ ) in the right-hand side of (6.3). To our knowledge, it is an open problem to solve (6.3) in general.

However, if we restrict ourselves to  $I^+ = J^+ = 1$  (with  $I^-$  and  $J^-$  arbitrarily large), then (6.3) is nothing else but (2.4), as  $F^1(x) = H(x, 0)$  and  $G^1(y) = H(0, y)$ . In particular, the approach developed in the present paper can be applied for any such random walk, provided that a conformal mapping  $w(x)$  can be found: if  $I^-$  or  $J^-$  are different from  $-1$ , then the kernel  $L(x, y)$  becomes more complicated, and it is an open problem to construct a suitable conformal mapping  $w(x)$  with properties (W1)–(W5).

#### APPENDIX A. PROOF OF LEMMA 2

By symmetry, it is enough to consider only  $H(x, 0)$ . We shall prove that the radius of convergence of  $H(x, 0)$  is at most 1. This turns out to be sufficient, since Lemma 3 directly implies that  $H(1, 0) = \infty$ . To prove that the radius of  $H(x, 0)$  is smaller than 1, we show that  $\limsup_{i \rightarrow \infty} (1/i) \log h(i, 1) \leq 0$ .

Let us define  $T_{i_1, j_1} = \inf\{n \geq 0 : (X(n), Y(n)) = (i_1, j_1)\}$ , the first hitting time of state  $(i_1, j_1)$ , as well as the Green functions  $G_{i_1, j_1}^{i_0, j_0} = \sum_{n \geq 0} \mathbf{P}_{(i_0, j_0)}[(X(n), Y(n)) = (i_1, j_1)]$ . The classical Harnack inequality yields

$$\frac{f(i, 1)}{f(1, 1)} \leq \frac{1}{\mathbf{P}_{(1,1)}[T_{i,1} < \infty]} = \frac{G_{i,1}^{i,1}}{G_{i,1}^{1,1}}.$$

First, we have that  $\sup_{i \geq 1} G_{i,1}^{i,1} < \infty$ . Indeed, the Green functions  $G_{i,1}^{i,1}$  for the quadrant  $\mathbf{Z}_+^2$  are smaller than the Green functions  $\tilde{G}_{i,1}^{i,1}$  associated with the half plane  $\mathbf{Z} \times \mathbf{Z}_+$ . The latter domain being invariant by horizontal translations, the Green functions  $\tilde{G}_{i,1}^{i,1}$  do not depend on  $i$ . Accordingly, we have (see [13] for the last inequality)

$$\sup_{i \geq 1} G_{i,1}^{i,1} \leq \sup_{i \geq 1} \tilde{G}_{i,1}^{i,1} = \tilde{G}_{1,1}^{1,1} < \infty.$$

To conclude, it is enough to prove that

$$(A.4) \quad \liminf_{i \rightarrow \infty} \frac{1}{i} \log G_{i,1}^{1,1} \geq 0.$$

To that purpose, let us first use the last inequality in [13, Proof of Proposition 4.2] (which is valid not only for the half plane—the topic of [13]—but also for the quarter plane); we obtain

$$(A.5) \quad \liminf_{i \rightarrow \infty} \frac{1}{i} \log G_{i,1}^{1,1} \geq -\inf_{\phi} I_{[0,T]}(\phi),$$

where the infimum is taken over all trajectories  $\phi : [0, T] \rightarrow \mathbf{R}_+^2$  such that  $\phi(0) = (0, 0)$  and  $\phi(T) = (1, 0)$ , and where the rate function  $I_{[0,T]}(\phi)$  equals  $\int_0^T L(\phi(t)) dt$ , with

$$L(v) = (\log R)^*(v) = \sup_{\alpha \in \mathbf{R}^2} (\langle \alpha, v \rangle - \log R(\alpha)), \quad R(\alpha) = \sum_{-1 \leq i, j \leq 1} p_{i,j} \exp(\langle \alpha, (i, j) \rangle).$$

Obviously, inequality (A.5) still holds if one takes the infimum over all linear paths  $\phi$ , i.e.,  $\phi(t) = (t/T)(1, 0)$  for  $t \in [0, T]$ . For such functions  $\phi$ , we have

$$\inf I_{[0,T]}(\phi) = T \cdot (\log R)^*(1/T, 0)$$

Then, as (see [23, Page 35])

$$\inf_{T>0} (T \cdot (\log R)^*(1/T, 0)) = 0,$$

we reach the conclusion that (A.4) holds. The proof of Lemma 2 is completed.

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