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Band structure of the Ruelle spectrum of contact Anosov flows

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Abstract

If X is a contact Anosov vector field on a smooth compact manifold M and $V \in C^\infty(M)$ it is known that the differential operator $A = -X + V$ has some discrete spectrum called Ruelle-Pollicott resonances in specific Sobolev spaces. We show that for $|\operatorname{Im}z| \rightarrow \infty$ the eigenvalues of A are restricted to vertical bands and in the gaps between the bands, the resolvent of A is bounded uniformly with respect to $|\operatorname{Im}(z)|$. In each isolated band the density of eigenvalues is given by the Weyl law. In the first band, most of the eigenvalues concentrate of the vertical line $\operatorname{Re}(z) = \langle D \rangle_M$, the space average of the function $D(x) = V(x) - \frac{1}{2} \operatorname{div} X|_{E_u(x)}$ where E_u is the unstable distribution.

1 Introduction

In this paper we announce some results concerning the Ruelle-Pollicott spectrum of transfer operators associated to contact Anosov flows [5]. Let X be a smooth vector field on a compact manifold M and suppose that X generates a contact Anosov flow.

The Ruelle-Pollicott spectrum of contact Anosov flows has been studied since a long time due to its importance to describe the precise behavior and decay of time correlation functions for large time. From this, one can deduce fine statistical properties of the dynamics of the flow such as exponential convergence towards equilibrium (i.e. mixing) or central limit theorem for the Birkhoff average of functions. The Ruelle-Pollicott spectrum is also useful to get some precise asymptotic counting of periodic orbits.

Recent results show that the Ruelle-Pollicott resonances are the discrete eigenvalues of the generator $(-X)$ seen as a differential operator in some specific Sobolev spaces of distributions $\mathcal{H} \subset \mathcal{D}'(M)$ [1, 3, 6]. A more precise description of the structure of this spectrum has been obtained in [9, 10] where it is shown that in the asymptotic limit $|\operatorname{Im}z| \rightarrow \infty$ the spectrum

is on the domain $\operatorname{Re}(z) \leq \gamma_0^+$ with some explicit “gap” $\gamma_0^+ < 0$ given below. More generally these results can be extended to the operator $A = -X + V$ where $V \in C^\infty(M)$ is a smooth function called “potential”.

In this paper we improve the description of the structure of this Ruelle-Pollicott spectrum. The main results are stated in Theorem 5.1. They show that the Ruelle-Pollicott spectrum of the first order differential operator $A = -X + V$ has some band structure in the asymptotic limit $|\operatorname{Im}z| \rightarrow \infty$, i.e. is contained in the union of vertical bands $\mathbf{B}_k = \{z \in \mathbb{C}, \operatorname{Re}(z) \in [\gamma_k^-, \gamma_k^+]\}$, $k \geq 0$ with $\gamma_{k+1}^\pm < \gamma_k^\pm$. The values γ_k^+, γ_k^- are given explicitly in (5.1) by the maximum (respect. minimum) of the time averaged along trajectories of a function $D \in C^\infty(M)$ called “damping function” given by $D = V - \frac{1}{2}\operatorname{div}X|_{E_u}$. If the band \mathbf{B}_k is isolated from the others by an asymptotic spectral gap (i.e. $\gamma_{k+1}^+ < \gamma_k^-$) then the norm of resolvent of A is bounded in this gap uniformly with respect to $|\operatorname{Im}(z)|$. Theorem 5.1 shows that the spectrum in every isolated band \mathbf{B}_k satisfies a Weyl law, i.e. the number of eigenvalues $z \in \mathbf{B}_k$ satisfying $\operatorname{Im}(z) \in [b, b + b^\varepsilon]$ is¹ $\mathcal{N}(b) \asymp b^{d+\varepsilon}$ as $b \rightarrow \infty$ for any $\varepsilon > 0$, where $\dim M = 2d + 1$. A previous result for the Weyl law is given in [2] and concerns an upper bound: it is shown that for any radius $C_0 > 0$, the number of resonances in the disk $D(ib, C_0)$ of center ib is $O(b^d)$.

Concerning the most interesting “external band” $\mathbf{B}_0 = \{z \in \mathbb{C}, \operatorname{Re}(z) \in [\gamma_0^-, \gamma_0^+]\}$, supposing that it is isolated ($\gamma_1^+ < \gamma_0^-$), it is shown in Theorem 5.3 that most of the resonances in the band \mathbf{B}_0 accumulate on the vertical line $\operatorname{Re}(z) = \langle D \rangle_M$ given by the space average of the function D . This is due to ergodicity. This problem is then closely related to the description of the spectrum of the damped wave equation [7].

In the forthcoming paper [5] we will consider the special case $V = V_0 = \frac{1}{2}\operatorname{div}X|_{E_u}$ for which the damping function vanishes $D = 0$, $\gamma_0^\pm = 0$, i.e. the Ruelle-Pollicott resonances of the external band accumulate on the imaginary axis. However, V_0 is not smooth and this requires an extension of the theory.

From Selberg theory and representation theory, this particular band structure is known for a long time in the case of homogeneous hyperbolic manifolds $M = \Gamma \backslash SO_{1,n}/SO_{n-1} \equiv \Gamma \backslash T_1^* \mathbb{H}^n$ where Γ is a discrete co-compact subgroup. In that case, the contact Anosov flow is the geodesic flow on the hyperbolic manifold surface $\mathcal{N} = \Gamma \backslash \mathbb{H}^n = \Gamma \backslash SO_{1,n}/SO_n$.

Technically we use semiclassical analysis to study the spectrum of the differential operator $A = -X + V$ [11]. We consider the associated “canonical dynamics” in the phase space T^*M which is simply the lifted flow. The key observation is that this canonical dynamics has a non wandering set or “trapped set” which is a smooth symplectic submanifold $K \subset T^*M$ and which is normally hyperbolic. This is the origin of the band structure of the spectrum. This approach has been presented in details in the paper [4] treating a very similar problem, the band structure of prequantum Anosov diffeomorphisms.

2 Contact Anosov flow

¹The notation $\mathcal{N}(b) \asymp |b|^{d+\varepsilon}$ means that $\exists C > 0$ independent of b s.t. $\frac{1}{C}|b|^{d+\varepsilon} \leq \mathcal{N}(b) \leq C|b|^{d+\varepsilon}$.

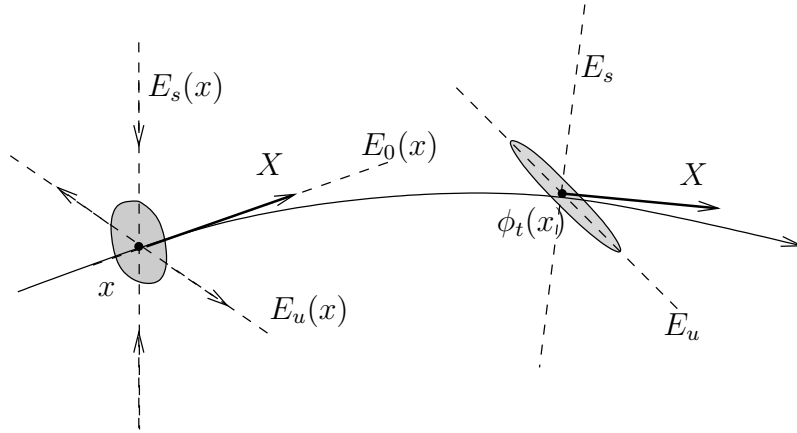


Figure 2.1: Anosov flow.

Definition 2.1. On a smooth Riemannian compact manifold (M, g) , a smooth vector field X generating a flow $\phi_t : M \rightarrow M, t \in \mathbb{R}$, is **Anosov**, if there exists an ϕ_t -invariant decomposition of the tangent bundle $TM = E_u \oplus E_s \oplus E_0$, where $E_0 = \mathbb{R}X$ and $C > 0, \lambda > 0$ such that for every $t \geq 0$

$$\|D\phi_t|_{E_s}\|_g \leq Ce^{-\lambda t}, \quad \|D\phi_{-t}|_{E_u}\|_g \leq Ce^{-\lambda t}, \quad (2.1)$$

Remark 2.2. In general the map $x \in M \rightarrow E_u(x), E_s(x)$ are only Hölder continuous. The “structural stability theorem” shows that Anosov vector field is a property robust under perturbation.

Definition 2.3. The Anosov one form $\alpha \in C(T^*M)$ is defined by $\text{Ker}\alpha = E_u \oplus E_s, \alpha(X) = 1$. X is a **contact Anosov vector field** if α is a smooth contact one form i.e. $(d\alpha)|_{E_u \oplus E_s}$ is non degenerate (symplectic).

Remark 2.4. In the case of a contact Anosov vector field we have that $\dim E_u = \dim E_s = d$, with $\dim M = 2d + 1$ and that $dx = \alpha \wedge (d\alpha)^d$ is smooth volume form on M preserved by the flow ϕ_t .

As an example, the geodesic flow on a compact manifold \mathcal{N} with negative sectional curvature (not necessary constant) defines a contact Anosov flow on $T_1^*\mathcal{N}$. In that case the Anosov one form α coincides with the canonical Liouville one form ξdx on $T^*\mathcal{N}$.

We will assume that X is a contact Anosov vector field on M in the rest of this paper.

3 The transfer operator

Let $V \in C^\infty(M)$ be a smooth function called “potential”.

Definition 3.1. The **transfer operator** is the group of operators

$$\hat{F}_t : \begin{cases} C^\infty(M) & \rightarrow C^\infty(M) \\ v & \rightarrow e^{tA}v \end{cases}, \quad t \geq 0$$

with the generator

$$A := -X + V$$

which is a first order differential operator.

Remark 3.2.

- Since X generates the flow ϕ_t we can write $\hat{F}_t v = \left(e^{\int_0^t V \circ \phi_{-s} ds} \right) v(\phi_{-t}(x))$, hence \hat{F}_t acts as transport of functions by the flow with multiplication by exponential of the function V averaged along the trajectory.
- In the case $V = 0$, the operator \hat{F}_t is useful in order to express “time correlation functions” between $u, v \in C^\infty(M)$, $t \in \mathbb{R}$:

$$C_{u,v}(t) := \int_M u \cdot (v \circ \phi_{-t}) dx = \langle u, \hat{F}_t v \rangle_{L^2} \quad (3.1)$$

The study of these time correlation functions permits to establish the mixing properties and other statistical properties of the dynamics of the Anosov flow. In particular $u = cste$ is an obvious eigenfunction of $A = -X$ with eigenvalue $z_0 = 0$. Since $\operatorname{div} X = 0$ we have that \hat{F}_t is unitary in $L^2(M, dx)$ and $iA = (iA)^*$ is self-adjoint but they have continuous spectrum. A has continuous spectrum on the imaginary axis $\operatorname{Re} z = 0$. In the next Theorem we consider more interesting functional spaces where the operator A has discrete spectrum but is non self-adjoint.

Theorem 3.3 ([1][3]). *“discrete spectrum”.* If X is an Anosov vector field and $V \in C^\infty(M)$ then for every $C > 0$, there exists an Hilbert space \mathcal{H}_C with $C^\infty(M) \subset \mathcal{H}_C \subset \mathcal{D}'(M)$, such that

$$A = -X + V \quad : \mathcal{H}_C \rightarrow \mathcal{H}_C$$

has discrete spectrum on the domain $\operatorname{Re}(z) > -C\lambda$, called **Ruelle-Pollicott resonances**, independent on the choice of \mathcal{H}_C .

Remark 3.4. Concerning the meaning of these eigenvalues, notice that with the choice $V = 0$, if $(-X)v = zv$, v is an invariant distribution with eigenvalue $z = -a + ib \in \mathbb{C}$, then $v \circ \phi_{-t} = e^{-tX}v = e^{-at}e^{ibt}v$, i.e. $a = -\operatorname{Re}(z)$ contributes as a damping factor and $b = \operatorname{Im}(z)$ as a frequency in time correlation function (3.1). Notice also the symmetry of the spectrum under complex conjugation that $Av = zv$ implies $A\bar{v} = \bar{z}\bar{v}$.

We introduce now the following function that will play an important role²

$$V_0(x) := \frac{1}{2} \operatorname{div} X|_{E_u}. \quad (3.2)$$

From (2.1) we have $V_0(x) \geq \frac{1}{2}d \cdot \lambda$. Since $E_u(x)$ is only Hölder in x so is $V_0(x)$. We will also consider the difference

$$D(x) := V(x) - V_0(x) \quad (3.3)$$

and called it the “**effective damping function**”. For simplicity we will write:

$$\left(\int_0^t D \right)(x) := \int_0^t (D \circ \phi_{-s})(x) ds, \quad x \in M,$$

for the Birkhoff average of D along trajectories.

Theorem 3.5 ([9, 10]). “*asymptotic gap*”. *If X is a contact Anosov vector field on M and $V \in C^\infty(M)$, then for any $\varepsilon > 0$ the Ruelle-Pollicott eigenvalues $(z_j)_j \in \mathbb{C}$ of $A = -X + V$ are contained in*

$$\operatorname{Re}(z) \leq \gamma_0^+ + \varepsilon$$

up to finitely many exceptions and with

$$\gamma_0^+ = \lim_{t \rightarrow \infty} \sup_{x \in M} \frac{1}{t} \left(\int_0^t D \right)(x). \quad (3.4)$$

Remark 3.6. See figure 5.1(b). Notice that in the case $V = 0$ we have $\gamma_0^+ \leq -\frac{1}{2}d \cdot \lambda < 0$.

4 Example of the geodesic flow on constant curvature surface

A simple and well known example of contact Anosov flow is provided by the geodesic flow on a surface \mathcal{S} with constant negative curvature. Precisely let $\Gamma < SL_2\mathbb{R}$ be a co-compact discrete subgroup of $G = SL_2\mathbb{R}$ (i.e. such that $M := \Gamma \backslash SL_2\mathbb{R}$ is compact). We suppose that $(-\operatorname{Id}) \in \Gamma$. Then we have a natural identification that $M \equiv T_1^*\mathcal{S}$ is the unit cotangent bundle of the hyperbolic surface $\mathcal{S} := \Gamma \backslash SL_2\mathbb{R}/SO_2 \equiv \Gamma \backslash \mathbb{H}^2$. Let X be the left invariant vector field on M given by the element $X_e = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in sl_2\mathbb{R} = T_eG$. Then X is an Anosov contact vector field on M and can be interpreted as the geodesic flow on the surface \mathcal{S} . Using representation theory, it is known that the Ruelle-Pollicott spectrum of the operator $(-X)$ coincides with the zeros of the dynamical Fredholm determinant. This dynamical Fredholm determinant is expressed as the alternative product of the Selberg zeta functions and gives the following result; see figure 5.1(a). We refer to [5] for further details.

²Let μ_g be the induced Riemann volume form on $E_u(x)$ defined from the choice of a metric g on M . As the usual definition in differential geometry [8, p.125], for tangent vectors $u_1, \dots, u_d \in E_u(x)$, $\operatorname{div} X|_{E_u}$ measures the rate of change of the volume of E_u and is defined by

$$(\operatorname{div} X|_{E_u}(x)) \cdot \mu_g(u_1, \dots, u_d) = \lim_{t \rightarrow 0} \frac{1}{t} (\mu_g(D\phi_t(u_1), \dots, D\phi_t(u_d)) - \mu_g(u_1, \dots, u_d))$$

Proposition 4.1. *If X is the geodesic flow on an hyperbolic surface $\mathcal{S} = \Gamma \backslash \mathbb{H}^2$ then the Ruelle-Pollicott eigenvalues z of $(-X)$ are of the form*

$$z_{k,l} = -\frac{1}{2} - k \pm i\sqrt{\mu_l - \frac{1}{4}} \quad (4.1)$$

where $k \in \mathbb{N}$ and $(\mu_l)_{l \in \mathbb{N}} \in \mathbb{R}^+$ are the discrete eigenvalues of the hyperbolic Laplacian Δ on the surface \mathcal{S} . There are also $z_n = -n$ with $n \in \mathbb{N}^*$. Each set $(z_{k,l})_l$ with fixed k will be called the line \mathbf{B}_k . The ‘‘Weyl law’’ for Δ gives the density of eigenvalues on each vertical line \mathbf{B}_k , for $b \rightarrow \infty$,

$$\#\{z_{k,l}, b < \text{Im}(z_{k,l}) < b + 1\} \asymp |b| \quad (4.2)$$

5 Band spectrum for general contact Anosov flow

Proposition 4.1 above shows that the Ruelle-Pollicott spectrum for the geodesic flow on constant negative surface has the structure of vertical lines \mathbf{B}_k at $\text{Re}z = -\frac{1}{2} - k$. In each line the eigenvalues are in correspondence with the eigenvalues of the Laplacian Δ . We address now the question if this structure persists somehow for geodesic flow on manifolds with negative (variable) sectional curvature and more generally for any contact Anosov flow. In the next Theorem, for a linear invertible map L , we note $\|L\|_{max} := \|L\|$ and $\|L\|_{min} := \|L^{-1}\|^{-1}$.

Theorem 5.1. [5]“*asymptotic band structure*”. If X is a contact Anosov vector field on M and $V \in C^\infty(M)$ then for every $C > 0$, there exists an Hilbert space \mathcal{H}_C with $C^\infty(M) \subset \mathcal{H}_C \subset \mathcal{D}'(M)$, such that for any $C > 0$, any $\varepsilon > 0$, the Ruelle-Pollicott eigenvalues $(z_j)_j \in \mathbb{C}$ of the operator $A = -X + V : \mathcal{H}_C \rightarrow \mathcal{H}_C$ on the domain $\text{Re}(z) > -C\lambda$ are contained, up to finitely many exceptions, in the union of finitely many bands

$$z \in \bigcup_{k \geq 0} \underbrace{[\gamma_k^- - \varepsilon, \gamma_k^+ + \varepsilon] \times i\mathbb{R}}_{\text{Band } \mathbf{B}_k}$$

with

$$\gamma_k^+ = \lim_{t \rightarrow \infty} \left| \sup_x \frac{1}{t} \left(\left(\int_0^t D \right) (x) - k \log \left\| D\phi_t(x)_{/E_u} \right\|_{\min} \right) \right|, \quad (5.1)$$

$$\gamma_k^- = \lim_{t \rightarrow \infty} \left| \inf_x \frac{1}{t} \left(\left(\int_0^t D \right) (x) - k \log \left\| D\phi_t(x)_{/E_u} \right\|_{\max} \right) \right| \quad (5.2)$$

and where $D = V - V_0$ is the damping function (3.3). In the gaps between the bands, the norm of the resolvent is controlled: For any $\varepsilon > 0$, there exists $c > 0$ such that for every $z \notin \bigcup_{k \geq 0} \mathbf{B}_k$ and $|\text{Im}(z)| > c$ then

$$\left\| (z - A)^{-1} \right\| \leq c$$

Also for any k the number of resonances in the band \mathbf{B}_k is controlled by a “**Weyl law**”: $\forall \varepsilon > 0, \exists c > 0, \forall b > c$,

$$\frac{1}{c} |b|^{d+\varepsilon} < \#\{z_j \in \mathbf{B}_k, b < \text{Im}(z_j) < b + b^\varepsilon\} < c |b|^{d+\varepsilon} \quad (5.3)$$

with $\dim M = 2d + 1$.

Remark 5.2. We can compare Theorem 5.1 with Proposition 4.1 in the special case of the geodesic flow on a constant curvature surface $\mathcal{S} = \Gamma \backslash \mathbb{H}^2$: we have $D\phi_t(x)_{/E_u} \equiv e^t$ hence $V_0 = \frac{1}{2}$. The choice of potential $V = 0$ gives the constant damping function $D = -\frac{1}{2}$, hence (5.1) gives $\gamma_k^+ = \gamma_k^- = -\frac{1}{2} - k$ as in Proposition 4.1.

Theorem 5.3. [5]If the external band \mathbf{B}_0 is isolated i.e. $\gamma_1^+ < \gamma_0^-$, then most of the resonances accumulate on the vertical line

$$\text{Re}(z) = \langle D \rangle := \frac{1}{\text{Vol}(M)} \int_M D(x) dx$$

in the precise sense that

$$\frac{1}{N_b} \sum_{z_i \in \mathbf{B}_0, |\text{Im}(z_i)| < b} |z_i - \langle D \rangle| \xrightarrow{b \rightarrow \infty} 0, \quad \text{with } N_b := \#\{z_i \in \mathbf{B}_0, |\text{Im}(z_i)| < b\}. \quad (5.4)$$

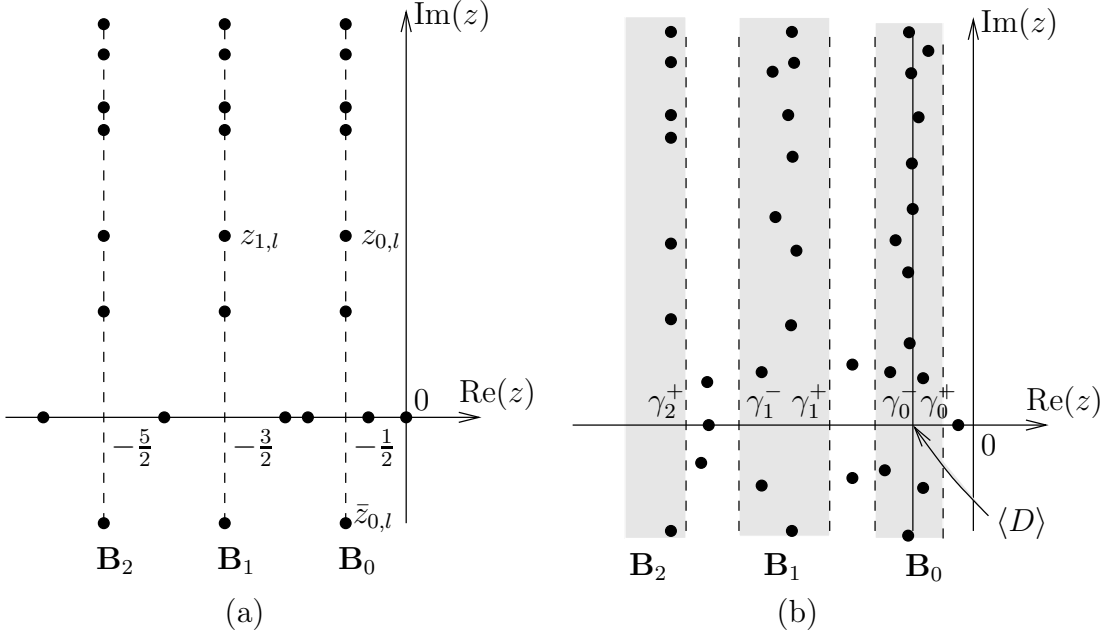


Figure 5.1: (a) Ruelle-Pollicott spectrum of $-X$ for the geodesic flow on an hyperbolic surface $\mathcal{S} = \Gamma \backslash \mathbb{H}^2$. (b) Spectrum of $A = -X + V$ and its asymptotic band structure for a general contact Anosov flow.

Outline of the proof: The proof of Theorem 5.1 relies on semiclassical analysis. The analysis is very similar to that done in [3] concerning the semiclassical analysis of Anosov flow (not necessary contact) but in fact we adapt very closely the results obtained in [4] concerning the band structure of prequantum maps. In the limit $|\text{Im}z| \rightarrow \infty$ under study, the semiclassical parameter is written $\hbar := 1/|\text{Im}z|$. $A = -X + V$ is a differential operator. Its principal symbol is the function $\sigma(A)(x, \xi) = X(\xi)$ on phase space T^*M (the cotangent bundle). It generates an Hamiltonian flow which is simply the canonical lift of the flow ϕ_t on M . Due to Anosov hypothesis on the flow of definition 2.1 the non wandering set of the Hamiltonian flow is the continuous sub-bundle $K = \mathbb{R}\alpha \subset T^*M$ where α is the Anosov one form. K is normally hyperbolic. With the hypothesis that α is a smooth contact one form, this makes $K \setminus \{0\}$ a smooth symplectic submanifold of T^*M , normally hyperbolic. Using a microlocal description of the trapped set together with linear normal forms the dynamics on the vicinity of K can be decomposed as an internal dynamics on K and a transverse hyperbolic dynamics on the symplectic orthogonal $(T_\rho K)^\perp$ around every point $\rho \in K$. At the level of operators, this local decomposition gives a local decomposition of the operator A in tensor product $A_{T_\rho K} \otimes A_{(T_\rho K)^\perp}$. With specific escape functions the operator $A_{(T_\rho K)^\perp}$ associated with the dynamics of a hyperbolic linear fixed point has a discrete spectrum of resonances indexed by integer $k \in \mathbb{N}$. This escape function is very sharp in the sense that it is present at a distance of order $O(\hbar^{1/2})$ of the trapped set. This allows to construct some approximate spectral projectors Π_k for every value of k , and patching these locals expression we get global operators. The positions γ_k^\pm of the band \mathbf{B}_k come from estimates of the operator A restricted by the projector Π_k .

The proof of the Weyl law is similar to the proof of J.Sjöstrand about the damped wave equation [7] but need more arguments. The accumulation of resonances on the value $\langle D \rangle$ in

Theorem 5.3 given by the spatial average of the damping function, Eq.(5.4), uses the ergodicity property of the Anosov flow and is also similar to the spectral results obtained in [7] for the damped wave equation.

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