HAL
open science

Well-posedness for density-dependent incompressible fluids with non-Lipschitz velocity

Boris Haspot

To cite this version:

Boris Haspot. Well-posedness for density-dependent incompressible fluids with non-Lipschitz velocity. Annales de l'Institut Fourier, 2012, 62 (5), pp.1717-1763. hal-00779844

HAL Id: hal-00779844

https://hal.science/hal-00779844v1

Submitted on 22 Jan 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Well-posedness for density-dependent incompressible fluids with non-Lipschitz velocity

Boris Haspot *†

Abstract

This paper is dedicated to the study of the initial value problem for density dependent incompressible viscous fluids in $\mathbb{R}^N$ with $N \geq 2$. We address the question of well-posedness for *large* data having critical Besov regularity and we aim at stating well-posedness in functional spaces as close as possible to the ones imposed in the incompressible Navier Stokes system by Cannone, Meyer and Planchon in [7] where $u_0 \in B^{\frac{N}{p}-1}_{p,\infty}$ with $1 \leq p < +\infty$. This improves the analysis of [13], [14] and [2] where u_0 is considered belonging to $B^{\frac{N}{p}-1}_{p,1}$ with $1 \leq p < 2N$. Our result relies on a new a priori estimate for transport equation introduce by Bahouri, Chemin and Danchin in [5] when the velocity u is not considered Lipschitz.

1 Introduction

In this paper, we are concerned with the following model of incompressible viscous fluid with variable density:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mathrm{div}(2\mu(\rho) Du) + \nabla \Pi = \rho f, \\ \mathrm{div} u = 0, \\ (\rho, u)_{/t=0} = (\rho_0, u_0). \end{cases} \tag{1.1}$$

Here $u = u(t,x) \in \mathbb{R}^N$ stands for the velocity field and $\rho = \rho(t,x) \in \mathbb{R}^+$ is the density, $Du = \frac{1}{2}(\nabla u +^t \nabla u)$ is the strain tensor. We denote by μ the viscosity coefficients of the fluid, which is assumed to satisfy $\mu > 0$. The term $\nabla \Pi$ (namely the gradient of the pressure) may be seen as the Lagrange multiplier associated to the constraint $\mathrm{div} u = 0$. We supplement the problem with initial condition (ρ_0, u_0) and an outer force f. Throughout the paper, we assume that the space variable $x \in \mathbb{R}^N$ or to the periodic box $\mathbb{T}^N_a$ with period a_i, in the i-th direction. We restrict ourselves the case $N \geq 2$.
The existence of global weak solution for (1.1) under the assumption that $\rho_0 \in L^\infty$ is

*Karls Ruprecht Universität Heidelberg, Institut for Applied Mathematics, Im Neuenheimer Feld 294, D-69120 Heildelberg, Germany. Tel. 49(0)6221-54-6112

†Ceremade UMR CNRS 7534 Université de Paris Dauphine, Place du Marchal DeLattre De Tassigny 75775 PARIS CEDEX 16 , haspot@ceremade.dauphine.fr

nonnegative, that divu_0, and that $\sqrt{\rho_0}u_0 \in L^2$ has been studied by different authors. It is based on the energy equality:

$$\|\sqrt{\rho}u(t)\|_{L^2}^2 + \int_0^t \|\sqrt{\mu(\rho)}Du(\tau)\|_{L^2}^2 d\tau = \|\sqrt{\rho_0}u_0\|_{L^2}^2 + \int 2(\rho f \cdot u)(\tau, x) d\tau\, dx. \qquad (1.2)$$

Using (1.2) and the fact that the density is advected by the flow of u so that the L^p norms of ρ are (at least formally) conserved during the evolution, it is then possible to use compactness methods to prove the existence of global weak solution. This approach has been introduced by J. Leray in 1934 in the homogeneous case (i.e $\rho = 1$) and no external force. For the non-homogeneous equation (1.1), we refer to [3] and to [28] for an overview of results on weak solution. Some recent improvements have been obtained by B. Desjardins in [19],[20] and [21].
The question of unique resolvability for (1.1) has been first addressed by O. Ladyzhenskaya and V. Solonnikov in the late seventies (see [27]). The authors consider system (1.1) in a bounded domain Ω with homogeneous Dirichlet boundary conditions for u. Under the assumption that $u_0 \in W^{2-\frac{2}{q},q}$ $(q > N)$ is divergence-free and vanishes on $\partial\Omega$ and that $\rho_0 \in C^1(\bar{\Omega})$ is bounded away from zero, the results are the following:

- Global well-posedness in dimension $N = 2$
- Local well-posedness in dimension $N = 3$. If in addition u_0 is small in $W^{2-\frac{2}{q},q}$, then global well-posedness holds true.

Let us mention by passing that for the dimension $N = 2$, O. Ladyzhenskaya and V. Solonnikov use a quasi-conservation law for the H^1 norm of the velocity and get global H^1 solutions.
The case of unbounded domains has been investigate by S. Itoh and A. Tani in [24]. In this framework, system (1.1) has been shown to be locally well-posed. In the present paper, we aim at proving similar qualitative results in the whole space $\mathbb{R}^N$ or in the torus $\mathbb{T}^N$ under weaker regularity assumptions.
Guided in our approach by numerous works dedicated to the incompressible Navier-Stokes equation (see z.g [29]):

$$(NS) \qquad \begin{cases} \partial_t v + v \cdot \nabla v - \mu\Delta v + \nabla\Pi = 0, \\ \text{div} v = 0, \end{cases}$$

we aim at solving in the case where the data (ρ_0, u_0, f) have critical regularity for the scaling of the equations and in particular when the initial velocity is in the same Besov spaces than Cannone, Meyer and Planchon in [7] for the incompressible Navier-Stokes system. It means that we want get strong solutions when u_0 belongs to $B_{p,\infty}^{\frac{N}{p}-1}$ with $1 \leq p < +\infty$. By critical, we mean that we want to solve the system functional spaces with norm in invariant by the changes of scales which leaves (1.1) invariant. That approach has been initiated by H. Fujita and T. Kato in [23]. In the case of incompressible fluids, it is easy to see that the transformations:

$$v(t, x) \longrightarrow lv(l^2 t, lx), \quad \forall\, l \in \mathbb{R}$$

have that property.
For density-dependent incompressible fluids, one can check that the appropriate transformation are

$$\begin{aligned}&(\rho_0(x), u_0(x)) \longrightarrow (\rho_0(lx), lu_0(lx)), \quad \forall\, l \in \mathbb{R}.\\ &(\rho(t,x), u(t,x), \Pi(t,x)) \longrightarrow (\rho(l^2t, lx), lu(l^2t, lx), l^2\Pi(l^2t, lx)).\end{aligned} \tag{1.3}$$

The use of critical functional frameworks led to several new well-posedness results for incompressible fluids (see [7], [26]). In the case od the density dependent incompressible fluids we want cite recent improvements by R. Danchin in [11], [12], [13], [14], [15], [16], [17] and of H. Abidi in [1] and Abidi, Paicu in [2]. All these works give results of strong solutions in finite time in critical spaces for the scaling of the equations. More precisely R. Danchin show the existence of strong solution in finite time in [16], [17], [18] when the initial data check $(\rho_0^{-1}-1, u_0) \in \left(B^{\frac{N}{2}}_{2,\infty} \cap L^\infty\right) \times B^{\frac{N}{2}-1}_{2,1}$ or $(\rho_0^{-1}-1, u_0) \in B^{\frac{N}{p}}_{p,1} \times B^{\frac{N}{p}}_{p,1}$ with $1 \leq p \leq N$. Moreover R. Danchin need of a condition of smallness on the initial data, it means that $\|\rho_0^{-1}-1\|_{B^{\frac{N}{p}}_{p,1}}$ is assumed small. More recently H. Abidi and M. Paicu in [2] have improved this result by working with initial data in $B^{\frac{N}{p_1}}_{p_1,1} \times B^{\frac{N}{p_2}-1}_{p_2,1}$ with p_1 and p_2 good choose and checking some relations. In particular they show that we have strong solution for initial data u_0 in $B^{\frac{N}{p_2}-1}_{p_2,1}$ with $1 \leq p_2 < 2N$ which improves the results of R. Danchin. All these results use of crucial way the fact that the solution are Lipschitz. In particular, it explains the choice of the third index for the Besov space where $r = 1$, indeed it allows a control of ∇u in $L^1_T(B^{\frac{N}{p}}_{p,1})$ which is embedded in $L^1_T(L^\infty)$. This control is imperative in these works to get estimate via the transport equation on the density.
However the scaling of (1.3) suggests choosing initial data (ρ_0, u_0) in $B^{\frac{N}{p_1}}_{p_1,r'} \times B^{\frac{N}{p_2}}_{p_2,r}$ (see the definition of Besov spaces in the section 2) with $(p_1, p_2) \in [1, +\infty)^2$ and $(r, r') \in [1, +\infty]^2$. Indeed it seems that it is not necessary by the study of the scaling of the equation to impose that $r, r' = 1$ as in the works of H. Abidi, R. Danchin and M. Paicu. The goal of this article is to reach the critical case with a general third index for the Besov spaces r and r'. More precisely in the sequel we will restrict our study to the case where the initial data (ρ_0, u_0) and external force f are such that, for some positive constant $\bar{\rho}$:

$$(\rho_0 - \bar{\rho}) \in B^{\frac{N}{p_1}+\varepsilon}_{p_1,\infty} \cap L^\infty,\ u_0 \in B^{\frac{N}{p_2}}_{p_2,r} \ \text{ and } \ f \in \widetilde{L}^1_{loc}(\mathbb{R}^+, \in B^{\frac{N}{p_2}-1}_{p_2,r}).$$

with $r \in [1, +\infty]$, $\varepsilon > 0$ and p_1, p_2 will verify appropriate inequalities (for a definition of $\widetilde{L}^1$ see section 2).
In this article we improve the result of H. Abidi, R. Danchin and M. Paicu by working with initial data in $B^{\frac{N}{p_2}}_{p_2,r}$ with $r \in [1, +\infty]$. In particular we generalize to the case of the Navier-Stokes incompressible dependent density the result of Cannone-Meyer-Planchon in [7]. For making we introduce a new idea to control the density via the transport equation when the velocity is not Lipschitz. We use some new a priori estimates on the transport equation when the velocity is only log Lipschitz. The difficulty is to deal with the loss of regularity on the density, that is why we need to ask more regularity on ρ_0 to compense this loss of regularity. The crucial point is that the density stay in a good

multiplier space. Moreover we improve the results of H. Abidi, R. Danchin and M. Paicu because we do not need to assume some condition of smallness on the initial density. In [11], [12], [13], [14], [15], [16], [17], [1] and [2] , we need to make the additional assumption that $\rho - \bar{\rho}$ is *small* in $B^{\frac{N}{p}}_{p,1}$. The reason why is that we handled the elliptic operator in the momentum equation of (1.1) as a constant coefficient second order operator plus a perturbation introduced by $\rho - \bar{\rho}$ which, if sufficiently small, may be treated as a harmless source term. For smoother data however, the additional regularity can compensate *large* perturbations of the constant coefficient operator. This fact has been used in [14] leads to local well-posedness results for smooth enough data with density bounded away from zero. The price to pay however is that assuming extra smoothness precludes from using a critical framework. In fact to treat the case of large initial data on the density, we follow a idea of R. Danchin in [**?**] use for the case of Navier-Stokes compressible where to get good estimate on the elliptic operator in the momentum equation, we need to split the behavior high frequencies and low-middle frequencies of the viscosity terms.
In the present paper, we address the question of local well-posedness in the critical functional framework under the sole assumption that the initial density is bounded away from 0 and tends to some positive constant $\bar{\rho}$ at infinity (or has average $\bar{\rho}$ if we consider the case of periodic boundary conditions). To simplify the notation, we assume from now on that $\bar{\rho} = 1$. Hence as long as ρ does not vanish, the equations for ($a = \rho^{-1} - 1$,u) read:

$$\begin{cases} \partial_t a + u \cdot \nabla a = 0, \\ \partial_t u + u \cdot \nabla u + (1+a)(\nabla \Pi - \mu \Delta u) = f, \\ \mathrm{div} u = 0, \\ (a,u)_{/t=0} = (a_0, u_0). \end{cases} \tag{1.4}$$

One can now state our main result. The first theorem generalize the work of Cannone,Meyer, Planchon in [7] when the third index of the Besov space for the initial data is in $[1, +\infty[$.

Theorem 1.1 *Let $1 \le r < \infty$, $1 \le p_1 < \infty$, $1 < p_2 < \infty$ and $\varepsilon > 0$ such that:*

$$\frac{N}{p_1} + \varepsilon < \frac{N}{p_2} + 1 \quad \textit{and} \quad \frac{N}{p_2} - 1 \le \frac{N}{p_1}.$$

Assume that $u_0 \in B^{\frac{N}{p_2}-1}_{p_2,r}$ with $\mathrm{div} u_0 = 0$, $f \in \widetilde{L}^1_{loc}(\mathbb{R}^+, B^{\frac{N}{p_2}-1}_{p_2,r})$ and $a_0 \in B^{\frac{N}{p_1}+\varepsilon}_{p_1,\infty} \cap L^\infty$, with $1 + a_0$ bounded away from zero and it exists $c > 0$ such that:

$$\|a_0\|_{B^{\frac{N}{p_1}}_{p_1,\infty}} \le c.$$

If $\frac{1}{p_1} + \frac{1}{p_2} > \frac{1}{N}$, there exists a positive time T such that system (1.4) has a solution (a, u) with $1 + a$ bounded away from zero and:

$$a \in \widetilde{C}([0,T], B^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_1,\infty}), \quad u \in \left(\widetilde{C}([0,T]; B^{\frac{N}{p_2}-1}_{p_2,r}) \cap \widetilde{L}^1(0,T, B^{\frac{N}{p_2}+1}_{p_2,r})\right)^N$$
$$\textit{and} \ \nabla \Pi \in \widetilde{L}^1(0,T, B^{\frac{N}{p_2}-1}_{p_2,r}).$$

This solution is unique when $\frac{2}{N} \le \frac{1}{p_1} + \frac{1}{p_2}$.

Remark 1 *This theorem can be applied for non degenerate viscosity, but for the simplicity in the sequel the viscosity is assumed to be a constant $\mu > 0$.*

Remark 2 *As in the work of H. Abidi and M. Paicu in [2], we are able to get strong solution when $u_0 \in B^{\frac{N}{p_2}-1}_{p_2,r}$ with $1 < p_2 \leq 2N$, it improves the result of R. Danchin in [16] and [17].*
Moreover we get weak solution with initial data very close from $(a_0, u_0) \in B^{1+\varepsilon}_{N,\infty} \times B^{-1}_{\infty,r}$ and $(a_0, u_0) \in B^{\varepsilon}_{\infty,\infty} \times B^{0}_{N,r}$. It means that in the first case we are not very far from the Koch-Tataru in [26] initial data for the velocity u_0, and in the second case we are close to just impose a condition of norm L^∞ on a_0 which is of great interest for the system bifluid.

Remark 3 *In the previous theorem, we need of a condition of smallness, because when $p_2 \neq 2$, we have extra term in our proposition 3.5 which requires a condition of smallness on a. It seems that there is a specific structure in the scalar case when $p_2 = 2$.*

In the following theorem, we improve the previous result in the specific case where $p_2 = 2$. In this case we don't need to impose condition of smallness on the initial data.

Theorem 1.2 *Let $1 \leq r < \infty$, $1 \leq p_1 < \infty$ and $\varepsilon > 0$ such that:*

$$\frac{N}{p_1} + \varepsilon < \frac{N}{2} + 1 \quad \textit{and} \quad \frac{N}{2} \leq 1 + \frac{N}{p_1}.$$

Assume that $u_0 \in B^{\frac{N}{2}-1}_{2,r}$ with $\mathrm{div} u_0 = 0$, $f \in \widetilde{L}^1_{loc}(\mathbb{R}^+, B^{\frac{N}{2}-1}_{2,r})$ and $a_0 \in B^{\frac{N}{p_1}+\varepsilon}_{p_1,\infty} \cap L^\infty$, with $1 + a_0$ bounded away from zero. There exists a positive time T such that system (1.4) has a solution (a, u) with $1 + a$ bounded away from zero and:

$$a \in \widetilde{C}([0,T], B^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_1,\infty}), \quad u \in \big(\widetilde{C}([0,T]; B^{\frac{N}{2}-1}_{2,r}) \cap \widetilde{L}^1(0,T, B^{\frac{N}{2}+1}_{2,r})\big)^N$$
$$\textit{and} \ \nabla \Pi \in \widetilde{L}^1(0,T, B^{\frac{N}{2}-1}_{2,r}).$$

This solution is unique when $\frac{2}{N} \leq \frac{1}{p_1} + \frac{1}{2}$.

The key of the theorems 1.1 and 1.2is new estimate for transport equation for u which is not considered Lipschitz. In this case we have to pay a loss of regularity on the density. ρ. The basic idea is to deal with this loss of regularity by conserving a in $\widetilde{C}_T(B^{\frac{N}{p_1}}_{p_1,+\infty}) \cap L^\infty$ which have good properties of multiplier. Moreover to avoid as in the work s of H. Abidi, R. Danchin and M. Paicu conditions of smallness on the initial density, the basic idea is that having coefficients in a in $\widetilde{C}_T(B^{\frac{N}{p_1}}_{p_1,+\infty})$ provides us with some uniform decay on the high frequencies of the variable coefficients so that the elliptic operator may be considered as an operator with *smooth* coefficients (of the type which has been investigated in [15]) plus a small error term.
Our paper is structured as follows. In the section 2, we give a few notation and briefly introduce the basic Fourier analysis techniques needed to prove our result. Section 3 and 4 are devoted to the proof of key estimates for the linearized system (1.4) in particular the

elliptic operator of the momentum equation with variable coefficients and the transport equation when the velocity is not assumed Lipschitz. In section 5, we prove the existence of solution for theorem 1.1 and 1.2 whereas section 6 is devoted to the proof of uniqueness. Elliptic and technical estimates commutator are postponed in an appendix.

2 Littlewood-Paley theory and Besov spaces

Throughout the paper, C stands for a constant whose exact meaning depends on the context. The notation $A \lesssim B$ means that $A \leq CB$. For all Banach space X, we denote by $C([0,T],X)$ the set of continuous functions on $[0,T]$ with values in X. For $p \in [1,+\infty]$, the notation $L^p(0,T,X)$ or $L^p_T(X)$ stands for the set of measurable functions on $(0,T)$ with values in X such that $t \to \|f(t)\|_X$ belongs to $L^p(0,T)$.

2.1 Littlewood-Paley decomposition

Littlewood-Paley decomposition corresponds to a dyadic decomposition of the space in Fourier variables. Let $\alpha > 1$ and (φ, χ) be a couple of smooth functions valued in $[0,1]$, such that φ is supported in the shell supported in $\{\xi \in \mathbb{R}^N / \alpha^{-1} \leq |\xi| \leq 2\alpha\}$, χ is supported in the ball $\{\xi \in \mathbb{R}^N / |\xi| \leq \alpha\}$ such that:

$$\forall \xi \in \mathbb{R}^N, \quad \chi(\xi) + \sum_{l \in \mathbb{N}} \varphi(2^{-l}\xi) = 1.$$

Denoting $h = \mathcal{F}^{-1}\varphi$, we then define the dyadic blocks by:

$$\begin{aligned}
&\Delta_l u = 0 \quad \text{if } \ l \leq -2, \\
&\Delta_{-1} u = \chi(D)u = \widetilde{h} * u \quad \text{with } \ \widetilde{h} = \mathcal{F}^{-1}\chi, \\
&\Delta_l u = \varphi(2^{-l}D)u = 2^{lN} \int_{\mathbb{R}^N} h(2^l y) u(x-y) dy \quad \text{with } \ h = \mathcal{F}^{-1}\chi, \ \text{ if } \ l \geq 0, \\
&S_l u = \sum_{k \leq l-1} \Delta_k u \,.
\end{aligned}$$

Formally, one can write that: $u = \sum_{k \in \mathbb{Z}} \Delta_k u$. This decomposition is called nonhomogeneous Littlewood-Paley decomposition.

2.2 Nonhomogeneous Besov spaces and first properties

Definition 2.1 *For $s \in \mathbb{R}$, $p \in [1,+\infty]$, $q \in [1,+\infty]$, and $u \in \mathcal{S}'(\mathbb{R}^N)$ we set:*

$$\|u\|_{B^s_{p,q}} = (\sum_{l \in \mathbb{Z}} (2^{ls} \|\Delta_l u\|_{L^p})^q)^{\frac{1}{q}}.$$

The Besov space $B^s_{p,q}$ is the set of temperate distribution u such that $\|u\|_{B^s_{p,q}} < +\infty$.

Remark 4 *The above definition is a natural generalization of the nonhomogeneous Sobolev and Hölder spaces: one can show that $B^s_{\infty,\infty}$ is the nonhomogeneous Hölder space C^s and that $B^s_{2,2}$ is the nonhomogeneous space H^s.*

Proposition 2.1 *The following properties holds:*

1. *there exists a constant universal C such that:*
   $C^{-1}\|u\|_{B^s_{p,r}} \leq \|\nabla u\|_{B^{s-1}_{p,r}} \leq C\|u\|_{B^s_{p,r}}$.

2. *If $p_1 < p_2$ and $r_1 \leq r_2$ then $B^s_{p_1,r_1} \hookrightarrow B^{s-N(1/p_1-1/p_2)}_{p_2,r_2}$.*

3. $(B^{s_1}_{p,r}, B^{s_2}_{p,r})_{\theta,r'} = B^{\theta s_1+(1-\theta)s_2}_{p,r'}$.

Let now recall a few product laws in Besov spaces coming directly from the paradifferential calculus of J-M. Bony (see [6]) and rewrite on a generalized form in [2] by H. Abidi and M. Paicu (in this article the results are written in the case of homogeneous spaces but it can easily generalize for the nonhomogeneous Besov spaces).

Proposition 2.2 *We have the following laws of product:*

- *For all $s \in \mathbb{R}$, $(p,r) \in [1,+\infty]^2$ we have:*

$$\|uv\|_{\widetilde{B}^s_{p,r}} \leq C(\|u\|_{L^\infty}\|v\|_{B^s_{p,r}} + \|v\|_{L^\infty}\|u\|_{B^s_{p,r}}). \tag{2.5}$$

- *Let $(p,p_1,p_2,r,\lambda_1,\lambda_2) \in [1,+\infty]^2$ such that:$\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}$, $p_1 \leq \lambda_2$, $p_2 \leq \lambda_1$, $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{\lambda_1}$ and $\frac{1}{p} \leq \frac{1}{p_2} + \frac{1}{\lambda_2}$. We have then the following inequalities:*
  if $s_1 + s_2 + N\inf(0, 1 - \frac{1}{p_1} - \frac{1}{p_2}) > 0$, $s_1 + \frac{N}{\lambda_2} < \frac{N}{p_1}$ and $s_2 + \frac{N}{\lambda_1} < \frac{N}{p_2}$ then:

$$\|uv\|_{B^{s_1+s_2-N(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})}_{p,r}} \lesssim \|u\|_{B^{s_1}_{p_1,r}}\|v\|_{B^{s_2}_{p_2,\infty}}, \tag{2.6}$$

  when $s_1 + \frac{N}{\lambda_2} = \frac{N}{p_1}$ (resp $s_2 + \frac{N}{\lambda_1} = \frac{N}{p_2}$) we replace $\|u\|_{B^{s_1}_{p_1,r}}\|v\|_{B^{s_2}_{p_2,\infty}}$ (resp $\|v\|_{B^{s_2}_{p_2,\infty}}$) by $\|u\|_{B^{s_1}_{p_1,1}}\|v\|_{B^{s_2}_{p_2,r}}$ (resp $\|v\|_{B^{s_2}_{p_2,\infty}\cap L^\infty}$), if $s_1 + \frac{N}{\lambda_2} = \frac{N}{p_1}$ and $s_2 + \frac{N}{\lambda_1} = \frac{N}{p_2}$ we take $r = 1$.
  If $s_1 + s_2 = 0$, $s_1 \in (\frac{N}{\lambda_1} - \frac{N}{p_2}, \frac{N}{p_1} - \frac{N}{\lambda_2}]$ and $\frac{1}{p_1} + \frac{1}{p_2} \leq 1$ then:

$$\|uv\|_{B^{-N(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})}_{p,\infty}} \lesssim \|u\|_{B^{s_1}_{p_1,1}}\|v\|_{B^{s_2}_{p_2,\infty}}. \tag{2.7}$$

  If $|s| < \frac{N}{p}$ for $p \geq 2$ and $-\frac{N}{p'} < s < \frac{N}{p}$ else, we have:

$$\|uv\|_{B^s_{p,r}} \leq C\|u\|_{B^s_{p,r}}\|v\|_{B^{\frac{N}{p}}_{p,\infty}\cap L^\infty}. \tag{2.8}$$

Remark 5 *In the sequel p will be either p_1 or p_2 and in this case $\frac{1}{\lambda} = \frac{1}{p_1} - \frac{1}{p_2}$ if $p_1 \leq p_2$, resp $\frac{1}{\lambda} = \frac{1}{p_2} - \frac{1}{p_1}$ if $p_2 \leq p_1$.*

Corollary 1 *Let $r \in [1,+\infty]$, $1 \leq p \leq p_1 \leq +\infty$ and s such that:*

- $s \in (-\frac{N}{p_1}, \frac{N}{p_1})$ *if* $\frac{1}{p} + \frac{1}{p_1} \leq 1$,

- $s \in (-\frac{N}{p_1} + N(\frac{1}{p} + \frac{1}{p_1} - 1), \frac{N}{p_1})$ *if* $\frac{1}{p} + \frac{1}{p_1} > 1$,

then we have if $u \in B^s_{p,r}$ *and* $v \in B^{\frac{N}{p_1}}_{p_1,\infty} \cap L^\infty$*:*

$$\|uv\|_{B^s_{p,r}} \leq C\|u\|_{B^s_{p,r}}\|v\|_{B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty}.$$

The study of non stationary PDE's requires space of type $L^\rho(0,T,X)$ for appropriate Banach spaces X. In our case, we expect X to be a Besov space, so that it is natural to localize the equation through Littlewood-Payley decomposition. But, in doing so, we obtain bounds in spaces which are not type $L^\rho(0,T,X)$ (except if $r=p$). We are now going to define the spaces of Chemin-Lerner in which we will work, which are a refinement of the spaces $L^\rho_T(B^s_{p,r})$.

Definition 2.2 *Let* $\rho \in [1,+\infty]$, $T \in [1,+\infty]$ *and* $s_1 \in \mathbb{R}$. *We set:*

$$\|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})} = \Big(\sum_{l\in\mathbb{Z}} 2^{lrs_1}\|\Delta_l u(t)\|^r_{L^\rho(L^p)}\Big)^{\frac{1}{r}}.$$

We then define the space $\widetilde{L}^\rho_T(B^{s_1}_{p,r})$ *as the set of temperate distribution* u *over* $(0,T)\times\mathbb{R}^N$ *such that* $\|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})} < +\infty$.

We set $\widetilde{C}_T(\widetilde{B}^{s_1}_{p,r}) = \widetilde{L}^\infty_T(\widetilde{B}^{s_1}_{p,r}) \cap \mathcal{C}([0,T],B^{s_1}_{p,r})$. Let us emphasize that, according to Minkowski inequality, we have:

$$\|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})} \leq \|u\|_{L^\rho_T(B^{s_1}_{p,r})} \text{ if } r \geq \rho, \quad \|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})} \geq \|u\|_{L^\rho_T(B^{s_1}_{p,r})} \text{ if } r \leq \rho.$$

Remark 6 *It is easy to generalize proposition 2.2, to* $\widetilde{L}^\rho_T(B^{s_1}_{p,r})$ *spaces. The indices* s_1, p, r *behave just as in the stationary case whereas the time exponent* ρ *behaves according to Hölder inequality.*

Here we recall a result of interpolation which explains the link of the space $B^s_{p,1}$ with the space $B^s_{p,\infty}$, see [11].

Proposition 2.3 *There exists a constant* C *such that for all* $s \in \mathbb{R}$, $\varepsilon > 0$ *and* $1 \leq p < +\infty$,

$$\|u\|_{\widetilde{L}^\rho_T(B^s_{p,1})} \leq C\frac{1+\varepsilon}{\varepsilon}\|u\|_{\widetilde{L}^\rho_T(B^s_{p,\infty})}\left(1+\log\frac{\|u\|_{\widetilde{L}^\rho_T(B^{s+\varepsilon}_{p,\infty})}}{\|u\|_{\widetilde{L}^\rho_T(B^s_{p,\infty})}}\right).$$

Definition 2.3 *Let* Γ *be an increasing function on* $[1,+\infty[$. *We denote by* $B_\Gamma(\mathbb{R}^N)$ *the set of bounded real valued functions* u *over* $\mathbb{R}^N$ *such that:*

$$\|u\|_{B_\Gamma} = \|u\|_{L^\infty} + \sup_{j\geq 0}\frac{\|\nabla S_j u\|_{L^\infty}}{\Gamma(2^j)} < +\infty.$$

We give here a proposition concerning these spaces showed by J-Y. Chemin, see [5].

Proposition 2.4 *Let* $\varepsilon > 0$ *and* $u \in \widetilde{L}^1_T(B^{\frac{N}{p}+1}_{p,r})$ *then we have* $u \in L^1_T(B_\Gamma(\mathbb{R}^N))$ *with* $\Gamma(t) = (-\log t)^{1+\varepsilon-\frac{1}{r}}$ *for* $0 \leq t \leq 1$.

3 Estimates for parabolic system with variable coefficients

In this section, the following linearization of the momentum equation is studied:

$$\begin{cases} \partial_t u + b(\nabla\Pi - \mu\Delta u) = f + g, \\ \mathrm{div} u = 0, \\ u_{/t=0} = u_0. \end{cases} \tag{3.9}$$

where b, f, g and u_0 are given. Above u is the unknown function. We assume that $u_0 \in B^s_{p,r}$ and $f \in \widetilde{L}^1(0,T;B^s_{p,r})$, that b is bounded by below by a positive constant $\underline{b}$ and that $a = b - 1$ belongs to $\widetilde{L}^\infty(0,T;B^{\frac{N}{p_1}+\alpha}_{p_1,\infty}) \cap L^\infty$. In the present subsection, we aim at proving a priori estimates for (3.9) in the framework of nonhomogeneous Besov spaces. Before stating our results let us introduce the following notation:

$$\mathcal{A}_T = 1 + \underline{b}^{-1}\|\nabla b\|_{\widetilde{L}^\infty(B^{\frac{N}{p_1}+\alpha-1}_{p_1,\infty})} \quad \text{with } \alpha > 0. \tag{3.10}$$

Proposition 3.5 *Let $\underline{\nu} = \underline{b}\mu$ and $(p,p_1) \in [1,+\infty]$.*

- *If $p_1 > p$ we assume that $s \in (-\frac{N}{p_1}, \frac{N}{p_1})$ if $\frac{1}{p}+\frac{1}{p_1} \leq 1$ and $s \in (-\frac{N}{p_1}+N(\frac{1}{p}+\frac{1}{p_1}-1), \frac{N}{p_1})$ if $\frac{1}{p}+\frac{1}{p_1} > 1$.*
- *If $p_1 \leq p$ then we suppose that $s \in (-\frac{N}{p}, \frac{N}{p})$ if $p \geq 2$ and $s \in (-\frac{N}{p'}, \frac{N}{p})$ if $p < 2$.*

If $p \neq 2$ we need to assume than there exists $c > 0$ such that:

$$\|a\|_{\widetilde{L}^\infty(B^{\frac{N}{p_1}+\alpha-1}_{p_1,\infty})} \leq c.$$

Let $m \in \mathbb{Z}$ be such that $b_m = 1 + S_m a$ satisfies:

$$\inf_{(t,x)\in[0,T)\times\mathbb{R}^N} b_m(t,x) \geq \frac{\underline{b}}{2}. \tag{3.11}$$

There exist three constants c, C and κ (with c, C, depending only on N and on s, and κ universal) such that if in addition we have:

$$\|a - S_m a\|_{\widetilde{L}^\infty(0,T;B^{\frac{N}{p_1}}_{p_1,\infty})\cap L^\infty} \leq c\frac{\underline{\nu}}{\mu} \tag{3.12}$$

then setting:

$$Z_m(t) = 2^{2m\alpha}\mu^2\underline{\nu}^{-1}\int_0^t \|a\|^2_{B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty} d\tau,$$

Let $\alpha' > 0$ checking $\alpha' \leq \min(1,\alpha,\frac{s-2+\frac{2}{m}}{2})$. We have for all $t \in [0,T]$ and $\kappa = \frac{s}{\alpha'}$:

$$\begin{aligned} \|u\|_{\widetilde{L}^\infty_T(B^s_{p,r})} + \kappa\underline{\nu}\|u\|_{\widetilde{L}^1_T(B^{s+2}_{p,r})} \leq e^{CZ_m(T)}\Big(&\|u_0\|_{B^s_{p,r}} + \mathcal{A}^\kappa_T(\|\mathcal{P}f\|_{\widetilde{L}^1_T(B^s_{p,r})} \\ &+ \underline{\mu}^{\frac{1}{m}}\|\mathcal{P}g\|_{\widetilde{L}^m_T(B^{s-2+\frac{2}{m}}_{p,r})} + \underline{\mu}^{\frac{1}{m}}(\frac{\underline{\nu}(p-1)}{p})\mathcal{A}_T\|u\|_{\widetilde{L}^1_T(B^{s+2-\alpha'}_{p,r})})\Big). \end{aligned} \tag{3.13}$$

Moreover we have $\nabla\Pi = \nabla\Pi_1 + \nabla\Pi_2$ *with:*

$$\begin{aligned}
&\underline{b}\|\nabla\Pi_1\|_{\widetilde{L}^1_T(B^s_{p,r})} \leq \mathcal{A}^\kappa_T\|\mathcal{P}f\|_{\widetilde{L}^1_T(B^s_{p,r})},\\
&\underline{b}\|\nabla\Pi_2\|_{\widetilde{L}^1_T(B^s_{p,r})} \leq \mathcal{A}^\kappa_T\big(\|\mathcal{Q}g\|_{\widetilde{L}^m_T(B^{s-2+\frac{2}{m}}_{p,r})} + \mu\|a\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p_1}+\alpha}_{p_1,+\infty})}\|\Delta u\|_{\widetilde{L}^m_T(B^{s-2+\frac{2}{m}}_{p,r})}\big).
\end{aligned} \tag{3.14}$$

Remark 7 *Let us stress the fact that if* $a \in \widetilde{L}^\infty((0,T)\times B^{\frac{N}{p_1}}_{p_1,\infty})$ *then assumption (3.11) and (3.12) are satisfied for* m *large enough. This will be used in the proof of theorem 1.2 and* **??**. *Indeed, according to Bernstein inequality for* m *large enough 93.11) and (3.12) are satisfied.*

Proving proposition 3.5 in the case $b = cste$ is not too involved as one can easily get rid of the pressure by taking advantage of the Leray projector $\mathcal{P}$ on solenoidal vector-fields. Then system (3.9) reduce to a linear ψDO which may be easily solved by mean of energy estimates. In our case where b is not assumed to be a constant, getting rid of the pressure will still be an appropriate strategy. This may be achieved by applying the operator div to (3.9). Indeed by doing so, we see that the pressure solves the elliptic equation:

$$\mathrm{div}(b\nabla\Pi) = \mathrm{div}F. \tag{3.15}$$

with $F = f + g + \mu a\Delta u$. Therefore denoting by $\mathcal{H}_b$ the linear operator $F \to \nabla\Pi$, system (3.9) reduces to a linear ODE in Banach spaces. Actually, due to the consideration of two forcing terms f and g with different regularities, the pressure has to be split into two parts, namely $\Pi = \Pi_1 + \Pi_2$ with:

$$\mathrm{div}(b\Pi_1) = \mathrm{div}f \tag{3.16}$$

$$\mathrm{div}(b\Pi_2) = \mathrm{div}H \ \text{ and } \ H = g + \mu a\Delta u. \tag{3.17}$$

Proof of proposition 3.5:

Let us first rewrite (3.9) as follows:

$$\begin{cases}
\partial_t u - b_m\mu\Delta u + b\nabla\Pi = f + g + E_m,\\
\mathrm{div}u = 0,\\
u_{t=0} = u_0.
\end{cases} \tag{3.18}$$

with $E_m = \mu\Delta u\,(\mathrm{Id} - S_m)a$ and $b_m = 1 + S_m a$. Note that by using corollary 1 and as $-\frac{N}{p_1} < s < \frac{N}{p_1}$ for $p \geq 2$ or $\frac{N}{p_1'} < s < \frac{N}{p_1}$ else, the error term E_m may be estimated by:

$$\|E_m\|_{B^s_{p,r}} \lesssim \|a - S_m a\|_{B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty}\|D^2u\|_{B^s_{p,r}}. \tag{3.19}$$

Now applying operator Δ_q and next operator of free divergence yield $\mathcal{P}$ to momentum equation (3.18) yields:

$$\frac{d}{dt}u_q - \mu\mathrm{div}(b_m\nabla u_q) = \mathcal{P}f_q + \mathcal{P}g_q + \Delta_q\mathcal{P}E_m + \widetilde{R}_q - \Delta_q\mathcal{P}(a\nabla\Pi), \tag{3.20}$$

where we denote by $u_q = \Delta_q u$ and with:

$$\widetilde{R}_q = \widetilde{R}_q^1 + \widetilde{R}_q^2.$$

where:

$$\begin{aligned}\widetilde{R}_q^1 &= \mu\big(\mathcal{P}\Delta_q(b_m\Delta u) - \mathcal{P}\mathrm{div}(b_m\nabla u_q)\big),\\ \widetilde{R}_q^2 &= \mu\big(\mathcal{P}\mathrm{div}(b_m\nabla u_q) - \mathrm{div}(b_m\nabla u_q)\big) = -\mu\mathcal{Q}\mathrm{div}(S_m a\nabla u_q).\end{aligned}$$

where $\mathcal{Q}$ is the gradient yield projector.

case $p \neq 2$

Next multiplying both sides by $|u_q|^{p-2}u_q$, and integrating by parts in the second, third and last term in the left-hand side, we get by using Bony decomposition (for the notation see [11]):

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\|u_q\|_{L^p}^p + \mu\int_{\mathbb{R}^N} b_m|\nabla u_q|^2|u_q|^{p-2}dx + \mu\int_{\mathbb{R}^N} b_m|u_q|^{p-4}|\nabla|u|^2|^2dx\\ &\leq \|u_q\|_{L^p}^{p-1}(\|\mathcal{P}f_q\|_{L^p} + \|\mathcal{P}g_q\|_{L^p} + \|\widetilde{R}_q\|_{L^p} + \|\Delta_q(T_{\nabla a}\Pi)\|_{L^p} + 2^q\|\Delta_q(T_a\Pi)\|_{L^p}\\ &\qquad + \|\Delta_q(T'_{\nabla\Pi}a)\|_{L^p} + \|\mathcal{P}\Delta_q E_m\|_{L^p}).\end{aligned} \tag{3.21}$$

Indeed we have as $\mathrm{div}u = 0$ and by using Bony's decomposition and by performing an integration by parts:

$$\begin{aligned}\int_{\mathbb{R}^N}\Delta_q(a\nabla\Pi)|u_q|^{p-2}u_q &= \int_{\mathbb{R}^N}\Delta_q(T'_{\nabla\Pi}a)|u_q|^{p-2}u_q dx - \int_{\mathbb{R}^N}\Delta_q(T_{\nabla a}\Pi)|u_q|^{p-2}u_q dx\\ &\quad - \int_{\mathbb{R}^N}\Delta_q(T_a\Pi)\mathrm{div}(|u_q|^{p-2}u_q)dx.\end{aligned}$$

Next we have:

$$\nabla(|u_q|^{p-2})\cdot u_q = (p-2)|u_q|^{p-4}\sum_{i,k} u_q^k\partial_i u_q^k u_q^i,$$

and by Hölder's and Berstein's inequalities:

$$\|\nabla(|u_q|^{p-2})\cdot u_q\|_{L^{\frac{p}{p-1}}} \leq C(p-2)2^q\|u_q\|_{L^p}^{p-1}.$$

Next from inequality (3.21), we get by using lemma A5 in [16]:

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|u_q\|_{L^p}^p + \frac{\underline{\nu}(p-1)}{p^2}2^{2q}\|u_q\|_{L^p}^p &\leq \|u_q\|_{L^p}^{p-1}\big(\|\mathcal{P}f_q\|_{L^p} + \|\mathcal{P}g_q\|_{L^p} + \|\mathcal{P}\Delta_q E_m\|_{L^p}\\ &\quad + \|\Delta_q(T_{\nabla a}\Pi)\|_{L^p} + 2^q\|\Delta_q(T_a\Pi)\|_{L^p} + \|\Delta_q(T'_{\nabla\Pi}a)\|_{L^p} + \|\widetilde{R}_q\|_{L^p}\big),\end{aligned}$$

Therefore, elementary computation yield (at least formally):

$$\begin{aligned}e^{-\frac{\underline{\nu}(p-1)}{p^2}2^{2q}t}\frac{d}{dt}\big(e^{\frac{\underline{\nu}(p-1)}{p^2}2^{2q}t}\|u_q\|_{L^p}\big) &\lesssim \|\mathcal{P}f_q\|_{L^p} + \|\mathcal{P}g_q\|_{L^p} + \|\mathcal{P}\Delta_q E_m\|_{L^p}\\ &\quad + \|\Delta_q(T_{\nabla a}\Pi)\|_{L^p} + 2^q\|\Delta_q(T_a\Pi)\|_{L^p} + \|\Delta_q(T'_{\nabla\Pi}a)\|_{L^p} + \|\widetilde{R}_q\|_{L^p}.\end{aligned}$$

We thus have:

$$\|u_q(t)\|_{L^p} \lesssim e^{-\frac{\underline{\nu}(p-1)}{p^2}2^{2q}t}\|\Delta_q u_0\|_{L^p} + \int_0^t e^{-\frac{\underline{\nu}(p-1)}{p^2}2^{2q}(t-\tau)}\big(\|\mathcal{P}f_q\|_{L^p} + \|\mathcal{P}g_q\|_{L^p} +$$
$$\|\mathcal{P}\Delta_q E_m\|_{L^p} + \|\Delta_q(T_{\nabla a}\Pi)\|_{L^p} + 2^q\|\Delta_q(T_a\Pi)\|_{L^p} + \|\Delta_q(T'_{\nabla\Pi}a)\|_{L^p} + \|\widetilde{R}_q\|_{L^p}\big)(\tau)d\tau,$$

which leads for all $q \geq -1$, after performing a time integration and using convolution inequalities to:

$$\begin{aligned}
&(\frac{\underline{\nu}(p-1)}{p^2})^{\frac{1}{m}}2^{\frac{2q}{m}}\|u_q\|_{L^m_T(L^p)} \lesssim \|\Delta_q u_0\|_{L^p} + \|\mathcal{P}f_q\|_{L^1_T(L^p)} + \|\Delta_q(T_{\nabla a}\Pi_1)\|_{L^1_T(L^p)} \\
&+ 2^q\|\Delta_q(T_a\Pi_1)\|_{L^1_T(L^p)} + \|\Delta_q(T'_{\nabla\Pi_1}a)\|_{L^1_T(L^p)} + \|\widetilde{R}_q\|_{L^1_T(L^p)} + \|\mathcal{P}\Delta_q E_m\|_{L^1_T(L^p)} \\
&\quad + (\frac{\underline{\nu}(p-1)}{p^2})^{\frac{1}{m}-1}2^{q(\frac{2}{m}-2)}\big(\|\Delta_q(T_{\nabla a}\Pi_2)\|_{L^1_T(L^p)} + 2^q\|\Delta_q(T_a\Pi_2)\|_{L^1_T(L^p)} \\
&\qquad + \|\Delta_q(T'_{\nabla\Pi_2}a)\|_{L^1_T(L^p)} + \|\mathcal{P}g_q\|_{L^m_T(L^p)}\big),
\end{aligned} \tag{3.22}$$

We are now interested by treating the commutator term $\widetilde{R}^1_q$, we have then by using lemma 1 in the appendix the following estimates with $\alpha < 1$:

$$\|\widetilde{R}^1_q\|_{L^p} \lesssim c_q\bar{\nu}2^{(-1+\alpha)qs}\|S_m a\|_{B^{\frac{N}{p_1}+\alpha}_{p_1,\infty}}\|Du\|_{B^s_{p,r}}, \tag{3.23}$$

where $(c_q)_{q\in\mathbb{Z}}$ is a positive sequence such that $c_q \in l^r$, and $\bar{\nu} = \mu$. Note that, owing to Bernstein inequality, we have:

$$\|S_m a\|_{B^{\frac{N}{p_1}+\alpha}_{p_1,\infty}} \lesssim 2^{m\alpha}\|a\|_{B^{\frac{N}{p_1}}_{p_1,\infty}}$$

Next we have by corollary 1:

$$\|\widetilde{R}^2_q\|_{L^p} \lesssim c_q\bar{\nu}2^{-qs}\|S_m a\|_{B^{\frac{N}{p_1}}_{p_1,r}\cap L^\infty}\|u\|_{B^{s+2}_{p,r}}, \tag{3.24}$$

Hence, plugging (3.23), (3.24) and (3.19) in (3.22), then multiplying by 2^{qs} and summing up on $q \in \mathbb{Z}$ in l^r, we discover that, for all $t \in [0,T]$:

$$\begin{aligned}
&\|u\|_{\widetilde{L}^\infty_T(B^s_{p,r})} + (\frac{\underline{\nu}(p-1)}{p})^{\frac{1}{m}}\|u\|_{\widetilde{L}^m_T(B^{s+\frac{2}{m}}_{p,r})} \leq \|u_0\|_{B^s_{p,r}} + \|\mathcal{P}f\|_{\widetilde{L}^1_T(B^s_{p,r})} \\
&\qquad + \|T_a\Pi_1\|_{\widetilde{L}^1_T(B^{s+1}_{p,r})} + \|T_{\nabla a}\Pi_1\|_{\widetilde{L}^1_T(B^s_{p,r})} + \|T'_{\nabla\Pi_1}a\|_{\widetilde{L}^1_T(B^s_{p,r})} \\
&+ C\bar{\nu}\|a - S_m a\|_{\widetilde{L}^\infty(B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty)}\|u\|_{\widetilde{L}^1(B^{s+2}_{p,r})} + 2^{m\alpha}\int_0^T \|a\|_{B^{\frac{N}{p_1}}_{p_1,\infty}}(\tau)\|u\|_{B^{s+1}_{p,r}}(\tau)d\tau \\
&+ \big(\frac{\underline{\nu}(p-1)}{p^2}\big)^{\frac{1}{m}}(\|\mathcal{P}g\|_{\widetilde{L}^m_T(B^{s-2+\frac{2}{m}}_{p,r})} + \|T_a\Pi_2\|_{\widetilde{L}^m_T(B^{s-1+\frac{2}{m}}_{p,r})} + \|T_{\nabla a}\Pi_2\|_{\widetilde{L}^m_T(B^{s-2+\frac{2}{m}}_{p,r})} \\
&\qquad + \|T'_{\nabla\Pi_2}a\|_{\widetilde{L}^m_T(B^{s-2+\frac{2}{m}}_{p,r})}\big),
\end{aligned} \tag{3.25}$$

for a constant C depending only on N and s. With our assumption on α, α' and s, the terms $\|T_{\nabla a}\Pi_1\|_{\widetilde{L}^1_T(B^s_{p,r})}$ and $\|T'_{\nabla\Pi_1}a\|_{\widetilde{L}^1_T(B^s_{p,r})}$ may be bounded by:

$$\|a\|_{\widetilde{L}^\infty(B^{\frac{N}{p_1}+\alpha}_{p,\infty}\cap L^\infty)}\|\nabla\Pi_1\|_{\widetilde{L}^1_T(B^{s-\alpha'}_{p,r})}$$

whereas $\|T_{\nabla a}\Pi_2\|_{\widetilde{L}^m_T(B^{s-2+\frac{2}{m}}_{p,r})}$ and $\|T'_{\nabla\Pi_2}a\|_{\widetilde{L}^m_T(B^{s-2+\frac{2}{m}}_{p,r})}$ may be bounded by:

$$\|a\|_{\widetilde{L}^\infty(B^{\frac{N}{p_1}+\alpha}_{p,\infty}\cap L^\infty)}\|\nabla\Pi_2\|_{\widetilde{L}^m_T(B^{s-2+\frac{2}{m}-\alpha'}_{p,r})}.$$

Moreover we control $\|T_a\Pi_1\|_{\widetilde{L}^1_T(B^{s+1}_{p,r})}$ and $\|T_a\Pi_2\|_{\widetilde{L}^m_T(B^{s-1+\frac{2}{m}}_{p,r})}$ by respectively:

$$\|a\|_{\widetilde{L}^\infty(B^{\frac{N}{p_1}}_{p,\infty}\cap L^\infty)}\|\nabla\Pi_1\|_{\widetilde{L}^1_T(B^s_{p,r})}.$$
$$\|a\|_{\widetilde{L}^\infty(B^{\frac{N}{p_1}}_{p,\infty}\cap L^\infty)}\|\nabla\Pi_2\|_{\widetilde{L}^m_T(B^{s-2+\frac{2}{m}}_{p,r})}.$$

Hence in view of proposition 7.8 and provided that $0<\alpha'<\min(1,\alpha,\frac{s}{2})$ and $s<\frac{N}{p_1}$ (which is assumed in the statement of proposition 3.5) and $\alpha''\in[0,\alpha']$,

$$\underline{b}\|\nabla\Pi_1\|_{\widetilde{L}^1_T(B^{s-\alpha''}_{p,r})}\lesssim\mathcal{A}_T^{\frac{s-\alpha''}{\alpha'}}\|\mathcal{Q}f\|_{\widetilde{L}^1(B^s_{p,r})}. \tag{3.26}$$

On the other hand, by virtue of proposition 2.2, and of assumption on α, α', α'' and s, we have:

$$\|\mathcal{Q}H\|_{\widetilde{L}^m_T(B^{s-2+\frac{2}{m}-\alpha''}_{p,r})}\lesssim\|\mathcal{Q}g\|_{\widetilde{L}^m_T(B^{s-2+\frac{2}{m}-\alpha''}_{p,r})}+\mu\|a\|_{\widetilde{L}^\infty(B^{\frac{N}{p_1}+\alpha}_{p_1,\infty}\cap L^\infty)}\|\Delta u\|_{\widetilde{L}^m_T(B^{s-2+\frac{2}{m}-\alpha''}_{p,r})}.$$

As $\alpha'\le\min(1,\alpha,\frac{1}{2}(s-2+\frac{2}{m}))$, proposition 7.8 with $\sigma=s-2+\frac{2}{m}-\alpha''$ (here comes $s>2-\frac{2}{m}$) applies, from which we get for all $\varepsilon>0$ ($\varepsilon=0$ does if $m\ge2$),

$$\begin{aligned}\underline{b}\|\nabla\Pi_2\|_{\widetilde{L}^m_T(B^{s-2+\frac{2}{m}-\alpha''}_{p,r})}\lesssim\mathcal{A}_T^{\frac{s-2+\frac{2}{m}+\varepsilon}{\alpha'}}\big(&\|\mathcal{Q}g\|_{\widetilde{L}^m_T(B^{s-2+\frac{2}{m}-\alpha''}_{p,r})}\\&+\mu\|a\|_{\widetilde{L}^\infty(B^{\frac{N}{p_1}+\alpha}_{p_1,\infty}\cap L^\infty)}\|\Delta u\|_{\widetilde{L}^m_T(B^{s-2+\frac{2}{m}-\alpha''}_{p,r})}\big).\end{aligned} \tag{3.27}$$

Let $X(t)=\|u\|_{L^\infty_t(B^s_{p,r})}+\nu\underline{b}\|u\|_{L^1_t(B^{s+2}_{p,r})}$. Assuming that m has been chosen so large as to satisfy:

$$C\bar{\nu}\|a-S_ma\|_{L^\infty_T(B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty)}\le\underline{\nu},$$

and by interpolation we get:

$$C\bar{\nu}2^{m\alpha}\|a\|_{B^{\frac{N}{p_1}}_{p_1,\infty}}\|u\|_{B^{s+2}_{p,r}}\le\kappa\underline{\nu}+\frac{C^2\bar{\nu}^2 2^{2m\alpha}}{4\kappa\underline{\nu}}\|a\|^2_{B^{\frac{N}{p_1}}_{p_1,\infty}}\|u\|_{B^s_{p,r}}, \tag{3.28}$$

Plugging (3.26), (3.27) and (3.28) in (3.25), we end up with:

$$\begin{aligned}X(T)\le\|u_0\|_{B^s_{p,r}}+\mathcal{A}_T^{\frac{s}{\alpha'}}\Bigg(&\|\mathcal{P}f\|_{\widetilde{L}^1_t(B^s_{p,r})}+\|\mathcal{P}g\|_{\widetilde{L}^m_t(B^{s-2+\frac{2}{m}}_{p,r})}+C\int_0^t\big(\frac{\bar{\nu}^2}{\underline{\nu}}2^{2m\alpha}\|a\|^2_{B^{\frac{N}{p_1}}_{p_1,\infty}}(\tau)\\&\times X(\tau)\big)d\tau+\big(\frac{\underline{\nu}(p-1)}{p}\big)^{\frac{1}{m}}\mathcal{A}_T\|u\|_{\widetilde{L}^1_T(B^{s+2-\alpha'}_{p,r})}\Bigg).\end{aligned}$$

Grönwall lemma then leads to the desired inequality.

Case $p=2$

In this case we don't need of condition of smallness on $\|a\|_{B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty}$, indeed the bad terms as $\widetilde{R}^2_q$ or $2^q\|\Delta_q(T_a\Pi)\|_{L^2}$ disapear in the integration by parts as $\mathrm{div}u_q=0$. So we can follow the same procedure and conclude. $\square$

4 The mass conservation equation

4.1 Losing estimates for transport equation

We now focus on the mass equation associated to (1.1):

$$\begin{cases}\partial_t a + v\cdot\nabla a = g,\\ a_{/t=0}=a_0.\end{cases}\tag{4.29}$$

We will precise in the sequel the regularity of a_0, v and g. In this section we intend to recall some result on transport equation associated to vector fields which are not Lipschitz with respect to the space variable. Since we still have in mind to get regularity theorems, those vector field cannot be to rough. In order to measure precisely the regularity of the vector field v, we shall introduce the following notation:

$$V'_{p_1,\alpha}(t)=\sup_{j\geq 0}\frac{2^{j\frac{N}{p_1}}\|\nabla S_j v(t)\|_{L^{p_1}}}{(j+1)^\alpha}<+\infty.\tag{4.30}$$

Let us remark that if $p_1=+\infty$ then $V'_{p_1,\alpha}$ is exactly the norm $\|B_\Gamma\|$ of definition 2.3.

4.1.1 Limited loss of regularity

In this section, we make the assumption that there exists some $\alpha\in]0,1[$ such that the function $V'_{p_1,\alpha}$ defined in (4.30) be locally integrable. We will show that in the case $\alpha=1$, then a linear loss of regularity may occur. In the theorem below, Bahouri, Chemin and Danchin show in [5] that if $\alpha\in]0,1[$ then the loss of regularity in the estimate is arbitrarily small.

Theorem 4.3 *Let (p,p_1) be in $[1,+\infty]^2$ such that $1\leq p\leq p_1$ and σ satisfying $\sigma>-1-N\min(\frac{1}{p_1},\frac{1}{p'})$. Assume that $\sigma<1+\frac{N}{p_1}$ and that $V'_{p_1,\alpha}\in]0,1[$ is in $L^1([0,T])$. Let $a_0\in B^\sigma_{p,\infty}$ and $g\in\widetilde{L}^1_T(B^\sigma_{p,\infty})$. Then the equation (4.29) has a unique solution $a\in C([0,T],\cap_{\sigma'<\sigma}B^{\sigma'}_{p,\infty})$ and the following estimate holds for all small enough ε:*

$$\|a\|_{\widetilde{L}^\infty_T(B^{\sigma-\varepsilon}_{p,\infty})}\leq C\big(\|a_0\|_{B^\sigma_{p,\infty}}+\|g\|_{\widetilde{L}^1_T(B^\sigma_{p,\infty})}\big)\exp\big(\frac{C}{\varepsilon^{\frac{\alpha}{1-\alpha}}}(V_{p_1,\alpha}(T))^{\frac{1}{1-\alpha}}\big),$$

where C depends only on α, p, p_1, σ and N.

In the following proposition, we are interested in showing a control of the high frequencies on the density when u is not Lipschitz. Indeed we recall that in the proposition 3.5 when $p=2$, we need to control the high frequencies of the density. In particular the following proposition is useful only in the case of theorem 1.2.

Proposition 4.6 *Let (p,p_1) be in $[1,+\infty]^2$ such that $1\leq p\leq p_1$ and σ satisfying $\sigma>-1-N\min(\frac{1}{p_1},\frac{1}{p'})$. Assume that $\sigma<1+\frac{N}{p_1}$ and that $V'_{p_1,\alpha}\in]0,1[$ is in $L^1([0,T])$. Let $a_0\in B^{\sigma}_{p,\infty}$ and $g\in\widetilde{L}^1_T(B^{\sigma}_{p,\infty})$, the equation (4.29) has a unique solution $a\in C([0,T],\cap_{\sigma'<\sigma}B^{\sigma'}_{p,\infty})$ and the following estimate holds for all small enough ε:*

$$\sum_{l\geq m}2^{(\sigma-\varepsilon)l}\|\Delta_l a(t')\|_{L^\infty(L^p)}\lesssim\sum_{l\geq m}(2^{\sigma l}\|\Delta_l a_0\|_{L^p})+C\eta^{\frac{\alpha}{1-\alpha}}\int_0^t V'_{p_1,\alpha}(t')$$
$$\times(\|a_0\|_{B^{\sigma}_{p,\infty}}+\|g\|_{\widetilde{L}^1_T(B^{\sigma}_{p,\infty})})\exp\big(\frac{C}{\varepsilon^{\frac{\alpha}{1-\alpha}}}(V_{p_1,\alpha}(t'))^{\frac{1}{1-\alpha}}\big)dt'.$$

where C depends only on α, p, p_1, σ and N.

Proof:

By using the proof of Bahouri, Chemin and Danchin in [5] one can write:

$$2^{(2+j)\sigma_t}\|\Delta_l a(t)\|_{L^p}\leq 2^{(2+l)\sigma}\|\Delta_l f_0\|_{L^p}+C(\frac{2C}{\eta\log 2})^{\frac{\alpha}{1-\alpha}}\int_0^t V'_{p_1,\alpha}(t')\|a(t')\|_{B^{\sigma_{t'}}_{p,\infty}}dt'. \quad (4.31)$$

Whence taking the supremum over $l\geq m$, we get

$$\sup_{t'\in[0,t]}\sup_{l\geq m}(2^{\sigma_{t'}l}\|\Delta_l a(t')\|_{L^p})\lesssim\sup_{l\geq m}(2^{\sigma l}\|\Delta_l a_0\|_{L^p})+C\eta^{\frac{\alpha}{1-\alpha}}\int_0^t V'_{p_1,\alpha}(t')\|a(t')\|_{B^{\sigma_{t'}}_{p,\infty}}dt'.$$

We apply Grönwall inequality in (4.31) and we insert in previous inequality:

$$\sup_{t'\in[0,t]}\sup_{l\geq m}(2^{\sigma_{t'}l}\|\Delta_l a(t')\|_{L^p})\lesssim\sup_{l\geq m}(2^{\sigma l}\|\Delta_l a_0\|_{L^p})+C\eta^{\frac{\alpha}{1-\alpha}}\|f_0\|_{B^{\sigma}_{p,\infty}}\int_0^t V'_{p_1,\alpha}(t')$$
$$\times\exp\big(\frac{C}{\varepsilon^{\frac{\alpha}{1-\alpha}}}(V_{p_1,\alpha}(t'))^{\frac{1}{1-\alpha}}\big)dt'.$$

which leads to the proposition. □

Remark 8 *In the sequel, we will use the theorem 4.3 and the proposition 4.6 when $p_1=\infty$ and $\alpha=1+\varepsilon'-\frac{1}{r}$. Indeed we will have u is in $\widetilde{L}^1(B^{\frac{N}{p_2}+1}_{p_2,r})$ (with $1\leq r<\infty$) and according proposition 2.4 and definition 2.3 we get:*

$$\int_0^t V'_{\infty,\alpha}(t')dt'\lesssim\|u\|_{\widetilde{L}^1(B^{\frac{N}{p_2}+1}_{p_2,r})}.$$

So it will allow to get estimates on the density with an arbitrarly small loss of regularity.

4.1.2 Linear loss of regularity in Besov spaces

Tis section is devoted to the estimates with linear loss of regularity. Remind that u is log-lipschitz only if only if there some constant C such that:

$$\|\nabla S_j\|_{L^\infty}\leq C(j+1)\quad\text{for all } j\geq -1.$$

So Bahouri, Chemin, Danchin show in [5] the following theorem.

Theorem 4.4 *Let $1 \leq p \leq p_1 \leq \infty$ and $s_1 \in \mathbb{R}$ satisfies $\sigma > -N\min(\frac{1}{p_1}, \frac{1}{p'})$. Let σ in $]s_1, 1+\frac{N}{p_1}[$ and v a vector field. There exists a constant C depending only on p, p_1, σ, s_1 and N such that for any $\lambda > C$, $T > 0$ and any nonnegative integrable function W over $[0,T]$ such that if $\sigma_T \geq s_1$ with:*

$$\sigma_t = \sigma - \lambda \int_0^t \big(V'_{p_1,1}(t') + W(t')\big)dt'$$

then the following property holds true.
Let $a_0 \in B^{\sigma}_{p,\infty}$ and $g = g_1 + g_2$ with, for all $t \in [0,T]$, $g_1(t) \in B^{\sigma_t}_{p,\infty}$ and:

$$\forall j \geq -1,\ \|\Delta_j g_2(t)\|_{L^p} \leq 2^{-j\sigma_t}(2+j)W(t)\|f(t)\|_{B^{\sigma_t}_{p,\infty}}.$$

Let $a \in \mathcal{C}([0,T]; B^{s_1}_{p,\infty})$ be a solution of (4.29). Then the following estimate holds:

$$\sup_{t\in[0,T]} \|a(t)\|_{B^{\sigma_t}_{p,\infty}} \leq \frac{\lambda}{\lambda - C}\big(\|a_0\|_{B^{\sigma}_{p,\infty}} + \int_0^T \|g_1(t)\|_{B^{\sigma}_{p,\infty}}dt\big).$$

In the following proposition, we generalize this result to the high frequencies of the density ρ.

Proposition 4.7 *Let $1 \leq p \leq p_1 \leq \infty$ and $s_1 \in \mathbb{R}$ satisfies $\sigma > -N\min(\frac{1}{p_1}, \frac{1}{p'})$. Let σ in $]s_1, 1+\frac{N}{p_1}[$ and v a vector field. There exists a constant C depending only on p, p_1, σ, s_1 and N such that for any $\lambda > C$, $T > 0$ and any nonnegative integrable function W over $[0,T]$ such that if $\sigma_T \geq s_1$ with:*

$$\sigma_t = \sigma - \lambda \int_0^t \big(V'_{p_1,1}(t') + W(t')\big)dt'$$

then the following property holds true.
Let $a_0 \in B^{\sigma}_{p,\infty}$ and $g = g_1 + g_2$ with, for all $t \in [0,T]$, $g_1(t) \in B^{\sigma_t}_{p,\infty}$ and:

$$\forall j \geq -1,\ \|\Delta_j g_2(t)\|_{L^p} \leq 2^{-j\sigma_t}(2+j)W(t)\|f(t)\|_{B^{\sigma_t}_{p,\infty}}.$$

Let $a \in \mathcal{C}([0,T]; B^{s_1}_{p,\infty})$ be a solution of (4.29). Then the following estimate holds:

$$\sup_{t\in 0,T} \sum_{l\geq m} 2^{l\sigma_t}\|\Delta_l a\|_{L^p} \leq \frac{\lambda}{\lambda - C}\big(\|a_0\|_{B^{\sigma}_{p,\infty}} + \int_0^T \|g_1(t)\|_{B^{\sigma}_{p,\infty}}dt\big).$$

Proof:

The proof follows the same line as in proposition 4.6. □

Remark 9 *In the sequel, we will use the theorem 4.4 and the proposition 4.7 when $p_1 = \infty$. Indeed we will have u in $\widetilde{L}^1(B^{\frac{N}{p_2}+1}_{p_2,\infty})$, then according proposition 2.1, ∇u is in $\widetilde{L}^1(B^0_{\infty,\infty})$so we obtain the following control:*

$$\int_0^t \sup_{l\geq 0}\big(\frac{\|\nabla S_l u(t',\cdot)\|_{L^\infty}}{l+1}\big)dt' \lesssim \|u\|_{\widetilde{L}^1(B^{\frac{N}{p_2}+1}_{p_2,\infty})}.$$

So it will allow to get estimates on the density with a small loss of regularity on a small time interval.

5 Proof of the existence for theorem 1.1

We use a standard scheme:

1. We smooth out the data and get a sequence of smooth solutions $(a^n, u^n)_{n\in\mathbb{N}}$ to (1.4) on a bounded interval $[0, T^n]$ which may depend on n.

2. We exhibit a positive lower bound T for T^n, and prove uniform estimates in the space:
$$E_T = \widetilde{C}_T(B^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_1,\infty}) \times \big(\widetilde{C}_T(B^{\frac{N}{p_2}-1}_{p_2,r}) \cap \widetilde{L}^1(B^{\frac{N}{p}+1}_{p_2,r}))\big)^N.$$
for the smooth solution (a^n, u^n).

3. We use compactness to prove that the sequence converges, up to extraction, to a solution of (1.4).

Construction of approximate solutions

We smooth on the data as follows:

$$a_0^n = S_n a_0, \quad u_0^n = S_n u_0 \quad \text{and} \quad f^n = S_n f.$$

Note that we have:

$$\forall l \in \mathbb{Z}, \quad \|\Delta_l a_0^n\|_{L^{p_1}} \leq \|\Delta_l a_0\|_{L^{p_1}} \quad \text{and} \quad \|a_0^n\|_{B^{\frac{N}{p_1}+\varepsilon'}_{p,\infty}} \leq \|a_0\|_{B^{\frac{N}{p_1}+\varepsilon'}_{p,\infty}},$$

and similar properties for u_0^n and f^n, a fact which will be used repeatedly during the next steps. Now, according [2], one can solve (1.4) with the smooth data (a_0^n, u_0^n, f^n). We get a solution (a^n, u^n) on a non trivial time interval $[0, T_n]$ such that:

$$\begin{aligned} a^n \in \widetilde{C}([0,T_n), H^{N+\varepsilon}), \quad u^n \in \mathcal{C}([0,T_n), H^N) \cap \widetilde{L}^1_{T_n}(H^{N+2}) \\ \text{and } \nabla\Pi^n \in \widetilde{L}^1_{T_n}(H^N). \end{aligned} \tag{5.32}$$

Uniform bounds

Let T_n be the lifespan of (a_n, u_n), that is the supremum of all $T > 0$ such that (1.4) with initial data (a_0^n, u_0^n) has a solution which satisfies (5.32). Let T be in $(0, T_n)$, we aim at getting uniform estimates in E_T for T small enough. For that, we need to introduce the solution $(u_L^n, \nabla\Pi_L^n)$ to the nonstationary Stokes system:

$$(L) \qquad \begin{cases} \partial_t u_L^n - \mu\Delta u_L^n + \nabla\Pi_L^n = f^n, \\ \mathrm{div} u_L^n = 0, \\ (u_L^n)_{t=0} = u_0^n. \end{cases}$$

Now, the vectorfields $\tilde{u}^n = u^n - u_L^n$ and $\nabla\Pi^n = \nabla\Pi_L^n + \nabla\tilde{\Pi}^n$ satisfy the parabolic system:

$$\begin{cases} \partial_t \tilde{u}^n - \mu(1+a^n)\Delta\tilde{u}^n + (1+a^n)\nabla\tilde{\Pi}^n = H(a^n, u^n, \nabla\Pi^n), \\ \mathrm{div}\tilde{u}^n = 0, \\ \tilde{u}^n(0) = 0, \end{cases} \tag{5.33}$$

which has been studied in proposition 3.5 with:

$$H(a^n, u^n, \nabla\Pi^n) = a^n(\mu\Delta u^n_L - \nabla\Pi^n_L) - u^n \cdot \nabla u^n.$$

Define $m \in \mathbb{Z}$ by:

$$m = \inf\{p \in \mathbb{Z}/ \ \ 2\bar{\nu}\sum_{l\geq q} 2^{l\frac{N}{p_1}}\|\Delta_l a_0\|_{L^{p_1}} \leq c\bar{\nu}\} \tag{5.34}$$

where c is small enough positive constant (depending only N) to be fixed hereafter.
Let:

$$\bar{b} = 1 + \sup_{x\in\mathbb{R}^N} a_0(x),\ A_0 = 1 + 2\|a_0\|_{B^{\frac{N}{p_1}+\varepsilon'}_{p_1,\infty}},\ U_0 = \|u_0\|_{B^{\frac{N}{p_2}-1}_{p_2,r}} + \|f\|_{\widetilde{L}^1(B^{\frac{N}{p_2}-1}_{p_2,r})},$$

and $\widetilde{U}_0 = 2CU_0 + 4C\bar{\nu}A_0$ (where C stands for a large enough constant depending only N which will be determined when applying proposition 2.2, 3.5 and 4.6 in the following computations.) We assume that the following inequalities are fulfilled for some $\eta > 0$, $\alpha > 0$:

$$(\mathcal{H}_1) \qquad \|a^n - S_m a^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p_1}}_{p_1,\infty})\cap L^\infty} \leq c\underline{\nu}\bar{\nu}^{-1},$$

$$(\mathcal{H}_2) \qquad C\bar{\nu}^2 T\|a^n\|^2_{\widetilde{L}^\infty_T(B^{\frac{N}{p_1}}_{p_1,\infty})\cap L^\infty} \leq 2^{-2m}\underline{\nu},$$

$$(\mathcal{H}_3) \qquad \frac{1}{2}\underline{b} \leq 1 + a^n(t,x) \leq 2\bar{b} \ \text{ for all } \ (t,x) \in [0,T]\times\mathbb{R}^N,$$

$$(\mathcal{H}_4) \qquad \|a^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_1,\infty})\cap L^\infty} \leq A_0,$$

$$(\mathcal{H}_5) \qquad \|u^n_L\|_{\widetilde{L}^1(B^{\frac{N}{p_2}+1}_{p_2,r})} \leq \eta,$$

$$(\mathcal{H}_6) \qquad \|\widetilde{u}^n\|_{\widetilde{L}^\infty(B^{\frac{N}{p_2}-1}_{p_2,r})} + \underline{\nu}\|\widetilde{u}^n\|_{\widetilde{L}^1(B^{\frac{N}{p_2}+1}_{p_2,r})} \leq \widetilde{U}_0\eta,$$

$$(\mathcal{H}_7) \qquad \|\nabla\Pi^n_L\|_{\widetilde{L}^1(B^{\frac{N}{p_2}-1}_{p_2,r})} \leq \eta,$$

$$(\mathcal{H}_8) \qquad \|\nabla\widetilde{\Pi}^n\|_{\widetilde{L}^1(B^{\frac{N}{p_2}-1}_{p_2,r})} \leq \widetilde{\Pi}_0\eta,$$

Remark that since:

$$1 + S_m a^n = 1 + a^n + (S_m a^n - a^n),$$

assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_3)$ insure that:

$$\inf_{(t,x)\in[0,T]\times\mathbb{R}^N}(1 + S_m a^n)(t,x) \geq \frac{1}{4}\underline{b}, \tag{5.35}$$

provided c has been chosen small enough (note that $\frac{\underline{\nu}}{\bar{\nu}} \leq \bar{b}$).
We are going to prove that under suitable assumptions on T and η (to be specified below) if condition $(\mathcal{H}_1)$ to $(\mathcal{H}_8)$ are satisfied, then they are actually satisfied with strict inequalities. Since all those conditions depend continuously on the time variable and are strictly satisfied initially, a basic boobstrap argument insures that $(\mathcal{H}_1)$ to $(\mathcal{H}_8)$ are indeed satisfied for a small $T'_n \leq T^n$. First we shall assume that η satisfies:

$$C(1 + \underline{\nu}^{-1}\widetilde{U}_0)\eta \leq \log 2 \tag{5.36}$$

so that denoting:

$$(\tilde{V}^n)_{p_2,1-\frac{1}{r}}(t)=\int_0^t(\tilde{V}^n)'_{p_2,1-\frac{1}{r}}(s)ds \quad \text{and} \quad (V_L^n)_{p_2,1-\frac{1}{r}}(t)=\int_0^t(V_L^n)'_{p_2,1-\frac{1}{r}}(s)ds,$$

with:

$$(\tilde{V}^n)'_{p_2,1-\frac{1}{r}}(s)=\sup_{l\geq 0}\big(\frac{2^{l\frac{N}{p_2}}\|\nabla S_l\tilde{u}^n(s)\|_{L^{p_2}}}{(l+1)^{1-\frac{1}{r}}}\big)ds \quad \text{and}$$

$$(V_L^n)'_{p_2,1-\frac{1}{r}}(s)=\sup_{l\geq 0}\big(\frac{2^{l\frac{N}{p_2}}\|\nabla S_l u_L^n(s)\|_{L^{p_2}}}{(l+1)^{1-\frac{1}{r}}}\big)ds,$$

We recall now that according proposition 2.4:

$$(\tilde{V}^n)_{p_2,1-\frac{1}{r}}(t)\leq C\|\tilde{u}^n\|_{\widetilde{L}^1_t(B^{\frac{N}{p_2}+1}_{p_2,r})} \quad \text{and} \quad (V_L^n)_{p_2,1-\frac{1}{r}}(t)\leq C\|u_L^n\|_{\widetilde{L}^1_t(B^{\frac{N}{p_2}+1}_{p_2,r})},$$

we have, according to $(\mathcal{H}_5)$ and $(\mathcal{H}_6)$:

$$e^{C(\frac{\varepsilon}{2})^{1-r}\left(\|\tilde{u}^n\|_{\widetilde{L}^1_t(B^{\frac{N}{p_2}+1}_{p_2,r})}+\|u_L^n\|_{\widetilde{L}^1_t(B^{\frac{N}{p_2}+1}_{p_2,r})}\right)}<2. \tag{5.37}$$

In order to bound a^n in $\widetilde{L}^\infty_T(B^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_1,\infty})$, we apply theorem 4.3 and get:

$$\|a^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_1,\infty})}<e^{C(\frac{\varepsilon}{2})^{1-r}}\big(\|\tilde{u}^n\|_{\widetilde{L}^1_t(B^{\frac{N}{p_2}}_{p_2,r})}+\|u_L^n\|_{\widetilde{L}^1_t(B^{\frac{N}{p_2}}_{p_2,r})}\big)\|a_0\|_{B^{\frac{N}{p_1}+\varepsilon}_{p_1,\infty}}. \tag{5.38}$$

Moreover as we know that $\|a^n\|_{L^\infty}\leq\|a_0\|_{L^\infty}$, $(\mathcal{H}_4)$ is satisfied with a strict inequality. Next, applying classical proposition on heat equation (see [11]) yields:

$$\|u_L^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p_2}-1}_{p_2,r})}\leq U_0, \tag{5.39}$$

$$\kappa\nu\|u_L^n\|_{\widetilde{L}^1_T(B^{\frac{N}{p_2}+1}_{p_2,r})}\leq\big(\sum_{l\in\mathbb{Z}}2^{lr(\frac{N}{p_2}-1)}(1-e^{-\kappa\nu 2^{2l}T})^r(\|\Delta_l u_0\|^r_{L^{p_2}}+\|\Delta_l f\|^r_{L^1(\mathbb{R}^+,L^{p_2})})\big)^{\frac{1}{r}}. \tag{5.40}$$

Hence taking T such that:

$$\big(\sum_{l\in\mathbb{Z}}2^{lr(\frac{N}{p_2}-1)}(1-e^{-\kappa\nu 2^{2l}T})^r(\|\Delta_l u_0\|^r_{L^2}+\|\Delta_l f\|^r_{L^1(\mathbb{R}^+,L^2)})\big)^{\frac{1}{r}}<\kappa\eta\nu, \tag{5.41}$$

insures that $(\mathcal{H}_5)$ is strictly verified.
Since $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and (5.35) are satisfied, proposition 3.5 may be applied with $\alpha=\frac{\varepsilon}{2}$. We get:

$$\|\widetilde{u}^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p_2}-1}_{p_2,r})}+\underline{\nu}\|\widetilde{u}^n\|_{\widetilde{L}^1_T(B^{\frac{N}{p_2}+1}_{p_2,r})}\leq Ce^{CZ^n_m(T)}\Big(\|u_0\|_{B^{\frac{N}{p_2}-1}_{p_2,r}}+\mathcal{A}^\kappa_{T,n}\times$$
$$\big(\|a^n(\Delta u_L^n-\nabla\Pi_L^n)\|_{\widetilde{L}^1(B^{\frac{N}{p_2}-1}_{p_2,r})}+\|u^n\cdot\nabla u^n\|_{\widetilde{L}^1(B^{\frac{N}{p_2}-1}_{p_2,r})}+\mathcal{A}_{T,n}\|u\|_{\widetilde{L}^1_T(B^{\frac{N}{p_2}+1-\alpha'}_{p_2,r})}\big)\Big).$$

with $Z_m^n(T) = 2^{m\varepsilon}\bar{\nu}^2\underline{\nu}^{-1}\int_0^T \|a^n\|^2_{B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty} d\tau$. Next using Bony's decomposition and $\mathrm{div} u^n = 0$, one can write:

$$\mathrm{div}(u^n \otimes u^n) = T_{\partial_j v}(u^n)^j + T_{(u^n)^j}\partial_j u^n + \partial_j R(u^n, (u^n)^j),$$

with the summation convention over repeated indices.

Hence combining proposition 1.4.1 and 1.4.2 in [11] with the fact that $\widetilde{L}^\rho_T(B^{\frac{N}{p_2}-\frac{1}{2}}_{p_2,r}) \hookrightarrow \widetilde{L}^\rho_T(B^{-\frac{1}{2}}_{\infty,\infty})$ for $\rho = \frac{4}{3}$ or $\rho = 4$, we get:

$$\|\mathrm{div}(u^n \otimes u^n)\|_{\widetilde{L}^1_T(B^{\frac{N}{p_2}-1}_{p_2,r})} \leq C\|u^n\|_{\widetilde{L}^{\frac{4}{3}}_T(B^{\frac{N}{p_2}+\frac{1}{2}}_{p_2,r})}\|u^n\|_{\widetilde{L}^4_T(B^{\frac{N}{p_2}-\frac{1}{2}}_{p_2,r})}.$$

By taking advantage of proposition 3.5, 2.2, 2.1 and Young'inequality, we end up with:

$$\|\widetilde{u}^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p_2}-1}_{p_2,r})} + \underline{\nu}\|\widetilde{u}^n\|_{\widetilde{L}^1_T(B^{\frac{N}{p_2}+1}_{p_2,r})} \leq e^{CZ^n_m(T)}\Big(\|u_0\|_{B^{\frac{N}{p_2}-1}_{p_2,r}} + \big(\|a^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_1,\infty}\cap L^\infty)} + 1\big)^\kappa$$
$$\times\big(C\|u^n_L\|_{\widetilde{L}^1_T(B^{\frac{N}{p_2}+1}_{p_2,r})}(\bar{\nu}\|a^n\|_{L^\infty_T(B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty)} + \|u^n_L\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p_2}-1}_{p_2,r})}) + \|\nabla\Pi^n_L\|_{\widetilde{L}^1_T(B^{\frac{N}{p_2}-1}_{p_2,r})}\times$$
$$\|a^n\|_{L^\infty_T(B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty)}\big) + (\|a^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_1,\infty}\cap L^\infty)} + 1)T^{\frac{\varepsilon}{2}}\|\widetilde{u}^n\|_{\widetilde{L}^1_T(B^{\frac{N}{p_2}+1}_{p_2,r})}\Big).$$

with $C = C(N)$ and $C_g = (N, g, \underline{b}, \bar{b})$. Now, using assumptions $(\mathcal{H}_2)$, $(\mathcal{H}_4)$, $(\mathcal{H}_5)$, $(\mathcal{H}_6)$ and $(\mathcal{H}_7)$, and inserting (5.37) in the previous inequality and choosing T enough small gives:

$$\|\widetilde{u}^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p_2}-1}_{p_2,r})} + \|\widetilde{u}^n\|_{\widetilde{L}^1_T(B^{\frac{N}{p_2}+1}_{p_2,r})} \leq 2C(\bar{\nu}A_0 + U_0)\eta + 2C_gTA_0,$$

hence $(\mathcal{H}_6)$ is satisfied with a strict inequality provided:

$$C_gT < C\bar{\nu}\eta. \tag{5.42}$$

To show that $(\mathcal{H}_7)$ and $(\mathcal{H}_8)$ are strictly verified on $(0, T_n')$, we proceed similarly as for $(\mathcal{H}_5)$ and $(\mathcal{H}_6)$. We now have to check whether $(\mathcal{H}_1)$ is satisfied with strict inequality. For that we apply proposition 4.6 which yields for all $m \in \mathbb{Z}$,

$$\sup_{l\geq m} 2^{l\frac{N}{p_1}}\|\Delta_l a^n\|_{L^\infty_T(L^{p_1})} \leq C\big(\sup_{l\geq m} 2^{l\frac{N}{p_1}}\|\Delta_l a_0\|_{L^{p_1}})\big)\big(\|\widetilde{u}^n\|_{\widetilde{L}^1_t(B^{\frac{N}{p_2}}_{p_2,r})} + \|u^n_L\|_{\widetilde{L}^1_t(B^{\frac{N}{p_2}}_{p_2,r})}\big) \tag{5.43}$$

Using (5.36) and $(\mathcal{H}_5)$, $(\mathcal{H}_6)$, we thus get:

$$\|a^n - S_m a^n\|_{L^\infty_T(B^{\frac{N}{p_1}}_{p_1,\infty})\cap L^\infty} \leq \sup_{l\geq m} 2^{l\frac{N}{p_1}}\|\Delta_l a_0\|_{L^{p_1}} + \frac{C}{\log 2}(1 + \|a_0\|_{B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty})(1 + \underline{\nu}^{-1}\widetilde{L}_0)\eta.$$

Hence $(\mathcal{H}_1)$ is strictly satisfied provided that η further satisfies:

$$\frac{C}{\log 2}(1 + \|a_0\|_{B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty})(1 + \underline{\nu}^{-1}\widetilde{L}_0)\eta < \frac{c\underline{\nu}}{2\bar{\nu}}. \tag{5.44}$$

So $\mathcal{H}_1$ is strictly verified.

$(\mathcal{H}_3)$ is trivially verified by the transport equation as we assume that $1 + a_0$ is bounded

away and that $a_0 \in L^\infty$.
Next, according to $(\mathcal{H}_4)$ condition $(\mathcal{H}_2)$ is satisfied provide:

$$T < \frac{2^{-2m}\underline{\nu}}{C\bar{\nu}^2 A_0^2} \tag{5.45}$$

One can now conclude that if $T < T^n$ has been choosen so that conditions (5.41), (5.42) and (5.45) are satisfied (with η verifying (5.36) and (5.44), and m defined in (5.34) and $n \geq m$ then (a^n, u^n, Π^n) satisfies $(\mathcal{H}_1)$ to $(\mathcal{H}_8)$, thus is bounded independently of n on $[0, T]$.
We still have to state that T^n may be bounded by below by the supremum $\bar{T}$ of all times T such that (5.41), (5.42) and (5.45) are satisfied. This is actually a consequence of the uniform bounds we have just obtained, and of a theorem of blow-up of R. Danchin in [14]. Indeed, by combining all these informations, one can prove that if $T^n < \bar{T}$ then $(a^n, u^n, \nabla\Pi^n)$ is actually in:

$$\widetilde{L}^\infty_{T^n}(B^{\frac{N}{p_1}+\varepsilon'}_{p_1,\infty} \cap B^{\frac{N}{2}+1}_{2,1}) \times \big(\widetilde{L}^\infty_{T^n}(B^{\frac{N}{p_2}-1}_{p_2,r} \cap B^{\frac{N}{2}}_{2,1}) \cap \widetilde{L}^1_{T^n}(B^{\frac{N}{p_2}+1}_{p_2,r} \cap B^{\frac{N}{2}+2}_{2,1})\big)^N \\ \times \widetilde{L}^1_{T^n}(B^{\frac{N}{p_2}-1}_{p_2,r} \cap B^{\frac{N}{2}}_{2,1}).$$

hence may be continued beyond $\bar{T}$ (see the remark on the lifespan in [14] where a control of ∇u in $\widetilde{L}^1(B^0_{\infty,\infty})$ is required). We thus have $T^n \geq \bar{T}$.

Compactness arguments

We now have to prove that $(a^n, u^n)_{n\in\mathbb{N}}$ tends (up to a subsequence) to some function (a, u) which belongs to E_T and satisfies (1.4). The proof is based on Ascoli's theorem and compact embedding for Besov spaces. As similar arguments have been employed in [12] or [13], we only give the outlines of the proof.

- Convergence of $(a^n)_{n\in\mathbb{N}}$:
  We use the fact that $\widetilde{a}^n = a^n - a^n_0$ satisfies:

  $$\partial_t \widetilde{a}^n = -u^n \cdot \nabla a^n.$$

  Since $(u^n)_{n\in\mathbb{N}}$ is uniformly bounded in $\widetilde{L}^1_T(B^{\frac{N}{p_2}+1}_{p_2,r}) \cap \widetilde{L}^\infty_T(B^{\frac{N}{p_2}-1}_{p_2,r})$, it is, by interpolation, also bounded in $\widetilde{L}^{r'}_T(B^{\frac{N}{p_2}-1+\frac{2}{r'}}_{p_2,r})$ for any $r' \in [1, +\infty]$. By taking $r = 2$ and using the standard product laws in Besov spaces, we thus easily gather that $(\partial_t \widetilde{a}^n)$ is uniformly bounded in $\widetilde{L}^2_T(B^{\frac{N}{p_1}-1}_{p_1,\infty})$.

  $$\|\partial_t \widetilde{a}^n\|_{\widetilde{L}^2_T(B^{\frac{N}{p_1}-1}_{p_1,\infty})} \lesssim \|u^n\|_{\widetilde{L}^2_T(B^{\frac{N}{p_2}}_{p_2,r})} \|\nabla a^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p_1}-1}_{p_1,\infty})}.$$

  Hence $(\widetilde{a}^n)_{n\in\mathbb{N}}$ is bounded in $\widetilde{L}^\infty_T(B^{\frac{N}{p_1}-1}_{p_1,\infty} \cap B^{\frac{N}{p_1}}_{p_1,\infty})$ and equicontinuous on $[0, T]$ with values in $B^{\frac{N}{p_1}-1}_{p_1,\infty}$. Since the embedding $B^{\frac{N}{p_1}-1}_{p_1,\infty} \cap B^{\frac{N}{p_1}}_{p_1,\infty} \hookrightarrow B^{\frac{N}{p_1}-1}_{p_1,\infty}$ is (locally) compact, and $(a^n_0)_{n\in\mathbb{N}}$ tends to a_0 in $B^{\frac{N}{p_1}}_{p_1,\infty}$, we conclude that $(a^n)_{n\in\mathbb{N}}$ tends (up to

extraction) to some distribution a. Given that $(a^n)_{n\in\mathbb{N}}$ is bounded in $\widetilde{L}^\infty_T(B^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_1,r})$, we actually have $a \in \widetilde{L}^\infty_T(B^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_1,r})$.

- Convergence of $(u^n_L)_{n\in\mathbb{N}}$:
  From the definition of u^n_L and classical proposition on Stokes equation, it is clear that $(u^n_L)_{n\in\mathbb{N}}$ and $(\nabla\Pi^n_L)_{n\in\mathbb{N}}$ tend to solution u_L and $\nabla\Pi_L$ to:

$$\partial_t u_L - \mu\Delta u_l + \nabla\Pi_L = f, \;\; u_L(0) = u_0$$

  in $\widetilde{L}^\infty_T(B^{\frac{N}{p_2}-1}_{p_2,r}) \cap \widetilde{L}^1_T(B^{\frac{N}{p_2}+1}_{p_2,r})$ for $(u^n_L)_{n\in\mathbb{N}}$ and $\widetilde{L}^1_T(B^{\frac{N}{p_2}-1}_{p_2,r})$ for $(\nabla\Pi^n_L)_{n\in\mathbb{N}}$.

- Convergence of $(\widetilde{u}^n)_{n\in\mathbb{N}}$:
  We use the fact that:

$$\partial_t\widetilde{u}^n = -u^n_L\cdot\nabla\widetilde{u}^n - \widetilde{u}^n\cdot\nabla u^n + (1+a^n)\Delta\widetilde{u}^n + a^n\Delta u^n_L - u^n_L\cdot\nabla u^n_L - \nabla\widetilde{\Pi}^n.$$

  As $(a^n)_{n\in\mathbb{N}}$ is uniformly bounded in $\widetilde{L}^\infty_T(B^{\frac{N}{p_1}}_{p_1,\infty})$ and $(u^n)_{n\in\mathbb{N}}$ is uniformly bounded in $\widetilde{L}^\infty_T(B^{\frac{N}{p_2}-1}_{p_2,r}) \cap \widetilde{L}^{\frac{4}{3}}(B^{\frac{N}{p_2}+\frac{1}{2}}_{p_2,r})$, it is easy to see that the last term of the right-hand side is uniformly bounded in $\widetilde{L}^\infty_T(B^{\frac{N}{p_2}-1}_{p_2,r})$ and that the other terms are uniformly bounded in $\widetilde{L}^{\frac{4}{3}}(B^{\frac{N}{p_2}-\frac{3}{2}}_{p_2,r})$.
  Hence $(\widetilde{u}^n)_{n\in\mathbb{N}}$ is bounded in $\widetilde{L}^\infty_T(B^{\frac{N}{p_2}-1}_{p_2,r})$ and equicontinuous on $[0,T]$ with values in $B^{\frac{N}{p_2}-1}_{p_2,r} + B^{\frac{N}{p_2}-\frac{3}{2}}_{p_2,r}$. This enables to conclude that $(\widetilde{u}^n)_{n\in\mathbb{N}}$ converges (up to extraction) to some function $\widetilde{u} \in \widetilde{L}^\infty_T(B^{\frac{N}{p_2}-1}_{p_2,r}) \cap \widetilde{L}^1_T(B^{\frac{N}{p_2}+1}_{p_2,r})$.

We proceed similarly for $(\Pi^n_L)_{n\in\mathbb{N}}$ and $(\widetilde{\Pi}^n)_{n\in\mathbb{N}}$. By interpolating with the bounds provided by the previous step, one obtains better results of convergence so that one can pass to the limit in the mass equation and in (5.33). Finally by setting $u = \widetilde{u} + u_L$ and $\Pi = \widetilde{\Pi} + \Pi_L$, we conclude that (a, u, Π) satisfies (1.4).
In order to prove continuity in time for a it suffices to make use of proposition 4.3. Indeed, a_0 is in $B^{\frac{N}{p_1}+\varepsilon}_{p_1,\infty} \cap L^\infty$, and having $a \in \widetilde{L}^\infty_T(B^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_1,\infty}) \cap L^\infty$ and $u \in \widetilde{L}^1_T(B^{\frac{N}{p_2}+1}_{p_2,r})$ insure that $\partial_t a + u\cdot\nabla a$ belongs to $\widetilde{L}^1_T(B^{\frac{N}{p_1}}_{p_1,\infty})$. Similarly, continuity for u may be proved by using that $u_0 \in B^{\frac{N}{p_2}}_{p_2,r}$ and that $(\partial_t u - \mu\Delta u) \in \widetilde{L}^1_T(B^{\frac{N}{p_2}-1}_{p_2,r})$.

6 The proof of the uniqueness

6.1 Uniqueness when $1 \le p_2 < 2N$, $\frac{2}{N} < \frac{1}{p_1} + \frac{1}{p_2}$ and $N \ge 3$.

In this section, we focus on the case $N \ge 3$ and postpone the analysis of the other cases (which turns out to be critical) to the next section. Throughout the proof, we assume that we are given two solutions (a^1, u^1) and (a^2, u^2) of (1.4) which belongs to:

$$\big(\widetilde{C}([0,T]; B^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_1,\infty}) \cap L^\infty\big) \times \big(\widetilde{C}([0,T]; B^{\frac{N}{p_2}-1}_{p_2,r}) \cap \widetilde{L}^1(0,T; B^{\frac{N}{p_2}+1}_{p_2,r})\big)^N.$$

Let $\delta a = a^2 - a^1$, $\delta u = u^2 - u^1$ and $\delta\Pi = \Pi^2 - \Pi^1$. The system for $(\delta a, \delta u)$ reads:

$$\begin{cases} \partial_t \delta a + u^2 \cdot \nabla \delta a = -\delta u \cdot \nabla a^1, \\ \partial_t \delta u - (1 + a^2)(\mu \Delta \delta u - \nabla \delta \Pi) = F(a^i, u^i, \Pi^i). \end{cases} \tag{6.46}$$

with:

$$F(a^i, u^i, \Pi^i) = u^1 \cdot \nabla \delta u + \delta u \cdot \nabla u^2 + \delta a(\mu \Delta u^1 - \nabla \Pi^1).$$

The function δa may be estimated by taking advantage of proposition 4.3 with $s = \frac{N}{p_1} - 1 + \frac{\varepsilon}{2}$. We get for all $t \in [0, T]$,

$$\|\delta a(t)\|_{B^{\frac{N}{p_1}-1}_{p_1,\infty}} \leq C \|\delta u \cdot \nabla a^1\|_{\widetilde{L}^1_T(B^{\frac{N}{p_1}-1+\frac{\varepsilon}{2}}_{p_1,\infty})} \exp\big(\frac{C}{\varepsilon^{r-1}}(V_{p_1,1-\frac{1}{r}}(t))^r\big),$$

We have then by proposition 2.2 and 2.4:

$$\|\delta a(t)\|_{B^{\frac{N}{p_1}-1}_{p_1,\infty}} \leq C \|\delta u\|_{\widetilde{L}^1_T(B^{\frac{N}{p_2}}_{p_2,r})} \|a^1\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_1,\infty})} \exp\big(\frac{C}{\varepsilon^{r-1}}(\|u^2\|_{\widetilde{L}^1_T(B^{\frac{N}{p_2}+1}_{p_2,r})})^r\big), \tag{6.47}$$

For bounding δu, we aim at applying proposition 3.5 to the second equation of (6.46). So let us fix an integer m such that:

$$1 + \inf_{(t,x)\in[0,T]\times\mathbb{R}^N} S_m a^2 \geq \frac{b}{2} \ \text{ and } \ \|a^2 - S_m a^2\|_{L^\infty_T(B^{\frac{N}{p_1}}_{p_1,\infty})} \leq c\frac{\nu}{\bar{\nu}}. \tag{6.48}$$

Now applying proposition 3.5 with $s = \frac{N}{p_2} - 2$ insures that for all time $t \in [0, T]$, we have:

$$\begin{aligned} \|u\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p_2}-2}_{p_2,r})} + \kappa\underline{\nu}\|u\|_{\widetilde{L}^1_T(B^{\frac{N}{p_2}}_{p_2,r})} + \|\nabla\delta\Pi\|_{\widetilde{L}^1_T(B^{\frac{N}{p_2}-2}_{p_2,r})} &\leq e^{CZ_m(T)} \times \\ \big(\|\mathcal{P}F(a^i, u^i, \Pi^i)\|_{\widetilde{L}^1_T(B^{\frac{N}{p_2}-2}_{p_2,r})} &+ (\frac{\underline{\nu}(p_2-1)}{p_2})\mathcal{A}_T\|u\|_{\widetilde{L}^1_T(B^{\frac{N}{p_2}-\alpha'}_{p,r})}\big). \end{aligned} \tag{6.49}$$

with $Z_m(t) = 2^m \mu^2 \underline{\nu}^{-1} \int_0^t \|a(\tau)\|^2_{B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty} d\tau$.

Hence, applying proposition 2.2, corollary 1 and the fact that $\mathrm{div}\delta u = 0$, we get as exemple:

$$\|\delta u \cdot \nabla u^1\|_{\widetilde{L}^1_T(B^{\frac{N}{p_2}-2}_{p_2,r})} \lesssim \|u^1\|^{\frac{1}{2}}_{\widetilde{L}^\infty_T(B^{\frac{N}{p_2}-1}_{p_2,r})} \|u^1\|^{\frac{1}{2}}_{\widetilde{L}^1_T(B^{\frac{N}{p_2}+1}_{p_2,r})} \big(\|\delta u\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p_2}-2}_{p_2,r})} + \|\delta u\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p_2}}_{p_2,r})}\big).$$

By the fact that $\frac{2}{N} < \frac{1}{p_1} + \frac{1}{p_2}$, $N \geq 3$ and $\frac{1}{p_2} \leq \frac{1}{N} + \frac{1}{p_1}$ imply that:

$$\begin{aligned} \|\delta a(\mu\Delta u_1 - \nabla\delta\Pi)\|_{\widetilde{L}^1_T(B^{\frac{N}{p_2}-2}_{p_2,r})} \lesssim \|\delta a\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p_1}-1}_{p_1,\infty})} \big(&\|\Delta u_1\|_{\widetilde{L}^1_T(B^{\frac{N}{p_2}-1}_{p_1,r})} \\ &+ \|\nabla\Pi^1\|_{\widetilde{L}^1_T(B^{\frac{N}{p_2}-1}_{p_2,r})}\big). \end{aligned} \tag{6.50}$$

Now let choose T_1 enough small to controll in (6.49) $\mathcal{A}_T\|u\|_{\widetilde{L}^1_T(B^{\frac{N}{p_2}-\alpha'}_{p,r})}$ and in (6.50) $\|\Delta u_1\|_{\widetilde{L}^1_T(B^{\frac{N}{p_2}-1}_{p_1,r})} + \|\nabla\Pi^1\|_{\widetilde{L}^1_T(B^{\frac{N}{p_2}-1}_{p_2,r})}$ and by the fact that $\|a_1\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p_1}+\frac{\varepsilon}{2}}_{p_1,\infty})} \leq c$ with c small, we obtain finally:

$$\|(\delta a, \delta u, \nabla\delta\Pi)\|_{F_{T_1}} \leq cC\|(\delta a, \delta u, \nabla\delta\Pi)\|_{F_{T_1}},$$

with:

$$F_T = \widetilde{C}_T([0,T], B^{\frac{N}{p_1}}_{p_1,\infty}) \times \big(\widetilde{L}^1_T(B^{\frac{N}{p_2}}_{p_2,r}) \cap \widetilde{L}^\infty_T(B^{\frac{N}{p_2}-2}_{p_2,r})\big) \times \widetilde{L}^1_T(B^{\frac{N}{p_2}-2}_{p_2,r}).$$

We obtain so $(\delta a, \delta u, \nabla\delta\Pi) = 0$ on $[0, T_1]$ for T_1 enough small. By connectivity we obtain that $(\delta a, \delta u, \delta\nabla\Pi) = 0$ on $[0, T]$. This conclude this case.

6.2 Uniqueness when: $\frac{2}{N} = \frac{1}{p_1} + \frac{1}{p_2}$ or $p_2 = 2N$ or $N = 2$.

The above proof fails in dimension two. One of the reasons why is that the product of functions does not map $B^{\frac{N}{p_1}-1}_{p_1,\infty} \times B^{\frac{N}{p_2}-1}_{p_2,r}$ in $B^{\frac{N}{p_2}-2}_{p_2,r}$ but only in the larger space $B^{-1}_{2,\infty}$. This induces us to bound δa in $\widetilde{L}^\infty_T(B^{\frac{N}{p_1}-1}_{p_1,\infty})$ and δu in $\widetilde{L}^\infty_T(B^{\frac{N}{p_2}-2}_{p_2,\infty}) \cap \widetilde{L}^1_T(B^{\frac{N}{p_2}}_{p_2,\infty})$. In fact, it is enough to study only the case $\frac{2}{N} = \frac{1}{p_1} + \frac{1}{p_2}$. Indeed the other cases deduct from this case. If $p_2 = 2N$ then $p_1 = \frac{2N}{3}$ as $\frac{1}{p_1} \leq \frac{1}{N} + \frac{1}{p_2}$ and $\frac{2}{N} \leq \frac{1}{p_1} + \frac{1}{p_2}$. So it is a particular case of $\frac{2}{N} = \frac{1}{p_1} + \frac{1}{p_2}$. For $N = 2$, we begin with $p_2 = 4$ and $p_1 = \frac{4}{3}$ and by embedding we get the result for $1 \leq p_1 \leq \frac{4}{3}$, $1 \leq p_2 \leq 4$ and for $1 \leq p_1 \leq 4$, $1 \leq p_2 \leq \frac{4}{3}$.
Moreover in your case, it exists two possibilities, one when $1 < p_2 < 2$ and when $p_2 \geq 2$. The first case is resolved by embedding so we have just to treat the case $p_2 \geq 2$. We want show that $(\delta a, \delta u, \nabla\delta\Pi) \in G_T$ where:

$$G_T = \widetilde{C}_T([0,T], B^{\frac{N}{p_1}-1+\frac{\varepsilon}{2}}_{p_1,\infty}) \times \big(\widetilde{L}^1_T(B^{\frac{N}{p_2}}_{p_2,\infty}) \cap \widetilde{L}^\infty_T(B^{\frac{N}{p_2}-2}_{p_2,\infty})\big) \times \widetilde{L}^1_T(B^{\frac{N}{p_2}-2}_{p_2,\infty}).$$

In fact we proceed exactly as in the previous proof but we benefit that a is in $\widetilde{C}_T([0,T], B^{\frac{N}{p_1}-1+\frac{\varepsilon}{2}}_{p_1,\infty})$ which give sense to the product $\delta\Delta u$. It conclude the proof of uniqueness.

6.3 Proof of theorem 1.2

The proof follow strictly the same lines than the proof of theorem 1.1 except that we profits of the fact that in the proposition 3.5 in the case of $p_2 = 2$ we do not need of conditions of smallness on the initial density a_0.

7 Appendix

7.1 Elliptic estimates

This section is devoted to the study of the elliptic equation:

$$\mathrm{div}(b\nabla\Pi) = \mathrm{div}F. \tag{7.51}$$

with $b = 1 + a$.
Let us first study the stationary case where F and b are independent of the time:

Proposition 7.8 *Let $0 < \alpha < 1$, $(p, r) \in [1, +\infty]^2$ and $\sigma \in \mathbb{R}$ satisfy $\alpha \leq \sigma \leq \alpha + \frac{N}{p_1}$. Then the operator $\mathcal{H}_b : F \to \nabla\Pi$ is a linear bounded operator in $B^\sigma_{p,r}$ and the following estimate holds true:*

$$\underline{b}\|\nabla\Pi\|_{B^\sigma_{p,r}} \lesssim \mathcal{A}^{\frac{|\sigma|}{\min(1,\alpha)}} \|\mathcal{Q}F\|_{B^\sigma_{p,r}}, \tag{7.52}$$

with if $\alpha \neq 1$:

$$\mathcal{A} = 1 + \underline{b}^{-1} \|\nabla b\|_{B^{\frac{N}{p_1}+\alpha-1}_{p_1,r}}.$$

Proof:

Let us first rewrite (7.51) as follows:

$$\mathrm{div}(b_m \nabla \Pi) = \mathrm{div} F - E_m, \tag{7.53}$$

with $E_m = \mathrm{div}((Id - S_m) a \nabla \Pi)$.
Apply Δ_q to (7.53) we get:

$$\mathrm{div}(b_m \nabla \Delta_q \Pi) = \mathrm{div} F_q - \Delta_q(E_m) + R_q, \tag{7.54}$$

with $R_q = \mathrm{div}(b_m \nabla \Pi_q) - \Delta_q \mathrm{div}(b_m \nabla \Pi)$. Multiplying (7.54) by $\Delta_q \Pi |\Delta_q \Pi|^{p-2}$ and integrate, we gather:

$$\begin{aligned} &\int_{\mathbb{R}^N} b_m |\nabla \Pi_q|^2 |\Pi_q|^{p-2} dx + \int_{\mathbb{R}^N} b_m |\nabla |\Pi_q|^2|^p dx \leq (\|\mathrm{div} F_q\|_{L^p} + \|R_q\|_{L^p}) \|\Pi_q\|^{p-1}_{L^p} \\ &+ \int_{\mathbb{R}^N} |(Id - S_m) a| \, |\nabla \Pi_q|^2 |\Pi_q|^{p-2} dx + \int_{\mathbb{R}^N} |(Id - S_m) a| \, |\nabla |\Pi_q|^2|^p dx, \end{aligned} \tag{7.55}$$

Assuming that m has been choose so large as to satisfy:

$$\|a - S_m a\|_{L^\infty_T (B^{\frac{N}{p_1}+\alpha}_{p_1,\infty}) \cap L^\infty} \leq \frac{\underline{b}}{2},$$

an by using lemma A5 in [16]:

$$2^{2q} \|\Pi_q\|_{L^p} \lesssim 2^q \|F_q\|_{L^p} + \|R_q\|_{L^p}.$$

By multipliyng by $2^{q(s-1)}$ and by integrating on l^r we get:

$$\underline{b} \|\nabla \Pi\|_{B^s_{p,r}} \lesssim \|\mathcal{Q} f\|_{B^s_{p,r}} + \|R_q\|_{B^s_{p,r}}.$$

The commutator may be bounded thanks to lemma 1 with $0 < \alpha < 1$, $\sigma = s - 1$. We have get:

$$\underline{b} \|\nabla \Pi\|_{B^s_{p,r}} \lesssim \|\mathcal{Q} F\|_{B^s_{p,r}} + \|\nabla S_m a\|_{B^{\frac{N}{p_1}+\alpha}_{p_1,r}} \|\nabla \Pi\|_{B^{s-\alpha}_{p,r}}.$$

Therefore complex interpolation entails:

$$\|\nabla \Pi\|_{B^{s-\alpha}_{p,r}} \leq \|\nabla \Pi\|^{\frac{s-\alpha}{s}}_{B^s_{p,r}} \|\nabla \Pi\|^{\frac{\alpha}{s}}_{B^0_{p,\infty}}.$$

Note that, owing to Bersntein inequality, we have:

$$\|\nabla S_m a\|_{B^{\frac{N}{p_1}+\alpha}_{p_1,r}} \lesssim 2^{m+\alpha} \|a\|_{B^{\frac{N}{p_1}}_{p_1,r}}.$$

We have then:

$$\underline{b} \|\nabla \Pi\|_{B^s_{p,r}} \lesssim \|\mathcal{Q} f\|_{B^s_{p,r}} + 2^{m+\alpha} \|a\|_{B^{\frac{N}{p_1}}_{p_1,\infty}} \|\nabla \Pi\|^{\frac{s-\alpha}{s}}_{B^s_{p,r}} \|\nabla \Pi\|^{\frac{\alpha}{s}}_{B^0_{p,\infty}}.$$

And we conclude by Young's inequality with $p_1 = \frac{s}{s-\alpha}$ and $p_2 = \frac{s}{\alpha}$. And we recall

7.2 Commutator estimates

This section is devoted to the proof of commutator estimates which have been used in section 2 and 3. They are based on paradifferentiel calculus, a tool introduced by J.-M. Bony in [6]. The basic idea of paradifferential calculus is that any product of two distributions u and v can be formally decomposed into:

$$uv = T_u v + T_v u + R(u,v) = T_u v + T_v^{'} u$$

where the paraproduct operator is defined by $T_u v = \sum_q S_{q-1} u \Delta_q v$, the remainder operator R, by $R(u,v) = \sum_q \Delta_q u(\Delta_{q-1} v + \Delta_q v + \Delta_{q+1} v)$ and $T_v^{'} u = T_v u + R(u,v)$.
Inequality (3.23) is a consequence of the following lemma:

Lemma 1 *Let $p_1 \in [1,+\infty]$, $p \in [1,+\infty]$, $\alpha \in (1-\frac{N}{p}, 1[$, $k \in \{1,\cdots,N\}$ and $R_q = \Delta_q(a\partial_k w) - \partial_k(a\Delta_q w)$. There exists $c = c(\alpha, N, \sigma)$ such that:*

$$2^{q\sigma}\|\widetilde{R}_q\|_{L^p} \leq C c_q \|a\|_{B^{\frac{N}{p_1}+\alpha}_{p_1,r}} \|w\|_{B^{\sigma+1-\alpha}_{p,r}} \tag{7.56}$$

whenever $-\frac{N}{p_1} < \sigma \leq \frac{N}{p_1} + \alpha$ and where $c_q \in L^r$.
In the limit case $\sigma = -\frac{N}{p_1}$, we have for some constant $C = C(\alpha, N)$:

$$2^{-q\frac{N}{p_1}}\|\widetilde{R}_q\|_{L^p} \leq C\|a\|_{B^{\alpha+\frac{N}{p_1}}_{p,1}} \|w\|_{B^{-\frac{N}{p_1}+1-\alpha}_{p,\infty}}. \tag{7.57}$$

Proof

The proof is based on Bony's decomposition which enables us to split R_q into:

$$R_q = \underbrace{\partial_k[\Delta_q, T_a]w}_{R_q^1} - \underbrace{\Delta_q T_{\partial_k a} w}_{R_q^2} + \underbrace{\Delta_q T_{\partial_k w} a}_{R_q^3} + \underbrace{\Delta_q R(\partial_k w, a)}_{R_q^4} - \underbrace{\partial_k T_{\Delta_q w}^{'} a}_{R_q^5}.$$

By using the fact that:

$$R_q^1 = \sum_{q'=q-4}^{q+4} \partial_k[\Delta_q, S_{q'-1}a]\Delta_{q'} w,$$

Using the definition of the operator Δ_q leads to:

$$[\Delta_q, S_{q'-1}a]\Delta_{q'} w(x) = -\int h(y)\big(S_{q'-1}a(x) - S_{q'-1}a(x-2^{-q}y)\big)\Delta_{q'} w(x-2^{-q}y)dy.$$

and:

$$\begin{aligned}\big|[\Delta_q, S_{q'-1}a]\Delta_{q'} w(x)\big| &\leq \|\nabla S_{q'-1}a\|_{L^\infty} 2^{-q}\int 2^{qN}|h(2^q u)||2^q u||\Delta_{q'} w|(x-u)du,\\ &\leq 2^{qN}\|\nabla S_{q'-1}a\|_{L^\infty}\big|\big(h(2^q\cdot)|\cdot| * \Delta_{q'} w\big)\big|(x).\end{aligned}$$

So we get:

$$\|[\Delta_q, S_{q'-1}a]\Delta_{q'} w\|_{L^p} \leq \|\nabla S_{q'-1}a\|_{L^\infty}\|\Delta_{q'} w\|_{L^p}$$

we readily get under the hypothesis that $\alpha < 1$,

$$2^{q\sigma}\|R_q^1\|_{L^p} \lesssim \sum_{q'=q-4}^{q+4} 2^{q\sigma}\|\nabla S_{q'-1}a\|_{L^\infty}\|\Delta_{q'}w\|_{L^p}. \tag{7.58}$$

We have then:

$$2^{q\sigma}\|R_q^1\|_{L^p} \lesssim c_q\|\nabla a\|_{B^{\alpha-1}_{\infty,\infty}}\|w\|_{B^{\sigma+1-\alpha}_{p,r}}. \tag{7.59}$$

In the case $\alpha = 1$, we get:

$$2^{q\sigma}\|R_q^1\|_{L^p} \lesssim c_q\|\nabla a\|_{B^{0}_{\infty,1}}\|w\|_{B^{\sigma+1-\alpha}_{p,r}}. \tag{7.60}$$

For bounding R_q^2, standard continuity results for the paraproduct insure that if $\alpha < 1$, R_q^2 satisfies that:

$$2^{q\sigma}\|R_q^2\|_{L^p} \leq c_q\|\nabla a\|_{B^{\alpha-1}_{\infty,\infty}}\|w\|_{B^{\sigma+1-\alpha}_{p,r}}.$$

and if $\alpha = 1$

$$2^{q\sigma}\|R_q^2\|_{L^p} \leq c_q\|\nabla a\|_{B^{\alpha-1}_{\infty,1}}\|w\|_{B^{\sigma+1-\alpha}_{p,r}}.$$

Standard continuity results for the paraproduct insure that R_q^3 satisfies:

$$2^{q\sigma}\|R_q^3\|_{L^p} \lesssim c_q\|\nabla w\|_{B^{\sigma-\alpha-\frac{N}{p}}_{\infty,\infty}}\|a\|_{B^{\frac{N}{p}+\alpha}_{p,r}}. \tag{7.61}$$

provided $\sigma - \alpha - \frac{N}{p} < 0$.
If $\sigma - \alpha - \frac{N}{p} = 0$ then:

$$2^{q\sigma}\|R_q^3\|_{L^p} \lesssim c_q\|\nabla w\|_{B^{0}_{\infty,1}}\|a\|_{B^{\frac{N}{p}+\alpha}_{p,r}}. \tag{7.62}$$

Next, standard continuity result for the remainder insure that under the hypothesis $\sigma > -\frac{N}{p}$, we have:

$$2^{q\sigma}\|R_q^4\|_{L^p} \lesssim c_q\|\nabla w\|_{B^{\sigma-\alpha}_{p,r}}\|a\|_{B^{\frac{N}{p}+\alpha}_{p,\infty}}. \tag{7.63}$$

For bounding R_q^5 we use the decomposition:

$$R_q^5 = \sum_{q'\geq q-3} \partial_k(S_{q'+2}\Delta_q w\Delta_{q'}a),$$

which leads (after a suitable use of Bernstein and Hölder inequalities) to:

$$\begin{aligned}
2^{q\sigma}\|R_q^5\|_{L^p} &\lesssim \sum_{q'\geq q-3} 2^{q\sigma}2^{q'}\|\Delta_{q'}a\|_{L^\infty}\|S_{q'+2}\Delta_q w\|_{L^p} \\
&\lesssim \sum_{q'\geq q-2} 2^{(q-q')(\alpha+\frac{N}{p}-1)}2^{q(\sigma+1-\alpha)}\|\Delta_q w\|_{L^p}2^{q'(\frac{N}{p}+\alpha)}\|\Delta_{q'}a\|_{L^p}.
\end{aligned}$$

Hence, since $\alpha + \frac{N}{p} - 1 > 0$, we have:

$$2^{q\sigma}\|R_q^5\|_{L^p} \lesssim c_q\|\nabla w\|_{B^{\sigma+1-\alpha}_{p,r}}\|a\|_{B^{\frac{N}{p}+\alpha}_{p,\infty}}.$$

Combining this latter inequality with (7.59), (7.61) and (7.63), and using the embedding $B^{\frac{N}{p}}_{p,r} \hookrightarrow B^{r-\frac{N}{p}}_{\infty,\infty}$ for $r = \frac{N}{p} + \alpha - 1$, $\sigma - \alpha$ completes the proof of (7.56).

The proof of (7.57) is almost the same: for bounding R^1_q, R^2_q, R^3_q and R^5_q, it is just a matter of changing $\sum_q$ into $\sup_q$. We proceed similarly for R^4_q. $\square$

References

[1] H. ABIDI. Équation de Navier-Stokes avec densité et viscosité variables dans l'espace critique. *Thèse de l'Université Paris VI.*

[2] H. ABIDI and M. PAICU. Équation de Navier-Stokes avec densité et viscosité variables dans l'espace critique. *Annales de l'institut Fourier*, 57 no. 3 (2007), p. 883-917.

[3] S. ANTONTSEV, A. KAZHIKOV and V. MONAKOV. Boundary value problems in mechanics of nonhomogeneous fluids. Translated from the Russian. *Studies in mathematics and its applications, 22. North-Holland publishing co. Amsterdam*, 1990.

[4] H. BAHOURI and J.-Y. CHEMIN. Équations d'ondes quasilinéaires et estimation de Strichartz, *Amer. J. Mathematics 121*, (1999), 1337-1377.

[5] H. BAHOURI, J.-Y. CHEMIN and R. DANCHIN. Fourier analysis and nonlinear partial differential equations, *to appear in Springer.*

[6] J.-M. BONY. Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires, *Annales Scientifiques de l'école Normale Supérieure 14*, (1981), 209-246.

[7] M. CANNONE, Y. MEYER and F. PLANCHON. Solutions auto-similaires des équations de Navier-Stokes. *Séminaire sur les équations aux dérivées partielles, 1993-1994*, exp. No12 pp. École polytech, palaiseau, 1994.

[8] J-Y CHEMIN. Fluides parfaits incompressibles. *Astérisque, 230*, 1995.

[9] J.-Y. CHEMIN. Théorèmes d'unicité pour le système de Navier-Stokes tridimensionnel, *J. d'Analyse Math.*, 77, (1999), 25-50.

[10] J.-Y. CHEMIN and N. LERNER. Flot de champs de vecteurs non lipschitziens et équations de Navier-Stokes, *J. Differential Equations.*, 121, (1995), 247-286.

[11] R. DANCHIN. Fourier analysis method for PDE's, *Preprint Novembre 2005.*

[12] R. DANCHIN. Local and global well-posedness results for flows of inhomogeneous viscous fluids. *Advances in differential equations*, 9 (2004), 353-386.

[13] R. DANCHIN. Density-dependent incompressible viscous fluids in critical spaces. *Proc. Roy. Soc. Edinburgh Sect. A*, 133(6), 1311-1334, 2003.

[14] R. DANCHIN. The inviscid limit for density-dependent incompressible fluids. *Ann. Fac. Sci. Toulouse Math.*, Sér. 6, 15 no. 4 (2006), p. 637-688.

[15] R. DANCHIN. Well-posedness in critical spaces for barotropic viscous fluids with truly not constant density. *Communications in Partial Differential Equations*,32:9,1373 1397, 2007.

[16] R. DANCHIN. Local Theory in critical Spaces for Compressible Viscous and Heat-Conductive Gases,[*Communication in Partial Differential Equations.* 26 (2001), 7-8, 1183-1233]. *Commun. Partial differential equations.* 27, (2002), 11-12, 2531-2532.

[17] R. DANCHIN. Global Existence in Critical Spaces for Flows of Compressible Viscous and Heat-Conductive Gases, *Arch. Rational Mech. Anal.*, 160, (2001), 1-39.

[18] R. DANCHIN. On the uniqueness in critical spaces for compressible Navier-Stokes equations. *NoDEA Nonlinear Differentiel Equations Appl*, 12(1), 111-128, 2005.

[19] B. DESJARDINS. Linear transport equations with initial values in Sobolev spaces and application to the Navier-Stokes equations, *Differential and Integral Equations*, 10 (1997), 587-598.

[20] B. DESJARDINS. Global existence results for the incompressible density-dependent Navier-Stokes equations in the whole space. *Differential and integral equations*, 10 (1997), 587-598.

[21] B. DESJARDINS. Regularity results for two-dimensional flows of multiphase viscous fluids. *Arch. Rational mech. Anal.*, 137, (1997), 135-158.

[22] E. FERNANDEZ-CARA and F. GUILLEN. The existence of nonhomogeneous viscous and incompressible flow in unbounded domains. *Comm. in Partial Differential Equation*, 17, (1992), 1253-1265.

[23] H. FUJITA and T. KATO. On the Navier-Stokes initial value problem I, *Archive for Rational Mechanics and Analysis*, 16, (1964), 269-315.

[24] S. ITOH and A. TANI. Solvability of nonstationnary problems for nonhomogeneous incompressible fluids and the convergence with vanishing viscosity, *Tokyo Journal of Mathematics*, 22, (1999), 17-42.

[25] V. KAZHIKOV. Resolution of boundary value problems for nonhomogeneous viscous fluids. *Dokl. Akad. Nauh*, 216, (1974), 1008-1010.

[26] H. KOCH and D. TATARU. Well-posedness for the Navier-Stokes equations, *Advances in Math.* 157, 920010, 22-35.

[27] O. LADYZHENSKAYA and V. SOLONNIKOV. The unique solvability of an initial-boundary value problem for viscous incompressible inhomogeneous fluids. *J. Soviet Math.* 9, (1978), 697-749.

[28] P-. L. LIONS, Mathematical topics in fluid dynamics. Vol 1 incompressible models, *Oxford university press* 1996.

[29] Y. MEYER. Wavelets, paraproducts, and Navier-Stokes equation. In Current developments in mathematics, 1996 (Cambridge, MA), page 105-212. *Int. Press, Boston*, MA, 1997.

[30] T. RUNST and W. SICKEL, Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations. *de Gruyter Series in Nonlinear Analysis and Applications, 3. Walter de Gruyter and Co.*, Berlin (1996)