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# ON LEFT REGULAR BANDS AND REAL CONIC-LINE ARRANGEMENTS

MICHAEL FRIEDMAN AND DAVID GARBER

ABSTRACT. As is well-known, every real (and complex) hyperplane arrangement divides the space into a set of faces, on which one can define a natural product, inducing a structure of a left regular band on this set. One can ask whether the same is possible for other arrangements of hypersurfaces. In this paper, we try to answer this question for the simplest generalization of hyperplane arrangements, that is, conic-line arrangements in the plane.

Investigating the different algebraic structures induced on the face poset of a conic-line arrangement, we present two possibilities for the product and its associated structures. We also study the structure of sub left regular bands induced by these arrangements. We finish with some combinatorial properties of conic-line arrangements.

## 1. INTRODUCTION

Hyperplane arrangements are one of the richest subjects in mathematics, as they lie in the intersection of several mathematical areas: algebra, topology and combinatorics. For example, a hyperplane arrangement in  $\mathbb{R}^n$  defines a partition of the space into a collection of faces, denoted by  $\mathcal{L}$ . One can define a product on  $\mathcal{L}$ , making this set into a left regular band  $S$ , that is, a semigroup where every element is an idempotent and for every  $x, y \in S$ ,  $x \cdot y \cdot x = x \cdot y$  (see [14, Section 3] for a survey on bands and examples of left regular bands; see also [4, 5] for a similar description for complex hyperplane arrangements). Moreover, the subset of chambers is the unique two-sided ideal of this band, which leads to the investigation of the structure and combinatorics associated to these chambers. Thus, on one hand, one finds Zaslavsky's restriction-deletion principle [19] for chamber counting, and on the other hand, there exists a description of random walks on the faces of a hyperplane arrangement [6]. One also finds a description of the structure of the algebra  $kS$  in terms of quivers (see [16, 17]) or the structure of the sub-left regular bands associated to every hyperplane which is a boundary of such a chamber.

A natural question is what happens to these simple structures when one deals, instead of hyperplane arrangements, with arrangements of hypersurfaces. Restricting ourselves to  $\mathbb{R}^2$  (or to  $\mathbb{C}^2$ ), this investigation already took place to some extent. Zaslavsky generalized the deletion-restriction criterion

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in several directions (e.g. see [20]). Note that when looking at a smooth real curve arrangement in  $\mathbb{R}^2$ , topologically we look at a subclass of conic-line arrangements.

Moreover, some properties of the fundamental group of the complement of arrangements of lines in  $\mathbb{C}^2$  were found to be determined by, or very closely linked to, the combinatorics of the arrangement: for example, conjugation-free arrangements (see e.g. [9, 10]) or real arrangements with up to 9 lines (e.g. [3, 12, 21]). These properties were found to be similar to properties associated to conic-line arrangements (see [11]).

However, an explicit investigation of the algebraic structure of the face poset of an arrangement of real curves or an investigation of the combinatorics of conic-line arrangements was not carried out. This paper is making a first step in understanding these structures by dealing with the simplest generalization of line arrangements: real conic-line arrangements, defined as follows:

**Definition 1.1.** *A real conic-line (CL) arrangement  $\mathcal{A}$  is a collection of conics and lines defined by the equations  $\{f_i = 0\}$  in  $\mathbb{C}^2$  (or in  $\mathbb{CP}^2$ ), where  $f_i \in \mathbb{R}[x, y]$ . Moreover, for every conic  $C \in \mathcal{A}$ ,  $C \cap \mathbb{R}^2$  is not an empty set, neither a point nor a (double) line.*

The first two sections of this paper investigate which algebraic structures (e.g. left regular bands) can be defined on the face poset of a CL arrangement in  $\mathbb{R}^2$ , in a way that will naturally generalize the structure of a face semigroup of a line arrangement. Section 2 surveys the problems one encounters when performing this generalization and proposes two possibilities for a well-defined product on this set. The most obvious problem is that while the face set of an arrangement of  $n$  lines can be identified with a subset of  $(\{+, -, 0\})^n$ , where each face is given a sign sequence according to its position with respect to the lines, this might not be possible (in a monomorphic way) for a CL arrangement. Thus, the first definition of a product on this set leads to view the face set of a CL arrangement as an alternative left regular band; the second induces the structure of aperiodic semigroup on this face set. Section 3 investigates the structure of sub-left regular bands for a given band, induced by a CL arrangement. Connections between the band, induced by restricting the CL arrangement to a conic or to a line, and the band induced by the whole arrangement, are presented. Section 4 presents some restrictions on the combinatorics of CL arrangements. Moreover, we generalize the restriction-deletion principle to the case of CL arrangements.

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## 2. CONIC-LINE ARRANGEMENTS: THE FACE SEMIGROUP

**2.1. Preliminaries: The left regular band and the face semigroup of hyperplane arrangements.** In this section, we recall the notion of a *left regular band* and its connections to hyperplane arrangements (see a survey in [14]).

**Definition 2.1.** A left regular band (or LRB) is a semigroup  $(S, \cdot)$  that satisfies the identities:

$$x \cdot x = x \quad \text{and} \quad x \cdot y \cdot x = x \cdot y \quad \text{for all } x, y \in S.$$

Let  $\mathcal{A} = \cup H_i$  be a hyperplane arrangement consists of  $n$  hyperplanes, where  $H_i$  is defined by the equation  $\{f_i = 0\}$ , where  $f_i \in \mathbb{R}[x_1, \dots, x_N]$ . Recall that for  $H \in \mathcal{A}$ , the arrangement  $\mathcal{A}^H = \mathcal{A} - \{H\}$  is called the *deleted* arrangement and  $\mathcal{A}_H = \{K \cap H | K \in \mathcal{A}^H\}$  is called the *restricted* arrangement. Let  $\mathcal{C}(\mathcal{A})$  be the set of chambers of  $\mathcal{A}$ , i.e. the components of  $\mathbb{R}^N - \mathcal{A}$ .

Define the collection of faces as:

$$\mathcal{L}(\mathcal{A}) = \bigcup_{X \in \mathcal{A}} \mathcal{C}(\mathcal{A}_X).$$

Let  $\mathcal{L} = \mathcal{L}(\mathcal{A})$ , and define a (monomorphic) function  $i : \mathcal{L} \rightarrow (\{+, -, 0\})^n$ , as follows:

$$(i(P))_k = \text{sign}(f_k(P)), \quad \forall P \in \mathcal{L},$$

where  $(\ )_k$  denotes the value of the  $k^{\text{th}}$  coordinate of the  $n$ -tuple  $i(P)$ .

Recall that on  $(\{+, -, 0\})^n$ , one can define an associative product, extending componentwise the product on  $\{+, -, 0\}$ , given by  $x \cdot y = x$  if  $x \neq 0$ , and  $y$  otherwise. This gives  $\text{Image}(i)$ , as a subset of  $(\{+, -, 0\})^n$ , the structure of a left regular band, and therefore also for  $\mathcal{L}$ , when identifying it with  $\text{Image}(i)$ .

For hyperplane arrangements, this product has a geometric meaning:  $F \cdot K$  is the face that we are in after moving a small positive distance from a point of a generic face  $F$  towards a generic point of a face  $K$  along a straight line (see e.g. [1, Section 1.4.6]).

**2.2. The semigroups  $\mathcal{L}$  and  $\mathcal{L}_0$  for CL arrangements.** Let  $\mathcal{A} = \cup H_i \subset \mathbb{R}^2$  be a real CL arrangement with  $n$  components, and let  $f_i \in \mathbb{R}[x, y]$  be the corresponding forms of the components. Let  $L = L(\mathcal{A})$  be the semi-lattice of non-empty intersections of elements of  $\mathcal{A}$ , and define the collection of faces as:

$$\mathcal{L} = \mathcal{L}(\mathcal{A}) = \bigcup_{X \in L} \mathcal{C}(\mathcal{A}_X).$$

**Definition 2.2.** (a) Define the map

$$\text{supp} : \mathcal{L} \rightarrow L,$$

sending each face to its support (i.e. the corresponding element in the intersection semi-lattice).

(b) As before, define a function:

$$(1) \quad i : \mathcal{L} \rightarrow (\{+, -, 0\})^n$$

as:  $(i(P))_k = \text{sign}(f_k(P))$ , where  $(\ )_k$  is the value of the  $k^{\text{th}}$  coordinate of the  $n$ -tuple  $i(P)$ .

**2.3. Properties of  $\text{Image}(i)$ .** Note that for real hyperplane arrangements, the function  $i$  is monomorphic: every face  $P$  is defined exactly by its set of  $n$  (in)equalities, i.e. for every  $1 \leq i \leq n$ ,  $P$  is either in  $\{f_i > 0\}$ , in  $\{f_i = 0\}$  or in  $\{f_i < 0\}$ . However, for real CL arrangements, this function might not be monomorphic. For example, given a line and a circle tangent to it, the two parts of the line have the same pair of signs. Another example is presented in Figure 1, where we have that:

$$i(P_1) = i(P_2) = (+, -, +).$$

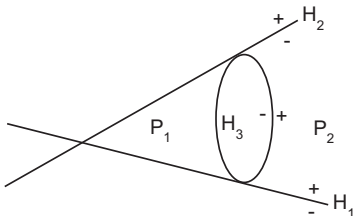


FIGURE 1. An example of a real CL arrangement with two faces  $P_1, P_2$  having the same image  $(+, -, +)$  under  $i$ .

Recall that one can define an associative product on  $(L_2^1)^n = (\{+, -, 0\})^n$  (see Section 2.1). This raises the following question: does this give  $\text{Image}(i)$  the structure of a sub-semigroup of  $(L_2^1)^n$ ? For hyperplane arrangements, the answer is positive and one identifies  $\mathcal{L}$  with  $\text{Image}(i)$ ; thus  $\mathcal{L}$  is endowed with a semigroup structure. But for real CL arrangements, as  $i$  is not necessarily monomorphic, we cannot identify  $\mathcal{L}$  with  $\text{Image}(i)$  (and thus we need to redefine the product on  $\mathcal{L}$ ). A more serious problem is presented in the following example.

**Example 2.3.** (1) There are real CL arrangements whose  $\text{Image}(i)$  is not even closed under the action induced by  $(L_2^1)^n$ , and thus it is not even a semigroup. For example, take three generic lines  $H_1, H_2, H_3$  (i.e. not passing through one point) and a circle  $C$  passing through the three intersection points; see Figure 2. Let  $\alpha, \beta \in \text{Image}(i) \subset (L_2^1)^4$  be two quadruples associated to two different intersection points (see Figure 2; the points are  $a, b$ ). Though  $\alpha, \beta \in \text{Image}(i)$ ,  $\alpha\beta \notin \text{Image}(i)$ , since there is no face which corresponds to the quadruple  $\alpha\beta$ , as there is no element in  $\text{Image}(i)$  that has exactly two zeros in its presentation as a quadruple in  $(L_2^1)^4$ . Explicitly,  $\alpha = i(a) = (0, +, 0, 0), \beta = i(b) = (0, 0, -, 0)$ , but  $\alpha\beta = (0, +, -, 0) \notin \text{Image}(i)$ .

Note that this is the minimal degree example for this phenomenon to occur: one can verify that for any real CL arrangement with up to degree 4,  $\text{Image}(i)$  is always closed under the action induced by  $(L_2^1)^n$ .

(2) The above example can be generalized: take a regular  $n$ -gon,  $n > 3$ , draw a circle passing through all the vertices of the  $n$ -gon, and extend the edges of the polygon into straight lines. One can check that the product of the corresponding signs of any pair of consecutive triple points of this arrangement does not represent any face of this arrangement.

(3) Moreover, taking three generic lines and a circle passing through two intersection points, one can check that the product of the corresponding signs of the pair of triple points of this arrangement does not represent any face of this arrangement.

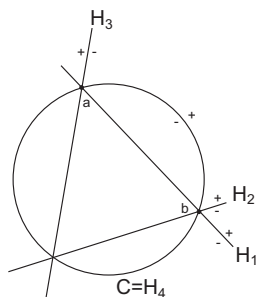


FIGURE 2.  $\alpha = i(a) = (0, +, 0, 0)$ ,  $\beta = i(b) = (0, 0, -, 0)$ , but  $\alpha\beta = (0, +, -, 0) \notin \text{Image}(i)$ .

**Definition 2.4.** Let  $\mathcal{L}_0 = \mathcal{L}_0(\mathcal{A}) = \text{Image}(i) \subseteq (L_2^1)^n$ .

When is  $\mathcal{L}_0(\mathcal{A})$  a semigroup? Obviously, if  $\mathcal{A}$  is a line arrangement, then  $\mathcal{L}_0(\mathcal{A})$  is a semigroup. Moreover, we have the following proposition regarding real CL arrangements:

**Proposition 2.5.** Let  $\mathcal{A}$  be a real CL arrangement. Assume that there is no singular point  $p$  such that there are more than two components passing through  $p$ . Then  $\mathcal{L}_0(\mathcal{A})$  is a semigroup.

*Proof.* Obviously, we have to consider only the arrangements whose singular points are either nodes or tangent points (or both). We need to check that  $\mathcal{L}_0(\mathcal{A})$  is closed under the product induced by  $(L_2^1)^n$ . For each face  $c \in \mathcal{L}(\mathcal{A})$ , we go over all the products of the form  $i(c)i(a)$ , where  $a \in \mathcal{L}(\mathcal{A})$ , and check that  $i(c)i(a) \in \mathcal{L}_0(\mathcal{A})$ .

If  $\dim(c) = 2$ , there is nothing to check, as  $i(c)i(a) = i(c)$  for every  $a \in \mathcal{L}(\mathcal{A})$ , since all the coordinates of  $i(c)$  are non-zero.

If  $\dim(c) = 1$ , let  $H = \text{supp}(c)$ , where  $H = \{f = 0\}$ . Then  $i(c)i(a)$  is either  $i(c)$  or one of the faces that has  $c$  in its boundary (which lies inside the domain  $\{f > 0\}$  or  $\{f < 0\}$ ), which exist as elements in  $\mathcal{L}_0(\mathcal{A})$ .

If  $\dim(c) = 0$ , then  $c$  is either a node or a tangent point. If it is a node, then locally, in the neighborhood of  $c$ , the arrangement is of the form  $\{xy = 0\}$ . Note that as an arrangement in  $\mathbb{R}^2$ ,  $\mathcal{L}_0(\{xy = 0\}) = (L_2^1)^2$ . This means that  $i(c)i(a)$  is in  $\mathcal{L}_0(\mathcal{A})$  for every  $a \in \mathcal{L}(\mathcal{A})$ .

If  $c$  is a tangent point, then locally, in the neighborhood of  $c$ , the arrangement is of the form  $\{y(y - x^2) = 0\}$  (obviously, the arrangement can consist of two tangent conics, however, locally, from the point of view of  $\mathcal{L}_0$ , the resulting set of signs will be the same), and thus, as an arrangement in  $\mathbb{R}^2$ ,

$$L_0 \doteq \mathcal{L}_0(\{y(y - x^2) = 0\}) = (L_2^1)^2 - \{(-, +), (-, 0), (0, +)\},$$

where the first coordinate corresponds to the line  $\{y = 0\}$  and the second to the conic. As can be easily checked,  $L_0$  is closed under multiplication, which means that  $i(c)i(a)$  is in  $\mathcal{L}_0(\mathcal{A})$  for every  $a \in \mathcal{L}(\mathcal{A})$ .  $\square$

**Remark 2.6.** Following the previous proof, we actually have that  $\mathcal{L}_0(\mathcal{A})$  not being a semigroup is a codimension 2 phenomenon. If  $\text{codim}(\text{supp}(\alpha)) = 0$ , then there is no  $j$  such that  $(\alpha)_j = 0$  and thus  $\alpha\beta = \alpha$  for all  $\beta \in \text{Image}(i)$ . For  $\alpha \in \text{Image}(i) \subseteq (L_2^1)^n$  such that  $\text{codim}(\text{supp}(\alpha)) = 1$ , then for every other element  $\beta \in \text{Image}(i)$ ,  $\alpha\beta$  exists as an element in  $\text{Image}(i)$ . Indeed, the fact that  $\text{codim}(\text{supp}(\alpha)) = 1$  means that there is only one coordinate  $j$  such that  $(\alpha)_j = 0$ . Then, either  $(\alpha\beta)_j = 0$  (in this case  $\alpha\beta = \alpha$ ), or  $(\alpha\beta)_j = +$  or  $(\alpha\beta)_j = -$  (note that for  $k \neq j$ ,  $(\alpha)_k = (\alpha\beta)_k$ ). In the last two cases, there exists an element  $\gamma \in \text{Image}(i)$  such that  $\gamma = \alpha\beta$ , since if  $(\alpha\beta)_j = +$ , then  $\gamma$  is defined (on the  $j^{\text{th}}$  coordinate) by  $f_j > 0$ , which exists as a face (as  $\alpha$  is in its boundary). A similar argument works for  $(\alpha\beta)_j = -$  too.

**2.4. Redefining the product.** We want to use the same geometric intuition of the product for hyperplane arrangements (see Section 2.1) for defining the corresponding product on the face poset  $\mathcal{L}$  for real CL arrangements. Explicitly, we want to maintain the following properties for every  $x, y, z \in \mathcal{L}$ :

- (1) For every  $x, y \in \mathcal{L}$ ,  $x^2 = x$  and  $x \cdot y \cdot x = x \cdot y$  (the LRB properties).
- (2) If  $x \cdot y = z$ , then  $i(x)i(y) = i(z)$  (if there are faces with sign sequence  $i(x)i(y)$ ). Explicitly, if  $\mathcal{L}_0(\mathcal{A})$  is a semigroup, then the surjective map  $\mathcal{L}(\mathcal{A}) \rightarrow \mathcal{L}_0(\mathcal{A})$  is a homomorphism.
- (3) If  $x \cdot y = z$ , then  $x \subseteq \bar{z}$  (where  $\bar{z}$  is the closure of  $z$ ).
- (4) If  $x \subseteq \bar{y}$ , then  $x \cdot y = y$ .
- (5)  $(x \cdot y) \cdot z = x \cdot (y \cdot z)$  (Associativity).

We offer two definitions for this product. The first definition, appearing in Section 2.4.1, preserves properties (1), (3) and (4) and thus will be more geometric; the second, appearing in Section 2.4.2, preserves properties (2), (5) and a weaker version of property (1), and thus  $(\mathcal{L}, \cdot)$  will be a semigroup.

2.4.1. *The geometric product.* We start with the more geometric definition, which will be given in two parts. The first part includes the basic requirements of this product. We start with examining the CL arrangement in Figure 3, which shows that requirement (3) (if  $x \cdot y = z$  then  $x \subseteq \bar{z}$ ) is not entirely based on the definition of  $i$ . Explicitly, we want that if  $x \cdot y = z$ , then  $z$  is a face intersecting a small neighborhood of  $x$ . The example in Figure 3 shows that this is not always the case when working with the product induced by  $(L_2^1)^n$ .

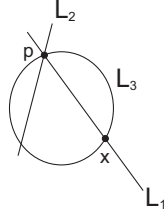


FIGURE 3. As  $i(p) = (0, 0, 0)$ ,  $i(x) = (0, -, 0)$ , we have that  $i(p)i(x) = i(x)$ , but  $x$  is not in the neighborhood of  $p$ .

**Definition 2.7.** (Part I: Geometric product on  $\mathcal{L}(\mathcal{A})$ )

Let  $\mathcal{A}$  be a real CL arrangement,  $P_1, P_2 \in \mathcal{L}(\mathcal{A})$  and choose a generic point  $y \in P_1$ . Look at

$$F_\varepsilon(P_1, P_2) \doteq \{P \in \mathcal{L}(\mathcal{A}) : i(P) = i(P_1)i(P_2) \text{ and } \forall \varepsilon, 0 < \varepsilon \ll 1, P \cap B(y, \varepsilon) \neq \emptyset\}.$$

If  $|F_\varepsilon(P_1, P_2)| = 0$ , then  $P_1 \cdot P_2 \doteq P_1$ .

If  $|F_\varepsilon(P_1, P_2)| = 1$ , i.e.  $F_\varepsilon(P_1, P_2) = \{P\}$ , then  $P_1 \cdot P_2 \doteq P$ .

Otherwise, we know that  $|F_\varepsilon(P_1, P_2)| > 1$ .

If  $P_2 \in F_\varepsilon(P_1, P_2)$ , then  $P_1 \cdot P_2 \doteq P_2$ .

Let us stop for a moment in defining the product. Obviously requirement (3) holds (note that in the example presented in Figure 3, when we use the above product, then  $p \cdot x = p$ ). Also, if  $x, y, z \in \mathcal{L}$ , then  $x \cdot (y \cdot z)$  is a face  $\alpha \in \mathcal{L}(\mathcal{A})$  such that  $x \subseteq \bar{\alpha}$ ,  $x \cdot y$  is a face  $\beta' \in \mathcal{L}(\mathcal{A})$  such that  $x \subseteq \bar{\beta}'$  and  $(x \cdot y) \cdot z$  is a face  $\beta \in \mathcal{L}(\mathcal{A})$  such that  $\beta' \subseteq \bar{\beta}$ ; thus  $x \subseteq \bar{\beta}$ . This means that even if the product is not associative, then

$$(2) \quad x \subseteq \overline{x \cdot (y \cdot z)} \cap \overline{(x \cdot y) \cdot z}.$$

Note also that if for every singular point  $p \in \text{Sing}(\mathcal{A})$  there are two components of  $\mathcal{A}$  intersecting each other transversally at  $p$ , then  $|F_\varepsilon(P_1, P_2)| = 1$  for all  $P_1, P_2 \in \mathcal{L}$  and thus the above product is well-defined for every pair of faces.

**Remark 2.8.** (1) Note that if  $P_1 \in F_\varepsilon(P_1, P_2)$ , then  $P_1 \cdot P_2 = P_1$ . The proof is by considering the dimension of  $P_1$ : if  $\dim(P_1) = 2$ , it is obvious, as already  $i(P_1)i(P_2) = i(P_1)$  and  $P_1$  is the only face in the neighborhood of a generic point of  $P_1$ . If  $\dim(P_1) < 2$ , then in the neighborhood of  $P_1$ , the only



face with the same vector of signs as  $P_1$  is  $P_1$  (note that if  $P_1 \in F_\varepsilon(P_1, P_2)$  then by definition  $i(P_1)i(P_2) = i(P_1)$ ).

(2) Note that  $|F_\varepsilon(P_1, P_2)| \leq 2$ . Indeed, if  $\dim(P_1) \in \{1, 2\}$ , then  $|F_\varepsilon(P_1, P_2)| = 1$ . This is obvious for a chamber. For a section of a curve, the faces of  $F_\varepsilon(P_1, P_2)$  can be  $P_1$  or one of the two chambers having  $P_1$  in their boundary. But each of the three faces has a different vector of signs, and thus  $|F_\varepsilon(P_1, P_2)| = 1$ .

If  $P_1$  is a point, then one can get two faces with the same vector of signs (i.e.  $|F_\varepsilon(P_1, P_2)| = 2$ ) in the neighborhood of  $P_1$  if, for example,  $P_1$  is a tangent point of two curves (either a line and a conic or two conics; this is the simplest case). Indeed, if there is a transversal intersection at  $P_1$ , then  $|F_\varepsilon(P_1, P_2)| = 1$ . Moreover, adding more lines or conics passing through the tangent point will not enlarge  $|F_\varepsilon(P_1, P_2)|$ .

(3) Assume that  $P_1 \neq P_2$ . Then,  $P_1$  and  $P_2$  cannot be together in the set  $F_\varepsilon(P_1, P_2)$ . Indeed, if  $P_1, P_2 \in F_\varepsilon(P_1, P_2)$ , then  $i(P_1) = i(P_2)$ . Also  $P_2$  intersects a neighborhood of any generic point  $y \in P_1$ . The last claim can happen if  $P_1$  is contained in the boundary of  $P_2$  (but then  $i(P_1) \neq i(P_2)$ ), or that  $P_1$  and  $P_2$  will have the same dimension, i.e.  $P_1$  must be equal to  $P_2$ , which is a contradiction.

(4) Assume that  $P_1 \subseteq \overline{P_2}$  and  $P_1 \neq P_2$ . Thus  $\dim(P_2) - 1 = \dim(P_1)$ ,  $i(P_1)i(P_2) = i(P_2)$  and obviously  $P_2 \in F_\varepsilon(P_1, P_2)$ . By definition,  $P_1 \cdot P_2 = P_2$ .

Note that requirement (4) holds, by the definition of  $F_\varepsilon$  and the last case in Definition 2.7. Now, we give a sufficient condition for the associativity of the above product:

**Proposition 2.9.** *If for every two faces  $P_1, P_2 \in \mathcal{L}$ ,  $|F_\varepsilon(P_1, P_2)| = 1$ , then the above product is associative.*

Note that if  $|F_\varepsilon(P_1, P_2)| = 1$  for every  $P_1, P_2 \in \mathcal{L}$ , then the product is already well-defined by Definition 2.7.

*Proof.* Let  $x, y, z \in \mathcal{L}$ . We know that a neighborhood of  $x$  intersects both  $w \doteq (x \cdot y) \cdot z$  and  $v \doteq x \cdot (y \cdot z)$  (by equation (2)), and  $w$  and  $v$  have the same vector of signs (indeed, note that since  $|F_\varepsilon(p, q)| = 1$  for every  $p, q \in \mathcal{L}$ ,  $i(p \cdot q) = i(p)i(q)$ , i.e.  $i$  is a homomorphism and thus  $i(w) = i(x \cdot y)i(z) = (i(x)i(y))i(z) = i(x)(i(y)i(z)) = i(v)$ ). If  $\dim(x) > 0$ , then a neighborhood of  $x$  can intersect only one face with a given vector of signs (see Remark 2.8(2)), which implies that  $v = w$ . If  $\dim(x) = 0$ , a neighborhood of  $x$  may intersect two different faces with the same vector of signs. That is,  $x$  is in the boundary of  $w$  and  $v$ , and thus  $F_\varepsilon(x, w) = \{w, v\}$ , as both  $w$  and  $v$  intersect a neighborhood of  $x$  and  $i(x)i(w) = i(w) = i(v)$  (the first equality is derived from requirement (4)). Thus,  $|F_\varepsilon(x, w)| = 2$ , which is a contradiction. This means that  $v = w$ .  $\square$

Now, as requirement (4) holds, we can prove requirement (1):  $\mathcal{L}$  is an *alternative* left regular band.

**Proposition 2.10.** *Assume that there is a product  $\cdot$  on  $\mathcal{L}$  satisfying the requirements of Definition 2.7. Let  $x, y \in \mathcal{L}$ . Then  $(\mathcal{L}, \cdot)$  is an alternative left regular band, i.e.:*

- (1)  $x^2 = x$ ,
- (2)  $x \cdot (x \cdot y) = (x \cdot x) \cdot y$ ,  $x \cdot (y \cdot y) = (x \cdot y) \cdot y$ ,
- (3)  $x \cdot y \cdot x = x \cdot y$ .

*Proof.* As  $x \in F_\varepsilon(x, x)$ , we get that  $x^2 = x$  (by Remark 2.8(1)). Thus we have to prove that  $x \cdot (x \cdot y) = x \cdot y$  and  $x \cdot y = (x \cdot y) \cdot y$ . If we denote  $z = x \cdot y$ , then  $x \subseteq \bar{z}$ . Thus  $x \cdot (x \cdot y) = x \cdot z = z = x \cdot y$  (the second equality is by Remark 2.8(4), since we can assume that  $x \neq y$ ). Note that if  $|P_\varepsilon(x, y)| > 0$  then  $i(z) = i(x)i(y)$  and thus  $i(z) = i(x)i(y) = i(x)i(y)i(y) = i(z)i(y)$  (since this holds in  $(L_2^1)^n$ ) and so  $z \in F_\varepsilon(z, y)$ ; thus  $z \cdot y = z$  (by Remark 2.8(1)). Otherwise,  $|P_\varepsilon(x, y)| = 0$  and thus  $x \cdot y = x$ , i.e.  $z = x$ . Thus  $z \cdot y = x \cdot y = x = z$ , i.e.  $z \cdot y = z$  in any case.

Therefore,  $(\mathcal{L}, \cdot)$  is an alternative magma. Thus the *flexible identity*

$$(x \cdot y) \cdot x = x \cdot (y \cdot x)$$

holds for any two faces  $x, y \in \mathcal{L}$  and the expression  $x \cdot y \cdot x$  is well-defined.

Again, if  $|P_\varepsilon(x, y)| > 0$  then  $i(x \cdot y) = i(x)i(y)$  and so  $i(x \cdot y) = i(x)i(y) = i(x)i(y)i(x) = i(x \cdot y)i(x)$  and so we have that  $x \cdot y \in F_\varepsilon(x \cdot y, x)$  and thus by Remark 2.8(1),  $x \cdot y = x \cdot y \cdot x$ . Otherwise,  $|P_\varepsilon(x, y)| = 0$  and thus  $x \cdot y = x$  and so  $x \cdot y \cdot x = x \cdot x \cdot x = x = x \cdot y$   $\square$

**Remark 2.11.** Note that the fact that the geometric product (as already described in Definition 2.7) is alternative (rather than associative) is not unnatural. Indeed, this kind of phenomena appears also when looking at the poset of the faces of a building, and more generally, in a projection poset (see [2, p. 26] for its definition).

There are several ways to complete the definition of the product. In this subsection, we will show one way to do so, though we cannot guarantee that the product will be associative (see Example 2.13 for a real CL arrangement inducing a non-associative product).

**Definition 2.12.** (*Part II: Geometric product on  $\mathcal{L}(\mathcal{A})$ )*

*With the notation of Definition 2.7, we continue the definition of the geometric product on  $\mathcal{L}(\mathcal{A})$ . Explicitly, we have that  $|F_\varepsilon(P_1, P_2)| = 2$ ,  $P_1 \not\subseteq \overline{P_2}$ , i.e.  $P_2 \notin F_\varepsilon(P_1, P_2)$  (by Remarks 2.8(1) and 2.8(4)). This situation can only happen when  $P_1$  is a point (see Remark 2.8(2)),  $P_1 \not\subseteq \overline{P_2}$ , and all the components of  $\mathcal{A}$  passing through  $P_1$  are tangent to each other (at  $P_1$ ), see Remark 2.8(2).*

*If  $P_1$  and  $P_2$  are on the same unbounded 1-dimensional component  $H$ , then  $P_1 \cdot P_2$  will be the face (in  $F_\varepsilon(P_1, P_2)$ ) we get after moving from  $P_1$  on  $H$  in the direction of  $P_2$  (see Figure 4(a)).*

*Otherwise, either  $P_1 \cdot P_2$  is a chamber or that  $P_1$  and  $P_2$  are on the same bounded 1-dimensional component (i.e. an ellipse). For each  $P \in$*

$F_\varepsilon(P_1, P_2)$ , let  $\ell_P$  be the minimal length of an arc passing through the point  $P_1$ , a generic point in  $P$  and a generic point in  $P_2$ . If the minimum of the set  $\{\ell_P\}_{P \in F_\varepsilon(P_1, P_2)}$  is achieved only once, say, at a face  $P_0$ , then define  $P_1 \cdot P_2 \doteq P_0$  (see Figure 4(b)). However, if there exist two faces  $P', P''$  such that:

$$\min_{P \in F_\varepsilon(P_1, P_2)} \{\ell_P\} = \ell_{P'} = \ell_{P''},$$

then draw a circle  $C$  through  $P_1$ , a generic point in  $P'$  (or in  $P''$ ) and a generic point in  $P_2$  and define  $P_1 \cdot P_2 \doteq P$ , where  $P \in \{P', P''\}$  is the face we are in after moving slightly clockwise on  $C$  from  $P_1$  (see Figure 4(c)).

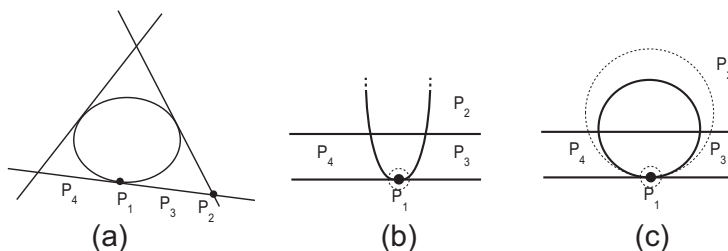


FIGURE 4. Different situations for the geometric product on  $\mathcal{L}(\mathcal{A})$ : In part (a),  $F_\varepsilon(P_1, P_2) = \{P_3, P_4\}$  and  $P_1$  and  $P_2$  are on the same unbounded 1-dimensional component. Thus,  $P_1 \cdot P_2 = P_3$ . In part (b),  $P_2 \notin F_\varepsilon(P_1, P_2) = \{P_3, P_4\}$ . Moreover,  $\ell_{P_3} < \ell_{P_4}$ , so we have:  $P_1 \cdot P_2 = P_3$ . In part (c), again  $P_2 \notin F_\varepsilon(P_1, P_2) = \{P_3, P_4\}$ , but in this case  $\ell_{P_3} = \ell_{P_4}$ , so we draw a circle  $C$  through  $P_1$ , a generic point in  $P' = P_3$  and a generic point in  $P_2$ , and move on it clockwise to get:  $P_1 \cdot P_2 = P_4$ .

**Example 2.13** (Non-associative product). Look at the real CL arrangement  $\mathcal{A}_0$  presented in Figure 5. All the labeled faces in Figure 5 are faces on the

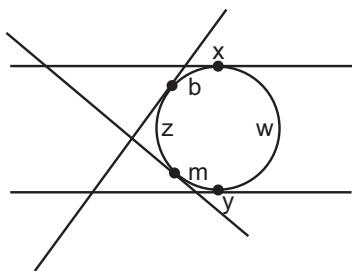


FIGURE 5. An example of a non-associative geometric product:

$$b = x \cdot (y \cdot z) \neq (x \cdot y) \cdot z = w$$

circle, where  $x, y$  are tangent points and  $b, w, m$  and  $z$  are 1-dimensional faces. We use Definition 2.12 in order to compute  $(x \cdot y) \cdot z$  and  $x \cdot (y \cdot z)$ .

Note that  $F_\varepsilon(x, y) = \{b, w\}$  and  $\ell_b = \ell_w$ . Thus, we should go clockwise on the circle  $C$  from  $x$  to  $y$  and therefore  $x \cdot y = w$ ,  $(x \cdot y) \cdot z = w \cdot z = w$ . However,  $F_\varepsilon(y, z) = \{m, w\}$  and  $\ell_m < \ell_w$ . Thus  $y \cdot z = m$  and by the same reasoning,  $x \cdot m = b$ . Therefore  $x \cdot (y \cdot z) = x \cdot m = b$ . Thus the geometric product is not associative for this CL arrangement.

Note that  $\mathcal{L}_0(\mathcal{A}_0)$  is an associative LRB, by Proposition 2.5.

**Remark 2.14.** Let  $\mathcal{A}$  be a real CL arrangement.

(1) If  $\mathcal{L}_0$  is a semigroup, the construction of  $\mathcal{L}$  implies that there are CL arrangements which induce two non-isomorphic LRBs (i.e.  $\mathcal{L}$  and  $\mathcal{L}_0$ ) with the same associated semi-lattice  $L$ . A simple example for this is a line and a circle tangent to it:  $\mathcal{L}_0$  has 6 elements, while  $\mathcal{L}$  has 7 elements, where the product in  $\mathcal{L}$  is taken as the one defined in Definition 2.12 (in this case,  $\mathcal{L}$  is indeed an associative LRB, see Example 2.18).

(2) Given a magma  $(\mathcal{L}, \cdot)$  (here we do not require that the operation  $\cdot$  will be specifically defined), one can define another semi-lattice  $L'$  such that there exists an epimorphism  $\mathcal{L} \rightarrow L'$  in the following way: for  $x, y \in \mathcal{L}$  define  $x \sim y$  if  $x \cdot y = x$  and  $y \cdot x = y$ . As this relation is reflexive, symmetric and transitive,  $L' \doteq \mathcal{L} / \sim$  is a semi-lattice (see e.g. [18, p. 153]). Taking  $\mathcal{L}$  to be the associated face set of a CL arrangement, we note that  $L'$  should not be the intersection lattice  $L$ . For example, the arrangement consisting of a conic intersecting transversally a line has the property that  $|F_\varepsilon(P_1, P_2)| = 1$  for every  $P_1, P_2 \in \mathcal{L}$  and thus the geometric product, as defined in Definition 2.7, is associative. However, the two intersection points are identified in  $L'$  but remain distinct in  $L$ . However, if  $\mathcal{A}$  is a line arrangement (or, more generally, a hyperplane arrangement), then  $L = L'$ , where the product in  $\mathcal{L} = \mathcal{L}_0$  is induced by the product in  $(L_2^1)^n$  (see e.g. [17]).

**2.4.2. The associative product.** As we saw in Example 2.13, the product defined in Definition 2.12 is not necessarily associative. Moreover, it does not satisfy requirement (2), i.e., if  $x \cdot y = z$  then  $i(x)i(y) = i(z)$ , where  $i : \mathcal{L} \rightarrow \mathcal{L}_0$  is the sign function. In this section, we define a different product on  $\mathcal{L}$  that will be associative and satisfy requirement (2). However, in order for that to happen, we have to assume that  $\mathcal{L}_0$  is closed under multiplication (induced by  $(L_2^1)^n$ ; see Example 2.3(1) for an example of an arrangement whose Image( $i$ ) is not closed under this multiplication).

**Definition 2.15.** Let  $\mathcal{A}$  be a real CL arrangement such that  $\mathcal{L}_0$  is closed under the product induced by  $(L_2^1)^n$ . Define a function  $j : \mathcal{L}_0 \rightarrow \mathcal{L}$  as follows. For every  $a \in \mathcal{L}_0$ , if  $|i^{-1}(a)| = 1$ , then  $j(a) \doteq i^{-1}(a)$ . Otherwise, choose an element  $a_0 \in i^{-1}(a)$  and define  $j(a) \doteq a_0$ .

For any two faces  $x, y \in \mathcal{L}$  define  $x \cdot y \doteq j(i(x)i(y))$ .

**Proposition 2.16.** *Let  $(\mathcal{L}, \cdot)$  be the face poset of a real CL arrangement, when the product is defined as in Definition 2.15 (i.e. the function  $j$  is already given). Then:*

- (1)  $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ .
- (2)  $x \cdot y \cdot x = x \cdot y$ .
- (3)  $x^2$  is not necessarily equal to  $x$ .

*Proof.* Properties (1) and (2) are immediate, since these identities are already satisfied in  $\mathcal{L}_0$  (as a subset of  $(L_2^1)^n$ ), i.e.  $i(x)i(y)i(z) = (i(x)i(y))i(z)$  and  $i(x)i(y)i(x) = i(x)i(y)$ . For property (3), look at the arrangement consists of a line intersecting transversally a circle. Let  $p_1, p_2$  be the two intersection points, and denote  $\alpha = i(p_1)$ . Note that  $i(p_1) = i(p_2) = \alpha$ . We may choose  $j(\alpha) = p_1$ . and thus,  $p_2^2 = p_1$ . For the other choice, i.e.  $j(\alpha) = p_2$ , we get that  $p_1^2 = p_2$ .  $\square$

**Remark 2.17.** (1) The product defined in Definition 2.15 satisfies  $x^2 = x^3$  (this is a specific case of Proposition 2.16(2), when taking  $x = y$ ). Thus  $(\mathcal{L}, \cdot)$  is an *aperiodic* semigroup, i.e. for every  $x \in \mathcal{L}$ ,  $x^2$  is an idempotent and the set  $\{x^2 : x \in \mathcal{L}\}$  is an LRB, isomorphic to  $\mathcal{L}_0$ .

(2) Note that once there are different faces in  $\mathcal{L}$  having the same image under  $i$ , then Definition 2.15 does not define a unique product on  $\mathcal{L}$ , as it depends on the choice made by the function  $j$  in this definition.

**2.5. LRBs coming from hyperplane arrangements and CL arrangements.** We give an example of an LRB, induced by a CL arrangement, which cannot be embedded in  $(L_2^1)^n$  for any  $n \in \mathbb{N}$ . This immediately implies that this LRB is not isomorphic to the face LRB of a hyperplane arrangement.

**Example 2.18.** Consider the real CL arrangement  $\mathcal{A}$  which consists of a line and a circle tangent to it (see Figure 6). This arrangement has 7 faces, and we denote the two parts (i.e. faces) of the line by  $b$  and  $a$ , the circle by  $c$ , and the face below the line and outside the circle by  $d$  (note that  $i(a) = i(b)$ ). Let  $e$  be the tangency point. As usual, denote the set of faces by  $\mathcal{L}$ .

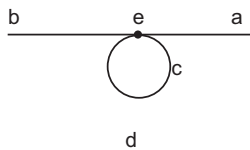


FIGURE 6

The product on  $\mathcal{L}$  is defined using Definition 2.12 (actually, in this example, Definition 2.7 is enough). Note that  $e$  is the identity element and one can check that  $(\mathcal{L}, \cdot)$  is an associative LRB. We have the following multiplication table for  $\{a, b, c\}$ :

$$\begin{aligned} a^2 &= a, & b^2 &= b, & c^2 &= c, \\ a \cdot b &= a, & a \cdot c &= c \cdot a = d, \\ b \cdot a &= b, & b \cdot c &= c \cdot b = d. \end{aligned}$$

Assume that we have a monomorphism  $h : \mathcal{L} \rightarrow (L_2^1)^n$  for some  $n$ . Note that the equalities

$$a \cdot b = a, \quad b \cdot a = b$$

imply that  $h(a)$  and  $h(b)$  have zeros in the same coordinates.

As  $a \neq b$ ,  $h(a) \neq h(b)$ , which means that there exists  $j$ ,  $1 \leq j \leq n$ , such that  $(h(a))_j \neq (h(b))_j$  and both coordinates are not zero (so without loss of generality, one is  $+$  and the other is  $-$ ). But

$$\begin{aligned} (h(d))_j &= (h(a \cdot c))_j = (h(a))_j(h(c))_j = (h(a))_j \neq \\ &(h(b))_j = (h(b))_j(h(c))_j = (h(b \cdot c))_j = (h(d))_j, \end{aligned}$$

by the multiplication laws in  $L_2^1$ , which is a contradiction.

**Remark 2.19.** We note here that  $\mathcal{L}(\mathcal{A})$  (with the product defined using Definition 2.7) is an example of a left regular band which is not *geometric* (i.e. it cannot be embedded in  $(L_2^1)^n$  for any  $n$ , see [14, Section 3.7]).

Moreover, note that for the LRB  $\mathcal{L}_0(\mathcal{A})$  (which is contained in  $(L_2^1)^2$ ), one cannot find a hyperplane arrangement  $\mathcal{A}' \subset \mathbb{R}^N$  such that  $\mathcal{L}_0(\mathcal{A}) \cong \mathcal{L}(\mathcal{A}') = \mathcal{L}_0(\mathcal{A}')$ . Indeed,  $\mathcal{L}_0(\mathcal{A})$  has 6 elements, has a unit  $(0, 0)$  and the three elements  $(+, +)$ ,  $(-, -)$ ,  $(-, +)$  form the unique two-sided ideal. Thus, if such a hyperplane arrangement  $\mathcal{A}'$  exists, it should be a central hyperplane arrangement with three chambers, which is impossible.

In fact, this can be generalized: given a line and a parabola tangent to it, then the LRB of faces is not a sub-LRB of  $(L_2^1)^n$ , using the same method (again, we use the product as defined in Definition 2.12, which is also associative in this case). This means that whenever a CL arrangement has a singular point which is locally of the form  $y(y - x^2) = 0$ , then, if we define the product on the set of faces using Definition 2.12 and this product is associative, then it can not be a sub-LRB of  $(L_2^1)^n$ , for any  $n$ .

**Remark 2.20.** (*Steinberg and Margolis, private communication*)

Note that if  $S$  is an LRB and  $k$  is a field, then there exists  $r \in \mathbb{N}$  such that the semigroup algebra  $kS$  can be embedded in  $k[(L_2^1)^r]$ , even if  $S$  cannot be embedded in  $(L_2^1)^m$  for any  $m$ . Indeed,  $kS$  is a basic algebra (see [16]) and thus it is a subalgebra of some triangular matrix algebra. Note that the algebra of  $n \times n$  triangular matrices can be embedded in the tensor product of  $n - 1$  copies of the algebra of  $2 \times 2$  triangular matrices. Since the algebra of  $(L_2^1)^r$  is the tensor product of  $r$  copies of the algebra of  $L_2^1$  which is the algebra of  $2 \times 2$  triangular matrices, this shows that every triangular matrix algebra is embedded in  $(L_2^1)^r$  for some  $r$ .

### 3. PLANE CURVE ARRANGEMENTS: STRUCTURE OF SUB-LRBs

Note that for any arrangement of smooth hypersurfaces  $\{f_1 = 0, \dots, f_m = 0\}$  in  $\mathbb{R}^N$  (where  $f_i \in \mathbb{R}[x_1, \dots, x_N]$ ,  $N > 1$ ), one can associate an  $m$ -tuple in  $(\{+, -, 0\})^m$  to any face in the arrangement, describing the position of this face with respect to the hypersurfaces  $f_i$ . Explicitly, for any such arrangement  $\mathcal{A} = \{f_1 = 0, \dots, f_m = 0\}$  we have, as before, an associated subset  $\mathcal{L}_0$  of  $(L_2^1)^m$  induced by the vectors of signs.

*Note:* From now on, we assume that  $\mathcal{L}_0$  is an LRB, i.e. it is closed under the multiplication induced by  $(L_2^1)^m$ . Moreover, to simplify notations, we assume that each  $H_i$  is connected in  $\mathbb{R}^N$ .

In this section, we want to study the explicit structure of sub-LRBs of  $\mathcal{L}_0$  for a given arrangement  $\mathcal{A}$  induced by the hypersurfaces  $\{H_1, \dots, H_m\}$  in  $\mathbb{R}^N$ , where  $H_i$  is defined by the hypersurface  $\{f_i = 0\}$ .

**3.1. The deletion-restriction principle.** Let  $H \doteq H_i \in \mathcal{A}$  be a given hypersurface. As before, one can define two associated LRBs. The first is the *deleted LRB*, corresponding to the deletion of the hypersurface  $H$  from  $\mathcal{A}$ , and the second is the *restricted LRB*, corresponding to the restriction of the arrangement  $\mathcal{A}$  to  $H$ . Explicitly,  $\mathcal{A}^H = \mathcal{A} - \{H\}$  is the *deleted* arrangement and  $\mathcal{A}_H = \{K \cap H \mid K \in \mathcal{A}^H\}$  is the *restricted* arrangement. Then,  $\mathcal{L}_0(\mathcal{A}^H)$  is obtained from  $\mathcal{L}_0(\mathcal{A})$  by deleting the  $i^{\text{th}}$  coordinate.

We also define:

$$\mathcal{L}_0(\mathcal{A})|_H \doteq \{x \in \mathcal{L}_0(\mathcal{A}) : (x)_i = 0\} \subset (L_2^1)^m.$$

$\mathcal{L}_0(\mathcal{A})|_H$  is a sub-LRB of  $\mathcal{L}_0(\mathcal{A})$ , to which corresponds the restricted arrangement  $\mathcal{A}_H$  as a sub-LRB. Indeed, it is a subset of  $\mathcal{L}_0(\mathcal{A})$  and thus the associativity and the properties  $x^2 = x, xyx = xy$  are immediately satisfied. The closure under multiplication is obvious. Note that  $\#\mathcal{L}_0(\mathcal{A}_H) = \#(\mathcal{L}_0(\mathcal{A})|_H)$ .

As  $H$  is not necessarily a hyperplane,  $\mathcal{A}_H$  is an arrangement in  $H$ , but it is not necessarily an arrangement in  $\mathbb{R}^k$  for some  $k$  (i.e. though  $H$  can be embedded in  $\mathbb{R}^N$ , we look at the arrangement in  $H$ ). However, when the arrangement is in  $\mathbb{R}^2$ , one can ask what are the relations between  $\mathcal{L}_0(\mathcal{A}_H)$  and  $\mathcal{L}_0(\mathcal{A})|_H$ . Note that when either  $H$  is a bounded component or an unbounded one,  $\mathcal{A}_H$  is a collection of points  $\{p_1, \dots, p_k\}$  on  $H$ . There are two cases to consider - where  $H$  is an unbounded component and where  $H$  is a bounded one.

If  $H$  is an unbounded component, we can set a direction on  $H$ , defined by choosing a continuous family of tangent vectors at each point on  $H$ . Then for each point  $p \in H$ , each point  $p_j$  (for  $1 \leq j \leq k$ ) determines a triple of signs  $\{+, -, 0\}$  describing the position of  $p$  with respect to  $p_j$ . Obviously, if  $q$  is another point in the same face of  $p$ , then the  $k$ -tuples of signs of  $p$  and  $q$  are the same. As usual, we get that the set  $\mathcal{L}_0(\mathcal{A}_H)$  is contained in  $(L_2^1)^k$ .

Moreover, the LRB structure of  $\mathcal{L}_0(\mathcal{A}_H)$ , induced by the LRB structure of  $(L_2^1)^k$ , describes the movement of a generic point along the chosen tangent vector field.

On the other hand, if  $H$  is a bounded component in  $\mathbb{R}^2$ , i.e. an oval, we can consider  $\mathcal{A}_H$  as an arrangement of points  $\{p_1, \dots, p_k\}$  on a circle. However, we cannot treat as in the former case, since there is no meaning to the phrase "the point  $q$  is to the right of the point  $p$ " when we are on an oval. However, there is another way to associate an LRB structure to  $\mathcal{A}_H$ . Let  $H = C$  be a bounded oval and  $\{p_1, \dots, p_k\}$  points on it numerated consecutively. We show now that the associated LRB structure is independent of the numeration, i.e. of the choice of the initial point  $p_1$ . Let  $p'_1$  (resp.  $p'_2$ ) be a point to the right of  $p_1$  (resp.  $p_2$ ), very close to  $p_1$  (resp.  $p_2$ ). Let  $C_i = C - \{p'_i\}$ ,  $i \in \{1, 2\}$ , see Figure 7(a).

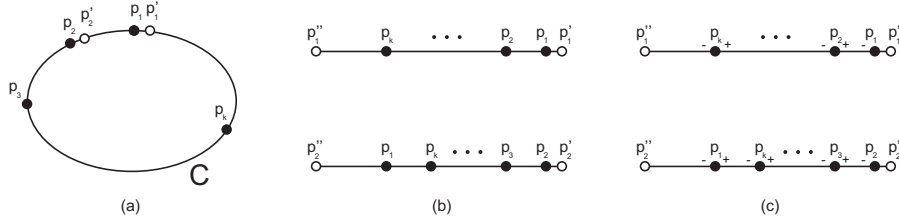


FIGURE 7. The LRB structure associated to an arrangement of points on a conic

We think of  $C_1$  (resp.  $C_2$ ) as a straight segment that starts at the point  $p'_1$  (resp.  $p'_2$ ), when the section that starts at  $p_k$  (resp.  $p_1$ ) ends at a point  $p''_1$  (resp.  $p''_2$ ), which, on  $C_1$  (resp.  $C_2$ ), is identified with  $p'_1$  (resp.  $p'_2$ ), see Figure 7(b). Given a point  $p \in C_i$  ( $i = 1, 2$ ), every point  $p_j$ ,  $1 \leq j \leq k$ , determines a sign for  $p$  chosen from the triple of signs  $\{+, -, 0\}$ , depending whether  $p$  is to the right (or to the left) of  $p_j$ , to the left of it (or to the right, resp.) or equal to it respectively. Note that  $\mathcal{A}_H$  has  $2k$  faces. For sign assignments on the different realizations of  $\mathcal{A}_H$ , see Figure 7(c). Thus, when ignoring the infinitesimally-small section between  $p'_1$  and  $p_1$  (resp. between  $p'_2$  and  $p_2$ ), we get that  $C_1$  (resp.  $C_2$ ) consists of  $2k$  faces. Note that the most left and the most right sections on  $\mathcal{A}_H$  are identified; this is the reason that on  $C_1$  (resp.  $C_2$ ), the right-most section is ignored. Thus, for every face on  $C_1$  (resp.  $C_2$ ) one can assign a  $k$ -tuple in  $(L_2^1)^k$ , as described above. This gives  $\mathcal{L}_0(C_1)$  and  $\mathcal{L}_0(C_2)$  a structure of an LRB.

**Proposition 3.1.** *The LRBs  $\mathcal{L}_0(C_1)$  and  $\mathcal{L}_0(C_2)$  are isomorphic. Therefore, the LRB structure on  $\mathcal{A}_H$  is independent of the choice of the point  $p'_1$ .*

*Proof.* First, note that  $\mathcal{L}_0(C_1)$  consists of the following  $2k$  elements (starting from the right-most point (in this case  $p_1$ ) and going left over all the other faces):

$$\mathcal{L}_0(C_1) = \{(0, +, +, \dots, +), (-, +, +, \dots, +), (-, 0, +, \dots, +),$$



$$(-, -, +, \dots, +), \dots, (-, -, -, \dots, -)\}.$$

$\mathcal{L}_0(C_2)$  consists of the following  $2k$  elements (starting from the right-most point (in this case  $p_2$ ) and going left over all the other faces):

$$\begin{aligned} \mathcal{L}_0(C_2) = \{ & (+, 0, +, +, \dots, +), (+, -, +, +, \dots, +), (+, -, 0, +, \dots, +), \\ & (+, -, -, +, \dots, +), \dots, (+, -, -, \dots, -, 0), (+, -, -, \dots, -, -), \\ & (0, -, -, \dots, -, -), (-, -, -, \dots, -, -)\}. \end{aligned}$$

Both LRBs describe the movement over the  $2k$  faces along a bounded straight segment with  $k$  marked points, i.e. given faces  $x$  and  $y$ , then  $x \cdot y$  is the face we enter in after the movement from  $x$  to  $y$  on this line and thus they are isomorphic. Thus the explicit isomorphism from  $\mathcal{L}_0(C_1)$  to  $\mathcal{L}_0(C_2)$  maps the points  $p_i \mapsto p_{i \pmod{k} + 1}$  and the sections of  $C_1$  are mapped to the corresponding sections of  $C_2$ , according to the mapping of the points.  $\square$

Thus, we can choose a point  $p$  infinitesimally-close to the point  $p_1$  and we delete it. By this way, we can consider the LRB associated to  $\mathcal{A}_H - \{p\}$ , as in the case of unbounded component (when ignoring the infinitesimally-small section between  $p$  and  $p_1$ ). As was shown, this LRB does not depend on the location of  $p$  (when the only condition is that  $p \neq p_j$  for all  $j$ ) up to an isomorphism. Denote this associated LRB by  $\mathcal{L}_0(\mathcal{A}_H)$ , which is a sub-LRB of  $(L_2^1)^k$ . For an explicit example, see Figure 10 and Example 3.7(3) below.

We are ready to describe the structure of the sub-LRBs of  $\mathcal{L}_0(\mathcal{A})$  induced by the components of  $\mathcal{A} \subset \mathbb{R}^2$ . Recall that for  $\alpha \in (L_2^1)^k$ ,  $(\alpha)_i$  denotes the value of the  $i^{\text{th}}$  coordinate of  $\alpha$ .

We start with the trivial case of line arrangements in  $\mathbb{R}^2$  (which can be easily generalized to hyperplane arrangements), as a preparation for the main proposition, which deals with arrangements of smooth real curves (see Proposition 3.6).

**Lemma 3.2.** *Let  $\mathcal{A} = \{H_1, \dots, H_m\}$  be an arrangement of lines in  $\mathbb{R}^2$ , such that  $H_i = \{f_i = 0\}$ . Denote  $H = H_1$  and let  $H \cap \{H_2, \dots, H_m\} = \{p_1, \dots, p_k\} \subset H$ . Then, there is an isomorphism of LRBs:*

$$\varphi : \mathcal{L}_0(\mathcal{A}_H) \rightarrow \mathcal{L}_0(\mathcal{A})|_H \subseteq (L_2^1)^m,$$

satisfying the following properties:

- (1)  $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_1 = 0$ .
- (2) For every  $j > 1$ :
  - (a) If  $H \cap H_j = \emptyset$ , then  $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_j$  is constant (either  $+$  or  $-$ , depending on the mutual position of the parallel lines  $H$  and  $H_j$ ). Explicitly, all the vectors in  $\varphi(\mathcal{L}_0(\mathcal{A}_H))$  have the same sign in the  $j^{\text{th}}$  coordinate.
  - (b) If  $H \cap H_j = \{p_s\}$ , then  $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_j = (\mathcal{L}_0(\mathcal{A}_H))_s$ , up to a constant scalar multiplication in  $\{\pm 1\}$ . The index of the right hand side is the index of  $p_s$  in the arrangement of points on  $H = H_1$ , i.e. in  $\mathcal{L}_0(\mathcal{A}_H)$ .

Before the proof, we illustrate this lemma by an example.

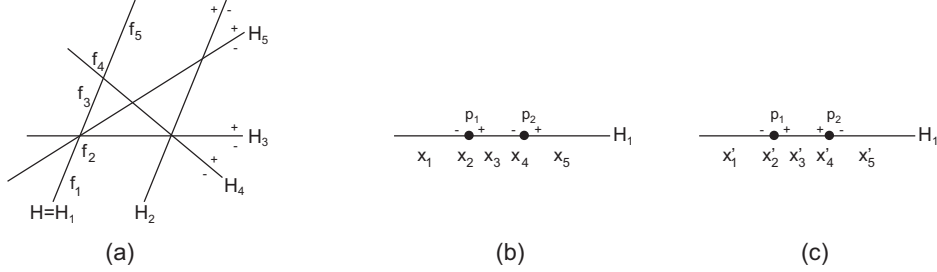


FIGURE 8. An example for illustrating Lemma 3.2:  $f_i$  are the faces contained in  $H_1$  in the face set  $\mathcal{L}(\mathcal{A})$ .  $x_i$  (and  $x'_i$ ) are the faces of the face set  $\mathcal{L}_0(\mathcal{A}_{H_1})$ .

**Example 3.3.** Figure 8(a) presents an arrangement  $\mathcal{A}$  and Figures 8(b) and 8(c) present two arrangements of points on a line, both can be thought of as the restricted arrangement  $\mathcal{A}_{H_1}$ . Note that the difference between the arrangements in Figures 8(b) and 8(c) is that the signs assigned with respect to the point  $p_2$  are opposite.

- (1) Considering the arrangement in Figure 8(b), the faces of  $\mathcal{A}_{H_1}$  are denoted by  $x_1, \dots, x_5$ ; their corresponding images, i.e. these faces in the arrangement  $\mathcal{A}$  are denoted by  $f_1, \dots, f_5$ . Let  $H = H_1$ . Then, the corresponding LRBs are

$$\begin{aligned} \mathcal{L}_0(\mathcal{A}_H) = \{i(x_1) = (-, -), i(x_2) = (0, -), i(x_3) = (+, -), \\ i(x_4) = (+, 0), i(x_5) = (+, +)\}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}_0(\mathcal{A})|_H = \varphi(\mathcal{L}_0(\mathcal{A}_H)) = \{i(f_1) = (0, +, -, -, -), i(f_2) = (0, +, 0, -, 0), \\ i(f_3) = (0, +, +, -, +), i(f_4) = (0, +, +, 0, +), i(f_5) = (0, +, +, +, +)\}. \end{aligned}$$

- (a) First, note that  $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_1 = 0$  (case (1) of the lemma).  
 (b) Since  $H \cap H_2 = \emptyset$ ,  $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_2 = +$ , i.e., by case (2)(a), the second coordinate in all the vectors of  $\varphi(\mathcal{L}_0(\mathcal{A}_H))$  is +.  
 (c) Since  $H_3 \cap H = H_5 \cap H = \{p_1\}$ ,

$$(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_3 = (\varphi(\mathcal{L}_0(\mathcal{A}_H)))_5 = (\mathcal{L}_0(\mathcal{A}_H))_1$$

(by case (2)(b)).

- (d) Since  $H_4 \cap H = \{p_2\}$ ,  $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_4 = (\mathcal{L}_0(\mathcal{A}_H))_2$  (again by case (2)(b)).

- (2) Considering the arrangement in Figure 8(c), the faces of  $\mathcal{A}_{H_1}$  are denoted by  $x'_1, \dots, x'_5$ . In this case:

$$\begin{aligned} \mathcal{L}_0(\mathcal{A}_H) = \{i(x'_1) = (-, +), i(x'_2) = (0, +), i(x'_3) = (+, +), \\ i(x'_4) = (+, 0), i(x'_5) = (+, -)\}. \end{aligned}$$

As before, since  $H_3 \cap H = H_5 \cap H = \{p_1\}$ ,

$$(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_3 = (\varphi(\mathcal{L}_0(\mathcal{A}_H)))_5 = (\mathcal{L}_0(\mathcal{A}_H))_1.$$

On the other hand, as  $H_4 \cap H = \{p_2\}$ ,  $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_4 = -(\mathcal{L}_0(\mathcal{A}_H))_2$ . Explicitly, in contrast to Example (1)(d) above, one has to multiply by the scalar  $-1$  to all the values in  $(\mathcal{L}_0(\mathcal{A}_H))_2$  in order to obtain  $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_4$ .

**Remark 3.4.** Note that  $\mathcal{L}_0(\mathcal{A}_H) \subseteq (L_2^1)^k$  and  $\mathcal{L}_0(\mathcal{A})|_H \subseteq (L_2^1)^m$ . In order not to confuse between the different vectors of signs when we talk on a corresponding face which can be thought of as both in  $\mathcal{L}(\mathcal{A}_H)$  and in  $\mathcal{L}(\mathcal{A})$ , we denote

$$i_{\mathcal{A}} : \mathcal{L}(\mathcal{A}) \rightarrow (L_2^1)^m, \text{ Image}(i_{\mathcal{A}}) = \mathcal{L}_0(\mathcal{A})$$

and

$$i_H : \mathcal{L}(\mathcal{A}_H) \rightarrow (L_2^1)^k, \text{ Image}(i_H) = \mathcal{L}_0(\mathcal{A}_H),$$

where both functions describe the vectors of signs in  $\mathcal{L}_0(\mathcal{A})$  (resp.  $\mathcal{L}_0(\mathcal{A}_H)$ ) of a face in  $\mathcal{L}(\mathcal{A})$  (resp.  $\mathcal{L}(\mathcal{A}_H)$ ).

*Proof of Lemma 3.2.* Case (1) is obvious (since we are on  $H_1$ ).

For case (2)(a), note that if  $H \cap H_j = \emptyset$ , then  $H_j$  is parallel to  $H$  and all the faces of  $\mathcal{A}$  with support in  $H$  are either on the halfplane  $\{f_j > 0\}$  (in this case  $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_j = +$ ) or on  $\{f_j < 0\}$  (in this case  $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_j = -$ ).

As for case (2)(b): assume that  $H \cap H_j = \{p_s\}$  and let  $c$  be a face of  $\mathcal{A}$  with  $\text{supp}(c) \subseteq H$ . As  $c$  goes over all the faces such that  $\text{supp}(c) \subseteq H$ , it passes over all the set  $\text{Image}(\varphi)$ . Then, either  $c \subset \{f_j > 0\}$ ,  $c \subset \{f_j < 0\}$  or  $c \subset \{f_j = 0\}$ . In the third case,  $c = p_s \in H$  and thus  $(i_H(c))_s = 0$  and as  $c \in H_j$ ,  $(i_{\mathcal{A}}(c))_j = 0$ . As for the first two cases, the fact that  $c$  is in one of the two halfplanes is determined by the position of  $H$  with respect to  $H_j$  (as  $c \subset H$ ), which reduced to checking if  $c$  is located to the right of  $\{p_s\} = H \cap H_j$  or to its left. Therefore, up to a constant scalar multiplication by  $\{\pm 1\}$  (for all faces  $c$  such that  $\text{supp}(c) \subseteq H$ ),  $c \subset \{f_j > 0\}$  is equivalent to the fact that  $c$  is to the right of  $p_s$ . The (constant) scalar multiplication is needed, since a priori there is no connection between the sign in  $\mathcal{L}_0(\mathcal{A}_H)$  that is assigned to the faces to the right of  $p_s$  and the sign in  $\mathcal{L}_0(\mathcal{A})$  assigned to these faces in the halfplane above  $H_j$  (see Figures 8(b) and 8(c) for an example of two different sign-assignments for  $\mathcal{A}_H$ ).  $\square$

**Remark 3.5.** Note that there is a natural LRB structure on  $\mathcal{L}_0(\mathcal{A}_H)$ , that is induced by  $\mathcal{L}_0(\mathcal{A})$ , in the following way: if the sign in  $\mathcal{L}_0(\mathcal{A})$  that is given to the halfplane  $\{f_j > 0\}$  is  $+$  (in the  $j^{\text{th}}$  coordinate, where  $H_j = \{f_j = 0\}$  and  $H \cap H_j = \{p_s\}$ ), and the section  $\{x > p_s\}$  on the line  $H$  is contained in  $\{f_j > 0\}$ , then the sign in  $\mathcal{L}_0(\mathcal{A}_H)$  associated to  $\{x > p_s\}$  (i.e. in the  $s^{\text{th}}$  coordinate) will also be  $+$ . If  $\{x < p_s\} \subset \{f_j > 0\}$ , then the sign associated

to the faces contained in this section will be  $+$ . In the case of this naturally induced signs to  $\mathcal{A}_H$ , the scalar multiplication in Lemma 3.2 is not needed.

Note that Lemma 3.2 is more general, as we do not assume any a priori connection between the signs associated to the halfplanes in  $\mathcal{A}$  and the signs associated to the half-lines in  $\mathcal{A}_H$ .

We now pass to the general case of arrangements of smooth curves.

**Proposition 3.6.** *Let  $\mathcal{A} = \{H_i\}_{i=1}^m$  be an arrangement of smooth connected curves in  $\mathbb{R}^2$ , such that  $H_i = \{f_i = 0\}$ . Let  $H \doteq H_1$  and  $H \cap \{H_2, \dots, H_m\} = \{p_1, \dots, p_k\} \subset H$ . Then there is a bijective function, which is not necessarily an isomorphism of LRBs,*

$$\varphi : \mathcal{L}_0(\mathcal{A}_H) \rightarrow \mathcal{L}_0(\mathcal{A})|_H \subseteq (L_2^1)^m$$

satisfying:

- (1)  $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_1 = 0$ .
- (2) For every  $j > 1$ :
  - (a) If  $H \cap H_j = \emptyset$ , then  $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_j$  is constant. Explicitly, all the vectors in  $\varphi(\mathcal{L}_0(\mathcal{A}_H))$  have the same sign in the  $j^{\text{th}}$  coordinate.
  - (b) If  $H \cap H_j \neq \emptyset$ , let  $H \cap H_j = \{p_i\}_{i \in K_j}$ , where  $K_j$  is the set of indices of  $H \cap H_j$ . Then (up to a constant scalar multiplication by  $\{\pm 1\}$ ):

$$(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_j = \prod_{i \in K_j} ((\mathcal{L}_0(\mathcal{A}_H))_i)^{m_i}$$

where  $m_i = \text{mult}_{p_i}(H \cap H_j)$  is the intersection multiplicity at the point  $p_i$ , and the multiplication of signs (in the right hand side) is the usual product (explicitly,  $+\cdot+ = -\cdot- = +$ ,  $+\cdot- = -\cdot+ = -$ ,  $0 \cdot \{\pm\} = 0$ ). Note that the numeration of the indices of the right hand side is according to the numeration of the points in the arrangement of points in  $H = H_1$ .

As before, we illustrate this proposition by some examples before proving it.

**Example 3.7.** (1) Figure 9(a) presents an arrangement  $\mathcal{A}$  with three lines and a conic tangent to one of the lines, and Figure 9(b) presents the restricted arrangement  $\mathcal{A}_{H_1}$ . By Proposition 2.5,  $\mathcal{L}_0(\mathcal{A})$  is indeed a semigroup. The faces of  $\mathcal{A}_{H_1}$  are denoted by  $x_1, \dots, x_5$  and their corresponding faces in  $\mathcal{A}$  are denoted by  $f_1, \dots, f_5$ . Let  $H = H_1$ . Then, the corresponding LRBs are:

$$\begin{aligned} \mathcal{L}_0(\mathcal{A}_H) = \{ & i(x_1) = (-, -), i(x_2) = (0, -), i(x_3) = (+, -), \\ & i(x_4) = (+, 0), i(x_5) = (+, +) \} \end{aligned}$$

and

$$\mathcal{L}_0(\mathcal{A})|_H = \varphi(\mathcal{L}_0(\mathcal{A}_H)) = \{i(f_1) = (0, +, +, +), i(f_2) = (0, 0, +, +),$$

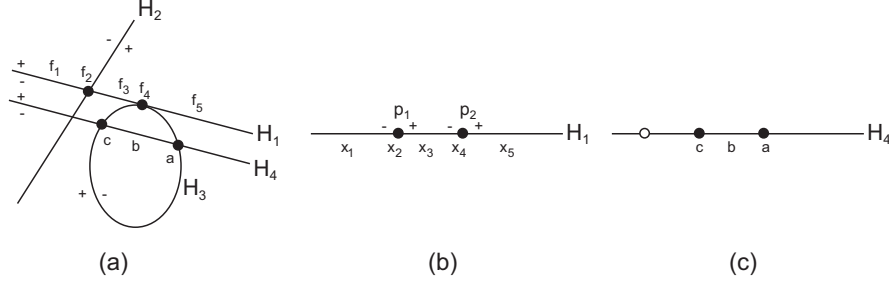


FIGURE 9. An example for illustrating Proposition 3.6:  $f_i$  are the faces contained in  $H_1$ , in the face set  $\mathcal{L}(\mathcal{A})$ .  $x_i$  are faces in the face set  $\mathcal{L}_0(\mathcal{A}_{H_1})$ .  $a, b, c$  are faces contained in  $H_4$ .

$$i(f_3) = (0, -, +, +), i(f_4) = (0, -, 0, +), i(f_5) = (0, -, +, +).$$

Then:

- (a) First, note that  $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_1 = 0$  (case (1) in the proposition).
  - (b) Since  $H \cap H_2 = \{p_1\} \in \mathcal{A}_H$ , by case (2)(b) of the proposition,  $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_2 = -(\mathcal{L}_0(\mathcal{A}_H))_1$  (note the scalar multiplication by  $-1$ ).
  - (c) Since  $H_3 \cap H = \{p_2\}$ , where  $\text{mult}_{p_2}(H \cap H_3) = 2$ , then again by case (2)(b),  $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_3 = ((\mathcal{L}_0(\mathcal{A}_H))_2)^2$ .
  - (d) Since  $H_4 \cap H = \emptyset$ , then  $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_4 = +$  (by case (2)(a)).
- (2) Let us show that  $\varphi$  is not necessarily a homomorphism. Let  $H = H_4$ . Let  $a, c$  be the intersection points of  $H$  with the conic and  $b$  be the 1-dimensional segment between them (see Figure 9(c)). When considering  $a, b, c$  as faces of  $\mathcal{A}_H$ , then in  $\mathcal{L}_0(\mathcal{A}_H)$ ,  $i(a)i(c) = i(b)$ . However, when considering  $a, b, c$  as faces of  $\mathcal{A}$  (see Figure 9(a)),  $i(a)$  and  $i(c)$  have a 0 value in the coordinate corresponding to the conic. However,  $i(b)$  does not have a 0 value in that coordinate. Thus, in  $\mathcal{L}_0(\mathcal{A})$ ,  $i(a)i(c) \neq i(b)$ .
- (3) Relabel the arrangement in Figure 9(a), such that the conic will be labeled as  $H_1$ , see Figure 10(a).

The faces of  $\mathcal{A}_{H_1}$  are denoted by  $x_1, \dots, x_6$  (see Figure 10(c)) and their corresponding faces in  $\mathcal{A}$  are denoted by  $f_1, \dots, f_6$  (see Figure 10(a)). Let  $H = H_1$ . As was explained in the paragraph before Proposition 3.1, one can induce an LRB structure on  $\mathcal{A}_H$ . Then, the corresponding LRBs are:

$$\begin{aligned} \mathcal{L}_0(\mathcal{A}_H) = \{ & i(x_1) = (-, -, -), i(x_2) = (-, -, 0), i(x_3) = (-, -, +), \\ & i(x_4) = (-, 0, +), i(x_5) = (-, +, +), i(x_6) = (0, +, +) \} \end{aligned}$$

and

$$\varphi(\mathcal{L}_0(\mathcal{A}_H)) = \{ i(f_1) = (0, -, -, +), i(f_2) = (0, -, -, 0), i(f_3) = (0, -, -, -),$$

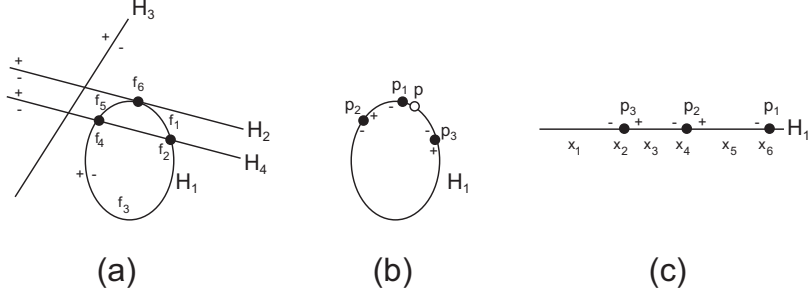


FIGURE 10. Another example for illustrating Proposition 3.6:  $f_i$  are the faces contained in  $H_1$ , in the face set  $\mathcal{L}(\mathcal{A})$  (see part (a)).  $x_i$  are the faces of the face set  $\mathcal{L}_0(\mathcal{A}_{H_1})$  (see part (c)). The three parts here illustrate the process of associating an LRB structure to  $H_1$  (the conic). First, we remove a point  $p$  from  $H_1$  to the right of  $p_1$  (see part (b)). Then, we consider  $H_1$  as a line with this point deleted, a line which starts from  $p_1$  (see part (c)).

$$i(f_4) = (0, -, -, 0), i(f_5) = (0, -, -, +), i(f_6) = (0, 0, -, +).$$

Then:

- (a) First, note that  $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_1 = 0$  (case (1) in the proposition).
- (b) Since  $H \cap H_2 = \{p_1\} \in \mathcal{A}_H$ , when  $m_1 = \text{mult}_{p_1}(H \cap H_2) = 2$ , then by case (2)(b),  $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_2 = -((\mathcal{L}_0(\mathcal{A}_H))_1)^2$  (note the scalar multiplication by  $-1$ ).
- (c) Since  $H_3 \cap H = \emptyset$ , then  $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_3 = -$  (by case (2)(a)).
- (d) Since  $H_4 \cap H = \{p_2, p_3\}$ , where

$$m_2 = \text{mult}_{p_2}(H \cap H_4) = \text{mult}_{p_3}(H \cap H_4) = 1,$$

then by case (2)(b),

$$(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_4 = (\mathcal{L}_0(\mathcal{A}_H))_2 \cdot (\mathcal{L}_0(\mathcal{A}_H))_3.$$

**Remark 3.8.** Note that if every singular point is locally a node (as in the case, for example, of a line arrangement), then one can easily see that Proposition 3.6 is indeed a generalization of Lemma 3.2.

We use the same notations introduced in Remark 3.4.

*Proof.* The proofs of cases (1) and (2)(a) are identical to the corresponding proofs in Lemma 3.2. The fact that  $\varphi$  is not necessarily an isomorphism is shown by Example 3.7(2).

We now prove case (2)(b). Let  $j > 1$  and assume that  $H \cap H_j = \{p_i\}_{i \in K_j}$ . Let  $c$  be a face of  $\mathcal{A}$  with  $\text{supp}(c) \subseteq H$ . Note that if  $c = p_a, a \in K_j$ , then  $(i_H(c))_i = 0$  in  $\mathcal{L}_0(\mathcal{A}_H)$  and  $(i_{\mathcal{A}}(c))_j = 0$  in  $\mathcal{L}_0(\mathcal{A})|_H = \varphi(\mathcal{L}_0(\mathcal{A}_H)) \subset \mathcal{L}_0(\mathcal{A})$ ; thus case (2)(b) is satisfied when  $c$  is 0-dimensional. Therefore, we can look only at a face  $c$  with  $\dim(c) = 1$ . Then, either  $c \subset \{f_j > 0\}$  or  $c \subset \{f_j < 0\}$ .

We claim that the corresponding vector of signs is determined by the relative position of  $c$  with respect to the points  $\{p_i\}$ : the (usual) product of the signs (of the  $i^{\text{th}}$  coordinates of  $\mathcal{L}_0(\mathcal{A}_H)$ , where  $i \in K_j$ ) describes whether  $c$  is in  $\{f_j > 0\}$  or in  $\{f_j < 0\}$ . Let us explicitly check a few cases:

- (1) If  $H \cap H_j = \{p_i\}$  is a single transversal intersection point ( $m_i = 1$ ), then, as  $H$  and  $H_j$  has only one connected component in  $\mathbb{R}^2$ , we can proceed as in case (2)(b) in Lemma 3.2.
- (2) If  $H \cap H_j = \{p_i\}$  is a single tangent point ( $m_i = 2$ ), then we claim that the  $j^{\text{th}}$  coordinate of  $\varphi(\mathcal{L}_0(\mathcal{A}_H))$  is constant: either  $+$  or  $-$  (except for the face  $x = p_i$ , whose sign in the  $j^{\text{th}}$  coordinate is  $0$ , as was described above for the case that  $\dim(c) = 0$ ). This is since  $H$  is either entirely outside or entirely inside  $\{H_j > 0\}$ , and the  $j^{\text{th}}$  coordinate is determined according to the signs attached to the two domains of the plane partitioned by the curve  $H_j$ . In the first case  $(i_{\mathcal{A}}(c))_j = +$  and in the second case  $(i_{\mathcal{A}}(c))_j = -$ . Also, in any case  $(i_H(c)_i)^2 = +$  and thus we proved that  $(i_{\mathcal{A}}(c))_j = \pm((i_H(c))_i)^2 = \{\pm 1\}$ , thus the  $j^{\text{th}}$  coordinate is indeed constant.
- (3) Generalizing case (2), if  $H \cap H_j = \{p_i\}$  is a single singular point with multiplicity  $m_i > 2$ , then all we are interested in is the parity of  $m_i$ . If  $m_i$  is even, then locally at  $p_i$ , the curve  $H_j$  does not pass to the “other side” of  $H$ , and thus the treatment of this case is as the case when  $m_i = 2$ . If  $m_i$  is odd, then locally at  $p_i$ , the curve  $H_j$  does pass to the “other side” of  $H$ , and thus the treatment of this case is as in case (1), where  $m_i = 1$ .
- (4) Assume now that  $H \cap H_j = \{p_{s_1}, p_{s_2}\}$  are two transversal intersection points ( $m_{s_1} = m_{s_2} = 1$ ; for example, when  $H$  is a line and  $H_j$  is a circle). Recall that the structure of the induced LRB on a pointed real curve  $C \cup \{p_1, \dots, p_k\}$  allows us to think on faces which are to the right (or to the left) of a point  $p_i$ ,  $1 \leq i \leq k$ . Assume without loss of generality that  $p_{s_2}$  is to the right of  $p_{s_1}$ .

If  $H$  is an unbounded curve, then the fact that  $c \subset \{f_j > 0\}$  is equivalent to the fact that  $c$  is to the right of  $p_{s_1}$  or to the left of  $p_{s_2}$ . In the first case,  $(i_H(c))_{s_1} \cdot (i_H(c))_{s_2} = + \cdot + = + = (i_{\mathcal{A}}(c))_j$ . In the second case,  $(i_H(c))_{s_1} \cdot (i_H(c))_{s_2} = - \cdot - = + = (i_{\mathcal{A}}(c))_j$ . We use a similar argument when  $c \subset \{f_j < 0\}$ .

If  $H$  is a bounded oval, then, as described in the paragraph before Proposition 3.1, one chooses a point  $p$  infinitesimally-close to a point  $p_i \in \{p_1, \dots, p_k\}$ . Thus an LRB structure on the set of faces of  $\mathcal{A}_H$  is induced, independent of the choice of  $p$ , when looking on  $H$  as a bounded segment. Therefore, we can use the same argument used in the case of an unbounded curve.

- (5) Generalizing case (4), assume that  $H \cap H_j = \{p_{s_1}, \dots, p_{s_n}\}$ , i.e. the intersection of  $H$  and  $H_j$  is a transversal intersection of  $n$  points ( $m_{s_i} = 1$ ,  $1 \leq i \leq n$ ).

Assume that  $H$  is an unbounded curve and thus without loss of generality, we can numerate the points  $\{p_{s_i}\}$  consecutively, such that the point  $p_{s_n}$  will be the most right point. Assume also that in  $\mathcal{L}_0(\mathcal{A})$ , the domain  $\{f_j > 0\}$  induces the sign  $+$  in the  $j^{\text{th}}$  coordinate. Let  $c$  be a 1-dimensional face in  $\mathcal{A}_H$ . Assume now that  $c$  is to the right of  $p_{s_n}$ . Thus  $(i_H(c))_{s_1} \cdot \dots \cdot (i_H(c))_{s_n} = + \cdot \dots \cdot + = +$  in  $\mathcal{L}_0(\mathcal{A}_H)$ . In addition, if  $c \subset \{f_j > 0\}$ , then in  $\mathcal{L}_0(\mathcal{A})$  (or, more accurately, in  $\mathcal{L}_0(\mathcal{A})|_H$ ),  $(i_{\mathcal{A}}(c))_j = +$  (otherwise  $(i_{\mathcal{A}}(c))_j = -$ ).

Now, if we move to the consecutive 1-dimensional face  $c'$ , adjacent to  $c$  (i.e. between  $p_{s_n}$  and  $p_{s_{n-1}}$ ), then in  $\mathcal{L}_0(\mathcal{A}_H)$ ,

$$(i_H(c'))_{s_1} \cdot \dots \cdot (i_H(c'))_{s_{n-1}} \cdot (i_H(c'))_{s_n} = \underbrace{+ \cdot \dots \cdot +}_{n-1 \text{ times}} \cdot - = -,$$

while in  $\mathcal{L}_0(\mathcal{A})$ , as  $c' \subset \{f_j < 0\}$  (if indeed  $c \subset \{f_j > 0\}$ ),  $(i_{\mathcal{A}}(c'))_j = -$ . Note that if  $c \subset \{f_j < 0\}$ , then  $c' \subset \{f_j > 0\}$ , so  $(i_{\mathcal{A}}(c'))_j = +$ , i.e. there a constant scalar multiplication by  $\{\pm 1\}$  of  $\prod_v (i_H(c))_{s_v}$ .

In this way, we can proceed to the next adjacent 1-dimensional face and so on, till we have reached to the leftmost face, i.e. to the face to the left of  $p_{s_1}$ , proving case (2)(b) for this type of intersection.

The treatment of the case when  $H$  is a bounded oval is similar to the former case (see also case (4)).

- (6) In other cases, i.e. when  $H \cap H_j = \{p_{s_1}, \dots, p_{s_n}\}$  and  $m_{s_i} \geq 1$ , then this case is treated as case (5) (i.e. treating each face separately, starting from the most right face and continuing to its adjacent face, and so on) combined with the insights of cases (1),(2),(3). □

**3.2. Embedding-deletion principle.** The sub-LRBs induced by a restriction to a connected component are of course not the only sub-LRB of  $\mathcal{L}_0(\mathcal{A})$ . In order to see the geometric meaning of different sub-LRBs, we combine the procedures of restriction and deletion described above.

Let  $\mathcal{A} = \{H_1, \dots, H_m\}$  be an arrangement of smooth connected curves in  $\mathbb{R}^2$ , such that  $\mathcal{L}_0(\mathcal{A})$  is an LRB. Let  $H_0$  be another smooth connected curve in  $\mathbb{R}^2$ . Denote  $\mathcal{A}' = \mathcal{A} \cup H_0$  and assume that  $\mathcal{L}_0(\mathcal{A}')$  is also an LRB. Then,  $\mathcal{L}_0(\mathcal{A})$  is a sub-LRB of  $(L_2^1)^m$ , and  $\mathcal{L}_0(\mathcal{A}')$  is a sub-LRB of  $(L_2^1)^{m+1}$ , where the first coordinate in each element in  $\mathcal{L}_0(\mathcal{A}')$  describes the position of the face with respect to  $H_0$ . Let  $d_1 : (L_2^1)^{m+1} \rightarrow (L_2^1)^m$  be the deletion epimorphism, deleting the first coordinate of an element in  $(L_2^1)^{m+1}$ .

**Lemma 3.9.**  $d_1(\mathcal{L}_0(\mathcal{A}')|_{H_0})$  is a sub-LRB of  $\mathcal{L}_0(\mathcal{A})$ .

*Proof.*  $\mathcal{L}_0(\mathcal{A}')|_{H_0}$  is an LRB and thus also  $d_1(\mathcal{L}_0(\mathcal{A}')|_{H_0})$  is an LRB, as  $d_1$  is a homomorphism. Note that one can also compute  $\mathcal{L}_0(\mathcal{A}')|_{H_0}$  by applying the function  $\varphi$  on  $\mathcal{L}_0(\mathcal{A}'_{H_0})$ , as described in Proposition 3.6. □

**Example 3.10.** Look at Figure 11.



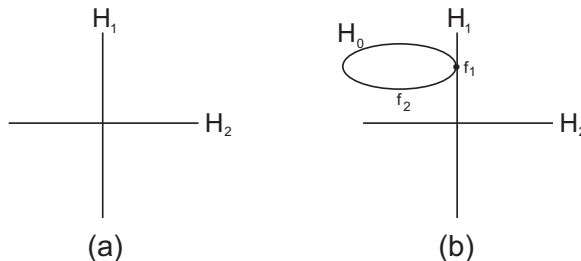


FIGURE 11. An example for Lemma 3.9.

Figure 11(a) presents an arrangement  $\mathcal{A}$ , consisting of two lines  $H_1, H_2$  intersecting transversally, and Figure 11(b) presents the arrangement  $\mathcal{A}' = \mathcal{A} \cup H_0$ , where  $H_0$  is a conic tangent to the line  $H_1$ . Let  $f_1, f_2$  be the faces of the arrangement  $\mathcal{A}'_{H_0}$  (see Figure 11(b)). Then,  $\mathcal{L}_0(\mathcal{A}) = (L_2^1)^2$  and  $\mathcal{L}_0(\mathcal{A}')|_{H_0} = \{i(f_1) = (0, 0, +), i(f_2) = (0, +, +)\}$ . Therefore, as can easily be checked,  $d_1(\mathcal{L}_0(\mathcal{A}')|_{H_0}) = \{(0, +), (+, +)\}$  is a sub-LRB of  $(L_2^1)^2$ .

#### 4. CONIC-LINE ARRANGEMENTS: COMBINATORICS

In this section, we present some restrictions on the combinatorics induced by CL arrangements, and especially by the fact that these arrangements form a partition of the plane. We start by recalling the deletion-restriction argument for hyperplane arrangements. The main references for the following two subsections are [6, 15].

**4.1. Preliminaries: The deletion-restriction argument.** Let  $\mathcal{A} = \{H_i\}_{i=1}^n \subset \mathbb{R}^N = V$  be a hyperplane arrangement, and let  $f_i \in \mathbb{R}[x, y]$  be the corresponding forms of the hyperplanes. Let also  $L = L(\mathcal{A})$  be the semi-lattice of non-empty intersections of elements of  $\mathcal{A}$ , called the *intersection poset* (ordered by inclusion).

As before, given  $H \in \mathcal{A}$ , let  $\mathcal{A}^H = \mathcal{A} - \{H\}$  be the *deleted* arrangement, and  $\mathcal{A}_H = \{K \cap H | K \in \mathcal{A}^H\}$  be the *restricted* arrangement. Recall that  $\mathcal{C}(\mathcal{A})$  is the set of chambers of  $\mathcal{A}$ , i.e. the components of  $\mathbb{R}^N - \mathcal{A}$ . Then, we have (see [19]):

$$(3) \quad |\mathcal{C}(\mathcal{A})| = |\mathcal{C}(\mathcal{A}^H)| + |\mathcal{C}(\mathcal{A}_H)|.$$

**4.2. Restrictions on the combinatorics.** Recall that for an arrangement of  $n$  lines in  $\mathbb{C}\mathbb{P}^2$ , defined by the equations  $\{f_i = 0\}_{i=1}^n$ , we have the following formula:

$$n(n-1) = \sum_{r>1} t_r \cdot r(r-1),$$

where  $t_r$  is the number of intersection points of multiplicity  $r$  in the arrangement (i.e. points which are locally of the form of an intersection of  $r$  lines at a point). This formula can be induced by Bézout's theorem for

the intersection of the curve  $C = \{f_1 f_2 \cdots f_n = 0\}$  and its polar curve with respect to a generic point  $x \in \mathbb{C}\mathbb{P}^2, x \notin C$ . That is, the local multiplicity of the intersection of  $C$  and  $\text{Pol}_x(C)$  at an intersection point of multiplicity  $r$  is  $r(r - 1)$ , where the *polar curve*  $\text{Pol}_x(C)$  is defined as follows: given a plane curve  $C = \{f = 0\}$  (in  $\mathbb{C}\mathbb{P}^2$ , with coordinates  $(x_0 : x_1 : x_2)$ ), take a generic point  $x = (p_0 : p_1 : p_2) \notin C$  in  $\mathbb{C}\mathbb{P}^2$  and define the *polar curve*  $\text{Pol}_x(C)$  as follows:

$$\text{Pol}_x(C) = \left\{ p_0 \frac{\partial f}{\partial x_0} + p_1 \frac{\partial f}{\partial x_1} + p_2 \frac{\partial f}{\partial x_2} = 0 \right\}.$$

We look now at CL arrangements in  $\mathbb{C}\mathbb{P}^2$ , which are not necessarily real CL arrangements (see Definition 1.1), i.e. these arrangements consist of conics and lines defined over  $\mathbb{C}$ . For such arrangements in  $\mathbb{C}\mathbb{P}^2$  we have other types of singular points, besides being locally an intersection of  $r$  lines. We depict in Figure 12 the different singular points for arrangements with one conic and also write the local multiplicity  $m_p$  of the intersection of  $C$  and  $\text{Pol}_p(C)$  at such points. We leave it to the reader to verify these simple calculations.

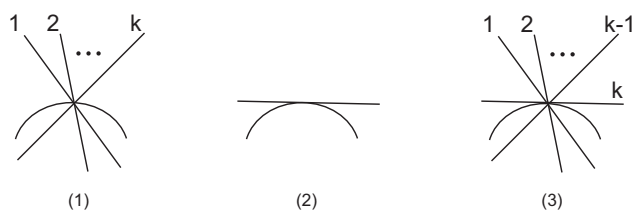


FIGURE 12. The local intersection multiplicities are: case (1):  $m_p = (k + 1)k$ ; case (2):  $m_p = 4$ ; case (3):  $m_p = k^2 + k + 2$ .

**Remark 4.1.** For an arrangement consisting of one conic and  $n$  lines in  $\mathbb{C}\mathbb{P}^2$ , we have:

$$(4) \quad (n + 2)(n + 1) = 2 + \sum_{p \in \text{Sing}(\mathcal{A})} m_p,$$

where the summand 2 appears as the intersection multiplicity of  $C$  and its polar curve at each of the two branch points is 1.

Formulas for an arrangement with  $m$  conics and  $n$  lines can be computed similarly.

4.2.1. *Real arrangements.* Note that equation (4) becomes an inequality (with the sign  $\geq$ ) if we are working with real CL arrangements in  $\mathbb{R}\mathbb{P}^2$ .

We follow Hirzebruch’s analysis [13], specifically for real CL arrangements. A real CL arrangement defines a cellular decomposition of  $\mathbb{R}\mathbb{P}^2$ . Denote by  $p_r$  the number of cells bounded by an  $r$ -gon, and by  $t_r$  the number of singular points  $p$  (of the arrangement) such that  $r$  curves pass through  $p$ . Let  $v, e$

and  $f$  be the number of vertices, edges and cells respectively. Using the fact that  $v - e + f = 1$  and that  $v = \sum t_r$ ,  $e = \sum r t_r = \frac{1}{2} \sum r p_r$ ,  $f = \sum p_r$ , we get that:

$$3 + \sum (r - 3)t_r + \sum (r - 3)p_r = 0,$$

or

$$\sum_{r \geq 4} (r - 3)p_r = p_2 - 3 - \sum_{r \geq 2} (r - 3)t_r.$$

As the left hand side is always non-negative, we get the following conclusion:

**Lemma 4.2.**

$$3 + \sum_{r \geq 4} (r - 3)t_r \leq p_2 + t_2.$$

Note that while for real line arrangements, when excluding the case of pencils,  $t_2 \geq 3$  (as  $p_2 = 0$ ), for CL arrangements the above lemma does not hold anymore: for the arrangement consists of three lines in generic position and a conic passing through the three intersection points (as in Figure 2),  $t_2 = 0$  and  $p_2 = 3$ .

**4.3. Chamber counting.** For a real CL arrangement, the deletion-restriction formula (3) for chamber counting does not hold anymore. For example, for the arrangement  $\mathcal{A}$  appearing in Figure 13,

$$|\mathcal{C}(\mathcal{A})| = 4, |\mathcal{C}(\mathcal{A}^H)| = 2, |\mathcal{C}(\mathcal{A}_H)| = 3 \Rightarrow |\mathcal{C}(\mathcal{A})| \neq |\mathcal{C}(\mathcal{A}^H)| + |\mathcal{C}(\mathcal{A}_H)|.$$

On the other hand,

$$|\mathcal{C}(\mathcal{A}^C)| = 2, |\mathcal{C}(\mathcal{A}_C)| = 2 \Rightarrow |\mathcal{C}(\mathcal{A})| = |\mathcal{C}(\mathcal{A}^C)| + |\mathcal{C}(\mathcal{A}_C)|.$$

This is in fact a general phenomena (see also Figure 14 for another example):



FIGURE 13. An example for the restriction-deletion formula for a CL arrangement.

**Proposition 4.3.** *Let  $\mathcal{A} \subset \mathbb{R}^2$  be a real CL arrangement and  $C \in \mathcal{A}$  be a conic. Then:*

$$|\mathcal{C}(\mathcal{A})| = |\mathcal{C}(\mathcal{A}^C)| + |\mathcal{C}(\mathcal{A}_C)|.$$

*Proof.* Define:

$$P = \{X \in \mathcal{C}(\mathcal{A}^C) : C \cap X \neq \emptyset\}, \quad Q = \{X \in \mathcal{C}(\mathcal{A}^C) : C \cap X = \emptyset\}.$$

Then:  $|\mathcal{C}(\mathcal{A}^C)| = |P| + |Q|$ .

Numerate the chambers in  $\mathcal{C}(\mathcal{A}_C)$ , i.e. the sections on the conic  $C$ :  $\mathcal{C}(\mathcal{A}_C) = \{s_i\}_{i=1}^k$ , separated by the points  $C \cap (\mathcal{A}^C)$ . Every section  $s_j$  divides a chamber  $Y \in P$  into two parts (maybe after adding a new chamber to  $P$ , induced by previous sections  $s_i$  for  $i < j$ , see Figure 14) and leaves the chambers of  $Q$  unchanged. That is, the number of new chambers added to the arrangement  $\mathcal{A}^C$  is exactly  $|\mathcal{C}(\mathcal{A}_C)|$ .  $\square$

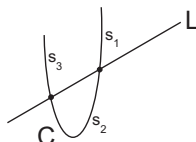


FIGURE 14. A parabola  $C$  and a line  $L$ , intersecting the parabola twice:

$$|\mathcal{C}(\mathcal{A})| = 5, |\mathcal{C}(\mathcal{A}_C)| = 3, |\mathcal{C}(\mathcal{A}^C)| = 2.$$

Note that the section  $s_1$  divides the chamber  $X$  (the half-plane above the line) in  $\mathcal{C}(\mathcal{A}^C)$  into two chambers, and then  $s_3$  divides one of these chambers, again, into two chambers.

**Remark 4.4.** Note that though this proof is similar to the proof of the deletion-restriction argument for chamber counting for line arrangements (see e.g. [15]), a line cannot “return” to a chamber once it has divided it.

However, the deletion-restriction argument is changed while deleting a line. In order to formulate this change accurately, we start by introducing some notations.

**Definition 4.5.** Let  $\mathcal{A} \subset \mathbb{R}^2$  be a real CL arrangement.

(1) Let  $H \in \mathcal{A}$  be a line. Define the function:

$$\text{bound} : \mathcal{C}(\mathcal{A}_H) \rightarrow \{Y \in P(\mathcal{C}(\mathcal{A})) : |Y| = 2\}$$

$$\text{bound}(E) = \{X_1, X_2\}, \text{ where } E \subset \overline{X_1} \cap \overline{X_2},$$

where  $P(\mathcal{C}(\mathcal{A}))$  is the power set of  $\mathcal{C}(\mathcal{A})$  and  $\overline{X}$  is the closure of  $X$ .

(2) For  $E_1, E_2 \in \mathcal{C}(\mathcal{A}_H)$ , define the following equivalence relation  $\sim$ :

$$E_1 \sim E_2 \Leftrightarrow \text{bound}(E_1) = \text{bound}(E_2),$$

and define:

$$b(H) = \mathcal{C}(\mathcal{A}_H) / \sim.$$

For example, for the arrangement in Figure 15,  $|b(H)| = 2$ .

**Remark 4.6.** It is easy to see that:

- (1)  $|b(H)| \leq |\mathcal{C}(\mathcal{A}_H)|$ .
- (2) If  $\mathcal{A}$  is a line arrangement, then  $b(H) = \mathcal{C}(\mathcal{A}_H)$  for any  $H \in \mathcal{A}$ .



FIGURE 15. An illustration of the elements in  $b(H)$ .

**Proposition 4.7.** *Let  $H$  be a line in a real CL arrangement  $\mathcal{A}$ . Then:*

$$|\mathcal{C}(\mathcal{A})| = |\mathcal{C}(\mathcal{A}^H)| + |b(H)|.$$

Note that by Remark 4.6(2), Proposition 4.7 is indeed a natural generalization of the situation for line arrangements to real CL arrangements.

*Proof.* For every chamber  $X \in \mathcal{C}(\mathcal{A}^H)$  such that  $H \cap X \neq \emptyset$ , the line  $H$  divides  $X$  into a certain number of chambers; we denote this number by  $k_X$ . Thus:

$$|\mathcal{C}(\mathcal{A})| = |\mathcal{C}(\mathcal{A}^H)| + \sum_{\substack{X \in \mathcal{C}(\mathcal{A}^H) \\ H \cap X \neq \emptyset}} (k_X - 1),$$

since every chamber  $X \in \mathcal{C}(\mathcal{A}^H)$  in the sum splits into  $k_X$  chambers, but we do not count  $X$  itself, as it is already counted in  $|\mathcal{C}(\mathcal{A}^H)|$ . For each  $X \in \mathcal{C}(\mathcal{A}^H)$  in the sum, denote:

$$X = \bigcup_{i=1}^{k_X} X_i, \quad H_X = H \cap X,$$

that is, (the interior of)  $X$  is divided into  $k_X$  chambers  $X_i$ , whose union (of their closure) is (the closure of)  $X$ .

Note that  $H_X$  is possibly a union of disjoint sections and  $H_X \subset \mathcal{C}(\mathcal{A}_H)$ . Therefore, we need to prove that  $1 + |b(H_X)| = k_X$ . For convenience, we rotate the arrangement in such a way that the line  $H$  will be horizontal, and we numerate the sections of  $H_X$  from right to left, which induces a numeration  $H_1, H_2, \dots$  of the sections of  $b(H_X)$  from right to left. For each  $H_i \in b(H_X)$ ,  $1 \leq i \leq |b(H_X)|$ , we look at the pair  $\text{bound}(H_i) = \{X_{i_1}, X_{i_2}\}$ ; see Figure 16 for an example.

We show that for each  $i < j$ , either  $|\text{bound}(H_i) \cap \text{bound}(H_j)| = 1$  or there is a sequence  $H_{i+1}, \dots, H_{j-1}$  such that for each  $k$ ,  $i \leq k < j$ ,  $|\text{bound}(H_k) \cap \text{bound}(H_{k+1})| = 1$ . Indeed,  $|\text{bound}(H_i) \cap \text{bound}(H_j)| < 2$ , otherwise  $H_i \sim H_j$ . If  $|\text{bound}(H_i) \cap \text{bound}(H_j)| = 0$ , look at  $\text{bound}(H_s)$  for  $s \in \{i, i+1, i+2\}$  (assuming that  $H_i \not\sim H_{i+1}$  and  $H_{i+1} \not\sim H_{i+2}$ ). Assume by contradiction that  $|\text{bound}(H_i) \cap \text{bound}(H_{i+1})| = 0$ . This means that we have the situation depicted in Figure 17.

However, this situation is impossible, since the sections are consecutive, and if  $\{X_1, X_2\} \cap \{X_3, X_4\} = \emptyset$ , then  $H_i, H_{i+1}$  will not dissect the same

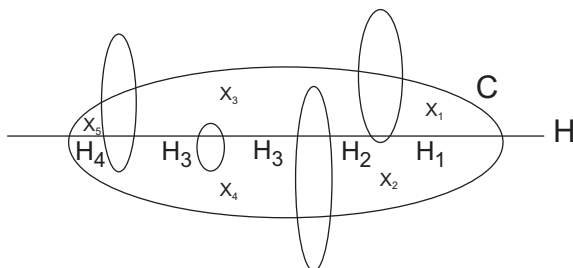


FIGURE 16. An example for the partition of  $H_X$ , where  $X = X_1 \cup \dots \cup X_5$  is a chamber contained inside the interior of the conic  $C$  and  $X_1, \dots, X_5$  are the chambers whose union is  $X$ . The sections of  $H_X = H \cap X$  are  $H_1, \dots, H_4$  under the equivalence relation  $\sim$ .

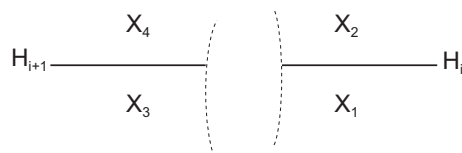


FIGURE 17.  $|\text{bound}(H_i) \cap \text{bound}(H_{i+1})| = 0$

(single) chamber  $X \in \mathcal{C}(\mathcal{A}_H)$  (since  $X_1, X_2$  and  $X_3, X_4$  will be contained in different chambers of  $\mathcal{C}(\mathcal{A}^H)$ ) – indeed, even before the operation of  $\sim$ , one can connect a generic point from  $H_i$  with a generic point from  $H_{i+1}$  with a continuous path which lies only in  $X$ , which mean that the above intersection is always non-empty.

Thus, we define recursively the following map  $\ell_X : (H_X/\sim) \rightarrow \{X_1, \dots, X_{k_X}\} :$   $\ell_X(H_1)$  is one of the chambers  $X$  such that  $H_1 \subset \overline{X}$ . Define  $\ell_X(H_i)$  for  $i > 1$  to be one of the chambers  $X'$  such that  $H_i \subset \overline{X'}$  and for every  $j < i$ ,  $\ell_X(H_j) \neq X'$ . Up to the choice of  $X$ , the map is well-defined, as for every  $1 < i$  there is only one option to choose (recall that for each  $i$ ,  $|\text{bound}(H_i) \cap \text{bound}(H_{i+1})| = 1$ ). By the definition of  $\ell_X$ , the map is injective. Therefore,  $|b(H_X)| = k_X - 1$ .  $\square$

**Remark 4.8.** *By the same arguments we have used above, one can easily see that Proposition 4.7 holds for arrangements in  $\mathbb{R}\mathbb{P}^2$  too. However, in Definition 4.5(1), the definition of the function bound should be changed as follows:*

$$\text{bound} : \mathcal{C}(\mathcal{A}_H) \rightarrow \{Y \in P(\mathcal{C}(A)) : |Y| \leq 2\}$$

$$\text{bound}(E) = \{X_1, X_2\} \text{ such that } E \subset \overline{X_1} \cap \overline{X_2} \text{ or } E \subset \overline{X_1}.$$

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