ON LEFT REGULAR BANDS AND REAL CONIC-LINE ARRANGEMENTS

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Abstract. An arrangement of curves in the real plane divides it into a collection of faces. Already in the case of line arrangements, this collection can be given a structure of a left regular band and one can ask whether the same is possible for other arrangements. In this paper, we try to answer this question for the simplest generalization of line arrangements, that is, conic–line arrangements.

Investigating the different algebraic structures induced on the face poset of a conic–line arrangement, we present two possibilities for generalizing the product and its associated structures. We also study the structure of sub left regular bands induced by these arrangements. We finish with some combinatorial properties of conic–line arrangements.

1. Introduction

An arrangement of curves $\mathcal{A}$ in $\mathbb{R}^2$ defines a partition of the plane into a collection of faces, denoted by $\mathcal{L}(\mathcal{A})$. For a line arrangement $\mathcal{A}$, the set $\mathcal{L}(\mathcal{A})$ already raises a variety of interesting questions, lying in the intersection of several mathematical areas: algebra, topology and combinatorics. Indeed, one can define a product on $\mathcal{L}(\mathcal{A})$, making this set into a left regular band (LRB) S, that is, a semigroup whose every element is an idempotent and for every $x, y \in S$, $x \cdot y \cdot x = x \cdot y$ (see [15, Section 3] for a survey on bands and examples of left regular bands; see also [6, 7] for a similar description for complex line and hyperplane arrangements).

Moreover, as $\mathcal{L}(\mathcal{A})$ determines the combinatorics of the arrangement, one can ask what are the connections between $\mathcal{L}(\mathcal{A})$ and other invariants of these arrangements. The relations between the face LRB on $\mathcal{A}$ and its combinatorics can be found in the numerous restriction-deletion principles: Zaslavsky's chamber counting formula [24], the deletion-restriction formula for the Poincaré polynomial $\pi(\mathcal{A}, t)$ and the addition-deletion theorem for the module of $\mathcal{A}$-derivations $D(\mathcal{A})$ (see e.g. [16]). Other applications can be found in the description of the algebra kS in terms of quivers (see [17, 18]), random walks on the faces of a hyperplane arrangement [8] and in the ongoing investigation of the connections between the fundamental group $\pi_1(\mathbb{C}^2 - \mathcal{A})$ and the combinatorics of $\mathcal{A}$ (see e.g. [4, 9, 10, 11, 13, 22, 23] and many more).

A natural question is what happens to these algebraic structures, associated to $\mathcal{L}(\mathcal{A})$, when one deals with arrangements of smooth curves in $\mathbb{R}^2$; i.e. topologically speaking, when we deal with real conic–line arrangements in $\mathbb{R}^2$. This investigation already took place to some extent. Zaslavsky [25] generalized the deletion-restriction formula in several directions and a research of $\pi_1(\mathbb{C}^2 - \mathcal{A})$ for some families of conic-line arrangements has taken place, see e.g., in [3, 5, 11, 12, 21]. Also, in [19], the existence of other restriction-deletion formulas with respect to the module of $\mathcal{A}$-derivations $D(\mathcal{A})$ for a quasihomogeneous free conic–line arrangement $\mathcal{A}$ was proven.

As in line arrangements, one can induce a restriction-deletion formula for the Poincaré polynomial $\pi(\mathcal{A}, t)$ from the corresponding theorem for $D(\mathcal{A})$. However, while for line arrangements, $\pi(\mathcal{A}, 1)$ is equal to the number of chambers of $\mathcal{A}$ (which enables us to induce Zaslavsky's chamber counting formula), this is not true anymore for conic-line arrangements (for more details, see Remark 4.8 below). Moreover, Schenck and Tohaneanu [19, Section 4] found a pair of combinatorially-equivalent conic-line arrangements with non-isomorphic modules of $\mathcal{A}$-derivations $D(\mathcal{A})$.

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Another motivation for a deeper investigation of the algebraic structure associated to $\mathcal{L}(\mathcal{A})$ for a conic-line arrangement $\mathcal{A}$, is that it is a natural candidate for an alternative product. This kind of phenomenon - an alternative product that replaces the associative one - is not unnatural: it appears also when looking at the poset of the faces of a building (see Tits [20, Section 3.19]), and more generally, in a *projection poset* (see [2, p. 26] for its definition). We will see that one of the generalized products we define for the face poset of a conic–line arrangement will be alternative.

Therefore, a first step in understanding the connections between the above mentioned structures is the investigation of the algebraic structure of the face poset $\mathcal{L}(\mathcal{A})$ associated to a real conic–line arrangement $\mathcal{A}$ and its applications.

Thus, the main purpose of this paper is to study the induced algebraic structures on the set of faces associated to real conic–line arrangements. We also deal with additional combinatorial aspects of real conic–line arrangements and prove several restriction-deletion and embedding principles. Let us introduce here explicitly the notion of a *real conic–line arrangement*:

Definition 1.1. *A* real conic–line (CL) arrangement $\mathcal{A}$ *is a collection of conics and lines defined by the equations* $\{f_i = 0\}$ *in* $\mathbb{C}^2$*, where* $f_i \in \mathbb{R}[x, y]$*. Moreover, for every conic* $C \in \mathcal{A}$*,* $C \cap \mathbb{R}^2$ *is not an empty set, neither a point nor a (double) line.*

The paper is organized as follows. The first two sections look for the natural generalization of the structure of $\mathcal{L}(\mathcal{A})$ to the case of a real CL arrangement $\mathcal{A}$. Based on the problems one encounters during this generalization, we propose in Section 2 two possibilities for a well-defined product on this set; the first turns $\mathcal{L}(\mathcal{A})$ into an alternative LRB and the second turns $\mathcal{L}(\mathcal{A})$ into an aperiodic semigroup. Section 3 investigates the embedding principles for sub-LRBs for a given band, induced by a real CL arrangement. Connections between the band, induced by restricting the real CL arrangement to a conic or to a line, and the band induced by the whole arrangement, are presented. Section 4 presents a generalization of the restriction-deletion principle for chamber counting for the case of CL arrangements.

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2. Real CL arrangements: The face semigroup

In this section, we concentrate on the structure of the face semigroup of real CL arrangements. We start in Section 2.1 by reviewing the corresponding known structure of the face semigroup $\mathcal{L}(\mathcal{A})$ associated to a hyperplane arrangement $\mathcal{A}$. In Section 2.2 we study the corresponding face semigroups in the case of real CL arrangements. The main results of this section appear in Section 2.3, where we introduce two possible generalizations for the corresponding product defined for hyperplane arrangements to real CL arrangements: one product turns $\mathcal{L}(\mathcal{A})$ into an alternative left regular band and the second product turns $\mathcal{L}(\mathcal{A})$ into an aperiodic semigroup.

2.1. **Preliminaries: The left regular band and the face semigroup of a hyperplane arrangement.** In this section, we recall the notion of a *left regular band* and its connections to the combinatorics of hyperplane arrangements (see also a survey in [15]).

Definition 2.1. *A* left regular band *(*LRB*) is a semigroup* $(S, \cdot)$ *that satisfies the identities:*

$$x \cdot x = x \quad \text{and} \quad x \cdot y \cdot x = x \cdot y \text{ for every } x, y \in S.$$

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a hyperplane arrangement in $\mathbb{R}^N$ consists of n hyperplanes, where H_i is defined by the equation $\{f_i = 0\}$, where $f_i \in \mathbb{R}[x_1, \ldots, x_N]$. Recall that for $H \in \mathcal{A}$, the arrangement $\mathcal{A}^H = \mathcal{A} - \{H\}$ is called the *deleted* arrangement and $\mathcal{A}_H = \left\{K \cap H | K \in \mathcal{A}^H\right\}$ is called the *restricted* arrangement. Let $\mathcal{C}(\mathcal{A})$ be the set of chambers of $\mathcal{A}$, i.e. the components of $\mathbb{R}^N - \mathcal{A}$, and let $L = L(\mathcal{A})$ be the semi-lattice of non-empty intersections of elements of $\mathcal{A}$.

Define the partially ordered set of faces as:

$$\mathcal{L} = \mathcal{L}(\mathcal{A}) = \bigcup_{X \in L} \mathcal{C}(\mathcal{A}_X),$$

where $\mathcal{L}$ is ordered by inclusion, which will be denoted by $\preceq$ (some authors order $\mathcal{L}$ by *reverse* inclusion). Note that $\mathcal{L}$ determines the combinatorics of the arrangement.

Define a (monomorphic) function $i : \mathcal{L} \to (\{+, -, 0\})^n$, as follows: for $P \in \mathcal{L}$, define

$$(i(P))_k = \text{sign}(f_k(P)),$$

where $()_k$ denotes the value of the k^{th} coordinate of the vector $i(P)$. The generalization of this function to complex hyperplane arrangements already appeared in [7], where the vector $i(P)$ is called there *a complex sign vector*.

Recall that one can define an associative product on $\{+, -, 0\}$, given by $x \cdot y = x$ if $x \neq 0$, and y otherwise. This product induces an LRB structure on $\{+, -, 0\}$, which is denoted by L_2^1. This product can be extended componentwise to a product on $\left(L_2^1\right)^n$. Thus, Image(i), as a subset of $(\{+, -, 0\})^n = \left(L_2^1\right)^n$, has the structure of an LRB, and therefore also $\mathcal{L}$, when identifying it with Image(i). Based on this LRB structure, one can associate a quiver to the semigroup algebra $k\mathcal{L}$, for a field k (see [17]).

For hyperplane arrangements, this product has a geometric meaning: for $F, K \in \mathcal{L}$, the product $F \cdot K$ is the face that we are in after moving a small positive distance from a generic point of the face F towards a generic point of the face K along a straight line connecting these points (see e.g. [1, Section 1.4.6]).

Remark 2.2. Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a hyperplane arrangement, where $H_i = \{f_i = 0\}$. Denote by $i(\mathcal{L}(\mathcal{A}))$ the embedding of $\mathcal{L}(\mathcal{A})$ into $(L_2^1)^n$. Let J be a nonempty subset of $\{1, \ldots, n\}$ and define $g_j = -f_j$ if $j \in J$ and $g_j = f_j$ otherwise. Let $H_i' \doteq \{g_i = 0\}$ and $\mathcal{A}' = \{H_1', \ldots, H_n'\}$. Obviously, $\mathcal{A} = \mathcal{A}'$. However, as the LRB structure on $\mathcal{L}(\mathcal{A})$ is defined by the sign function, the embedding of $\mathcal{L}(\mathcal{A}')$ into $(L_2^1)^n$ will be different than the embedding of $\mathcal{L}(\mathcal{A})$ (that is, as sets, $i(\mathcal{L}(\mathcal{A})) \neq i(\mathcal{L}(\mathcal{A}'))$; explicitly, for all $j \in J$, $(i(\mathcal{L}(\mathcal{A})))_j = -(i(\mathcal{L}(\mathcal{A}')))_j)$, but the two LRBs will still be isomorphic.

2.2. **The semigroups $\mathcal{L}$ and $\mathcal{L}_0$ for CL arrangements.** Let $\mathcal{A} = \{H_1, \ldots, H_n\} \subset \mathbb{R}^2$ be a real CL arrangement with n components, and let $f_i \in \mathbb{R}[x, y]$ be the corresponding forms of the components. Let $L = L(\mathcal{A})$ be as before, and define the partially ordered set of faces as:

$$\mathcal{L} = \mathcal{L}(\mathcal{A}) = \bigcup_{X \in L} \mathcal{C}(\mathcal{A}_X),$$

where $\mathcal{L}$ is ordered by inclusion. We denote the partial order by $\preceq$.

Definition 2.3. (a) Define the map

$$\text{supp} : \mathcal{L} \to L,$$

sending each face to its support, i.e. the corresponding element in the intersection semi-lattice.

(b) As before, define a function:

$$i : \mathcal{L} \to (\{+, -, 0\})^n \tag{1}$$

as: $(i(P))_k = \text{sign}(f_k(P))$, where $()_k$ is the value of the k^{th} coordinate of the vector $i(P)$.

We now deal with some properties of Image(i). Note that for real hyperplane arrangements, the function i is monomorphic: every face P is uniquely determined by its vector of n signs. However, for real CL arrangements, this function might not be monomorphic. For example, given a line and a circle tangent to it, the two parts of the line have the same pair of signs. Another example is presented in Figure 1, where we have that:

$$i(P_1) = i(P_2) = (+, -, +) \in \left(L^1_2\right)^3.$$

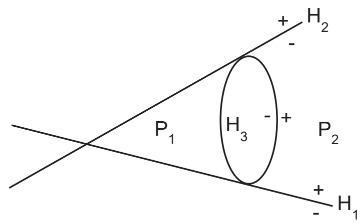

FIGURE 1. An example of a real CL arrangement consisting of three components H_1, H_2, H_3 with two 2-dimensional faces P_1, P_2 having the same vector of signs: $i(P_1) = i(P_2) = (+, -, +)$.

Recall that one can define an associative product on $(L^1_2)^n = (\{+, -, 0\})^n$ (see Section 2.1). This raises the following question: does this give Image(i) the structure of a sub-semigroup of $(L^1_2)^n$? For hyperplane arrangements, the answer is positive as one identifies $\mathcal{L}$ with Image(i); thus $\mathcal{L}$ is endowed with a semigroup structure. However, for real CL arrangements, as i is not necessarily monomorphic, we cannot identify $\mathcal{L}$ with Image(i) (and thus we need to redefine the product on $\mathcal{L}$). A more serious problem is presented in the following example.

Example 2.4. (1) There are real CL arrangements whose Image(i) is not even closed under the product induced by $(L^1_2)^n$, and thus it is not even a semigroup. For example, take three lines H_1, H_2, H_3 in general position (i.e. not passing through a single point) and a circle C passing through the three intersection points; see Figure 2. Let $\alpha, \beta \in$ Image(i) $\subset (L^1_2)^4$ be two quadruples associated to two different intersection points (see Figure 2; the points are a, b). Explicitly,

$$\alpha = i(a) = (0, +, 0, 0), \beta = i(b) = (0, 0, -, 0), \text{ but } \alpha\beta = (0, +, -, 0) \notin \text{Image}(i).$$

Though $\alpha, \beta \in$ Image(i), $\alpha\beta \notin$ Image(i), since there is no face which corresponds to the quadruple $\alpha\beta$, as there is no element in Image(i) that has exactly two zeros in its presentation as a quadruple in $(L^1_2)^4$.

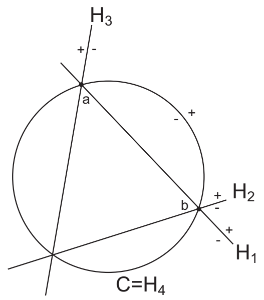

FIGURE 2. $\alpha = i(a) = (0, +, 0, 0), \beta = i(b) = (0, 0, -, 0)$, but $\alpha\beta = (0, +, -, 0) \notin$ Image(i).

Note that this is the minimal degree example for this phenomenon to occur: one can verify that for any real CL arrangement with up to degree 4, Image(i) is always closed under the product induced by $(L^1_2)^n$.

Moreover, taking three generic lines and a circle passing through two intersection points, one can check that the product of the corresponding vector of signs of the pair of triple points of this arrangement does not represent any face of this arrangement.

(2) The above example can be generalized: take a regular n-gon, where $n > 3$, draw a circle passing through all the vertices of the n-gon, and extend the edges of the polygon into straight lines. One can check that the product of the corresponding vector of signs of any pair of consecutive triple points of this arrangement does not represent any face of this arrangement.

Definition 2.5. *Let* $\mathcal{L}_0 = \mathcal{L}_0(\mathcal{A}) = \mathrm{Image}(i) \subseteq (L_2^1)^n$.

Following the previous example, the following question raises: when is $\mathcal{L}_0(\mathcal{A})$ closed under the product induced by $(L_2^1)^n$, and hence form a semigroup? Obviously, if $\mathcal{A}$ is a line arrangement, then $\mathcal{L}_0(\mathcal{A})$ is a semigroup. Moreover, we have the following proposition regarding real CL arrangements:

Proposition 2.6. *Let $\mathcal{A}$ be a real CL arrangement. Assume that there is no singular point p such that there are more than two components passing through p. Then $\mathcal{L}_0(\mathcal{A})$ is a semigroup.*

Proof. We consider only the arrangements whose singular points are either nodes or tangent points of order 2 (or both). Indeed, there are other types of singular points to consider, such as tangency points of order 3 or 4 between two conics, but from the point of view of the structure of the associated LRB, they are the same as nodes and tangency points of order 2, respectively, as we are interested only in the local structure in the neighborhood of the singular point.

We need to check that $\mathcal{L}_0(\mathcal{A})$ is closed under the product induced by $(L_2^1)^n$. For each face $c \in \mathcal{L}(\mathcal{A})$, we go over all the products of the form $i(c)i(a)$, where $a \in \mathcal{L}(\mathcal{A})$, and check that $i(c)i(a) \in \mathcal{L}_0(\mathcal{A})$.

If $\dim(c) = 2$, there is nothing to check, as $i(c)i(a) = i(c)$ for every $a \in \mathcal{L}(\mathcal{A})$, since all the entries of $i(c)$ are non-zero.

If $\dim(c) = 1$, let $H = \mathrm{supp}(c)$, where $H = \{f = 0\}$. Then $i(c)i(a)$ is either $i(c)$ or one of the faces that has c in its boundary (which lies inside the domain $\{f > 0\}$ or $\{f < 0\}$), which exist as elements in $\mathcal{L}_0(\mathcal{A})$.

If $\dim(c) = 0$, then, as assumed above, c is either a node or a tangent point of multiplicity 2. If it is a node, then locally, in the neighborhood of c, the arrangement is of the form $\{xy = 0\}$ (obviously, this approximation is also applied to two conics which intersect with multiplicity 3). Note that as an arrangement in $\mathbb{R}^2$, $\mathcal{L}_0(\{xy = 0\}) = (L_2^1)^2$. This means that $i(c)i(a) \in \mathcal{L}_0(\mathcal{A})$ for every $a \in \mathcal{L}(\mathcal{A})$.

If c is a tangent point, then locally, in the neighborhood of c, the arrangement is of the form $\{y(y - x^2) = 0\}$ (obviously, the arrangement can consist of two tangent conics with intersection multiplicity 2 or 4, but from the point of view of the LRB $\mathcal{L}_0(\mathcal{A})$, the resulting set of vectors of signs will be the same), and thus, as an arrangement in $\mathbb{R}^2$,

$$L_0 \doteq \mathcal{L}_0(\{y(y - x^2) = 0\}) = (L_2^1)^2 - \{(-,+),(-,0),(0,+)\},$$

where the first coordinate corresponds to the line $\{y = 0\}$ and the second to the conic. As can be easily checked, L_0 is closed under this product, which means that $i(c)i(a) \in \mathcal{L}_0(\mathcal{A})$ for every $a \in \mathcal{L}(\mathcal{A})$. □

2.3. **Redefining the product.** In this section, we introduce two possible generalizations for the product defined for hyperplane arrangements to CL arrangements, in two different directions: one product turns $\mathcal{L}(\mathcal{A})$ into an alternative LRB (i.e. an alternative magma such that $x^2 = x, xyx = xy$ for every $x, y \in \mathcal{L}$) and the second product turns $\mathcal{L}(\mathcal{A})$ into an aperiodic semigroup.

We want to use the same geometric intuition of the product for hyperplane arrangements (see Section 2.1) for defining the corresponding product on the face poset $(\mathcal{L}, \preceq)$ for real CL arrangements (where $\preceq$ is the partial order defined by inclusion). Explicitly, we want to maintain the following properties for every $x, y, z \in \mathcal{L}$:

(1) For every $x, y \in \mathcal{L}, x^2 = x$ and $x \cdot y \cdot x = x \cdot y$ (the LRB properties).
(2) If $x \cdot y = z$, then $i(x)i(y) = i(z)$ (if there are faces with a vector of signs $i(x)i(y)$). Explicitly, if $\mathcal{L}_0(\mathcal{A})$ is a semigroup, then the surjective map $\mathcal{L}(\mathcal{A}) \to \mathcal{L}_0(\mathcal{A})$ would be a homomorphism.
(3) If $x \cdot y = z$, then $x \preceq z$.
(4) If $x \preceq y$, then $x \cdot y = y$.
(5) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ (Associativity).

We present two possible definitions for this product in the case of real CL arrangements. The first definition, appearing in Section 2.3.1, preserves properties (1), (3) and (4) and thus will be more geometric, inducing a structure of an alternative LRB on $(\mathcal{L}, \cdot)$; the second, appearing in Section 2.3.2, preserves properties (2), (5) and a weaker version of property (1), and thus $(\mathcal{L}, \cdot)$ will be an aperiodic semigroup. Based on the second definition, one can associate a quiver to the semigroup algebra $k\mathcal{L}$, as was already done in the case of line arrangements (see [17]).

2.3.1. *The geometric product.* We start with the more geometric definition, which will be given in two parts. The first part includes the basic requirements of this product. We start with examining the CL arrangement in Figure 3, which shows that requirement (3) is not entirely based on the definition of i. Explicitly, we want that if $x \cdot y = z$, then $x \preceq z$, i.e. z is a face intersecting a small neighborhood of x. The example in Figure 3 shows that this is not always the case when working with the product induced by $(L_2^1)^n$. In the CL arrangement presented in Figure 3, $i(p)i(x) = i(x)$, but $p \not\preceq x$.

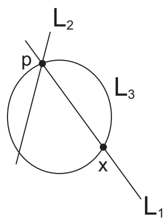

FIGURE 3. As $i(p) = (0, 0, 0), i(x) = (0, -, 0)$, we have that $i(p)i(x) = i(x)$, but $p \not\preceq x$.

Definition 2.7. *(Geometric product on $\mathcal{L}(\mathcal{A})$, Part I):*
Let $\mathcal{A}$ be a real CL arrangement, and let $P_1, P_2 \in \mathcal{L}(\mathcal{A})$. Define:

$$F(P_1, P_2) \doteq \{P \in \mathcal{L}(\mathcal{A}) : i(P) = i(P_1)i(P_2) \text{ and } P_1 \preceq P\}.$$

If $|F(P_1, P_2)| = 0$, then $P_1 \cdot P_2 \doteq P_1$.
If $|F(P_1, P_2)| = 1$, i.e. $F(P_1, P_2) = \{P\}$, then $P_1 \cdot P_2 \doteq P$.
Otherwise, we know that $|F(P_1, P_2)| > 1$.
If $P_2 \in F(P_1, P_2)$, then $P_1 \cdot P_2 \doteq P_2$.

Let us stop for a moment in defining this product. Obviously, requirement (3) holds (note that in the example presented in Figure 3, when we use the above product, then $p \cdot x = p$, since $F(p, x) = \emptyset$). Note that if $x, y \in \mathcal{L}$, then $x \preceq y$ actually means that $x \subseteq \bar{y}$, where $\bar{y}$ is the (topological) closure of y, i.e. any neighborhood of x intersects y.

Moreover, if $x, y, z \in \mathcal{L}$, then $x \cdot (y \cdot z)$ is a face $\alpha \in \mathcal{L}(\mathcal{A})$ satisfying $x \preceq \alpha$, $x \cdot y$ is a face $\beta' \in \mathcal{L}(\mathcal{A})$ satisfying $x \preceq \beta'$ and $(x \cdot y) \cdot z$ is a face $\beta \in \mathcal{L}(\mathcal{A})$ satisfying $\beta' \preceq \beta$; thus $x \preceq \beta$. This means that even if the product is not associative, then

$$x \subseteq (\overline{x \cdot (y \cdot z)} \cap \overline{(x \cdot y) \cdot z}) \quad \text{or} \quad x \preceq (x \cdot (y \cdot z) \wedge (x \cdot y) \cdot z). \tag{2}$$

Note that if every singular point $p \in \mathrm{Sing}(\mathcal{A})$ includes a transversal intersection of two components of $\mathcal{A}$, then $|F(P_1, P_2)| = 1$ for all $P_1, P_2 \in \mathcal{L}$ and thus the above product is well-defined for each pair of faces.

The following proposition presents some properties of the geometric product:

Proposition 2.8. *Let $P_1, P_2 \in \mathcal{L}$. Then the following properties hold:*
(1) If $P_1 \neq P_2$, then P_1 and P_2 cannot be together in the set $F(P_1, P_2)$.
(2) If $P_1 \in F(P_1, P_2)$, then $P_1 \cdot P_2 = P_1$.
(3) $|F(P_1, P_2)| \leq 2$.
(4) If $P_1 \preceq P_2$ and $P_1 \neq P_2$, then $P_1 \cdot P_2 = P_2$.
(5) If $|F(P_1, P_2)| = 1$ for every two faces $P_1, P_2 \in \mathcal{L}$, then the product is associative.

Proof. (1) If $P_1, P_2 \in F(P_1, P_2)$, then $i(P_1) = i(P_2)$. Moreover, since $P_2 \in F(P_1, P_2)$, $P_1 \preceq P_2$ implies that P_1 is contained in the boundary of P_2 (but then $i(P_1) \neq i(P_2)$), or that P_1 and P_2 have the same dimension, i.e. P_1 must be equal to P_2, which is a contradiction.

(2) Consider the dimension of P_1: if $\dim(P_1) = 2$, it is obvious, as already $i(P_1)i(P_2) = i(P_1)$ and P_1 is the only face X satisfying $P_1 \preceq X$. If $\dim(P_1) < 2$, then in the neighborhood of P_1, the only face with the same vector of signs as P_1 is P_1 (note that if $P_1 \in F(P_1, P_2)$, then by definition $i(P_1)i(P_2) = i(P_1)$).

(3) If $\dim(P_1) \in \{1, 2\}$, then $|F(P_1, P_2)| = 1$: this is obvious for a chamber, and for a section of a curve, the faces of $F(P_1, P_2)$ can be P_1 or one of the two chambers having P_1 in their boundary. But each of the three faces has a different vector of signs, and thus $|F(P_1, P_2)| = 1$.

If P_1 is a point, then one can get two faces with the same vector of signs (i.e. $|F(P_1, P_2)| = 2$) in the neighborhood of P_1 if, for example, P_1 is a tangent point of two components (either a line and a conic or two conics). This is the simplest case; that is, when only two components are passing through P_1. Indeed, if there is a transversal intersection at P_1, then $|F(P_1, P_2)| = 1$. Moreover, adding more lines or conics passing through P_1 will not increase $|F(P_1, P_2)|$.

(4) Since $P_1 \preceq P_2$ and $P_1 \neq P_2$, then $\dim(P_2) - 1 \geq \dim(P_1)$, $i(P_1)i(P_2) = i(P_2)$ and obviously $P_2 \in F(P_1, P_2)$. By definition, $P_1 \cdot P_2 = P_2$.

(5) Let $x, y, z \in \mathcal{L}$. We know that a neighborhood of x intersects both $w \doteq (x \cdot y) \cdot z$ and $v \doteq x \cdot (y \cdot z)$ (by Equation (2)), and w and v have the same vector of signs (indeed, note that since $|F(p, q)| = 1$ for every $p, q \in \mathcal{L}$, then $i(p \cdot q) = i(p)i(q)$, i.e. i is a homomorphism and thus $i(w) = i(x \cdot y)i(z) = (i(x)i(y))i(z) = i(x)(i(y)i(z)) = i(v)$).

If $\dim(x) > 0$, then a neighborhood of x can intersect only one face with a given vector of signs (see property (3) above), which implies that $v = w$. If $\dim(x) = 0$, a neighborhood of x may intersect two different faces with the same vector of signs (see property (3) again). That is, x is in the boundary of w and v, and thus $F(x, w) = \{w, v\}$ (as $x \preceq w$ and $x \preceq v$ and $i(x)i(w) = i(w) = i(v)$; the first equality is derived from requirement (4), which holds by the definition of F and the last case in Definition 2.7). Hence, $|F(x, w)| = 2$, which is a contradiction. This means that $v = w$. □

Since requirement (4) holds, we can prove requirement (1); i.e. $\mathcal{L}$ is an *alternative* left regular band.

Proposition 2.9. *Assume that $(\mathcal{L}, \cdot)$ satisfies the requirements of Definition 2.7. Let $x, y \in \mathcal{L}$. Then $(\mathcal{L}, \cdot)$ is an* alternative *left regular band, i.e.:*
(1) $x^2 = x$,
(2) $x \cdot (x \cdot y) = (x \cdot x) \cdot y$ and $x \cdot (y \cdot y) = (x \cdot y) \cdot y$,
(3) $x \cdot y \cdot x = x \cdot y$.

Proof. As $x \in F(x, x)$, we get that $x^2 = x$ (by Proposition 2.8(2)). Next, we have to prove that $x \cdot (x \cdot y) = x \cdot y$ and $x \cdot y = (x \cdot y) \cdot y$. If we denote $z = x \cdot y$, then $x \preceq z$. Thus $x \cdot (x \cdot y) = x \cdot z = z = x \cdot y$

(the second equality is by Proposition 2.8(4), since we can assume that $x \neq y$). Note that if $|F(x,y)| > 0$, then $i(z) = i(x)i(y)$ and thus $i(z) = i(x)i(y) = i(x)i(y)i(y) = i(z)i(y)$ (since this holds in $(L_2^1)^n$) and so $z \in F(z,y)$; thus $z \cdot y = z$ (by Proposition 2.8(2)). Otherwise, $|F(x,y)| = 0$ and thus $x \cdot y = x$, i.e. $z = x$. Thus $z \cdot y = x \cdot y = x = z$; i.e. $z \cdot y = z$ in any case.

Therefore, $(\mathcal{L}, \cdot)$ is an alternative magma. Thus the *flexible identity* $(x \cdot y) \cdot x = x \cdot (y \cdot x)$ holds for any two faces $x, y \in \mathcal{L}$ and the expression $x \cdot y \cdot x$ is well-defined.

As before, if $|F(x,y)| > 0$, then $i(x \cdot y) = i(x)i(y)$ and so $i(x \cdot y) = i(x)i(y) = i(x)i(y)i(x) = i(x \cdot y)i(x)$ and so we have that $x \cdot y \in F(x \cdot y, x)$ and thus by Proposition 2.8(2), $x \cdot y \cdot x = x \cdot y$. Otherwise, $|F(x,y)| = 0$ and thus $x \cdot y = x$ and so $x \cdot y \cdot x = x \cdot x = x = x \cdot y$ □

There are several ways to complete the definition of the product. In this subsection, we will show one way to do so, though we cannot guarantee that the product will be associative (see Example 2.11 below for an example of a real CL arrangement inducing a non-associative product).

Definition 2.10. *(Geometric product on $\mathcal{L}(\mathcal{A})$, Part II)*

With the notations of Definition 2.7, we continue the definition of the geometric product on $\mathcal{L}(\mathcal{A})$. Explicitly, we have that $|F(P_1, P_2)| = 2$ and $P_1 \not\preceq P_2$, and thus $P_2 \notin F(P_1, P_2)$ (by Propositions 2.8(2) and 2.8(4)). By Proposition 2.8(3), this situation can only happen when P_1 is a point, and all the components of $\mathcal{A}$ passing through P_1 are tangent to each other (at P_1).

If P_1 and P_2 are on the same unbounded 1–dimensional component H, then $P_1 \cdot P_2$ will be the face (in $F(P_1, P_2)$) we get after moving from P_1 on H in the direction of P_2 (see Figure 4(a)).

Otherwise, either $P_1 \cdot P_2$ is a chamber or that P_1 and P_2 are on the same bounded 1–dimensional component (i.e. an ellipse). For each $P \in F(P_1, P_2)$, let ℓ_P be the minimal length of an arc passing through the point P_1, a generic point in P and a generic point in P_2. If the minimum of the set $\{\ell_P\}_{P \in F(P_1,P_2)}$ is achieved only once, say, at a face P_0, then define $P_1 \cdot P_2 \doteq P_0$ (see Figure 4(b)). However, if there exist two faces P', P'' such that:

$$\min_{P \in F(P_1,P_2)} \{\ell_P\} = \ell_{P'} = \ell_{P''},$$

then draw a circle C through P_1, a generic point in P' (or in P'') and a generic point in P_2 and define $P_1 \cdot P_2 \doteq P$, where $P \in \{P', P''\}$ is the face we are in after moving slightly clockwise *on C from P_1 (see Figure 4(c)).*

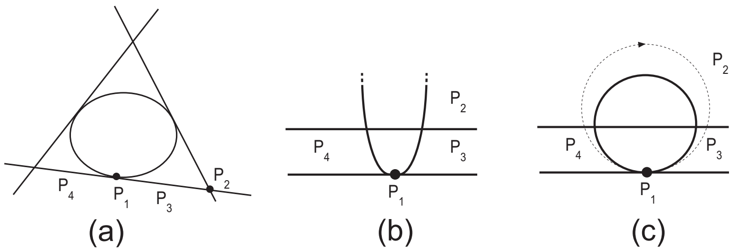

FIGURE 4. Different situations for part II of the geometric product on $\mathcal{L}(\mathcal{A})$: In part (a), $F(P_1, P_2) = \{P_3, P_4\}$ and P_1 and P_2 are on the same unbounded 1–dimensional component, so $P_1 \cdot P_2 = P_3$. In part (b), $P_2 \notin F(P_1, P_2) = \{P_3, P_4\}$. Moreover, $\ell_{P_3} < \ell_{P_4}$, so we have: $P_1 \cdot P_2 = P_3$. In part (c), again $P_2 \notin F(P_1, P_2) = \{P_3, P_4\}$, but in this case $\ell_{P_3} = \ell_{P_4}$, so we draw a dotted circle C through P_1, a generic point in $P' = P_3$ and a generic point in P_2, and move on it clockwise to get: $P_1 \cdot P_2 = P_4$.

The next example shows that the geometric product is not always associative:

Example 2.11 (Non-associative product)**.** Look at the real CL arrangement $\mathcal{A}_0$ presented in Figure 5, where the circle in $\mathcal{A}_0$ is denoted by C. All the labeled faces are on the circle, where x, y

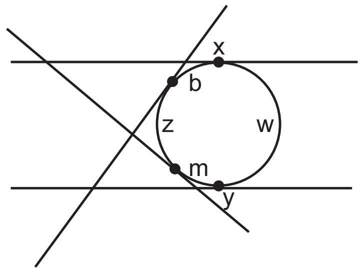

FIGURE 5. An example of a non-associative geometric product:

$$b = x \cdot (y \cdot z) \neq (x \cdot y) \cdot z = w$$

are tangent points and b, w, m and z are 1-dimensional faces. We use Definition 2.10 in order to compute $(x \cdot y) \cdot z$ and $x \cdot (y \cdot z)$.

Note that $F(x, y) = \{b, w\}$ and $\ell_b = \ell_w$. Thus, we should go clockwise on the circle C from x to y and therefore $x \cdot y = w$ and so: $(x \cdot y) \cdot z = w \cdot z = w$. However, $F(y, z) = \{m, w\}$ and $\ell_m < \ell_w$. Thus $y \cdot z = m$ and by the same reasoning, $x \cdot m = b$. Therefore $x \cdot (y \cdot z) = x \cdot m = b$. Thus the geometric product is not associative for this CL arrangement.

However, note that $\mathcal{L}_0(\mathcal{A}_0)$ is an associative LRB, by Proposition 2.6.

2.3.2. *The associative product.* As we saw in Example 2.11, the product introduced in Definition 2.10 is not necessarily associative. Moreover, it does not satisfy requirement (2), i.e., if $x \cdot y = z$ then $i(x)i(y) = i(z)$, where $i : \mathcal{L} \to \mathcal{L}_0$ is the sign function. In this section, we introduce a different product on $\mathcal{L}$ that will be associative and satisfy requirement (2). However, in order to obtain this, we have to assume that $\mathcal{L}_0$ is closed under the product induced by $(L_2^1)^n$ (see Example 2.4(1) above for a CL arrangement whose Image(i) is not closed under this product).

Definition 2.12. *(Associative product on $\mathcal{L}(\mathcal{A})$)*
Let $\mathcal{A}$ be a real CL arrangement such that $\mathcal{L}_0(\mathcal{A})$ is closed under the product induced by $(L_2^1)^n$. Define a function $j : \mathcal{L}_0 \to \mathcal{L}$ as follows. For every $a \in \mathcal{L}_0$, if $|i^{-1}(a)| = 1$, then $j(a) \doteq i^{-1}(a)$. Otherwise, choose an element $a_0 \in i^{-1}(a)$ and define $j(a) \doteq a_0$.

For any two faces $x, y \in \mathcal{L}$, define the product: $x \cdot y \doteq j(i(x)i(y))$.

In the following proposition, we present the properties of this product:

Proposition 2.13. *Let $(\mathcal{L}, \cdot)$ be the partially ordered set of faces of a real CL arrangement, where the product is defined as in Definition 2.12 (i.e. the function j is already given). Then:*

(1) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.
(2) $x \cdot y \cdot x = x \cdot y$.
(3) x^2 *is not necessarily equal to* x.

Proof. Properties (1) and (2) are immediate, since these identities are already satisfied in $\mathcal{L}_0$ (as a subset of $(L_2^1)^n$), i.e. $i(x)(i(y)i(z)) = (i(x)i(y))i(z)$ and $i(x)i(y)i(x) = i(x)i(y)$.

For property (3), look at the arrangement consisting of a line intersecting transversally a circle. Let p_1, p_2 be the two intersection points, and denote $\alpha = i(p_1)$. Note that $i(p_1) = i(p_2) = \alpha$. We may choose $j(\alpha) = p_1$ and thus, $p_2^2 = p_1$. For the other choice, i.e. $j(\alpha) = p_2$, we get that $p_1^2 = p_2$. $\square$

Remark 2.14. (1) The product defined in Definition 2.12 satisfies $x^2 = x^3$ (this is a specific case of Proposition 2.13(2), when taking $x = y$). Thus $(\mathcal{L}, \cdot)$ is an *aperiodic* semigroup, i.e. for every $x \in \mathcal{L}$, x^2 is an idempotent and the set $\{x^2 : x \in \mathcal{L}\}$ is an LRB, isomorphic to $\mathcal{L}_0$.

(2) Note that once there are different faces in $\mathcal{L}$ having the same image under i, then Definition 2.12 does not define a unique product on $\mathcal{L}$, as it depends on the choice made by the function j in this definition.

2.4. **Non-geometric LRBs coming from CL arrangements.** In this section, we present an example of an LRB, induced by a CL arrangement, which cannot be embedded in $(L_2^1)^n$ for any $n \in \mathbb{N}$. This immediately implies that this LRB is not isomorphic to the face LRB of any hyperplane arrangement, and that the family of LRBs associated to CL arrangements is broader than the corresponding family of LRBs associated to hyperplane arrangements.

Example 2.15. Consider the real CL arrangement $\mathcal{A}$ which consists of a line and a circle tangent to it (see Figure 6). This arrangement has 7 faces, and we denote the two 1-dimensional parts (i.e. faces) of the line by b and a, the circle by c, and the 2-dimensional face below the line and outside the circle by d. Let e be the tangency point. As usual, denote the set of faces by $\mathcal{L}$.

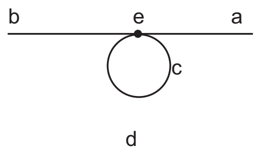

FIGURE 6

The product on $\mathcal{L}$ is defined using Definition 2.7. Note that e is the identity element and one can check that $(\mathcal{L}, \cdot)$ is indeed an LRB. We have the following multiplication table for $\{a, b, c\}$:

$\cdot$	a	b	c
a	a	a	d
b	b	b	d
c	d	d	c

Assume that we have a *monomorphism* $h : \mathcal{L} \to (L_2^1)^n$ for some n. Note that the equalities

$$a \cdot b = a, \qquad b \cdot a = b$$

imply that $h(a)$ and $h(b)$ have zeros in the same coordinates. Indeed, if $(h(a))_j = 0$ then $(h(a))_j \cdot (h(b))_j = (h(b))_j$ (by the multiplication laws in L_2^1), but since $(h(a))_j \cdot (h(b))_j = (h(a))_j = 0$, so $(h(b))_j = 0$. By the same reasoning, using the second equality, we get that $(h(b))_j = 0$ implies $(h(a))_j = 0$.

Since $a \neq b$, then $h(a) \neq h(b)$, which means that there exists a coordinate j, $1 \leq j \leq n$, such that $(h(a))_j \neq (h(b))_j$ and both coordinates are not zero (so without loss of generality, one is $+$ and the other is $-$). But

$$(h(d))_j = (h(a \cdot c))_j = (h(a))_j (h(c))_j = (h(a))_j \neq$$
$$(h(b))_j = (h(b))_j (h(c))_j = (h(b \cdot c))_j = (h(d))_j,$$

by the multiplication laws in L_2^1, which is a contradiction.

Therefore, $\mathcal{L}(\mathcal{A})$ (with the geometric product (Definition 2.7)) is an example of an LRB which is not *geometric* (i.e. it cannot be embedded in $(L_2^1)^n$ for any n, see [15, Section 3.7]).

Moreover, note that for the LRB $\mathcal{L}_0(\mathcal{A})$ (which is contained in $(L_2^1)^2$), one cannot find a hyperplane arrangement $\mathcal{A}' \subset \mathbb{R}^N$ such that $\mathcal{L}_0(\mathcal{A}) \cong \mathcal{L}(\mathcal{A}') \cong \mathcal{L}_0(\mathcal{A}')$. Indeed, $\mathcal{L}_0(\mathcal{A})$ has 6 elements, has a unit $i(e) = (0, 0) \in (L_2^1)^2$ and the three elements $(+, +), (-, -), (-, +)$ form the unique two-sided ideal of $\mathcal{L}_0(\mathcal{A})$. Thus, if such a hyperplane arrangement $\mathcal{A}'$ exists, it should be a central hyperplane arrangement with three chambers, which is impossible.

3. Plane curve arrangements: structure of sub-LRBs

Let $\mathcal{A}$ be a given arrangement of hypersurfaces $\{H_1,\dots,H_m\}$ in $\mathbb{R}^N$, where $H_i \doteq \{f_i = 0\}$ and $f_i \in \mathbb{R}[x_1,\dots,x_N]$. In this section, we study the explicit structure of sub-LRBs of $\mathcal{L}_0(\mathcal{A})$ induced by the embedding of a component H into the arrangement $\mathcal{A}$. In section 3.1, we examine the simple case of line arrangements, and in Sections 3.2 and 3.3 we examine the general case of real curve arrangements in $\mathbb{R}^2$.

Let $H \doteq H_i \in \mathcal{A}$ be a given hypersurface. As before, one can define two associated LRBs. The first is the *deleted* LRB $\mathcal{A}^H$, corresponding to the deletion of the hypersurface H from $\mathcal{A}$, and the second is the *restricted* LRB $\mathcal{A}_H$, corresponding to the restriction of the arrangement $\mathcal{A}$ to H. Explicitly, $\mathcal{A}^H = \mathcal{A} - \{H\}$ is the *deleted* arrangement in $\mathbb{R}^N$ and $\mathcal{A}_H = \left\{K \cap H | K \in \mathcal{A}^H\right\}$ is the *restricted* arrangement located in H. Obviously, $\mathcal{L}_0\left(\mathcal{A}^H\right)$ is obtained from $\mathcal{L}_0(\mathcal{A})$ by deleting the i^{th} coordinate.

For an arrangement $\mathcal{A}$, one can associate a vector of signs in $(\{+,-,0\})^m$ to any face in the arrangement, which describes the position of this face with respect to the hypersurfaces f_i. Explicitly, as before, one can associate to $\mathcal{A}$ a subset $\mathcal{L}_0(\mathcal{A})$ of $(L^1_2)^m$ induced by these vectors of signs.

Define:

$$\mathcal{L}_0(\mathcal{A})|_H \doteq \{x \in \mathcal{L}_0(\mathcal{A}) : (x)_i = 0\} \subset (L^1_2)^m.$$

$\mathcal{L}_0(\mathcal{A})|_H$ is a sub-LRB of $\mathcal{L}_0(\mathcal{A})$, to which corresponds the restricted arrangement $\mathcal{A}_H$ as a sub-LRB. Indeed, it is a subset of $\mathcal{L}_0(\mathcal{A})$ and thus the associativity and the LRB properties $x^2 = x, xyx = xy$ are immediately satisfied. The closure under the product is obvious. Note that $\sharp\mathcal{L}_0(\mathcal{A}_H) = \sharp(\mathcal{L}_0(\mathcal{A})|_H)$.

As H is not necessarily a hyperplane, $\mathcal{A}_H$ is an arrangement of points located on H, but it is not necessarily an arrangement in $\mathbb{R}^k$ for some k (though H can be embedded in $\mathbb{R}^N$, we look at the arrangement in H). However, when the arrangement is in $\mathbb{R}^2$, the question regarding the connections between $\mathcal{L}_0(\mathcal{A}_H)$ and $\mathcal{L}_0(\mathcal{A})|_H$ becomes more manageable, as one can try to define a structure of an LRB on $\mathcal{A}_H$ as is done in Definitions 3.5 and 3.7. Note that when either H is a bounded component or an unbounded one, $\mathcal{A}_H$ is a collection of points $\{p_1,\dots,p_k\}$ on H.

Note: From now on, we assume that $\mathcal{L}_0(\mathcal{A})$ is an LRB, i.e. it is closed under the product induced by $(L^1_2)^m$. Moreover, to simplify notations, we assume that each H_i is connected in $\mathbb{R}^N$, where H_i is defined by the hypersurface $\{f_i = 0\}$.

Remark 3.1. Note that $\mathcal{L}_0(\mathcal{A}_H) \subseteq (L^1_2)^k$ and $\mathcal{L}_0(\mathcal{A})|_H \subseteq (L^1_2)^m$. In order to distinguish between the different vectors of signs when we talk on a corresponding face, which can be thought of both as a face in $\mathcal{L}(\mathcal{A}_H)$ and in $\mathcal{L}(\mathcal{A})|_H \subseteq \mathcal{L}(\mathcal{A})$, we denote:

$$i_{\mathcal{A}} : \mathcal{L}(\mathcal{A}) \to (L^1_2)^m, \ \text{Image}(i_{\mathcal{A}}) = \mathcal{L}_0(\mathcal{A})$$

and

$$i_H : \mathcal{L}(\mathcal{A}_H) \to (L^1_2)^k, \ \text{Image}(i_H) = \mathcal{L}_0(\mathcal{A}_H),$$

where both maps describe the vectors of signs in $\mathcal{L}_0(\mathcal{A})$ (resp. $\mathcal{L}_0(\mathcal{A}_H)$) of a face in $\mathcal{L}(\mathcal{A})$ (resp. $\mathcal{L}(\mathcal{A}_H)$). See the exact definition of i_H in Definitions 3.5 and 3.7.

3.1. **Preliminaries: The embedding principle for the face LRB of line arrangements.** In this section, we present the embedding principle for the face LRB of line arrangements in $\mathbb{R}^2$ (which can be easily generalized to hyperplane arrangements), i.e. the connections between $\mathcal{L}_0(\mathcal{A}_H)$ and $\mathcal{L}_0(\mathcal{A})|_H$ for a line arrangement $\mathcal{A}$ and $H \in \mathcal{A}$. This is done as a preparation for Proposition 3.9, which deals with the embedding principle for arrangements of smooth real curves.

Lemma 3.2. *Let $\mathcal{A} = \{H_1,\dots,H_m\}$ be an arrangement of lines in $\mathbb{R}^2$, where $H_i = \{f_i = 0\}$. Denote $H = H_1$ and let $H \cap \{H_2,\dots,H_m\} = \{p_1,\dots,p_k\} \subset H$ be k points. Then, there is an isomorphism of LRBs:*

$$\varphi : \mathcal{L}_0(\mathcal{A}_H) \xrightarrow{\sim} \mathcal{L}_0(\mathcal{A})|_H \subseteq (L^1_2)^m,$$

satisfying the following properties:

(1) $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_1 = 0$.
(2) *For every* $j > 1$*:*
   (a) If $H \cap H_j = \emptyset$*, then* $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_j$ *is constant (either + or −, depending on the mutual position of the parallel lines* H *and* H_j*). Explicitly, all the vectors in* $\varphi(\mathcal{L}_0(\mathcal{A}_H))$ *have the same sign in the* j^{th} *coordinate.*
   (b) If $H \cap H_j = \{p_s\}$*, then* $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_j = (\mathcal{L}_0(\mathcal{A}_H))_s$*, up to a constant scalar multiplication in* $\{\pm 1\}$*. The index of the right hand side is the index of* p_s *in the arrangement of points in* $H = H_1$*, i.e. in* $\mathcal{L}_0(\mathcal{A}_H)$*.*

Before the proof, we illustrate the above lemma by an example.

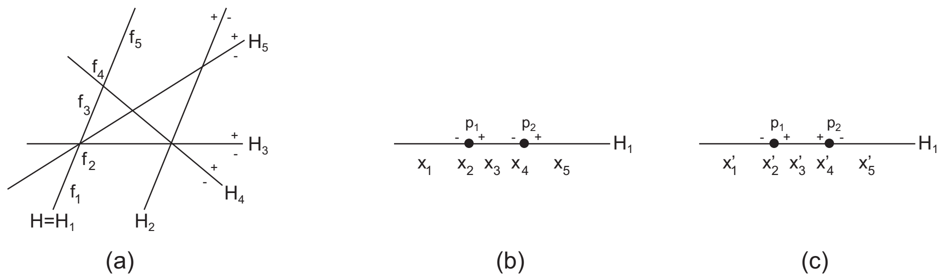

FIGURE 7. An example for illustrating Lemma 3.2: f_i are the faces contained in H_1 in the face set $\mathcal{L}(\mathcal{A})$. x_i (and x'_i) are the faces in the face set $\mathcal{L}_0(\mathcal{A}_{H_1})$.

Example 3.3. Figure 7(a) presents an arrangement $\mathcal{A}$ and Figures 7(b) and 7(c) present two arrangements of points on a line, both can be thought of as the restricted arrangement $\mathcal{A}_{H_1}$. Note that the difference between the arrangements in Figures 7(b) and 7(c) is that the signs assigned with respect to the point p_2 are opposite.

(1) Considering the arrangement in Figure 7(b), the faces of $\mathcal{A}_{H_1}$ are denoted by $x_1, \ldots, x_5$; their corresponding images by φ, i.e. these faces in the arrangement $\mathcal{A}$, are denoted by $f_1, \ldots, f_5$. Let $H = H_1$. Then, the corresponding LRBs are

$$\mathcal{L}_0(\mathcal{A}_H) = \left\{ \begin{array}{c} i_H(x_1) = (-,-), i_H(x_2) = (0,-), i_H(x_3) = (+,-), \\ i_H(x_4) = (+,0), i_H(x_5) = (+,+) \end{array} \right\},$$

and

$$\mathcal{L}_0(\mathcal{A})|_H = \varphi(\mathcal{L}_0(\mathcal{A}_H)) = \left\{ \begin{array}{c} i_{\mathcal{A}}(f_1) = (0,+,-,-,-), i_{\mathcal{A}}(f_2) = (0,+,0,-,0), \\ i_{\mathcal{A}}(f_3) = (0,+,+,-,+), i_{\mathcal{A}}(f_4) = (0,+,+,0,+), \\ i_{\mathcal{A}}(f_5) = (0,+,+,+,+) \end{array} \right\}.$$

   (a) First, note that $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_1 = 0$ (case (1) of the lemma).
   (b) Since $H \cap H_2 = \emptyset$, $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_2 = +$, i.e., by case (2)(a), the second coordinate in all the vectors of $\varphi(\mathcal{L}_0(\mathcal{A}_H))$ is $+$.
   (c) Since $H_3 \cap H = H_5 \cap H = \{p_1\}$,

$$(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_3 = (\varphi(\mathcal{L}_0(\mathcal{A}_H)))_5 = (\mathcal{L}_0(\mathcal{A}_H))_1$$

   (by case (2)(b)).
   (d) Since $H_4 \cap H = \{p_2\}$, $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_4 = (\mathcal{L}_0(\mathcal{A}_H))_2$ (again by case (2)(b)).

(2) Considering the arrangement in Figure 7(c), the faces of $\mathcal{A}_{H_1}$ are denoted by $x'_1, \ldots, x'_5$. In this case, we have:

$$\mathcal{L}_0(\mathcal{A}_H) = \left\{ \begin{array}{c} i_H(x'_1) = (-,+), i_H(x'_2) = (0,+), i_H(x'_3) = (+,+), \\ i_H(x'_4) = (+,0), i_H(x'_5) = (+,-) \end{array} \right\}.$$

As before, since $H_3 \cap H = H_5 \cap H = \{p_1\}$,

$$(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_3 = (\varphi(\mathcal{L}_0(\mathcal{A}_H)))_5 = (\mathcal{L}_0(\mathcal{A}_H))_1.$$

On the other hand, as $H_4 \cap H = \{p_2\}$, $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_4 = -(\mathcal{L}_0(\mathcal{A}_H))_2$. Explicitly, in contrast to Example (1)(d) above, in order to obtain $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_4$, one has to multiply all the values in $(\mathcal{L}_0(\mathcal{A}_H))_2$ by the scalar (-1).

Proof of Lemma 3.2. We use the notation introduced in Remark 3.1. Case (1) is obvious (since we are in H_1).

For case (2)(a), note that if $H \cap H_j = \emptyset$, then H_j is parallel to H and all the faces of $\mathcal{A}$ with support in H are either in the halfplane $\{f_j > 0\}$ (in this case $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_j = +$) or in $\{f_j < 0\}$ (in this case $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_j = -$).

As for case (2)(b): assume that $H \cap H_j = \{p_s\}$ for some $1 \leq s \leq k$ and let c be a face of $\mathcal{A}$ with $\mathrm{supp}(c) \subseteq H$. As c goes over all the faces such that $\mathrm{supp}(c) \subseteq H$, it passes over all the set $\mathrm{Image}(\varphi)$. Then, either $c \subset \{f_j > 0\}$, $c \subset \{f_j < 0\}$ or $c \subset \{f_j = 0\}$. In the third case, $c = p_s \in H$ and thus $(i_H(c))_s = 0$ and as $c \in H_j$, $(i_{\mathcal{A}}(c))_j = 0$. As for the first two cases, the fact that c is in one of the two halfplanes is determined by the position of H with respect to H_j (as $c \subset H$), which is reduced to checking if c is located to the right of $\{p_s\} = H \cap H_j$ or to its left. Therefore, up to a constant scalar multiplication by $\{\pm 1\}$ (for all faces c such that $\mathrm{supp}(c) \subseteq H$), $c \subset \{f_j > 0\}$ is equivalent to the fact that c is to the right of p_s. The (constant) scalar multiplication is needed, since a priori there is no connection between the sign in $\mathcal{L}_0(\mathcal{A}_H)$ that is assigned to the faces to the right of p_s and the sign in $\mathcal{L}_0(\mathcal{A})$ assigned to these faces in the halfplane above H_j (see Figures 7(b) and 7(c) for an example of two different assignments of signs for $\mathcal{A}_H$). □

Remark 3.4. Note that there is a natural assignment of signs for the elements in $\mathcal{A}_H$, that is induced by $\mathcal{L}_0(\mathcal{A})$, in the following way: if the sign in $\mathcal{L}_0(\mathcal{A})$ that is given to the halfplane $\{f_j > 0\}$ is + (in the j^{th} coordinate, where $H_j = \{f_j = 0\}$ and $H \cap H_j = \{p_s\}$), and the section $\{x > p_s\}$ on the line H is contained in $\{f_j > 0\}$, then the sign in $\mathcal{L}_0(\mathcal{A}_H)$ associated to $\{x > p_s\}$ (i.e. in the s^{th} coordinate) will also be +. If $\{x < p_s\} \subset \{f_j > 0\}$, then the sign associated to the faces contained in this section will be −. In the case of this natural assignment of signs to $\mathcal{A}_H$, the scalar multiplication in case (2)(b) of Lemma 3.2 is not needed.

However, note that Lemma 3.2 is more general, as we do not assume any a priori connection between the signs associated to the halfplanes in $\mathcal{A}$ and the signs associated to the half-lines in $\mathcal{A}_H$.

3.2. **The structure of the LRB on a real pointed curve.** Before passing to the embedding principle in the general case of arrangements of smooth curves (Section 3.3), one has to consider two cases with respect to the structure of the induced LRB of a real pointed curve $\mathcal{A}_H$ - where H is an unbounded component and where H is a bounded one. In this section, we deal with the structure of the induced LRB of a real pointed curve $\mathcal{A}_H$ (that is, an arrangement of points on a real curve) in the above two cases.

Given an arrangement of smooth curves $\mathcal{A}$ in $\mathbb{R}^2$ and a connected component $H \in \mathcal{A}$, the restricted arrangement $\mathcal{A}_H$ will be the real curve H with points on it corresponding to the intersection points of the deleted arrangement $\mathcal{A} - \{H\}$ with H, i.e. we get an arrangement of real points on a connected component H.

We start with the case of an unbounded component:

Definition 3.5 (LRB structure on an unbounded component)**.**
Let $H \subset \mathbb{R}^2$ be an unbounded smooth connected real plane curve with no self-intersections. Let $\{p_1, \dots, p_k\}$ be a collection of points on H and let $\mathcal{L}(H)$ be the set of faces of H with respect to these points; explicitly, the faces are the points themselves and the sections of the curve that are bounded by the points.

Each point $p_j \in H$, $1 \le j \le k$, divides the curve H into three distinct parts: the point itself and two other open sets: $H_{j,1}$ and $H_{j,2}$ such that $H = \{p_j\} \cup H_{j,1} \cup H_{j,2}$. Associate to the set $H_{j,1}$ the sign $+$, to the set $H_{j,2}$ the sign $-$ and to the set $\{p_j\}$ the sign 0. Obviously, one can rename the set $H_{j,1}$ as $H_{j,2}$ and $H_{j,2}$ as $H_{j,1}$ and thus induce a different assignment of signs, but once we assign these signs for each set, they are fixed.

For each face $P \in \mathcal{L}(H)$, we associate an element $i_H(P)$ in $(L_2^1)^k$, that is, a vector of signs, in the following way: for each j, $1 \le j \le k$, if $P \subseteq H_{j,1}$, then $(i_H(P))_j = +$; otherwise, if $P \subseteq H_{j,2}$, then $(i_H(P))_j = -$; otherwise, that is $P = \{x = p_j\}$, $(i_H(P))_j = 0$.

In this way, we get a monomorphic map $i_H : \mathcal{L}(H) \to (L_2^1)^k$ and we can identify $\mathcal{L}(H)$ with its image $i_H(\mathcal{L}(H)) \subseteq (L_2^1)^k$.

We have the following lemma:

Lemma 3.6. *(1) The set $i_H(\mathcal{L}(H))$ is closed under the product induced by the LRB $(L_2^1)^k$, so it is an LRB as well.*
(2) Different assignments of signs to $H_{j,1}$, $H_{j,2}$ (as described above) induce isomorphic LRBs.

Proof. (1) The set $i_H(\mathcal{L}(H))$ is closed under the product induced by the LRB $(L_2^1)^k$, since H is topologically equivalent to a line, and the assignment of the vectors of signs to $\mathcal{L}(H)$ is thus equivalent to associating an LRB structure to the set of faces of a pointed line, as a spcial case of a hyperplane arrangement (as described in Section 2.1).

(2) Since H is topologically equivalent to a line, different assignments of signs to $H_{j,1}$, $H_{j,2}$, will induce isomorphic LRBs, by Remark 2.2. □

We pass to the case of a bounded component. If H is a smooth bounded component in $\mathbb{R}^2$, i.e. an oval, we can consider an arrangement of points $\{p_1, \dots, p_k\}$ on an oval and look at the corresponding set of faces $\mathcal{L}(H)$. However, we cannot treat $\mathcal{L}(H)$ as in the former case, since there is no meaning to the phrase "every point divides the curve H into three distinct parts", when we are on an oval. We introduce here an alternative way to associate an LRB structure to $\mathcal{L}(H)$.

Definition 3.7 (LRB structure on a bounded component)**.**
Let $H = C$ be a smooth pointed bounded oval, where $\{p_1, \dots, p_k\}$ is the set of points on it numerated consecutively clockwise. As can easily be seen, the set of faces $\mathcal{L}(C)$ contains $2k$ faces: k points and k sections of the curve that are bounded by the points. Let p'_1 be a point to the left of p_1 which is infinitesimally close to p_1 (see Figure 8(a)), and let $C_1 = C - \{p'_1\}$. C_1 is topologically equivalent to an open segment $S = (a'_1, a''_1)$, that is, there exists a distance-preserving homeomorphism $f : C_1 \to S$, such that $f(p'_1) = a'_1 = a''_1$. Denote $f(p_i) = a_i$ for $1 \le i \le k$.

Explicitly, we think of C_1 as a straight segment that starts at the point a'_1, when the section that starts at a_k ends at a point a''_1, which, on C_1, is identified with p'_1 (see Figure 8(b)).

On the pointed segment $S \cup \{a_1, \dots, a_k\}$, the set of faces consists of $2k + 1$ faces. However, on C, the segments $f^{-1}(a'_1, a_1)$ and $f^{-1}(a_k, a''_1)$ are contained in the same face. As $a_1 - a'_1 = \varepsilon \ll 1$, we ignore this infinitesimally-small face and thus $\mathcal{L}(S)$, the set of faces of S, has only $2k$ faces: k points and k open sections of the curve. We now identify this set of faces with the set of faces $\mathcal{L}(C)$.

We can now associate an LRB structure to $\mathcal{L}(S)$, as it is done for a set of faces of a pointed line; that is, to every face $P \in \mathcal{L}(S)$, we associate a vector of signs $i_H(P) \in (L_2^1)^k$ in the following way: Given $1 < j \leq k$, the point a_j divides S into three distinct parts: the point itself $\{x = a_j\}$ and two other open sets: $H_{j,1} = \{x > a_j\}$ and $H_{j,2} = \{x < a_j\}$. Associate to the set $H_{j,1}$ the sign $+$, to the set $H_{j,2}$ the sign $-$ and to the set $\{x = a_j\}$ the sign 0. Obviously, as in the case of an unbounded component, one can rename the set $H_{j,1}$ as $H_{j,2}$ and $H_{j,2}$ as $H_{j,1}$ and thus induce a different assignment of signs, but once we assign these signs for each set, they are fixed.

For $j = 1$, since we ignore the section $\{a'_1 < x < a_1\}$, the point a_1 divides S into two distinct parts: the point $\{x = a_1\}$ itself and $H_{1,1} = \{x > a_1\}$. Associate to the set $H_{1,1}$ the sign $+$ (or $-$) and to the set $\{x = a_1\}$ the sign 0. Again, once we associated these signs for each set, they are fixed (see Figure 8(c)).

Thus, the map $i_H : \mathcal{L}(S) \to (L_2^1)^k$ is defined as in Definition 3.5: for each face, the j^{th} coordinate of $i_H(P)$ for $P \in \mathcal{L}(S)$, depends on whether $P = a_j$, $P \subseteq H_{1,j}$ or $P \subseteq H_{2,j}$.

Note that $\mathcal{L}(S)$ has an LRB structure (by the same arguments of Lemma 3.6(1)). Similar to the case of an unbounded component, as C_1 is topologically equivalent to an open segment, different assignments of signs to $H_{j,1}$, $H_{j,2}$, as described above, will induce isomorphic LRBs, by Remark 2.2.

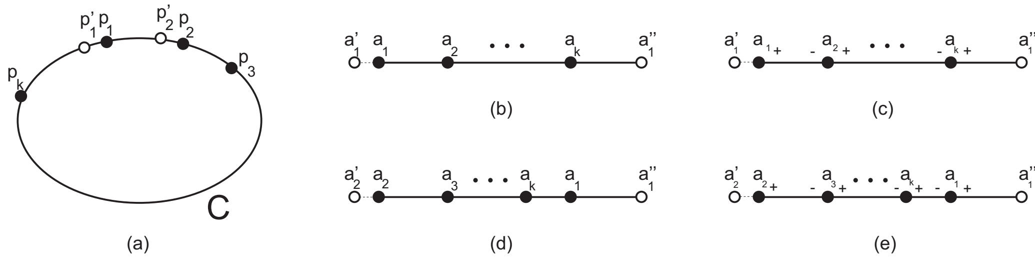

FIGURE 8. The LRB structure associated to an arrangement of points on an oval (for different choices of reference points p'_1 or p'_2). The section (a'_1, a_1) (or (a'_2, a_2)) is ignored.

We still have to prove that Definition 3.7 does not depend on the choice of the initial point p_1, when numerating the points on C. This is equivalent to prove that if we choose a point p'_2 as a point to the left of p_2 (being infinitesimally close to p_2) and consider the induced LRB structure on $C_2 = C - \{p'_2\}$ (see Figures 8(a) and 8(d)), then the LRBs $i_H(\mathcal{L}(C_1))$ and $i_H(\mathcal{L}(C_2))$ are isomorphic.

Proposition 3.8. *The LRBs $\mathcal{L}_0(C_1) = i_H(\mathcal{L}(C_1))$ and $\mathcal{L}_0(C_2) = i_H(\mathcal{L}(C_2))$ are isomorphic. Therefore, the LRB structure on $C \cup \{p_1, \dots, p_k\}$ is independent of the choice of the removed point.*

Proof. As noted after Definition 3.7, different sign assignments on $\mathcal{L}(C_1)$ (or on $\mathcal{L}(C_2)$) induce isomorphic LRB structures. Thus, we first set a fixed assignment of signs for $\mathcal{L}(C_1)$ and $\mathcal{L}(C_2)$ and then prove that the LRBs are isomorphic.

The sign assignment for $\mathcal{L}(C_1)$ is the following: for each $1 < j \leq k$, we assign the sign $+$ to $H_{j,1}$, the sign $-$ to $H_{j,2}$ and the sign 0 to $\{x = a_j\}$; for $j = 1$, we assign the sign $+$ to $H_{1,1}$ and the sign 0 to $\{x = a_1\}$ (see Figure 8(c)).

The sign assignment for $\mathcal{L}(C_2)$ is the following: for each $1 \leq j \leq k$ where $j \neq 2$, we assign the sign $+$ to $H_{j,1}$, the sign $-$ to $H_{2,2}$ and the sign 0 to $\{x = a_j\}$; for $j = 2$, we assign the sign $+$ to $H_{j,1}$ and the sign 0 to $\{x = a_2\}$ (see Figure 8(e)).

Thus, going over all the $2k$ faces of $\mathcal{L}(C_1)$ from left to right, we get that:

$$i_H(\mathcal{L}(C_1)) = \left\{ \begin{array}{c} (0, -, -, \dots, -), (+, -, -, \dots, -), (+, 0, -, \dots, -), \\ (+, +, -, \dots, -), \dots, (+, \dots, +) \end{array} \right\}.$$

In the same way, going over all the $2k$ faces of $\mathcal{L}(C_2)$ from left to right, we get that:

$$i_H(\mathcal{L}(C_2)) = \left\{ \begin{array}{c} (-,0,-,-,\dots,-),(-,+,-,-,\dots,-),(-,+,0,-,\dots,-), \\ (-,+,+,-,\dots,-),\dots,(-,+,\dots,+),(0,+,\dots,+),(+,+,\dots,+) \end{array} \right\}.$$

Both LRBs describe the movement over the $2k$ faces along a bounded open straight segment with k marked points, i.e. given faces x and y, then $x \cdot y$ is the face we enter in after the movement from x to y on this line and thus they are isomorphic. Thus the explicit isomorphism from $\mathcal{L}_0(C_1)$ to $\mathcal{L}_0(C_2)$ maps the points $p_i \mapsto p_{i(\text{mod } k)+1}$ and the sections of C_1 are mapped to the corresponding sections of C_2, according to the mapping of the points. □

Thus, given an arrangement $\mathcal{A}$ and a bounded component $H \in \mathcal{A}$, we can choose a point p infinitesimally close to the point p_1 and delete it. In this way, we can consider the LRB associated to $\mathcal{A}_H - \{p\}$, as in the case of an unbounded component (when ignoring the infinitesimally-small section between p and p_1). As was shown, this LRB does not depend on the location of p (when the only condition is that $p \neq p_j$ for all j) up to an isomorphism. Denote this associated LRB by $\mathcal{L}_0(\mathcal{A}_H)$, which is a sub-LRB of $(L_2^1)^k$. For other examples, see Example 3.10(3) and Figure 10 below.

3.3. **The embedding principle for the face LRB of CL arrangements.** We are ready to describe the main result of this section: the structure of the sub-LRBs of $\mathcal{L}_0(\mathcal{A})$ induced by the components of $\mathcal{A} \subset \mathbb{R}^2$.

Proposition 3.9. *Let $\mathcal{A} = \{H_1, \dots, H_m\}$ be an arrangement of smooth connected curves in $\mathbb{R}^2$, such that $H_i = \{f_i = 0\}$ where $f_i \in \mathbb{R}[x,y]$. Let $H \doteq H_1$ and*

$$H \cap \{H_2, \dots, H_m\} = \{p_1, \dots, p_k\} \subset H.$$

Then there is a bijective function, which is not necessarily an isomorphism, of LRBs:

$$\varphi : \mathcal{L}_0(\mathcal{A}_H) \to \mathcal{L}_0(\mathcal{A})|_H \subseteq (L_2^1)^m$$

satisfying:

(1) $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_1 = 0$.
(2) *For every $j > 1$:*
(a) If $H \cap H_j = \emptyset$, then $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_j$ is constant. Explicitly, all the vectors in $\varphi(\mathcal{L}_0(\mathcal{A}_H))$ have the same sign in the j^{th} coordinate.
(b) If $H \cap H_j \neq \emptyset$, let $H \cap H_j = \{p_i\}_{i \in K_j}$, where K_j is the set of indices of the points in $H \cap H_j$. Then (up to a constant scalar multiplication by $\{\pm 1\}$):

$$(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_j = \prod_{i \in K_j} ((\mathcal{L}_0(\mathcal{A}_H))_i)^{m_i}$$

where $m_i = \text{mult}_{p_i}(H \cap H_j)$ is the intersection multiplicity at the point p_i, and the multiplication of signs (in the right hand side) is the usual product (explicitly, $+ \cdot + = - \cdot - = +$, $+ \cdot - = - \cdot + = -$, $0 \cdot \{\pm\} = 0$). Note that the numeration of the indices in the right hand side is according to the numeration of the points in the arrangement of points in $H = H_1$.

As before, we illustrate this proposition by some examples before proving it.

Example 3.10. (1) Figure 9(a) presents an arrangement $\mathcal{A}$ with three lines and a conic tangent to one of the lines, and Figure 9(b) presents the restricted arrangement $\mathcal{A}_{H_1}$. By Proposition 2.6, $\mathcal{L}_0(\mathcal{A})$ is indeed a semigroup. The faces of $\mathcal{A}_{H_1}$ are denoted by $x_1, \dots, x_5$ and their

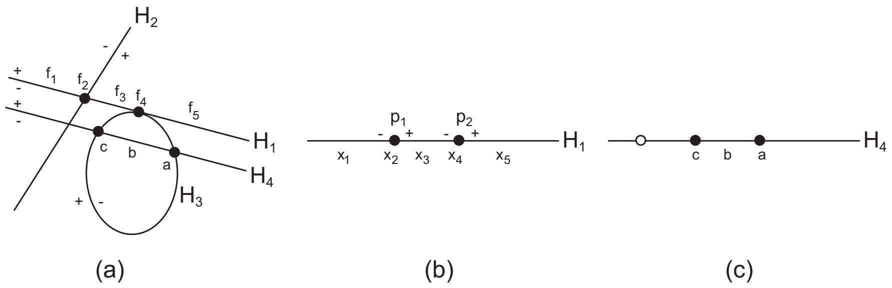

FIGURE 9. An example for illustrating Proposition 3.9: f_i are the faces contained in H_1 in the face set $\mathcal{L}(\mathcal{A})$. x_i are faces in the face set $\mathcal{L}_0(\mathcal{A}_{H_1})$. a, b, c are faces contained in H_4.

corresponding faces in $\mathcal{A}$ are denoted by $f_1, \dots, f_5$. Let $H = H_1$. Then, the corresponding LRBs are:

$$\mathcal{L}_0(\mathcal{A}_H) = \left\{ \begin{array}{c} i_H(x_1) = (-,-), i_H(x_2) = (0,-), i_H(x_3) = (+,-), \\ i_H(x_4) = (+,0), i_H(x_5) = (+,+) \end{array} \right\}$$

and

$$\mathcal{L}_0(\mathcal{A})|_H = \varphi(\mathcal{L}_0(\mathcal{A}_H)) = \left\{ \begin{array}{c} i_{\mathcal{A}}(f_1) = (0,+,+,+), i_{\mathcal{A}}(f_2) = (0,0,+,+), \\ i_{\mathcal{A}}(f_3) = (0,-,+,+), i_{\mathcal{A}}(f_4) = (0,-,0,+), \\ i_{\mathcal{A}}(f_5) = (0,-,+,+) \end{array} \right\}.$$

Then:

(a) First, note that $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_1 = 0$ (case (1) of the proposition).

(b) Since $H \cap H_2 = \{p_1\} \in \mathcal{A}_H$, by case (2)(b) of the proposition,

$$(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_2 = -(\mathcal{L}_0(\mathcal{A}_H))_1$$

(note the scalar multiplication by -1).

(c) Since $H_3 \cap H = \{p_2\}$, where $\text{mult}_{p_2}(H \cap H_3) = 2$, then again by case (2)(b),

$$(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_3 = ((\mathcal{L}_0(\mathcal{A}_H))_2)^2.$$

(d) Since $H_4 \cap H = \emptyset$, then $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_4 = +$ (by case (2)(a)).

(2) Relabel the arrangement in Figure 9(a), such that the conic will be now labeled as H_1, see Figure 10(a).

The faces of $\mathcal{A}_{H_1}$ are denoted by $x_1, \dots, x_6$ (see Figure 10(c); note that the section between p'_1 and p_1 is ignored) and their corresponding faces in $\mathcal{A}$ are denoted by $f_1, \dots, f_6$ (see Figure 10(a)). Let $H = H_1$. As was explained in Definition 3.7, one can induce an LRB structure on $\mathcal{A}_H$. Then, the corresponding LRBs are:

$$\mathcal{L}_0(\mathcal{A}_H) = \left\{ \begin{array}{c} i_H(x_1) = (0,-,-), i_H(x_2) = (+,-,-), i_H(x_3) = (+,0,-), \\ i_H(x_4) = (+,+,-), i_H(x_5) = (+,+,0), i_H(x_6) = (+,+,+) \end{array} \right\}$$

and

$$\mathcal{L}_0(\mathcal{A})|_H = \varphi(\mathcal{L}_0(\mathcal{A}_H)) = \left\{ \begin{array}{c} i_{\mathcal{A}}(f_1) = (0,0,-,+), i_{\mathcal{A}}(f_2) = (0,-,-,+), i_{\mathcal{A}}(f_3) = (0,-,-,0), \\ i_{\mathcal{A}}(f_4) = (0,-,-,-), i_{\mathcal{A}}(f_5) = (0,-,-,0), i_{\mathcal{A}}(f_6) = (0,-,-,+) \end{array} \right\}.$$

Then:

(a) First, note that $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_1 = 0$ (case (1) of the proposition).

(b) Since $H \cap H_2 = \{p_1\} \in \mathcal{A}_H$, where $m_1 = \text{mult}_{p_1}(H \cap H_2) = 2$, then by case (2)(b), $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_2 = -((\mathcal{L}_0(\mathcal{A}_H))_1)^2$ (note the scalar multiplication by -1).

(c) Since $H_3 \cap H = \emptyset$, then $(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_3 = -$ (by case (2)(a)).

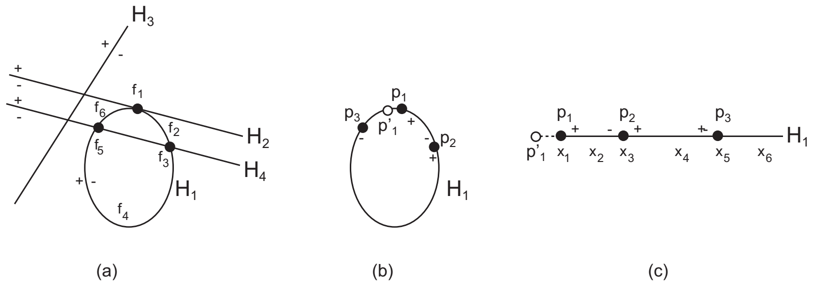

FIGURE 10. Another example for illustrating Proposition 3.9: f_i are the faces contained in the conic H_1 in the face set $\mathcal{L}(\mathcal{A})$ (see part (a)). x_i are the faces of the face set $\mathcal{L}_0(\mathcal{A}_{H_1})$ (see part (c)). The three parts illustrate the process of associating an LRB structure to the conic H_1. First, we remove a point p'_1 from H_1 to the left of p_1 (see part (b)). Then, we consider H_1 as a segment with this point deleted, i.e. a ray which starts from p_1 (see part (c)).

(d) Since $H_4 \cap H = \{p_2, p_3\}$, where
$$m_2 = \text{mult}_{p_2}(H \cap H_4) = \text{mult}_{p_3}(H \cap H_4) = 1,$$
then by case (2)(b),
$$(\varphi(\mathcal{L}_0(\mathcal{A}_H)))_4 = (\mathcal{L}_0(\mathcal{A}_H))_2 \cdot (\mathcal{L}_0(\mathcal{A}_H))_3.$$

Remark 3.11. Note that if every singular point is locally a transversal intersection of two components (as in the case, for example, of a line arrangement), then one can easily see that Proposition 3.9 is indeed a generalization of Lemma 3.2.

Proof of Proposition 3.9. We use the same notations introduced in Remark 3.1. The proofs of cases (1) and (2)(a) are identical to the corresponding proofs in Lemma 3.2.

First, we show that φ is not necessarily a homomorphism. Let $\mathcal{A}$ be the arrangement presented in Figure 9(a), and let $H = H_4$. Let a, c be the intersection points of H with the conic and b be the 1-dimensional segment between them (see Figure 9(c)). When considering a, b, c as faces of $\mathcal{A}_H$, then in $\mathcal{L}_0(\mathcal{A}_H)$, $i_H(a)i_H(c) = i_H(b)$. However, when considering a, b, c as faces of $\mathcal{A}$ (see Figure 9(a)), $i_{\mathcal{A}}(a)$ and $i_{\mathcal{A}}(c)$ have a zero value in the coordinate corresponding to the conic. However, $i_{\mathcal{A}}(b)$ does not have a zero value in that coordinate. Thus, in $\mathcal{L}_0(\mathcal{A})$, $i_{\mathcal{A}}(a)i_{\mathcal{A}}(c) \neq i_{\mathcal{A}}(b)$.

We now prove case (2)(b). Let $j > 1$ and assume that $H \cap H_j = \{p_i\}_{i \in K_j}$. Let c be a face of $\mathcal{A}$ with $\text{supp}(c) \subseteq H$. Note that if $c = p_k$ for $k \in K_j$, then $(i_H(c))_i = 0$ in $\mathcal{L}_0(\mathcal{A}_H)$ and $(i_{\mathcal{A}}(c))_j = 0$ in $\mathcal{L}_0(\mathcal{A})|_H = \varphi(\mathcal{L}_0(\mathcal{A}_H)) \subset \mathcal{L}_0(\mathcal{A})$; thus case (2)(b) is satisfied when c is 0-dimensional.

Therefore, we can assume that c is a face satisfying $\dim(c) = 1$. Then, either $c \subset \{f_j > 0\}$ or $c \subset \{f_j < 0\}$. We claim that the corresponding vector of signs is determined by the relative position of c with respect to the points $\{p_i\}$: the (usual) product of the signs (of the i^{th} coordinates of $\mathcal{L}_0(\mathcal{A}_H)$, where $i \in K_j$) describes whether c is in $\{f_j > 0\}$ or in $\{f_j < 0\}$. Let us explicitly check all the possible cases:

(1) If $H \cap H_j = \{p_i\}$ is a single transversal intersection point ($m_i = 1$), then, as H and H_j has only one connected component in $\mathbb{R}^2$, we can proceed as in case (2)(b) in Lemma 3.2.
(2) If $H \cap H_j = \{p_i\}$ is a single tangent point ($m_i = 2$), then we claim that the j^{th} coordinate of $\varphi(\mathcal{L}_0(\mathcal{A}_H))$ is constant: either $+$ or $-$ (except for the face $x = p_i$, whose sign in the j^{th} coordinate is 0, as was described above for the case that $\dim(c) = 0$). This is since H is either entirely outside or entirely inside the domain $\{f_j > 0\}$, and the j^{th} coordinate is determined according to the signs attached to the two domains of the plane partitioned by

the curve H_j. In the first case $(i_{\mathcal{A}}(c))_j = +$ and in the second case $(i_{\mathcal{A}}(c))_j = -$. Also, in any case $((i_H(c))_i)^2 = +$ and thus we proved that $(i_{\mathcal{A}}(c))_j = \pm((i_H(c))_i)^2 = \{\pm 1\}$, thus the j^{th} coordinate is indeed constant.

(3) Generalizing case (1) and (2), if $H \cap H_j = \{p_i\}$ is a single singular point of multiplicity $m_i > 2$, then we are only interested in the parity of m_i. If m_i is even, then locally at p_i, the curve H does not pass to the "other side" of H_j (i.e. it is only in the domain $\{f_j \geq 0\}$ or in $\{f_j \leq 0\}$), and thus the treatment of this case is as in case (2), where $m_i = 2$. If m_i is odd, then locally at p_i, the curve H_j does pass to the "other side" of H, and thus the treatment of this case is as in case (1), where $m_i = 1$.

(4) Assume now that $H \cap H_j = \{p_{s_1}, p_{s_2}\}$ is two transversal intersection points (i.e. $m_{s_1} = m_{s_2} = 1$; for example, when H is a line and H_j is a circle intersecting H twice transversally). Recall that the structure of the induced LRB on a pointed real curve $C \cup \{p_1, \dots, p_k\}$ (see Section 3.2) allows us to think on faces which are to the right (or to the left) of a point p_i, $1 \leq i \leq k$. Assume without loss of generality that p_{s_2} is to the right of p_{s_1}.

If H is an unbounded curve, then the fact that $c \subset \{f_j > 0\}$ is equivalent to the fact that c is to the right of p_{s_2} or to the left of p_{s_1}. In the first case, $(i_H(c))_{s_1} \cdot (i_H(c))_{s_2} = + \cdot + = + = (i_{\mathcal{A}}(c))_j$. In the second case, $(i_H(c))_{s_1} \cdot (i_H(c))_{s_2} = - \cdot - = + = (i_{\mathcal{A}}(c))_j$. We use a similar argument when $c \subset \{f_j < 0\}$.

If H is a bounded oval, then, as described in Definition 3.7, one chooses a point p infinitesimally close to a point $p_i \in \{p_1, \dots, p_k\}$. Thus, an LRB structure on the set of faces of $\mathcal{A}_H$ is induced independent of the choice of the point p, when looking on H as a bounded segment. Therefore, we can use the same argument used in the case of an unbounded curve.

(5) Generalizing case (4), assume that $H \cap H_j = \{p_{s_1}, \dots, p_{s_n}\}$, i.e. the intersection of H and H_j is a transversal intersection of n points ($m_{s_i} = 1$ for $1 \leq i \leq n$).

Assume that H is an unbounded connected curve and thus without loss of generality, we can numerate the points $\{p_{s_i}\}$ consecutively, such that the point p_{s_n} will be the rightmost point. Assume also that in $\mathcal{L}_0(\mathcal{A})$, the domain $\{f_j > 0\}$ induces the sign $+$ in the j^{th} coordinate. Let c be a 1-dimensional face in $\mathcal{A}_H$. Assume now that c is to the right of p_{s_n}. Thus $(i_H(c))_{s_1} \cdot \ldots \cdot (i_H(c))_{s_n} = + \cdot \ldots \cdot + = +$ in $\mathcal{L}_0(\mathcal{A}_H)$. In addition, if $c \subset \{f_j > 0\}$, then in $\mathcal{L}_0(\mathcal{A})$ (or, more accurately, in $\mathcal{L}_0(\mathcal{A})|_H$), $(i_{\mathcal{A}}(c))_j = +$ (otherwise $(i_{\mathcal{A}}(c))_j = -$).

Now, if we move to the consecutive 1-dimensional face c', adjacent to c (i.e. between p_{s_n} and $p_{s_{n-1}}$), then in $\mathcal{L}_0(\mathcal{A}_H)$,

$$(i_H(c'))_{s_1} \cdot \ldots \cdot (i_H(c'))_{s_{n-1}} \cdot (i_H(c'))_{s_n} = \underbrace{+ \cdot \ldots \cdot +}_{n-1 \text{ times}} \cdot - = -,$$

while in $\mathcal{L}_0(\mathcal{A})$, as $c' \subset \{f_j < 0\}$ (if indeed $c \subset \{f_j > 0\}$), $(i_{\mathcal{A}}(c'))_j = -$. Note that if $c \subset \{f_j < 0\}$, then $c' \subset \{f_j > 0\}$, so $(i_{\mathcal{A}}(c'))_j = +$, i.e. there a constant scalar multiplication by $\{\pm 1\}$ of $\prod_v (i_H(c))_{s_v}$.

In this way, we can proceed to the next adjacent 1–dimensional face and so on, till we have reached to the leftmost face, i.e. to the face to the left of p_{s_1}, proving case (2)(b) for this type of intersection.

The treatment of the case when H is a bounded oval is similar to the former case (see also case (4)).

(6) In other cases, i.e. when $H \cap H_j = \{p_{s_1}, \dots, p_{s_n}\}$ and $m_{s_i} \geq 1$, then this case is treated as case (5) (i.e. treating each face separately, starting from the rightmost face and continuing to its adjacent face, and so on) combined with the insights of cases (1),(2) and (3).

□

4. CL arrangements: Chamber counting

In this section, we present some restrictions on the combinatorics induced by CL arrangements, induced by the fact that these arrangements induce a partition of the plane (Section 4.2). We start by recalling the deletion-restriction argument for hyperplane arrangements (Section 4.1).

4.1. **Preliminaries: Chamber counting for hyperplane arrangements.**

The main references for this subsection are [8, 16].

Let $\mathcal{A} = \{H_1, \dots, H_n\} \subset \mathbb{R}^N$ be a hyperplane arrangement, and let $f_i \in \mathbb{R}[x_1, \dots, x_N]$ be the corresponding forms of the hyperplanes. Let also $L = L(\mathcal{A})$ be the semi-lattice of nonempty intersections of elements of $\mathcal{A}$.

As before, given $H \in \mathcal{A}$, let $\mathcal{A}^H = \mathcal{A} - \{H\}$ be the *deleted* arrangement, and $\mathcal{A}_H = \left\{K \cap H | K \in \mathcal{A}^H\right\}$ be the *restricted* arrangement. Let $\mathcal{C}(\mathcal{A})$ be the set of chambers of $\mathcal{A}$, i.e. the components of $\mathbb{R}^N - \mathcal{A}$. Then, we have Zaslavsky's chamber counting formula (see [24]):

$$|\mathcal{C}(\mathcal{A})| = \left|\mathcal{C}\left(\mathcal{A}^H\right)\right| + \left|\mathcal{C}\left(\mathcal{A}_H\right)\right|. \tag{3}$$

Remark 4.1. One can give a simple set-theoretic proof for this formula: Deleting a hyperplane H from the arrangement $\mathcal{A}$ induces a surjection of LRBs $f : \mathcal{L}_0(\mathcal{A}) \to \mathcal{L}_0\left(\mathcal{A}^H\right)$, which deletes the coordinate corresponding to the hyperplane H. Thus, the number of chambers in $\mathcal{A}$ is equal to the sum of the number of chambers in the deleted arrangement $\mathcal{A}^H$ plus the number of chambers which are identified by the map f. Given $C_1, C_2 \in \mathcal{C}(\mathcal{A})$, note that $f(C_1) = f(C_2)$ if and only if C_1 and C_2 share a common codimension-1 face contained in H, i.e. a chamber in the restricted arrangement $\mathcal{A}_H$. Hence the number of the identified chambers is equal to the number of the chambers of $\mathcal{A}_H$, and Equation (3) follows.

Note that if we denote by $I(\mathcal{A})$ the unique two-sided ideal of the LRB $\mathcal{L}(\mathcal{A})$ (which is the set of the chambers of $\mathcal{A}$), Equation (3) is equivalent to the following equation:

$$|I(\mathcal{A})| = \left|I\left(\mathcal{A}^H\right)\right| + \left|I\left(\mathcal{A}_H\right)\right|.$$

Remark 4.2. Other restrictions on the combinatorics of real and complex line arrangements can be found, for example, in Hirzebruch's seminal paper [14], but we do not deal with their generalizations here.

4.2. **Chamber counting for CL arrangements.**

For a real CL arrangement, the deletion-restriction formula (3) for chamber counting does not hold anymore. For example, for the arrangement $\mathcal{A}$ appearing in Figure 11,

$$|\mathcal{C}(\mathcal{A})| = 4,\ \left|\mathcal{C}\left(\mathcal{A}^H\right)\right| = 2,\ \left|\mathcal{C}\left(\mathcal{A}_H\right)\right| = 3 \ \Rightarrow\ |\mathcal{C}(\mathcal{A})| \neq \left|\mathcal{C}\left(\mathcal{A}^H\right)\right| + \left|\mathcal{C}\left(\mathcal{A}_H\right)\right|.$$

On the other hand,

$$\left|\mathcal{C}\left(\mathcal{A}^C\right)\right| = 2,\ \left|\mathcal{C}\left(\mathcal{A}_C\right)\right| = 2 \ \Rightarrow\ |\mathcal{C}(\mathcal{A})| = \left|\mathcal{C}\left(\mathcal{A}^C\right)\right| + \left|\mathcal{C}\left(\mathcal{A}_C\right)\right|.$$

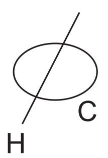

Figure 11. An example for the restriction-deletion formula for a CL arrangement.

Thus, the deletion-restriction argument needs to be changed. In order to formulate this change accurately, we start by introducing some notations.

Definition 4.3. *Let $\mathcal{A} \subset \mathbb{R}^2$ be a real CL arrangement.*
(1) Let $H \in \mathcal{A}$. Define the function:

$$\text{bound} : \mathcal{C}(\mathcal{A}_H) \to \{Y \in P(\mathcal{C}(\mathcal{A})) : |Y| = 2\}$$
$$\text{bound}(E) = \{X_1, X_2\} \textit{ where } E \subset \overline{X_1} \cap \overline{X_2},$$

where $P(\mathcal{C}(\mathcal{A}))$ is the power set of $\mathcal{C}(\mathcal{A})$ and $\overline{X}$ is the (topological) closure of X.
(2) For $E_1, E_2 \in \mathcal{C}(\mathcal{A}_H)$, define the following equivalence relation $\sim$:

$$E_1 \sim E_2 \;\Leftrightarrow\; \text{bound}(E_1) = \text{bound}(E_2),$$

and define:

$$b(H) = \mathcal{C}\left(\mathcal{A}_H\right) / \sim .$$

For example, for the arrangements in Figure 12, $|b(H)| = 2$.

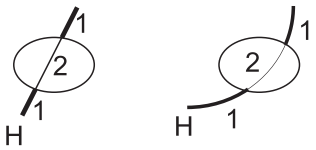

FIGURE 12. An illustration of the different elements in $b(H)$. Note that it does not matter if we delete either a line or a conic. The bold sections denoted by 1 are identified in $b(H)$.

Remark 4.4. It is easy to see that:

(1) $|b(H)| \leq |\mathcal{C}\left(\mathcal{A}_H\right)|$.
(2) If $\mathcal{A}$ is a line arrangement, then $b(H) = \mathcal{C}\left(\mathcal{A}_H\right)$ for any line $H \in \mathcal{A}$.

Proposition 4.5. *Let $H \in \mathcal{A}$ be a component in a real CL arrangement $\mathcal{A}$. Then:*

$$|\mathcal{C}(\mathcal{A})| = |\mathcal{C}\left(\mathcal{A}^H\right)| + |b(H)|.$$

Note that by Remark 4.4(2), Proposition 4.5 is indeed a natural generalization of the situation for line arrangements to the case of real CL arrangements.

Remark 4.6. By the same arguments we have used above, one can easily see that Proposition 4.5 holds for arrangements in $\mathbb{RP}^2$ too. However, in Definition 4.3(1), the definition of the function bound should be changed as follows:

$$\text{bound} : \mathcal{C}(\mathcal{A}_H) \to \{Y \in P(\mathcal{C}(A)) : |Y| \leq 2\}$$
$$\text{bound}(E) = \{X_1, X_2\} \text{ such that } E \subset \overline{X_1} \cap \overline{X_2} \text{ or } E \subset \overline{X_1}.$$

Proof of Proposition 4.5. For every chamber $X \in \mathcal{C}\left(\mathcal{A}^H\right)$ satisfying $H \cap X \neq \emptyset$, H divides X into a certain number of chambers; we denote this number by k_X. Thus:

$$|\mathcal{C}(\mathcal{A})| = |\mathcal{C}\left(\mathcal{A}^H\right)| + \sum_{\substack{X \in \mathcal{C}(\mathcal{A}^H) \\ H \cap X \neq \emptyset}} (k_X - 1),$$

since every chamber $X \in \mathcal{C}\left(\mathcal{A}^H\right)$ in the sum splits into k_X chambers, but we do not count X itself, as it is already counted in $\left|\mathcal{C}\left(\mathcal{A}^H\right)\right|$. For each $X \in \mathcal{C}\left(\mathcal{A}^H\right)$ in the sum, denote:

$$X = \bigcup_{i=1}^{k_X} X_i, \; H_X = H \cap X,$$

that is, (the interior of) X is divided into k_X chambers X_i, whose union (of their closure) is (the closure of) X.

Note that H_X is possibly a union of disjoint sections and $H_X \subset \mathcal{C}(\mathcal{A}_H)$. Therefore, we need to prove that $1 + |b(H_X)| = k_X$. We numerate the sections of H_X consecutively, which induces a numeration $H_1, H_2, \ldots$ of the sections of $b(H_X)$ from right to left. For each $H_i \in b(H_X)$, $1 \le i \le |b(H_X)|$, we look at the pair bound$(H_i) = \{X_{i_1}, X_{i_2}\}$; see Figure 13 for an example.

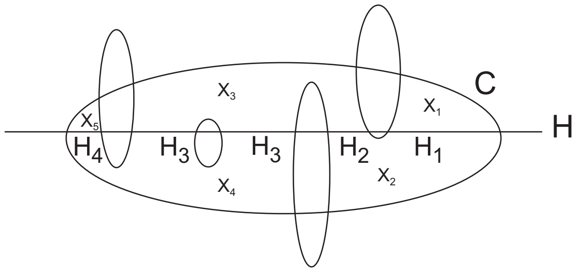

FIGURE 13. An example for the partition of H_X, where $X = X_1 \cup \cdots \cup X_5$ is a chamber contained inside the interior of the conic C and $X_1, \ldots, X_5$ are the chambers whose union is X. The equivalence classes of the sections of $H_X = H \cap X$ are $H_1, \ldots, H_4$.

We show that for each $i < j$, either $|\text{bound}(H_i) \cap \text{bound}(H_j)| = 1$ or there is a sequence $H_{i+1}, \ldots, H_{j-1}$ such that for each k, $i \le k < j$, $|\text{bound}(H_k) \cap \text{bound}(H_{k+1})| = 1$. Indeed, $|\text{bound}(H_i) \cap \text{bound}(H_j)| < 2$, otherwise $H_i \sim H_j$. If $|\text{bound}(H_i) \cap \text{bound}(H_j)| = 0$, look at bound$(H_s)$ for $s \in \{i, i+1, i+2\}$ (assuming that $H_i \nsim H_{i+1}$ and $H_{i+1} \nsim H_{i+2}$). Assume by contradiction that $|\text{bound}(H_i) \cap \text{bound}(H_{i+1})| = 0$. This means that we have the situation depicted in Figure 14.

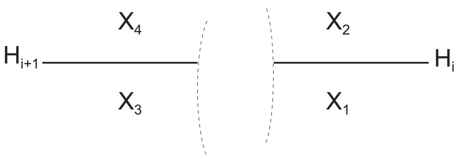

FIGURE 14. $|\text{bound}(H_i) \cap \text{bound}(H_{i+1})| = 0$

However, this situation is impossible, since the sections are consecutive, and if $\{X_1, X_2\} \cap \{X_3, X_4\} = \emptyset$, then H_i, H_{i+1} will not dissect the same (single) chamber $X \in \mathcal{C}(\mathcal{A}_H)$ (since X_1, X_2 and X_3, X_4 will be contained in different chambers of $\mathcal{C}\left(\mathcal{A}^H\right)$) – indeed, even before the equivalence relation $\sim$, one can connect a generic point from H_i with a generic point from H_{i+1} with a continuous path which lies only in X, which mean that the above intersection is always nonempty.

Thus, we define recursively the following map $\ell_X : b(H) \to \{X_1, \ldots, X_{k_X}\} : \ell_X(H_1)$ is one of the chambers X satisfying $H_1 \subset \overline{X}$. For $i > 1$ define $\ell_X(H_i)$ to be one of the chambers X' such that $H_i \subset \overline{X'}$ and for every $j < i$, $\ell_X(H_j) \neq X'$. Up to the choice of X, the map is well-defined, as for every $1 < i$ there is only one option to choose (recall that for each i, $|\text{bound}(H_i) \cap \text{bound}(H_{i+1})| = 1$). By its definition, the map ℓ_X is injective. Therefore, $|b(H_X)| = k_X - 1$ as requested. □

Remark 4.7. (1) Note that a set-theoretic proof to Proposition 4.5, which is parallel to the one given in Remark 4.1 for line arrangements, can be given in a similar way. As in the case of line arrangements, deleting a connected component H from the arrangement $\mathcal{A}$ induces a surjection of LRBs $f : \mathcal{L}_0(\mathcal{A}) \to \mathcal{L}_0(\mathcal{A}^H)$, which deletes the sign corresponding to the component H. Thus, the number of chambers in $\mathcal{A}$ is equal to the sum of the number of chambers in $\mathcal{A}^H$ plus the number

of chambers which are identified by the map f. We have shown in the proof of Proposition 4.5 that the map $\ell_X : b(H) \to \{X_1, \dots, X_{k_X}\}$ is a bijection between the number of chambers which are identified by the map f and the elements in $b(H)$, and therefore the result follows as before.
(2) Other deletion-restriction theorems with respect to chamber counting for arrangements of curves on surfaces can be found, for example, in Zaslevsky [25].

Remark 4.8. In a recent paper of Schenck and Tohaneanu [19], the existence of other restriction-deletion theorems with respect to the module of $\mathcal{A}$-derivations $D(\mathcal{A})$ for a CL arrangement $\mathcal{A}$ was proven. However, the connection between these theorems and the results we have obtained with respect to a deleted or restricted CL arrangement is not clear. First, the restriction-deletion theorems in [19, Theorem 2.5 and Theorem 3.4] can be applied only for free quasihomogeneous triples $\left(\mathcal{A}^H, \mathcal{A}, \mathcal{A}_H\right)$ (where $H \in \mathcal{A}$; note that line arrangements are always quasihomogeneous). However, the restriction-deletion proposition for chamber counting (see Proposition 4.5) works for any CL arrangement, and the restriction-deletion proposition for $\mathcal{L}_0(\mathcal{A})$ (see Section 3) can be applied only when $\mathcal{L}_0(\mathcal{A})$ is an LRB.

Second, for deleting a component H, the chamber counting restriction-deletion formula (Proposition 4.5) depends on the number of 1-dimensional faces in $\mathcal{L}(\mathcal{A})$ on this component having the same sign in $\mathcal{L}_0(\mathcal{A})$, a number which does not appear on the restriction-deletion theorem for $D(\mathcal{A})$ for deleting a component (see [19, Theorem 2.5]).

Moreover, while for line arrangements, the connection between these theorems is obvious, for CL arrangements the connection is more subtle. For a free line arrangement $\mathcal{L}$, the chamber counting formula can be induced by the restriction-deletion theorem with respect to $D(\mathcal{L})$: indeed, the addition-deletion formula for $D(\mathcal{L})$ implies the addition-deletion formula for the characteristic polynomial $\pi(\mathcal{L}, t)$ and $\pi(\mathcal{L}, 1) = |\mathcal{C}(\mathcal{L})|$. However, for free quasihomogeneous CL arrangements, the connections between the different restriction-deletion theorems (for $D(\mathcal{A})$, for $\pi(\mathcal{A}, t)$ and for $\mathcal{C}(\mathcal{A})$) are not clear; for example, $\pi(\mathcal{A}, 1) \neq |\mathcal{C}(\mathcal{A})|$ even for a CL arrangement $\mathcal{A}$ consists of a line intersecting a conic transversally. We leave this for further investigation.

Note also that while the characteristic polynomial is combinatorially determined (for any arrangement of curves in $\mathbb{C}^2$), the module of $\mathcal{A}$-derivations $D(\mathcal{A})$ for a CL arrangement $\mathcal{A}$ is not: in [19], a pair of combinatorially-equivalent CL arrangements having different modules of $\mathcal{A}$-derivations is presented.

References

[1] P. Abramenko and K. S. Brown, *Buildings: theory and applications*, Graduate Texts in Mathematics **248**, Springer, New York, 2008.

[2] M. Aguiar and S. Mahajan, *Coxeter groups and Hopf algebras*, Fields Institute Monographs **23**, AMS, Providence, RI, 2006.

[3] M. Amram, D. Garber and M. Teicher, *Fundamental groups of tangented conic-line arrangements with singularities up to order 6*, Math. Zeit. **256** (2007), 837–870.

[4] M. Amram, M. Teicher and F. Ye, *Moduli spaces of arrangements of 10 projective lines with quadruple points*, arXiv:1206.2486.

[5] M. Amram, M. Teicher and A. M. Uludag, *Fundamental groups of some quadric-line arrangements*, Topology Appl. **130**(2) (2003), 159–173 .

[6] A. Björner, *Random walks, arrangements, cell complexes, greedoids, and selforganizing libraries*, in: *Building bridges*, Bolyai Soc. Math. Stud. **19**, 165–203. Springer, Berlin, 2008.

[7] A. Björner and G. Ziegler, *Combinatorial stratification of complex arrangements*, J. Amer. Math. Soc. **5**(1) (1992), 105–149.

[8] K.S. Brown, *Semigroups, rings, and Markov chains*, J. Theoret. Probab. **13**(3) (2000), 871–938.

[9] M. Eliyahu, D. Garber and M. Teicher, *A conjugation-free geometric presentation of fundamental groups of arrangements*, Manuscripta Math. **133**(1–2) (2010), 247–271.

[10] M. Eliyahu, D. Garber and M. Teicher, *A conjugation-free geometric presentation of fundamental groups of arrangements II: Expansion and some properties*, Int. J. Alg. Comput. **21**(5) (2011), 775–792.

[11] M. Friedman and D. Garber, *On the structure of conjugation-free fundamental groups of conic–line arrangements*, submitted.

[12] M. Friedman and D. Garber, *On the structure of fundamental groups of conic–line arrangements having a cycle in their graph*, submitted.

[13] D. Garber, M. Teicher and U. Vishne, *Classes of wiring diagrams and their invariants*, J. Knot Theory Ramifications **11**(8) (2002), 1165–1191.

[14] F. Hirzebruch, *Arrangements of lines and algebraic surfaces*, in: *Arithmetic and geometry*, Vol. II, 113–140, Progr. Math. **36**, Birkhauser, Boston, Mass., 1983.

[15] S. Margolis, F.V. Saliola and B. Steinberg, *Combinatorial topology and the global dimension of algebras arising in combinatorics*, arXiv:1205.1159.

[16] P. Orlik and H. Terao, *Arrangements of Hyperplanes*, Grundlehren der Mathematischen Wissenschaften **300**, Springer–Verlag, 1992.

[17] F.V. Saliola, *The quiver of the semigroup algebra of a left regular band*, Internat. J. Alg. Comput. **17**(8) (2007), 1593–1610.

[18] F.V. Saliola, *The face semigroup algebra of a hyperplane arrangement*, Canad. J. Math., **61**(4) (2009), 904–929.

[19] H. Schenck and S.O. Tohaneanu, *Freeness of conic-line arrangements in* P^2, Comment. Math. Helv. **84** (2009), 235–258.

[20] J. Tits, *Buildings of spherical type and finite BN-pairs*, Springer-Verlag, Berlin, Lec. Notes in Math **386** (1974).

[21] H. Tokunaga, *Elliptic dihedral covers in dimension 2, geometry of sections of elliptic surfaces, and Zariski pairs for line-conic arrangements*, arXiv:1111.5924.

[22] S. Wang and S.S.-T. Yau, *Rigidity of differentiable structure for new class of line arrangements*, Comm. Anal. Geom. **13**(5) (2005), 1057–1075 .

[23] F. Ye, *Classification of moduli spaces of arrangements of 9 projective lines*, arXiv:1112.4306.

[24] T. Zaslavsky, *Facing up to arrangements: face-count formulas for partitions of space by hyperplanes*, Mem. Amer. Math. Soc. **1** (Issue 1, 154) , 1975, vii–102.

[25] T. Zaslavsky, *A combinatorial analysis of topological dissections*, Adv. Math. **25**(3) (1977), 267–285.

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