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# ISOPERIMETRIC CONTROL OF THE STEKLOV SPECTRUM

BRUNO COLBOIS, AHMAD EL SOUFI, AND ALEXANDRE GIROUARD

ABSTRACT. We prove that the normalized Steklov eigenvalues of a bounded domain in a complete Riemannian manifold are bounded above in terms of the isoperimetric ratio of the domain. Consequently, the normalized Steklov eigenvalues of a bounded domain in Euclidean space, hyperbolic space or a standard hemisphere are uniformly bounded above. On a compact surface with boundary, we obtain uniform bounds for the normalized Steklov eigenvalues in terms of the genus. We also establish a relationship between the Steklov eigenvalues of a domain and the eigenvalues of the Laplace-Beltrami operator on its boundary hypersurface.

## 1. INTRODUCTION

The goal of this paper is to obtain geometric upper bounds for the spectrum of the Dirichlet-to-Neumann map. Let  $N$  be a complete Riemannian manifold. Let  $\Omega$  be a relatively compact domain in  $N$  with smooth boundary  $\Sigma$ . The Dirichlet-to-Neumann map  $\Lambda : C^\infty(\Sigma) \rightarrow C^\infty(\Sigma)$  is defined by

$$\Lambda f = \partial_n(Hf)$$

where  $Hf$  is the harmonic extension of  $f$  to the interior of  $\Omega$  and  $\partial_n$  is the outward normal derivative. The Dirichlet-to-Neumann map is a first order elliptic pseudodifferential operator [25]. Because  $\Sigma$  is compact, the spectrum of  $\Lambda$  is positive, discrete and unbounded [1, p. 95]:

$$0 = \sigma_1 \leq \sigma_2(\Omega) \leq \sigma_3(\Omega) \leq \dots \nearrow \infty.$$

The spectrum of this operator is also called the Steklov spectrum of the domain  $\Omega$ .

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**1.1. Physical interpretation.** Prototypical in inverse problems, the Dirichlet-to-Neumann map is closely related to the Calderón problem [3] of determining the anisotropic conductivity of a body from current and voltage measurements at its boundary. This point of view makes it useful as a model for Electrical Impedance Tomography. A particularly striking related result [21] is that if the manifold  $M$  is real analytic of dimension at least 3, then the knowledge of  $\Lambda$  determines  $M$  up to isometry. The study of the spectrum of  $\Lambda$  was initiated by Steklov in 1902 [24]. Eigenvalues and eigenfunctions of this operator are used in fluid mechanics, heat transmission and vibration problems [12, 19].

**1.2. Optimization.** The general question we are interested in is to give upper bounds for the eigenvalues in terms of natural geometric quantities. Because the eigenvalues are not invariant under scaling of the Riemannian metric, we consider normalized eigenvalues

$$\bar{\sigma}_k(\Omega) := \sigma_k(\Omega)|\Sigma|^{\frac{1}{n}} \quad \text{with} \quad |\Sigma| = \int_{\Sigma} dv_{\Sigma}$$

where  $n$  is the dimension of the boundary  $\Sigma$  and  $dv_{\Sigma}$  is the measure induced by the Riemannian metric of  $N$  restricted to  $\Sigma$ .

**Question 1.1.** *Given a complete Riemannian manifold  $N$ , is  $\bar{\sigma}_k(\Omega)$  uniformly bounded above among bounded domains  $\Omega \subset N$ ?*

For the first non-zero eigenvalue, this question has been studied by many authors. See [22, 18] for early results in the planar case. The series of paper by J. Escobar [9, 10, 11] is influential. For more recent results, see [26, 13]. For higher eigenvalues in the planar situation, see [14, 15].

The main result of this paper (Theorem 2.2) is an upper bound for the eigenvalues of the Dirichlet-to-Neumann map on a domain in a complete Riemannian manifold satisfying a growth and a packing condition in terms of its isoperimetric ratio. We list here some applications.

*Domains in space forms.* The case of simply connected planar domain is well understood. See [27, 18] and especially [15] for a survey of this problem. If a bounded domain  $\Omega \subset \mathbb{R}^2$  is simply connected, then

$$\bar{\sigma}_k(\Omega) \leq 2\pi k. \tag{1.1}$$

This inequality is optimal. In higher dimensions, only few results about the first non-zero eigenvalue are known. See [2] for a different normalization.

Our first result is a generalization of the above to the case of arbitrary<sup>1</sup> domains in space forms.

**Theorem 1.2.** *There exists a constant  $C_n$  depending only on the dimension  $n$  such that, for each bounded domain  $\Omega$  in a space form  $\mathbb{R}^n$ ,  $\mathbb{H}^n$  or in an hemisphere of  $\mathbb{S}^n$ , we have*

$$\bar{\sigma}_k(\Omega) \leq C_n k^{2/n}. \quad (1.2)$$

This result follows from a more general result allowing control of the Steklov spectrum of a domain in a complete manifold in terms of its isoperimetric ratio.

*Domains in a complete manifold.* The following theorem shows that under an additional assumption on Ricci curvature, we can control the normalized Steklov eigenvalue  $\bar{\sigma}_k$  of a domain in terms of its isoperimetric ratio.

**Theorem 1.3.** *Let  $N$  be a complete manifold of dimension  $n + 1$ . If  $N$  is conformally equivalent to a complete manifold with non-negative Ricci curvature, then for each domain  $\Omega \subset N$ , we have*

$$\bar{\sigma}_k(\Omega) \leq \frac{\gamma(n)}{I(\Omega)^{\frac{n-1}{n}}} k^{2/(n+1)}, \quad (1.3)$$

where  $I(\Omega)$  is the classical isoperimetric ratio of  $\Omega$ , namely

$$I(\Omega) = \frac{|\Sigma|}{|\Omega|^{n/(n+1)}}.$$

A surprising corollary of this theorem is that if  $\dim N \geq 3$  then a large isoperimetric ratio  $I(\Omega)$  implies that the normalized eigenvalue  $\bar{\sigma}_k(\Omega)$  is small. This is false for surfaces ( $n = 1$ ), see Example 5.1.

**Remark 1.4.** *Since there exists a constant  $c_n$  such that*

$$\bar{\sigma}_k(\Omega) \sim c_n k^{1/n} \text{ as } k \rightarrow \infty,$$

*one may expect that a bound such as (1.3) should hold with exponents  $1/n$ . In fact, for  $n \geq 2$ , this is impossible because it would imply an upper bound on  $I(\Omega)$ . Naturally, if we remove  $I(\Omega)$ , such a bound might still be possible. For instance, we do not know if inequality (1.2) holds with exponent improved to  $1/n$ .*

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<sup>1</sup>i.e. not necessarily simply connected.

*Large eigenvalues.* The assumption of non-negative Ricci curvature is essential. In section 6 we will construct for each  $n \geq 2$  and each  $\kappa < 0$  a complete manifold  $N$  of dimension  $n + 1$  with Ricci curvature bounded below by  $\kappa$  admitting a sequence  $\Omega_j$  of domains such that the normalized eigenvalues  $\bar{\sigma}_2(\Omega_j) \rightarrow \infty$  and the isoperimetric ratio  $I(\Omega_j) \rightarrow \infty$ .

Under the assumption of non-negative Ricci curvature, we do not know if the presence of the isoperimetric ratio is essential. Namely, is there a constant  $C(n, k)$  such that for each domain  $\Omega \subset N$ ,  $\bar{\sigma}_k(\Omega) \leq C(n, k)$ ? Of course, this will be the case if we can give uniform lower bound on the isoperimetric ratio  $I(\Omega)$ . This situation will be discussed in Proposition 3.3 and in Corollary 3.6.

**1.3. Surfaces.** If  $N$  is two-dimensional the isoperimetric ratio disappears from inequality (1.3). This means that for any domain in a complete surface with conformally non-negative curvature we get a uniform bound similar to (1.1) :

$$\bar{\sigma}_k(\Omega) \leq \gamma(2)k.$$

In fact, in the case of surfaces, we don't need to assume our compact manifold to be a domain in a complete manifold with non-negative Ricci curvature. Let  $M$  be a compact surface with smooth boundary  $\Sigma$ . The Steklov spectrum of  $M$  is defined exactly as in the case of a domain.

**Theorem 1.5.** *There exists a constant  $C$  such that for any compact orientable Riemannian surface  $M$  of genus  $\gamma$  with non-empty smooth boundary,*

$$\bar{\sigma}_k(M) \leq C \left\lfloor \frac{\gamma + 3}{2} \right\rfloor k \quad (1.4)$$

where  $\lfloor \cdot \rfloor$  is the integer part.

This result is in the spirit of Korevaar [20] and generalizes a recent result of Fraser-Schoen [13].

**1.4. Relationships with the spectrum of the Laplacian for Euclidean hypersurface.** In section 4, we will use the result of our paper [7] to establish a relation between the spectrum of the Dirichlet-to-Neumann map and the spectrum of the Laplacian acting on smooth function of the boundary  $\Sigma$ . The main consequence of this estimate is that for a manifold embedded as hypersurface in Euclidean space, the presence of large normalized eigenvalue of the Laplacian will force the normalized eigenvalues  $\bar{\sigma}_k$  to be small.

## 2. STATEMENT AND PROOF OF THE MAIN THEOREM

We consider a slightly more general eigenvalue problem than that of the introduction. Let  $M$  be a sufficiently regular compact Riemannian manifold of dimension  $n + 1$  with boundary  $\Sigma$ . Let  $\delta$  is a smooth non-negative and non identically zero function on  $\Sigma$ . The Steklov eigenvalue problem is

$$\begin{aligned}\Delta f &= 0 \quad \text{in } M, \\ \partial_n f &= \sigma \delta f \quad \text{on } \Sigma.\end{aligned}$$

It has positive and discrete spectrum [1, p. 95]:

$$0 = \sigma_1 \leq \sigma_2(M, \delta) \leq \sigma_3(M, \delta) \leq \dots \nearrow \infty.$$

Because the eigenvalues are not invariant under scaling of the Riemannian metric or of the mass density  $\delta$ , we consider the normalized eigenvalues

$$\bar{\sigma}_k(M, \delta) := \sigma_k(M, \delta) m(\Sigma, \delta) |\Sigma|^{\frac{1}{n}},$$

with

$$|\Sigma| = \int_{\Sigma} dv_{\Sigma} \quad \text{and} \quad m(\Sigma, \delta) = \frac{1}{|\Sigma|} \int_{\Sigma} \delta dv_{\Sigma}.$$

Let  $(N, g_0)$  be a complete Riemannian manifold of dimension  $(n + 1)$ . We consider the Riemannian distance  $d_0$  induced by  $g_0$  and we assume:

**(P1)** There exists a constant  $C$  depending on  $d_0$  such that each ball of radius  $2r$  in  $N_0$  may be covered by at most  $C$  balls of radius  $r$ .

**(P2)** There exists a constant  $\omega$  depending only on  $g_0$  such that, for each  $x \in N_0$ , and  $r \geq 0$ ,  $|B(x, r)| \leq \omega r^{n+1}$ .

**Example 2.1.** *There is a large supply of complete Riemannian manifolds satisfying these conditions.*

- (1) *If  $N$  is compact, then (P1) and (P2) are satisfied. In this case the constants  $C$  and  $\omega$  depend on  $g_0$ .*
- (2) *If the Ricci curvature of  $g_0$  is non-negative then, by Bishop-Gromov comparison theorem, there exist constants  $C$  and  $\omega$  depending only on the dimension of  $N$  such that (P1) and (P2) are satisfied. This is in particular the case of the Euclidean space  $\mathbb{R}^{n+1}$ , and we will use this in the proof of Theorem 3.4 and Theorem 1.3.*

It follows from the previous example that Theorem 1.3 is a corollary of the following theorem.

**Theorem 2.2.** *Let  $(N, g_0)$  be a complete Riemannian manifold of dimension  $(n + 1)$  satisfying (P1) and (P2). Let  $g \in [g_0]$  be a metric in the conformal class of  $g_0$ . Then, there exists a constant  $\gamma(g_0)$  depending only on the constants  $C$  and  $\omega$  coming from (P1) and (P2) such that, for any bounded domain  $\Omega \subset N$  and any density  $\delta$  on  $\Sigma = \partial\Omega$ , we have*

$$\bar{\sigma}_k(\Omega, \delta) \leq \frac{\gamma(g_0)}{I(\Omega)^{\frac{n-1}{n}}} k^{2/(n+1)}. \quad (2.1)$$

The proof of Theorem 2.2 is based on the construction of a family of disjointly supported functions with controlled Rayleigh quotient

$$R(f) = \frac{\int_{\Omega} |\nabla_g f|^2 dv_g}{\int_{\Sigma} f^2 \delta dv_{\Sigma}}.$$

On  $N$  we consider the Borel measure  $\mu = \delta dv_{\Sigma}$ . That is, the measure of an open set  $\mathcal{O} \subset N$  is

$$\mu(\mathcal{O}) = \int_{\mathcal{O} \cap \Sigma} \delta dv_{\Sigma}. \quad (2.2)$$

In particular, we have

$$\mu(N) = \int_{\Sigma} \delta dv_{\Sigma} = |\Sigma| m(\Sigma, \delta).$$

**Definition 2.3.** *Let  $(X, d)$  be a metric space. An annulus  $A \subset X$  is a subset of the form  $\{x \in X : r < d(x, a) < R\}$  where  $a \in X$  and  $0 \leq r < R < \infty$ . The annulus  $2A$  is the annulus  $\{x \in X : r/2 < d(x, a) < 2R\}$ . In particular,  $A \subset 2A$ .*

Theorem 1.1 and Corollary 3.12 of [16] tell us that if a metric measured space  $(X, d, \nu)$  satisfy property (P1) and if the measure  $\nu$  is non-atomic, then there is a constant  $c > 0$  such that, for each positive integer  $k$ , there exist a family of  $2k$  annuli  $\{A_i\}_{i=1}^{2k}$  in  $X$  such that

$$\mu(A_i) \geq c \frac{\nu(X)}{k}.$$

and the annuli  $2A_i$  are disjoint.

The constant  $c$  depends only on the constant  $C$  of property (P1), that is only on the distance  $d$  and not on the measure  $\nu$ .

*Proof of Theorem 2.2.* Consider the metric measured space  $(N, d_0, \mu)$ , where  $d_0$  is the Riemannian distance associated to  $g_0$  and  $\mu$  is the measure induced by  $g_0$  and the density  $\delta$  as defined above in (2.2).

It follows from Theorem 1 and Corollary 3.12 of [16] mentioned above that there exist  $2k$  annuli  $A_1, \dots, A_{2k} \subset N$  with

$$\mu(A_i) \geq \frac{\mu(N)}{ck}, \quad c = c(g_0) > 0. \quad (2.3)$$

The annuli  $B_i = 2A_i$  are mutually disjoint. We can reorder them so that the first  $k$  of them satisfy

$$|B_i \cap \Omega|_g \leq \frac{|\Omega|_g}{k} \quad (i = 1, \dots, k). \quad (2.4)$$

Let  $A = \{x \in N : r < d(x, a) < R\}$  be one of these first  $k$  annuli and let  $h$  a function supported in  $2A$ . Taking (2.4) into account, it follows from Hölder's inequality and the conformal invariance of the generalized Dirichlet energy that

$$\begin{aligned} \int_{B \cap \Omega} |\nabla_g h|^2 dv_g &\leq \left( \int_{B \cap \Omega} |\nabla_g h|^{n+1} dv_g \right)^{2/(n+1)} |B \cap \Omega|_g^{1-2/(n+1)} \\ &\leq \left( \int_{2A} |\nabla_{g_0} h|^{n+1} dv_{g_0} \right)^{2/(n+1)} \left( \frac{|\Omega|_g}{k} \right)^{1-2/(n+1)} \end{aligned}$$

Choosing the function  $h$  that is identically 1 on  $A$  and proportional to the distance to  $A$  on  $2A \setminus A$ , we have

$$|\nabla_{g_0} h|^{n+1} \leq \begin{cases} \frac{2^{n+1}}{r^{n+1}} & \text{on } B(a, r) \setminus B(a, r/2), \\ \frac{1}{R^{n+1}} & \text{on } B(a, 2R) \setminus B(a, R). \end{cases}$$

It follows from  $(P_2)$  that

$$\int_{2A} |\nabla_{g_0} h|^{n+1} dv_{g_0} \leq 2^{n+2} \omega.$$

This leads to

$$\int_{B \cap \Omega} |\nabla_g h|^2 dv_g \leq (2^{n+2} \omega)^{2/(n+1)} \left( \frac{|\Omega|_g}{k} \right)^{(n-1)/(n+1)}$$

Moreover, using (2.3) we get

$$\int_{\Sigma} h^2 \delta dv_{\Sigma} \geq \mu(A) \geq \frac{\mu(N)}{ck}$$



By considering the Rayleigh quotient, this leads to

$$\sigma_k(\Omega, \delta) \leq \frac{2^{n+2}\omega ck}{\mu(N)} \left( \frac{|\Omega|_g}{k} \right)^{(n-1)/(n+1)}.$$

Using  $\mu(N) = |\Sigma|m(\Sigma, \delta)$ , we conclude

$$\bar{\sigma}_k(\Omega, \delta) = \sigma_k(\Omega, \delta)m(\Sigma, \delta)|\Sigma|^{1/n} \leq \frac{\gamma(g_0)}{I(\Omega)^{\frac{n-1}{n}}} k^{2/(n+1)},$$

with  $\gamma(g_0) = 2^{n+2}c\omega$ . □

### 3. APPLICATIONS OF THEOREM 2.2

In this section, we prove most of the results announced in the introduction as consequence of our Theorem 2.2.

**3.1. Domains in a manifold with conformally non-negative Ricci curvature.** It is difficult to estimate the packing constant  $C$  and the growth constant  $\omega$  of a general Riemannian manifold. Nevertheless, as was observed in Example 2.1, in the special situation where  $\Omega$  is a domain  $\Omega$  in a complete Riemannian manifold  $N$  with non-negative Ricci curvature, it follows from the Bishop-Gromov inequality that these constants can be estimated in terms of the dimension.

**Theorem 3.1.** *Let  $(N, g)$  be a complete Riemannian manifold of dimension  $(n+1)$  and assume that the metric  $g$  is conformally equivalent to a metric  $g_0$  with  $\text{Ric}(g_0) \geq 0$ . Then, for any bounded domain  $\Omega \subset N$ , and for any density  $\delta$  on  $\partial\Omega$ , we have*

$$\bar{\sigma}_k(\Omega, \delta) \leq \frac{\gamma(n)}{I(\Omega)^{\frac{n-1}{n}}} k^{2/(n+1)}, \quad (3.1)$$

where  $\gamma(n)$  is a constant depending only on  $n$ .

This theorem is a direct consequence of Theorem 2.2 and of Example 2.1. Theorem 1.3 is the special case when  $\delta \equiv 1$ .

If  $n \geq 2$ , large isoperimetric ratio  $I(\Omega)$  implies small eigenvalues  $\bar{\sigma}_k(\Omega, \delta)$ .

**Corollary 3.2.** *Under the assumptions of Theorem 3.1, if a family of domains  $\{\Omega_t\}_{0 < t < 1}$  is such that  $\lim_{t \rightarrow 0} I(\Omega_t) = \infty$ , then, if  $n \geq 2$  and for each density  $\delta_t$  on  $\partial\Omega_t$ , we have*

$$\lim_{t \rightarrow 0} \bar{\sigma}_k(\Omega_t, \delta_t) \rightarrow 0.$$

This is false for  $n = 2$ . See Example 5.1.

**3.2. Control of the isoperimetric ratio.** In general, it is difficult to estimate the isoperimetric ratio  $I(\Omega)$ . We give two special situations where we have a uniform lower estimate on it. This will be a consequence of the inequality of Croke [8] as presented by Chavel [4, p.136].

**Proposition 3.3.** *Let  $N$  be a complete Riemannian manifold. For each  $x \in N$ , let  $\text{inj}(x)$  the injectivity radius of  $N$  at  $x$ . Given  $p \in N$  and  $\rho > 0$ , consider*

$$r < \frac{1}{2} (\inf_{x \in B(p, \rho)} \text{inj}(x)). \quad (3.2)$$

*Then, for a each domain  $\Omega \subset B(p, r)$ , we have  $I(\Omega) \geq C(n)$  for a constant  $C(n)$  depending only on the dimension.*

If the injectivity radius of  $N$  is strictly positive, we can choose any  $r < \frac{\text{inj}(N)}{2}$ .

**3.2.1. Domains in space forms.** A special but very important case is when the ambient space  $N$  is a space form, that is the Euclidean space  $\mathbb{R}^{n+1}$ , the hyperbolic space  $\mathbb{H}^{n+1}$  or the sphere  $\mathbb{S}^{n+1}$  with their natural metric of curvature 0,  $-1$  and  $1$  respectively.

**Theorem 3.4.** *For any bounded domain  $\Omega$  with smooth boundary  $\Sigma = \partial\Omega$  in  $\mathbb{R}^{n+1}$ ,  $\mathbb{H}^{n+1}$  or on an hemisphere of the sphere  $\mathbb{S}^{n+1}$  and any  $k \geq 1$ , we have*

$$\bar{\sigma}_k(\Omega, \delta) \leq \frac{\gamma(n)}{I(\Omega)^{\frac{n-1}{n}}} k^{2/(n+1)} \leq C_n k^{2/(n+1)}$$

where  $C_n$  and  $\gamma_n$  are constants depending only on  $n$ .

*Proof.* The standard metrics on Euclidean space and on the sphere have non-negative Ricci curvature. The standard metric on the hyperbolic space is conformally equivalent to the Euclidean one. We can therefore apply Theorem 2.2, with  $g_0 = g$  one of these standard metric.

The injectivity radii of Euclidean and hyperbolic space are infinite. That of the unit sphere is  $\pi$ . The proof is completed by using Proposition 3.3.  $\square$

In particular, this proves Theorem 1.2.

**Remark 3.5.** *It is also classically known that any domain  $\Omega$  in Euclidean space, the hyperbolic space or an hemisphere, isoperimetric ratio bounded from below by a constant depending on the dimension. This can be used instead of Croke's result in the above proof.*

3.2.2. *Domains inside a ball.* In the case where the Ricci curvature of  $N$  is non-negative, we deduce the following

**Corollary 3.6.** *If the Ricci curvature of  $N$  is non-negative and if  $\Omega \subset B(p, r)$ , where  $r$  satisfy (3.2), then*

$$\bar{\sigma}_k(\Omega, \delta) \leq C_n k^{2/(n+1)}$$

for some constant  $C_n$  depending only on  $n$ .

#### 4. RELATION BETWEEN THE SPECTRUM OF THE DIRICHLET-TO-NEUMANN OPERATOR AND THE SPECTRUM OF THE LAPLACIAN.

Let  $\Delta_\Sigma$  be the Laplacian acting on smooth functions of the boundary  $\Sigma = \partial M$  of a compact Riemannian manifold with boundary. Let  $0 = \lambda_1 \leq \lambda_2(\Sigma) \leq \dots \nearrow \infty$  be the spectrum of  $\Delta_\Sigma$ . It is well known that  $\Lambda$  is a first order pseudodifferential operator and that its principal symbol is the square root of the principal symbol of  $\Delta_\Sigma$ . It follows that  $\sigma_k \sim \sqrt{\lambda_k}$  as  $k \rightarrow \infty$ . See for instance ([25, p. 38 and p. 453], [23]).

**Question 4.1.** *Can the eigenvalues  $\sigma_k$  and  $\lambda_l$  be compared to each other ?*

Recently, Wang and Xia studied this question [26] for the first non-zero eigenvalues of both operators. Under the assumption that Ricci curvature of  $M$  is non-negative and that the principal curvatures of  $\partial M$  are bounded below by a positive constant  $c$ , they proved that

$$\sigma_2 \leq \frac{\sqrt{\lambda_2}}{nc} (\sqrt{\lambda_2} + \sqrt{\lambda_2 - nc^2})$$

Note that Xia had previously proved [28], under the same hypothesis, that  $\lambda_2 \geq nc^2$ .

In [7], we study the control of the spectrum of the Laplacian on a closed hypersurface by the isoperimetric ratio. The following is a particular case of one of our results.

**Theorem 4.2.** *Let  $N$  be a complete Riemannian manifold with non-negative Ricci curvature. Let  $\Omega \subset N$  be a bounded domain with smooth boundary  $\Sigma = \partial\Omega$  contained in a ball of radius  $r < \frac{\text{inj}(N)}{2}$ . There is a constant  $B_n$  depending only on dimension such that for any  $k \geq 0$ ,*

$$\bar{\lambda}_k(\Sigma) \leq B_n I(\Omega)^{(n+2)/n} k^{2/n}$$

where  $\bar{\lambda}_k(\Sigma) = \lambda_k(\Sigma) |\Sigma|^{2/n}$  are the normalized eigenvalues of the Laplacian.

Combining Theorem 4.2 and Corollary 3.4, we get

**Theorem 4.3.** *Let  $N$  be a complete Riemannian manifold of dimension  $(n+1)$  with non-negative Ricci curvature. There exists a constant  $\kappa_n$  depending only on dimension such that for any bounded domain  $\Omega \subset N$  with boundary  $\Sigma = \partial\Omega$  contained in a ball of radius  $r < \frac{inj(N)}{2}$  the following holds:*

$$\bar{\lambda}_k(\Sigma)\bar{\sigma}_l(\Omega) \leq \kappa_n \left( \frac{|\Sigma|}{|\Omega|} \right)^{3/n} k^{2/n} l^{2/(n+1)}. \quad (4.1)$$

*In the special case where  $N$  is the Euclidean space  $\mathbb{R}^{n+1}$ , the injectivity radius in each point is  $\infty$ , so that there are no further restrictions on  $\Omega$ , and Inequalities (4.1) and (4.2) are true for all bounded domains.*

Without the normalization, we have

$$\lambda_k(\Sigma)\sigma_l(\Omega)m(\Sigma, \delta) \leq \kappa_n \frac{k^{2/n} l^{2/(n+1)}}{|\Omega|^{3/(n+1)}}. \quad (4.2)$$

**Remark 4.4.** *In comparison with [26], we make no assumption on the convexity of  $\Omega$ . We also have comparison for all eigenvalues. Note however that our method does not give any sharpness.*

A remarkable feature of this inequality is that large eigenvalues of the Laplacian are seen to impose small eigenvalues of the Dirichlet-to-Neumann map.

**Corollary 4.5.** *Under the assumptions of Theorem 4.3, if a family of domains  $\{\Omega_t\}_{0 < t < 1}$  of volume one with boundaries  $\Sigma_t = \partial\Omega_t$  is such that  $\lim_{t \rightarrow 0} \lambda_k(\Omega_t) = \infty$ , then, if  $n \geq 2$  we have for each  $l \geq 1$*

$$\lim_{t \rightarrow 0} \bar{\sigma}_l(\Omega_t) \rightarrow 0.$$

## 5. SURFACES

The situation for surface is special. We begin by a proof of the upper bound of  $\sigma_k$  in term of the genus.

*Proof of Theorem 1.5.* This is a modification of the proof of Theorem 2.2.

By gluing a disk on each boundary components of  $M$ , we can see  $M$  as a domain in a compact surface  $S$  of genus  $\gamma$ . This closed surface can be represented as a branched cover over  $\mathbb{S}^2$  with degree  $d = \lfloor \frac{\gamma+3}{2} \rfloor$  (See [17] for instance).

On  $\mathbb{S}^2$  we consider the usual spherical distance  $d$  and we define a Borel measure  $\mu = \psi_*(\delta dv_\Sigma)$ . That is, the measure of an open set  $\mathcal{O} \subset \mathbb{S}^2$  is

$$\mu(\mathcal{O}) = \int_{\psi^{-1}(\mathcal{O}) \cap \Sigma} \delta dv_\Sigma. \quad (5.1)$$

In particular,

$$\mu(\mathbb{S}^2) = |\Sigma| m(\Sigma, \delta).$$

It follows from Theorem 1 and Corollary 3.12 of [16] applied to the metric measured space  $(\mathbb{S}^2, d, \mu)$  that there exist  $2k$  annuli  $A_1, \dots, A_{2k} \subset \mathbb{S}^2$  with

$$\mu(A_i) \geq \frac{\mu(\mathbb{S}^2)}{ck}. \quad (5.2)$$

Because the annuli  $2A_i$  are mutually disjoint, so are the sets  $B_i = \psi^{-1}(2A_i)$ . These sets can be reordered so that the first  $k$  of them satisfy

$$|B_i|_g \leq \frac{|M|_g}{k} \quad (i = 1, \dots, k). \quad (5.3)$$

Let  $A = \{x \in \mathbb{S}^2 : r < d(x, a) < R\}$  be one of the above annuli and let  $h$  be a function supported in  $2A$ . Let  $f = h \circ \psi$  be the lift of this function to  $M$ . It is supported in the set  $B = \psi^{-1}(2A)$ .

Taking (5.3) into account, it follows from conformal invariance of the Dirichlet energy that

$$\int_{B \cap \Omega} |\nabla_g f|^2 dv_g \leq \left( \deg(\psi) \int_{2A} |\nabla_{g_0} h|^{n+1} dv_{g_0} \right)^{2/(n+1)} \left( \frac{|\Omega|_g}{k} \right)^{1-2/(n+1)}.$$

The rest of the proof is almost identical to that of Theorem 2.2 and is left to the reader.  $\square$

In Corollary 3.2 it was mentioned that for manifold of dimension at least three, a large isoperimetric ratio implies small Steklov eigenvalues. The next example shows that this is false for surfaces.

**Example 5.1.** *Let  $M$  be a compact Riemannian manifold with metric  $g$ . Let  $f \in C^\infty(\bar{M})$  be a smooth function vanishing on the boundary  $\partial M$ . Consider a conformal perturbation  $\tilde{g} = e^f g$  of the original metric. It is well known that the Laplacian is conformally invariant in dimension two. Moreover, because  $\tilde{g} = g$  on  $\partial M$ , the normal derivative is also preserved. It follows that the Dirichlet-to-Neumann map induced by  $\tilde{g}$  is the same as that induced by  $g$ . In particular, they have the same spectrum.*

On the other hand the measure of the surface is given by

$$|M|_{\bar{g}} = \int_M e^f dg.$$

By taking a function  $f$  that decays fast away from the boundary, we can make this quantity as small as we want. In other words, the isoperimetric ratio  $I(M) = \frac{|\partial M|}{\sqrt{|M|_g}}$  will become very large.

## 6. CONSTRUCTION OF LARGE EIGENVALUES

The behavior of the Steklov spectrum depends on the interior of the domain in an essential way. For a closed Riemannian manifold  $\Sigma$  with large eigenvalue  $\lambda_k$  of the Laplacian, embedding as an hypersurface in Euclidean space forces very small Steklov eigenvalues. This comes from the fact that, by [7] the isoperimetric ratio  $I(\Omega)$  has to be big, with  $\Sigma = \partial\Omega$ , and this implies the presence of small eigenvalues. If we embed  $\Sigma$  as the cross-section of a cylinder  $\Sigma \times \mathbb{R}$  with its product metric, we will see that exactly the opposite will happen. This shows that our geometric assumptions are necessary.

**Lemma 6.1.** *Let  $\Sigma$  be a closed Riemannian manifold of volume one. Let the spectrum of its Laplace operator  $\Delta_\Sigma$  be*

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \cdots \nearrow \infty$$

and let  $(u_k)$  be an orthonormal basis of  $L^2(\Sigma)$  such that

$$\Delta_\Sigma u_k = \lambda_k u_k.$$

Let  $N = \mathbb{R} \times \Sigma$ . On the domain  $\Omega = [-L, L] \times \Sigma \subset N$ , a complete system of orthogonal eigenfunctions of the Dirichlet-to-Neumann map is given by

$$1, t, \cosh(\sqrt{\lambda_k}t)f_k(x), \sinh(\sqrt{\lambda_k}t)f_k(x)$$

with eigenvalues

$$0, 1/L, \sqrt{\lambda_k} \tanh(\sqrt{\lambda_k}L) < \sqrt{\lambda_k} \coth(\sqrt{\lambda_k}L).$$

*Proof.* It is enough to check that these functions are Steklov eigenfunctions since their restriction to the boundary form a basis  $L^2$ .  $\square$

**Proposition 6.2.** *Let  $\Sigma$  be a closed manifold of dimension  $\geq 3$ . On the product manifold  $N = \Sigma \times \mathbb{R}$  there exists a complete Riemannian metric  $g$  and a sequence of bounded domains  $\Omega_i$  such that*

$$\lim_{i \rightarrow \infty} \bar{\sigma}_2(\Omega_i) = \infty, \text{ and } \lim_{i \rightarrow \infty} I(\Omega_i) = \infty.$$

*Proof.* Let  $\Sigma$  be a closed manifold of dimension  $\geq 3$ . The first author and Dodziuk [5] proved the existence of a sequence  $h_i$  of Riemannian metrics of volume one such that  $\lim_{i \rightarrow \infty} \lambda_2(\Sigma, h_i) = \infty$ . Without loss of generality, we assume for each  $i$  that  $\lambda_2(\Sigma, h_i) > 1$ .

Consider the cylinder  $\Omega_i = \Sigma \times [i, i + L_i]$  with

$$1 > L_i = \frac{1}{\sqrt{\lambda_2(\Sigma, h_i)}} \rightarrow 0 \text{ as } i \rightarrow \infty.$$

Let  $g$  be a complete Riemannian metric on  $\Sigma \times \mathbb{R}$  such that the restriction of  $g$  to  $\Omega_i$  is the product of  $h_i$  with the Euclidean metric on  $\mathbb{R}$ . It follows from Lemma 6.1 that

$$\sigma_2(\Omega_i) = \min \left( \sqrt{\lambda_2(\Sigma, h_i)}, \sqrt{\lambda_2(\Sigma, h_i)} \tanh(1) \right) = \sqrt{\lambda_2(\Sigma, h_i)} \tanh(1).$$

In particular

$$\lim_{i \rightarrow \infty} \bar{\sigma}_2(\Omega_i) = \infty.$$

□

**Proposition 6.3.** *There exists a complete three-dimensional Riemannian manifold  $N$  admitting a sequence of bounded domains  $\Omega_i \subset N$  such that*

$$\lim_{i \rightarrow \infty} \bar{\sigma}_2(\Omega_i) = \infty, \text{ and } \lim_{i \rightarrow \infty} I(\Omega_i) = \infty.$$

*Proof.* It is well known that there exists a sequence of Riemann surfaces of volume one  $\Sigma_i$  such that  $\lambda_2(\Sigma_i) \rightarrow \infty$  (see [6]). We consider the complete Riemannian manifold  $N_i = \Sigma_i \times \mathbb{R}$  (with the product Riemannian metric) and the subset

$$\Omega_i = \Sigma_i \times [0, L_i]$$

with  $L_i = \frac{1}{\sqrt{\lambda_2(\Sigma_i)}}$ . As before, we see that  $\lim_{i \rightarrow \infty} \bar{\sigma}_2(\Omega_i) = \infty$ . The manifold  $N$  is obtained by joining the  $N_i$ 's by tubes. □

**Proposition 6.4.** *Let  $M$  be a compact manifold of dimension  $\geq 4$ . There exists a sequence of Riemannian metric  $g_i$  and a domain  $\Omega \subset M$  such that*

$$\lim_{i \rightarrow \infty} \bar{\sigma}_2(\Omega, g_i) = \infty, \text{ and } \lim_{i \rightarrow \infty} I(\Omega, g_i) = \infty.$$

*Proof.* Let  $\Omega$  be any domain of  $M$  that is diffeomorphic to the cylinder  $(0, 1) \times \mathbb{S}^n$ . Because  $n \geq 3$ , there exists a sequence of Riemannian metric  $h_i$  on  $\mathbb{S}^n$  such that  $\lim_{i \rightarrow \infty} \lambda_2(\mathbb{S}^n, h_i) = \infty$ . Let  $g_i$  be a Riemannian metric on  $M$  such that the restriction of  $g_i$  to  $\Omega$  is isometric to product  $\mathbb{S}^n \times (0, L_i)$  with  $L_i = \frac{1}{\sqrt{\lambda_2(\mathbb{S}^n, h_i)}}$ . □

It follows from scaling invariance of the normalized eigenvalues that in each of the three previous examples, the Riemannian metrics can be chosen to have Ricci curvature arbitrarily close to zero.

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