WELL-COMPOSED IMAGES AND RIGID TRANSFORMATIONS

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ABSTRACT

We study the conditions under which a 2D well-composed binary image preserves its topological properties under any rigid transformations. This work initiates a more global study of digital image topological properties under such transformations, which is a crucial but unconsidered issue in the context of image processing, *e.g.*, for image registration and warping.

Index Terms— Well-composed images, rigid transformation, digital topology.

1. INTRODUCTION

The preservation of digital image topological properties is a crucial issue for several application fields, involving 3D data (*e.g.*, medical imaging [1]) or 2D ones (*e.g.*, remote sensing [2]). In particular, topology preservation has been considered in the context of image transformations, both from the viewpoints of registration [3] and warping [4]. The main efforts have been devoted to deal with the case of complex transformations, while more simple ones have been unconsidered.

Indeed, simple transformations are generally assumed to be trivially handled. This can be explained by the fact that, in the continuous case (*i.e.*, in $\mathbb{R}^n$), such transformations, *e.g.*, translations or rotations, are generally topology-preserving, while this is not necessarily the case for more complex ones, *e.g.*, those induced by nonrigid registration [5]. Based on this "continuous" assertion, it is often thought that simple transformations still lead to easy handling of topological properties in the discrete case (*i.e.*, in $\mathbb{Z}^n$). This is a wrong belief.

In the case of discrete rigid transformations [6] (composed of translations and rotations, followed by a digitisation), some topological issues have already been identified [7]. These issues are directly induced by the sampling that are mandatory to guarantee the stability of the transformations inside $\mathbb{Z}^n$.

In this article, we propose a preliminary study that deals with necessary and sufficient conditions under which *binary digital images in $\mathbb{Z}^2$ preserve their topological properties under any rigid transformations*. We develop this study in the framework of well-composed images [8], by considering transformations in the backwards model. We show that such conditions can be checked in linear time with respect to the size of images. They also lead to procedures for "correcting" the images before transformations.

The article is organised as follows. Sec. 2 provides background notions. Sec. 3 establishes the conditions for topological invariance of well-composed images under rigid transformations. Sec. 4 presents experiments and methodological remarks. Sec. 5 concludes the article by perspective works.

The research leading to these results has received funding from the French Agence Nationale de la Recherche (Grant Agreement ANR-2010-BLAN-0205 03).

2. BACKGROUND NOTIONS

2.1. Discrete rigid transformations

In $\mathbb{R}^2$, a rigid transformation is a function $\mathcal{U} : \mathbb{R}^2 \to \mathbb{R}^2$ defined, for any $\boldsymbol{x} = (x, y) \in \mathbb{R}^2$ by

$$\mathcal{U}(\boldsymbol{x}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} \quad (1)$$

where $a, b \in \mathbb{R}$ and $\theta \in [0, 2\pi[$. The function $\mathcal{U}$ is a bijection, and we note $\mathcal{T} = \mathcal{U}^{-1}$ its inverse function, which is also a rigid transformation. These definitions cannot be directly applied to the case of digital images, since there is no guarantee that $\mathcal{U}(\mathbb{Z}^2) \subseteq \mathbb{Z}^2$. The handling of *discrete* rigid transformations then requires to consider a discretisation operator $D : \mathbb{R}^2 \to \mathbb{Z}^2$ (in the most common cases, and in the current one, the standard rounding function), to define discrete analogues of $\mathcal{U}$ and $\mathcal{T}$, namely $U = D \circ \mathcal{U}_{|\mathbb{Z}^2}$ and $T = D \circ \mathcal{T}_{|\mathbb{Z}^2}$.

In this work, we consider the Eulerian (backwards) transformation model. More precisely, given an image $I : \mathbb{Z}^2 \to \mathbb{V}$ and a rigid transformation $\mathcal{U} : \mathbb{R}^2 \to \mathbb{R}^2$, the transformed image $I' : \mathbb{Z}^2 \to \mathbb{V}$, with respect to I and $\mathcal{U}$, is defined as

$$I' = I \circ D \circ (\mathcal{U}^{-1})_{|\mathbb{Z}^2} = I \circ T \quad (2)$$

In the case of binary images (namely, $\mathbb{V} = \{0, 1\}$) the image I (resp. I') can be modeled by the set $S = \{\boldsymbol{p} \mid I(\boldsymbol{p}) = 1\} \subset \mathbb{Z}^2$ (resp. $S' = \{\boldsymbol{p} \mid I'(\boldsymbol{p}) = 1\} \subset \mathbb{Z}^2$), and its complement $\overline{S} = \mathbb{Z}^2 \setminus S$ (resp. $\overline{S'} = \mathbb{Z}^2 \setminus S'$). Then, Eq. (2) is rewritten as

$$S' = \mathbb{Z}^2 \cap \mathcal{U}(S \oplus \square) \quad (3)$$

$$\overline{S'} = \mathbb{Z}^2 \cap \mathcal{U}(\overline{S} \oplus \square) \quad (4)$$

where $\oplus$ is the dilation operator (Minkowski addition) [9] and $\square = [-\frac{1}{2}, \frac{1}{2}]^2 \subset \mathbb{R}^2$ is the unit square (*i.e.*, pixel) element.

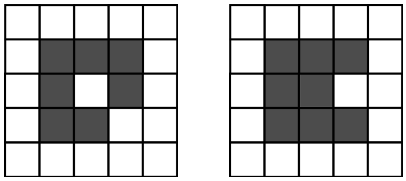


Fig. 1: Left: S (in black) is weakly well-composed, while $\overline{S}$ (in white) is not. Right: both S and $\overline{S}$ are well-composed.

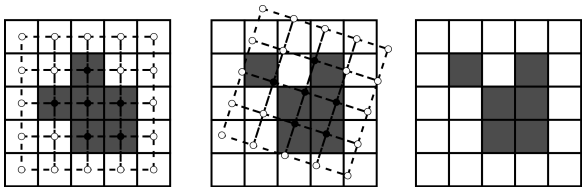


Fig. 2: Left: a well-composed image and the grid modelling its discrete structure. Middle: a rigid transformation applied on this grid. Right: the transformed image, which is not well-composed, and whose adjacency tree differs from the one of the initial image (with the 4-adjacency model).

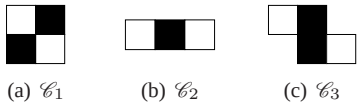
Remark 1. *In this work, we assume that $\mathcal{U}$ is such that $\mathbb{Z}^2$ does not intersect any transformed pixel border. In other words, Eqs.* (2)–(4) *lead to same results for $\square = [-\frac{1}{2}, \frac{1}{2}]^2$ or $]-\frac{1}{2}, \frac{1}{2}[^2$. From a theoretical viewpoint, this allows us to develop a general discussion without confusing variants related to the definition of D. From a pratical viewpoint, this assumption is compliant with computer-based applications, that generally rely on floating point arithmetic.*

2.2. Digital topology and well-composed sets

Digital topology [10] provides an efficient framework for handling the topology of binary images in $\mathbb{Z}^n$. It relies on a graph structure that models the adjacency relations between the points of S and $\overline{S}$. To deal with topological paradoxes related to the discrete version of the Jordan theorem, a couple of dual adjacencies [11] are defined from the L^1 and L^∞ norms, leading to the well-known 4- and 8-adjacencies in $\mathbb{Z}^2$ [10].

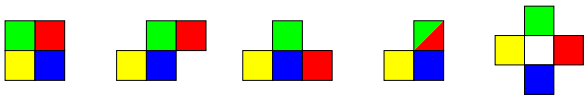
From these (irreflexive and symmetric) 4-/8-adjacency relations over the points of $\mathbb{Z}^2$, we obtain by reflexive/transitive closure, some equivalence relations (the 4-/8-connectedness), and their equivalence classes (the 4-/8-connected components), associated either to S or $\overline{S}$. The "nested" relation between all these connected components can be modeled by a tree structure, called the adjacency tree [12]. More precisely, a component A of S and a component B of $\overline{S}$ are *4-/8-adjacent* if either A 4-/8-surrounds B or B 8-/4-surrounds A. In particular, in $\mathbb{Z}^2$, if two binary images have isomorphic adjacency trees, they can be seen as topologically equivalent [13].

Based on the digital topology framework, some notions of well-composed sets/images [8] have been introduced (see Fig. 1), to characterise the binary images whose structure intrinsically avoids the topological issues of the Jordan theorem.



(a) $\mathscr{C}_1$ (b) $\mathscr{C}_2$ (c) $\mathscr{C}_3$

Fig. 3: (a) Configuration forbidden in well-composed images (see Th. 3). (a,b,c) Configurations forbidden in regular images (see Def. 9). (These configurations are given up to $\pi/2$ rotations, symmetries and values inversion.)



(a) $\boxplus$, $\mathscr{P}_1$ (b) $\mathscr{P}_2$ (c) $\mathscr{P}_3$ (d) $\mathscr{P}_4$ (e) $\mathscr{P}_5$

Fig. 4: (a) The pattern $\boxplus \subset \mathbb{Z}^2$. (a–e) The patterns $\mathscr{P}_i$ that can verify $T(\boxplus) = \mathscr{P}_i$ (the colours represent the point correspondance between $\boxplus$ and $\mathscr{P}_i$).(See Prop. 6.)

Definition 2 (Well-composed sets [8])**.** *We say that S is* weakly well-composed *if any 8-connected component of S is also a 4-connected component. We say that S (resp. I) is* well-composed *if both S and $\overline{S}$ are weakly well-composed.*

When interpreting digital topology in a continuous framework [14] that relies, *e.g*, on cubical complexes [15], a binary digital image is well-composed iff the boundaries shared by the foreground and background pixels constitute a manifold [16].

It has been proved that well-composed sets can be locally characterized by observing only 2×2 samples of an image.

Theorem 3 ([8])**.** *A 2D image is well-composed iff it does not contain the configuration $\mathscr{C}_1$ illustrated in Fig. 3(a).*

3. TOPOLOGICAL INVARIANCE UNDER RIGID TRANSFORMATIONS

Well-composed images present nice topological properties, that may however be altered by discrete rigid transformations (see Fig. 2). We propose to investigate conditions that guarantee the *topological invariance* of a well-composed image.

Definition 4 (Topological invariance)**.** *We say that a well-composed image I is* topologically invariant *if any transformed image $I' = I \circ T$: (i) is well-composed, and (ii) has an adjacency tree isomorphic to the one of I.*

3.1. Preliminary properties

Let $\boxminus = \{x\} \times \{y, y+1\}$ or $\{x, x+1\} \times \{y\} \subset \mathbb{Z}^2$. Let $\boxplus = \{x, x+1\} \times \{y, y+1\} \subset \mathbb{Z}^2$. The following properties (given for any T, $\boxminus$ and $\boxplus$) derive from Eq. (2).

Property 5. *There exists $\boldsymbol{p} \in \mathbb{Z}^2$ such that $T(\boldsymbol{p}) \in \boxminus$.*

Property 6 ([17])**.** *There exist 5 patterns that can be equal to $T(\boxplus)$, up to $\pi/2$ rotations and symmetries (see Fig. 4).*

Property 7. *If $\boxplus_1 \cap \boxplus_2 \neq \emptyset$, then any set $P \subset \mathbb{Z}^2$, such that $T(P) = \boxplus_1 \cup \boxplus_2$, is 4-connected.*

3.2. Necessary condition for topological invariance

The next result is easy to prove by combinatorial analysis.

Lemma 8. *If I contains $\mathscr{C}_2$ (resp. $\mathscr{C}_3$), then it does not satisfy condition (ii) (resp. (i)) of Def. 4.*

We now introduce a notion of *regularity*, that strenghtens the notion of well-composition.

Definition 9 (Regularity)**.** *We say that I is* regular *if it does not contain any of $\mathscr{C}_1$, $\mathscr{C}_2$ and $\mathscr{C}_3$.*

From Defs. 4 and 9, and Lem. 8, we then derive the following necessary condition for topological invariance.

Proposition 10. *If an image I is topologically invariant, then it is regular.*

3.3. Sufficient condition for well-composition

We now show that regularity is a sufficient condition to satisfy condition (i) of Def. 4.

Proposition 11. *If I is a regular image, then any transformed image $I' = I \circ T$ is well-composed.*

Proof Let T be such that $I' : I \circ T$ is not well-composed. Then, I' has a pattern $\boxplus$ that is a $\mathscr{C}_1$ (Th. 3) and that derives from a $\mathscr{P}_i$ in I (Prop. 6). But then, $\mathscr{P}_1$ is a $\mathscr{C}_1$, $\mathscr{P}_2$ is a $\mathscr{C}_3$, $\mathscr{P}_3$ and $\mathscr{P}_5$ contain a $\mathscr{C}_2$, and $\mathscr{P}_4$ is impossible. Thus, I is non-regular. The result follows by contraposition. ■

3.4. Regularity and 4-adjacency

We now introduce a new adjacency relation, that strenghtens the standard 4-adjacency in $\mathbb{Z}^2$.

Definition 12 ($\boxplus$-adjacency)**.** *The points $\boldsymbol{p}, \boldsymbol{q} \in S$ (resp. $\overline{S}$) are* $\boxplus$-adjacent *if (i) they are 4-adjacent, and (ii) there exists $\boxplus \subseteq S$ (resp. $\overline{S}$) such that $\boldsymbol{p}, \boldsymbol{q} \in \boxplus$.*

We then derive the notions of $\boxplus$-connectedness (equivalence relation) and $\boxplus$-connected components (equivalence classes).

It is plain that the $\boxplus$-adjacency implies the 4-adjacency. In the context of regular images, the counterpart is also true.

Lemma 13. *If I is a regular image, then the 4-adjacency is equal to the $\boxplus$-adjacency in S and $\overline{S}$.*

Proof Let $\boldsymbol{p}, \boldsymbol{q} \in S$ be 4-adjacent (the same holds for $\overline{S}$). We set $\boldsymbol{p} = (x, y)$ and $\boldsymbol{q} = (x, y+1)$ (the same holds up to $\pi/2$ rotation). We have $\boldsymbol{r'} = (x-1, y) \in S$ or $\boldsymbol{r} = (x+1, y) \in S$, otherwise $\{\boldsymbol{r'}, \boldsymbol{p}, \boldsymbol{r}\}$ is a $\mathscr{C}_2$; for instance $\boldsymbol{r} \in S$. Let $\boldsymbol{s} = (x+1, y+1)$. If $\boldsymbol{s} \in S$, then we have $\{\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}\} = \boxplus \in S$. If $\boldsymbol{s} \notin S$, then we have $\boldsymbol{s'} = (x-1, y+1) \in S$, otherwise $\{\boldsymbol{s'}, \boldsymbol{q}, \boldsymbol{s}\}$ is a $\mathscr{C}_2$. Thus, we have $\boldsymbol{r'} = (x-1, y) \in S$, otherwise $\{\boldsymbol{r'}, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{s}\}$ is a $\mathscr{C}_3$. Finally, we have $\{\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{s'}, \boldsymbol{r'}\} = \boxplus \in S$. ■

3.5. Sufficient condition for topological invariance

We note $\frown_S$ (resp. $\frown_{S'}$) the 4-adjacency relation on S (resp. S'), and $\sim_S$ (resp. $\sim_{S'}$) the 4-connectedness relation on S (resp. S'). We recall that $T_{|S'}$ is the restriction of T to S'. (Due to the symmetry of Def. 2, the results of Lem. 14, Cor. 15 and Lem. 16, stated for S/S' are also valid for $\overline{S}/\overline{S'}$.)

Lemma 14. *If I is a well-composed image, then $T_{|S'}$ establishes a homomorphism from $(S', \frown_{S'})$ to $(S, \sim_S)$.*

Proof Let $\boldsymbol{p'} \frown_{S'} \boldsymbol{q'}$, $\boldsymbol{p} = T_{|S'}(\boldsymbol{p'})$ and $\boldsymbol{q} = T_{|S'}(\boldsymbol{q'})$. From Eq. (2), we can have: (i) $\boldsymbol{p} = \boldsymbol{q}$, and then $\boldsymbol{p} \sim_S \boldsymbol{q}$; (ii) $\boldsymbol{p} \frown_S \boldsymbol{q}$, and then $\boldsymbol{p} \sim_S \boldsymbol{q}$; or (iii) $\boldsymbol{p}$, $\boldsymbol{q}$ are 8-adjacent, and thus 8-connected, and then $\boldsymbol{p} \sim_S \boldsymbol{q}$ derives from Def. 2. ■

Corollary 15. *Let $C' \in [S']_{\sim_{S'}}$ be a 4-connected component of S'. If I is well-composed, then there exists one 4-connected component $C \in [S]_{\sim_S}$ of S such that $T_{|S'}(C') \subseteq C$.*

We can then set the function $\mathfrak{T}_{S'} : [S']_{\sim_{S'}} \to [S]_{\sim_S}$, defined, for any $C' \in [S']_{\sim_{S'}}$, by $T_{|S'}(C') \subseteq \mathfrak{T}_{S'}(C')$.

Lemma 16. *If I is a regular image, then $\mathfrak{T}_{S'}$ is a bijection.*

Proof *(Surjectivity)* Let $C \in [S]_{\sim_S}$ and $\boldsymbol{p} \in C$. There exists $\boldsymbol{q} \in C$ such that $\boldsymbol{p} \frown_S \boldsymbol{q}$, otherwise, I contains $\mathscr{C}_2$ centered on $\boldsymbol{p}$. Then $\{\boldsymbol{p}, \boldsymbol{q}\} = \boxminus$. There exists $\boldsymbol{r} \in S'$ such that $T_{|S'}(\boldsymbol{r}) \in \{\boldsymbol{p}, \boldsymbol{q}\} \subseteq C$ (Prop. 5). Then $\mathfrak{T}_S$ is a surjection (Cor. 15). *(Injectivity)* Let $\boldsymbol{p'}, \boldsymbol{q'} \in S'$. Let $\boldsymbol{p} = T_{|S'}(\boldsymbol{p'})$, $\boldsymbol{q} = T_{|S'}(\boldsymbol{q'})$. Suppose that $\boldsymbol{p}, \boldsymbol{q} \in C \in [S]_{\sim_S}$, then $\boldsymbol{p}, \boldsymbol{q}$ are 4-connected, and therefore $\boxplus$-connected in S (Lem. 13). In other words, we have $\boldsymbol{p}, \boldsymbol{q} \in \bigcup_{i=0}^{k} \boxplus_i$ ($k \geq 0$), where $\forall i \in [\![0, k]\!], \boxplus_i \subset S$ and $\forall i \in [\![0, k-1]\!], \boxplus_i \cap \boxplus_{i+1} \neq \emptyset$. For any $i \in [\![0, k]\!]$, we define $P_i \subset S'$ such that $T_{|S'}(P_i) = \boxplus_i$. Any P_i is 4-connected (Prop. 7, for $\boxplus_1 = \boxplus_2 = \boxplus_i$), and then $\bigcup_{i=0}^{k} P_i$ is also 4-connected (Props. 5 and 7). Consequently, there exists $C' \in [S']_{\sim_{S'}}$ such that $\boldsymbol{p'}, \boldsymbol{q'} \in \bigcup_{i=0}^{k} P_i \subseteq C'$. It follows that $\mathfrak{T}_S$ is an injection. ■

Let $\mathcal{C}_I = [S]_{\sim_S} \cup [\overline{S}]_{\sim_{\overline{S}}}$ and $\mathcal{C}_{I'} = [S']_{\sim_{S'}} \cup [\overline{S'}]_{\sim_{\overline{S'}}}$. We define the function $\mathfrak{T}_{S'} \cup \mathfrak{T}_{\overline{S'}} = \mathfrak{T}_{I'} : \mathcal{C}_{I'} \to \mathcal{C}_I$, which is a bijection between the 4-connected components of I' and those of I. We say that $C'_1 \neq C'_2 \in \mathcal{C}_{I'}$ (resp. $C_1 \neq C_2 \in \mathcal{C}_I$) are *adjacent*, and we note $C'_1 \frown_{I'} C'_2$ (resp. $C_1 \frown_I C_2$), if there exist $\boldsymbol{p'} \in C'_1$, $\boldsymbol{q'} \in C'_2$ (resp. $\boldsymbol{p} \in C_1$, $\boldsymbol{q} \in C_2$) that are 4-adjacent. The adjacency tree [12] of I' (resp. I) is then defined by $(\mathcal{C}_{I'}, \frown_{I'})$ (resp. $(\mathcal{C}_I, \frown_I)$).

Proposition 17. *If I is a regular image, then $\mathfrak{T}_{I'}$ establishes an isomorphism between $(\mathcal{C}_{I'}, \frown_{I'})$ and $(\mathcal{C}_I, \frown_I)$.*

Proof Let $C'_1 \frown_{I'} C'_2$. Up to reindexing, we have $C'_1 \in [S']_{\sim_{S'}}$, $C'_2 \in [\overline{S'}]_{\sim_{\overline{S'}}}$. Let $\boldsymbol{p'} \in C'_1$, $\boldsymbol{q'} \in C'_2$ be 4-adjacent. Let $\boldsymbol{p} = T_{|S'}(\boldsymbol{p'}) \in \mathfrak{T}_{I'}(C'_1)$, $\boldsymbol{q} = T_{|\overline{S'}}(\boldsymbol{q'}) \in \mathfrak{T}_{I'}(C'_2)$. From Eq. (2), we can have: (i) $\boldsymbol{p}$, $\boldsymbol{q}$ are 4-adjacent, and then $\mathfrak{T}_{I'}(C'_1) \frown_I \mathfrak{T}_{I'}(C'_2)$; or (ii) $\boldsymbol{p}$, $\boldsymbol{q}$ are 8-adjacent (but not 4-adjacent). In that second case, let $\boldsymbol{r}$ be 4-adjacent with both

$\boldsymbol{p}$ and $\boldsymbol{q}$. We have either $\boldsymbol{r} \in \mathfrak{T}_{I'}(C'_1)$ or $\mathfrak{T}_{I'}(C'_2)$, and then $\mathfrak{T}_{I'}(C'_1) \frown_I \mathfrak{T}_{I'}(C'_2)$.

Let $C_1 \frown_I C_2$. Up to reindexing, we have $C_1 \in [S]_{\sim_S}$, $C_2 \in [\overline{S}]_{\sim_{\overline{S}}}$. Let $\boldsymbol{p} \in C_1$, $\boldsymbol{q} \in C_2$ be 4-adjacent. Up to $\pi/2$ rotation, we have $\boldsymbol{p} = (x, y)$, $\boldsymbol{q} = (x, y+1)$. Case (i): we have $\boldsymbol{r} = (x+1, y) \in C_1$ and $\boldsymbol{s} = (x+1, y+1) \in C_2$. Then we have $\{\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}\} = \boxplus$, $\{\boldsymbol{p}, \boldsymbol{r}\} = \boxminus \subset C_1$ and $\{\boldsymbol{q}, \boldsymbol{s}\} = \boxminus \subset C_2$. Let $P \subset I'$ such that $T(P) = \{\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}\}$. We have $P \cap \mathfrak{T}_{I'}^{-1}(C_1) \neq \emptyset$ and $P \cap \mathfrak{T}_{I'}^{-1}(C_2) \neq \emptyset$ (Prop. 5). Moreover, P is 4-connected (Prop. 7). Then, we have $\mathfrak{T}_{I'}^{-1}(C_1) \frown_I \mathfrak{T}_{I'}^{-1}(C_2)$. Case (ii): we have $\boldsymbol{r}, \boldsymbol{s} \in C_1$. Let $\boxplus_1 = \{\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s}\}$ and $\boxplus_2 = \boldsymbol{q} + \{-1, 0\} \times \{0, 1\}$. We have $\boxplus_2 \subseteq C_2$ (by a similar reasonning as for Prop. 13). Let $P \subset I'$ such that $T(P) = \boxplus_1 \cup \boxplus_2$. We have $P \cap \mathfrak{T}_{I'}^{-1}(C_1) \neq \emptyset$ and $P \cap \mathfrak{T}_{I'}^{-1}(C_2) \neq \emptyset$ (Prop. 5). Moreover, P is 4-connected (Prop. 7). Then, we have $\mathfrak{T}_{I'}^{-1}(C_1) \frown_I \mathfrak{T}_{I'}^{-1}(C_2)$. The other cases are impossible or equivalent to the above two cases, by symmetry between S and $\overline{S}$. ■

By gathering Props. 10, 11 and 17, we obtain our main result.

Theorem 18. *The image $I : \mathbb{Z}^2 \to \{0, 1\}$ is topologically invariant iff it is regular.*

This result enables, in particular, to check the topological invariance of an image I by simply scanning it once to identify potential occurrences of $\mathscr{C}_1$, $\mathscr{C}_2$ and $\mathscr{C}_3$.

4. EXPERIMENTS AND METHODOLOGY

From Sec. 3, we can determine –in linear time– whether a 2D binary well-composed image is topologically invariant using the introduced notion of regularity. The analytic usefulness of these results is illustrated in Fig. 5(a–d), which emphasises the ability to predict the "topological behaviour" of an image.

Beyond the possibility to analyse images, these results also lead to solutions for correcting them, in order to improve their topological robustness. Indeed, once the pixels that forbid topological invariance have been identified (see Fig. 5(c)), the image can be processed to locally modify the corresponding areas, in order to fulfill the regularity requirements (see Fig. 5(e,f)). This process has to be performed iteratively until convergence, and can then fail in pathological cases, *e.g.*, when the images present a fine texture (see Fig. 6).

In such cases, a solution consists of considering a super-resolution strategy. Indeed, by increasing the resolution by a factor 2, *i.e.*, by associating a 2×2 pixel square to each initial pixel, a well-composed image becomes regular. Moreover, this strategy can be associated to other super-resolution paradigms based, *e.g.*, on Khalimsky grids [18], to transform any $(4, 8)$ or $(8, 4)$-adjacency binary image into a well-composed, and finally regular one.

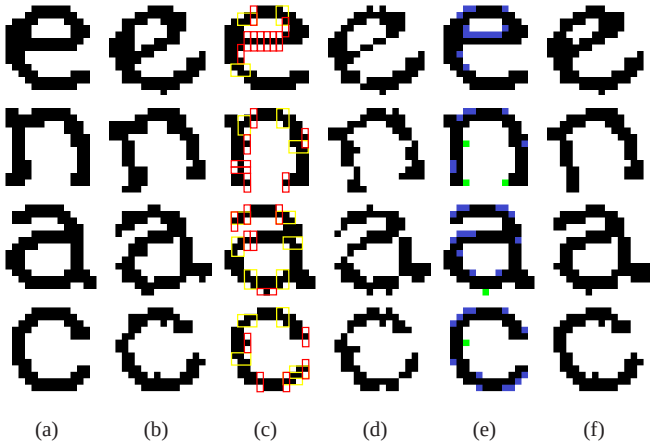


Fig. 5: (a) Regular images. (b) Rigid transformations of (a), with topological preservation. (c) Well-composed but non-regular images (the occurrences of critical configurations $\mathscr{C}_2$ and $\mathscr{C}_3$ are depicted by red and yellow frames, respectively). (d) Rigid transformations of (c), with topological alterations. (e) Corrected (regular) images obtained from (c) (in blue: added pixels; in green: removed pixels). (f) Rigid transformations of (e), with topological preservation.

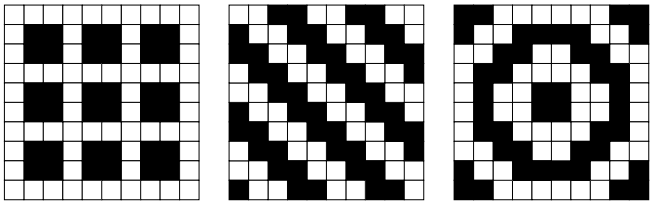


Fig. 6: Well-composed images that cannot be modified (without a super-resolution approach) to fulfil regularity requirements, due to fine texture effects.

5. CONCLUSION

We have explored necessary and sufficient conditions (namely, regularity) under which 2D binary well-composed images preserve their topological properties under rigid transformations, with solutions for favouring such preservation in application cases.

From a methodological viewpoint, this preliminary study opens the way to further extentions in terms of dimensions (from 2D to 3D [19]), and value spaces (from binary to grey-scale [20] and colour [21]), with applicative perspectives in image processing (registration, image warping, etc.). In addition, the case of standard dual $(8, 4)$- and $(4, 8)$-adjacencies, commonly used in digital topology may also be investigated.

From a theoretical viewpoint, the relations that may exist between these results and various (continuous and/or discrete) geometrical and topological concepts will also be explored. In particular, some links may be found with notions such as the r-regularity [22, 23] and the connectivity classes [24, 25].

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