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# Sequent calculi with procedure calls

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## Abstract

In this paper, we study 3 focussed sequent calculi that are based on Miller-Liang’s LKF system [LM09] for polarised classical logic, and integrate the possibility to call a decision procedure for some background theory  $\mathcal{T}$ .

The main sequent calculus out of the three is  $\text{LK}(\mathcal{T})$ , in which we prove cut-elimination. Changing the polarities of literals and connectives does not change the provability of formulae, only the shape of proofs.

In order to prove this, we introduce a second sequent calculus,  $\text{LK}^+(\mathcal{T})$  that extends  $\text{LK}(\mathcal{T})$  with a relaxed focussing discipline.

We then prove completeness of  $\text{LK}(\mathcal{T})$  (and therefore of  $\text{LK}^+(\mathcal{T})$ ) with respect to first-order reasoning modulo the ground propositional lemmas of the background theory  $\mathcal{T}$ .

A third sequent calculus is introduced,  $\text{LK}^p(\mathcal{T})$ , that extends  $\text{LK}(\mathcal{T})$  with the possibility to polarise literals "on the fly" during proof-search. This is used in other works [FLM12, FGLM13] to simulate the DPLL( $\mathcal{T}$ ) procedure [NOT06], and we show here its completeness.

Encodings of  $\text{LK}^+(\mathcal{T})$  and  $\text{LK}^p(\mathcal{T})$  into the most restrictive system  $\text{LK}(\mathcal{T})$  are presented along the way.

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# 1 The sequent calculus $LK(\mathcal{T})$

The sequent calculus  $LK(\mathcal{T})$  manipulates the formulae of first-order logic, with the specificity that every predicate symbol is classified as either positive or negative, and boolean connectives come in two versions: positive and negative.

## Definition 1 (Formulae)

$$\begin{array}{lll} \text{Positive formulae} & P & ::= p \mid A \wedge^+ B \mid A \vee^+ B \mid \exists x A \\ \text{Negative formulae} & N & ::= p^\perp \mid A \wedge^- B \mid A \vee^- B \mid \forall x A \\ \text{Formulae} & A, B & ::= P \mid N \end{array}$$

where  $p$  ranges over a set of elements called *positive literals*. Formulae of the form  $p^\perp$  are called *negative literals*.

## Definition 2 (Negation) Negation is extended from literals to all formulae:

$(p)^\perp$	$:= p^\perp$	$(p^\perp)^\perp$	$:= p$
$(A \wedge^+ B)^\perp$	$:= A^\perp \vee^- B^\perp$	$(A \wedge^- B)^\perp$	$:= A^\perp \vee^+ B^\perp$
$(A \vee^+ B)^\perp$	$:= A^\perp \wedge^- B^\perp$	$(A \vee^- B)^\perp$	$:= A^\perp \wedge^+ B^\perp$
$(\exists x A)^\perp$	$:= \forall x A^\perp$	$(\forall x A)^\perp$	$:= \exists x A^\perp$

## Definition 3 ( $LK(\mathcal{T})$ ) The sequent calculus $LK(\mathcal{T})$ manipulates two kinds of sequents:

$$\begin{array}{ll} \text{Focused sequents} & \Gamma \vdash [A] \\ \text{Unfocused sequents} & \Gamma \vdash \Delta \end{array}$$

where  $\Gamma$  is a multiset of negative formulae and positive literals,  $\Delta$  is a multiset of formulae, and  $A$  is said to be in the *focus* of the (focused) sequent. By  $\text{lit}(\Gamma)$  we denote the sub-multiset of  $\Gamma$  consisting of its literals.

The rules of  $LK(\mathcal{T})$ , given in Figure 1, are of three kinds: synchronous rules, asynchronous rules, and structural rules. These correspond to three alternating phases in the proof-search process that is described by the rules.

If  $S$  is a set of literals,  $\mathcal{T}(S)$  is the call to the decision procedure on the conjunction of all literals of  $S$ . It holds if the procedure returns UNSAT.

# 2 Admissible rules

## Definition 4 (Assumptions on the procedure)

We assume that the procedure calls satisfy the following properties:

- Weakening If  $\mathcal{T}(S)$  then  $\mathcal{T}(S, S')$ .
- Contraction If  $\mathcal{T}(S, A, A)$  then  $\mathcal{T}(S, A)$ .
- Instantiation If  $\mathcal{T}(S)$  then  $\mathcal{T}(\{\not{x}\}_x S)$ .
- Consistency If  $\mathcal{T}(S, p)$  and  $\mathcal{T}(S, p^\perp)$  then  $\mathcal{T}(S)$ .
- Inconsistency  $\mathcal{T}(S, p, p^\perp)$ .  
where  $S$  is a set of literals.

## Lemma 1 (Admissibility of weakening and contraction)

The following rules are admissible in  $LK(\mathcal{T})$ .

$$\begin{array}{cc} \frac{\Gamma \vdash [B]}{\Gamma, A \vdash [B]} & \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \\ \frac{\Gamma, A, A \vdash [B]}{\Gamma, A \vdash [B]} & \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \end{array}$$

**Proof:** By induction on the derivation of the premiss. □

Synchronous rules			
$\frac{\Gamma \vdash [A] \quad \Gamma \vdash [B]}{\Gamma \vdash [A \wedge^+ B]}$	$\frac{\Gamma \vdash [A_i]}{\Gamma \vdash [A_1 \vee^+ A_2]}$	$\frac{\Gamma \vdash [\{^t_x\} A]}{\Gamma \vdash [\exists x A]}$	
$\frac{}{\Gamma, p \vdash [p]} p \text{ positive literal}$	$\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp)}{\Gamma \vdash [p]} p \text{ positive literal}$		
$\frac{\Gamma \vdash N}{\Gamma \vdash [N]} N \text{ negative}$			
Asynchronous rules			
$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge^- B, \Delta}$	$\frac{\Gamma \vdash A_1, A_2, \Delta}{\Gamma \vdash A_1 \vee^- A_2, \Delta}$	$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash (\forall x A), \Delta} x \notin \text{FV}(\Gamma, \Delta)$	
$\frac{\Gamma, A^\perp \vdash \Delta}{\Gamma \vdash A, \Delta} A \text{ positive or literal}$			
Structural rules			
$\frac{\Gamma, P^\perp \vdash [P]}{\Gamma, P^\perp \vdash} P \text{ positive}$		$\frac{\mathcal{T}(\text{lit}(\Gamma))}{\Gamma \vdash}$	

Figure 1: System  $\text{LK}(\mathcal{T})$

**Lemma 2 (Admissibility of instantiation)** *The following rules are admissible in  $\text{LK}(\mathcal{T})$ .*

$$\frac{\Gamma \vdash [B]}{\{^t_x\} \Gamma \vdash [\{^t_x\} B]} \quad \frac{\Gamma \vdash \Delta}{\{^t_x\} \Gamma \vdash \{^t_x\} \Delta}$$

**Proof:** By induction on the derivation of the premiss. □

### 3 Invertibility of the asynchronous phase

**Lemma 3 (Invertibility of asynchronous rules)** *All asynchronous rules are invertible in  $\text{LK}(\mathcal{T})$ .*

**Proof:** By induction on the derivation proving the conclusion of the asynchronous rule considered.

- Inversion of  $A \wedge^- B$ : by case analysis on the last rule actually used

$$\frac{\Gamma \vdash A \wedge^- B, C, \Delta' \quad \Gamma \vdash A \wedge^- B, D, \Delta'}{\Gamma \vdash A \wedge^- B, C \wedge^- D, \Delta'}$$

By induction hypothesis we get

$$\frac{\Gamma \vdash A, C, \Delta' \quad \Gamma \vdash A, D, \Delta'}{\Gamma \vdash A, C \wedge^- D, \Delta'} \quad \text{and} \quad \frac{\Gamma \vdash B, C, \Delta' \quad \Gamma \vdash B, D, \Delta'}{\Gamma \vdash B, C \wedge^- D, \Delta'}$$

$$\frac{\Gamma \vdash A \wedge^- B, C, D, \Delta'}{\Gamma \vdash A \wedge^- B, C \vee^- D, \Delta'}$$

By induction hypothesis we get

$$\frac{\Gamma \vdash A, C, D, \Delta'}{\Gamma \vdash A, C \vee^- D, \Delta'} \quad \text{and} \quad \frac{\Gamma \vdash B, C, D, \Delta'}{\Gamma \vdash B, C \vee^- D, \Delta'}$$

$$\frac{\Gamma \vdash A \wedge^- B, C, \Delta'}{\Gamma \vdash A \wedge^- B, (\forall x C), \Delta'} x \notin \text{FV}(\Gamma, \Delta', A \wedge^- B)$$

- By induction hypothesis we get
- $$\frac{\Gamma \vdash A, C, \Delta'}{\Gamma \vdash A, (\forall x C), \Delta'} x \notin \text{FV}(\Gamma, \Delta', A) \text{ and } \frac{\Gamma \vdash B, C, \Delta'}{\Gamma \vdash B, (\forall x C), \Delta'} x \notin \text{FV}(\Gamma, \Delta', B)$$
- $\frac{\Gamma, C^\perp \vdash A \wedge^- B, \Delta'}{\Gamma \vdash A \wedge^- B, C, \Delta'} C \text{ positive or literal}$
- By induction hypothesis we get
- $$\frac{\Gamma, C^\perp \vdash A, \Delta'}{\Gamma \vdash A, C, \Delta'} C \text{ positive or literal and } \frac{\Gamma, C^\perp \vdash B, \Delta'}{\Gamma \vdash B, C, \Delta'} C \text{ positive or literal}$$
- Inversion of  $A \vee^- B$ 

$$\frac{\Gamma \vdash A \vee^- B, C, \Delta' \quad \Gamma \vdash A \vee^- B, D, \Delta'}{\Gamma \vdash A \vee^- B, C \wedge^- D, \Delta'}$$

By induction hypothesis we get  $\frac{\Gamma \vdash A, B, C, \Delta' \quad \Gamma \vdash A, B, D, \Delta'}{\Gamma \vdash A, B, C \wedge^- D, \Delta'}$

$$\frac{\Gamma \vdash A \vee^- B, C, D, \Delta'}{\Gamma \vdash A \vee^- B, C \vee^- D, \Delta'}$$

By induction hypothesis we get  $\frac{\Gamma \vdash A, B, C, D, \Delta'}{\Gamma \vdash A, B, C \vee^- D, \Delta'}$

$$\frac{\Gamma \vdash A \vee^- B, C, \Delta'}{\Gamma \vdash A \vee^- B, (\forall x C), \Delta'} x \notin \text{FV}(\Gamma, \Delta')$$

By induction hypothesis we get  $\frac{\Gamma \vdash A, B, C, \Delta'}{\Gamma \vdash A, B, (\forall x C), \Delta'} x \notin \text{FV}(\Gamma, \Delta')$

$$\frac{\Gamma, C^\perp \vdash A \vee^- B, \Delta'}{\Gamma \vdash A \vee^- B, C, \Delta'} C \text{ positive or literal}$$

By induction hypothesis we get  $\frac{\Gamma, C^\perp \vdash A, B, \Delta'}{\Gamma \vdash A, B, C, \Delta'} C \text{ positive or literal}$
  - Inversion of  $\forall x A$ 

$$\frac{\Gamma \vdash (\forall x A), C, \Delta' \quad \Gamma \vdash (\forall x A), D, \Delta'}{\Gamma \vdash (\forall x A), C \wedge^- D, \Delta'}$$

By induction hypothesis we get  $\frac{\Gamma \vdash A, C, \Delta' \quad \Gamma \vdash A, D, \Delta'}{\Gamma \vdash A, C \wedge^- D, \Delta'} x \notin \text{FV}(\Gamma, \Delta')$

$$\frac{\Gamma \vdash (\forall x A), C, D, \Delta'}{\Gamma \vdash (\forall x A), C \vee^- D, \Delta'}$$

By induction hypothesis we get  $\frac{\Gamma \vdash A, C, D, \Delta'}{\Gamma \vdash A, C \vee^- D, \Delta'}$

$$\frac{\Gamma \vdash (\forall x A), D, \Delta'}{\Gamma \vdash (\forall x A), (\forall x D), \Delta'} x \notin \text{FV}(\Gamma, \Delta')$$

By induction hypothesis we get  $\frac{\Gamma \vdash A, D, \Delta'}{\Gamma \vdash A, (\forall x D), \Delta'} x \notin \text{FV}(\Gamma, \Delta')$

$$\frac{\Gamma, C^\perp \vdash (\forall x A), \Delta'}{\Gamma \vdash (\forall x A), C, \Delta'} C \text{ positive or literal}$$

By induction hypothesis we get  $\frac{\Gamma, C^\perp \vdash A, \Delta'}{\Gamma \vdash A, C, \Delta'} C \text{ positive or literal}$
  - Inversion of literals and positive formulae (A)
$$\frac{\Gamma \vdash A, C, \Delta' \quad \Gamma \vdash A, D, \Delta'}{\Gamma \vdash A, C \wedge^- D, \Delta'}$$

By induction hypothesis we get  $\frac{\Gamma, A^\perp \vdash C, \Delta' \quad \Gamma, A^\perp \vdash D, \Delta'}{\Gamma, A^\perp \vdash C \wedge^- D, \Delta'}$

$$\begin{array}{l}
- \frac{\Gamma \vdash A, C, D, \Delta'}{\Gamma \vdash A, C \vee^- D, \Delta'} \\
\text{By induction hypothesis } \frac{\Gamma, A^\perp \vdash C, D, \Delta'}{\Gamma, A^\perp \vdash C \vee^- D, \Delta'} \\
- \frac{\Gamma \vdash A, D, \Delta'}{\Gamma \vdash A, (\forall x D), \Delta'} \quad x \notin \text{FV}(\Gamma, \Delta') \\
\text{By induction hypothesis we get } \frac{\Gamma, A^\perp \vdash D, \Delta'}{\Gamma, A^\perp \vdash (\forall x D), \Delta'} \quad x \notin \text{FV}(\Gamma, \Delta') \\
- \frac{\Gamma, B^\perp \vdash A, \Delta'}{\Gamma \vdash A, B, \Delta'} \quad B \text{ positive or literal} \\
\text{By induction hypothesis we get } \frac{\Gamma, A^\perp, B^\perp \vdash \Delta'}{\Gamma, A^\perp \vdash B, \Delta'} \quad B \text{ positive or literal}
\end{array}$$

□

## 4 Cut-elimination

**Theorem 4 (cut<sub>1</sub> and cut<sub>2</sub>)** *The following rules are admissible in LK(T).*

$$\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash \Delta}{\Gamma \vdash \Delta} \text{cut}_1 \quad \frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash [B]}{\Gamma \vdash [B]} \text{cut}_2$$

**Proof:** By simultaneous induction on the derivation of the right premiss.

We reduce cut<sub>8</sub> by case analysis on the last rule used to prove the right premiss.

$$\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \frac{\Gamma, p \vdash B, \Delta \quad \Gamma, p \vdash C, \Delta}{\Gamma, p \vdash B \wedge^- C, \Delta}}{\Gamma \vdash B \wedge^- C, \Delta} \text{cut}_1$$

reduces to

$$\frac{\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash B, \Delta}{\Gamma \vdash B, \Delta} \text{cut}_1 \quad \frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash C, \Delta}{\Gamma \vdash C, \Delta} \text{cut}_1}{\Gamma \vdash B \wedge^- C, \Delta}$$

$$\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \frac{\Gamma, p \vdash B_1, B_2, \Delta}{\Gamma, p \vdash B_1 \vee^- B_2, \Delta} \text{cut}_1}{\Gamma \vdash B_1 \vee^- B_2, \Delta} \text{cut}_1 \quad \text{reduces to} \quad \frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash B_1, B_2, \Delta}{\Gamma \vdash B_1, B_2, \Delta} \text{cut}_1$$

$$\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \frac{\Gamma, p \vdash B, \Delta}{\Gamma, p \vdash \forall x B, \Delta} \text{cut}_1}{\Gamma \vdash \forall x B, \Delta} \text{cut}_1 \quad \text{reduces to} \quad \frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash B, \Delta}{\Gamma \vdash B, \Delta} \text{cut}_1$$

$$\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \frac{\Gamma, p, B^\perp \vdash \Delta}{\Gamma, p \vdash B, \Delta} \text{cut}_1}{\Gamma \vdash B, \Delta} \text{cut}_1 \quad \text{reduces to} \quad \frac{\mathcal{T}(\text{lit}(\Gamma, B^\perp), p^\perp) \quad \Gamma, p, B^\perp \vdash \Delta}{\Gamma, B^\perp \vdash \Delta} \text{cut}_1$$

We have  $\mathcal{T}(\text{lit}(\Gamma), p^\perp, B^\perp)$  as we assume the procedure to satisfy weakening.

If  $P^\perp \in (\Gamma, p)$ ,

$$\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \frac{\Gamma, p \vdash [P]}{\Gamma, p \vdash} \text{cut}_1}{\Gamma \vdash} \text{cut}_1 \quad \text{reduces to} \quad \frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \Gamma, p^\perp \vdash [P]}{\Gamma \vdash [P]} \text{cut}_2$$

as  $P^\perp \in (\Gamma)$ .

$$\frac{\frac{\mathcal{T}(\text{lit}(\Gamma), p)}{\Gamma, p \vdash} \text{cut}_1}{\Gamma \vdash} \text{reduces to } \frac{\mathcal{T}(\text{lit}(\Gamma))}{\Gamma \vdash}$$

using the assumption of consistency.

We reduce  $\text{cut}_2$  again by case analysis on the last rule used to prove the right premiss.

$$\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \frac{\Gamma, p \vdash [B] \quad \Gamma, p \vdash [C]}{\Gamma, p \vdash [B \wedge^+ C]} \text{cut}_2}{\Gamma \vdash [B \wedge^+ C]} \text{cut}_2$$

reduces to

$$\frac{\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash [B]}{\Gamma \vdash [B]} \text{cut}_2 \quad \frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash [C]}{\Gamma \vdash [C]} \text{cut}_2}{\Gamma \vdash [B \wedge^+ C]}$$

$$\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \frac{\Gamma, p \vdash [B_i]}{\Gamma, p \vdash [B_1 \vee^+ B_2]} \text{cut}_2}{\Gamma \vdash [B_1 \vee^+ B_2]} \text{cut}_2 \text{ reduces to } \frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash [B_i]}{\Gamma \vdash [B_i]} \text{cut}_2$$

$$\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \frac{\Gamma, p \vdash [\{\check{y}_x\} B]}{\Gamma, p \vdash [\exists x B]} \text{cut}_2}{\Gamma \vdash [\exists x B]} \text{cut}_2 \text{ reduces to } \frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash [\{\check{y}_x\} B]}{\Gamma \vdash [\{\check{y}_x\} B]} \text{cut}_2$$

$$\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \frac{\Gamma, p \vdash N}{\Gamma, p \vdash [N]} \text{cut}_2}{\Gamma \vdash [N]} \text{cut}_2 \text{ reduces to } \frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash N}{\Gamma \vdash N} \text{cut}_1$$

If  $p' \in \Gamma, p$ ,

$$\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \frac{\Gamma, p \vdash [p']}{\Gamma \vdash [p']} \text{cut}_2}{\Gamma \vdash [p']} \text{ reduces to } \frac{\Gamma \vdash [p']}{\Gamma \vdash [p']} \text{ if } p' \in \Gamma$$

$$\text{reduces to } \frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp)}{\Gamma \vdash [p']} \text{ if } p' = p$$

Finally,

$$\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \frac{\mathcal{T}(\text{lit}(\Gamma), p, p'^\perp)}{\Gamma, p \vdash [p']} \text{cut}_2}{\Gamma \vdash [p']} \text{ reduces to } \frac{\mathcal{T}(\text{lit}(\Gamma), p'^\perp)}{\Gamma \vdash [p']}$$

since weakening gives  $\mathcal{T}(\text{lit}(\Gamma), p^\perp, p'^\perp)$  and consistency then gives  $\mathcal{T}(\text{lit}(\Gamma), p'^\perp)$ .  $\square$

**Theorem 5 ( $\text{cut}_3$ ,  $\text{cut}_4$  and  $\text{cut}_5$ )** *The following rules are admissible in  $LK(\mathcal{T})$ .*

$$\frac{\Gamma \vdash [A] \quad \Gamma \vdash A^\perp, \Delta}{\Gamma \vdash \Delta} \text{cut}_3$$

$$\frac{\Gamma \vdash N \quad \Gamma, N \vdash \Delta}{\Gamma \vdash \Delta} \text{cut}_4 \quad \frac{\Gamma \vdash N \quad \Gamma, N \vdash [B]}{\Gamma \vdash [B]} \text{cut}_5$$

**Proof:** By simultaneous induction on the following lexicographical measure:

- the size of the cut-formula ( $A$  or  $N$ )
- the fact that the cut-formula ( $A$  or  $N$ ) is positive or negative (if of equal size, a positive formula is considered smaller than a negative formula)
- the height of the derivation of the right premiss

Weakenings and contractions (as they are admissible in the system) are implicitly used throughout this proof.

In order to eliminate  $\text{cut}_3$ , we analyse which rule is used to prove the left premiss. We then use invertibility of the negative phase so that the last rule used in the right premiss is its dual one.

$$\frac{\frac{\Gamma \vdash [A] \quad \Gamma \vdash [B]}{\Gamma \vdash [A \wedge^+ B]} \quad \frac{\Gamma \vdash A^\perp, B^\perp, \Delta}{\Gamma \vdash A \vee^- B, \Delta} \text{cut}_3}{\Gamma \vdash \Delta} \text{cut}_3 \quad \text{reduces to} \quad \frac{\Gamma \vdash [A] \quad \Gamma \vdash A^\perp, B^\perp, \Delta}{\Gamma \vdash B^\perp, \Delta} \text{cut}_3}{\Gamma \vdash \Delta} \text{cut}_3$$

$$\frac{\frac{\Gamma \vdash [A_i]}{\Gamma \vdash [A_1 \vee^+ A_2]} \quad \frac{\Gamma \vdash A_1^\perp, \Delta \quad \Gamma \vdash A_2^\perp, \Delta}{\Gamma \vdash A_1 \wedge^- A_2, \Delta} \text{cut}_3}{\Gamma \vdash \Delta} \text{cut}_3 \quad \text{reduces to} \quad \frac{\Gamma \vdash [A_i] \quad \Gamma \vdash A_i^\perp, \Delta}{\Gamma \vdash \Delta} \text{cut}_3$$

$$\frac{\frac{\Gamma \vdash [\{\not{x}\}A]}{\Gamma \vdash [\exists x A]} \quad \frac{\Gamma \vdash A^\perp, \Delta}{\Gamma \vdash (\forall x A^\perp), \Delta} \text{cut}_3}{\Gamma \vdash \Delta} \text{cut}_3 \quad \text{reduces to} \quad \frac{\Gamma \vdash [\{\not{x}\}A] \quad \frac{\Gamma \vdash A^\perp, \Delta}{\Gamma \vdash (\{\not{x}\}A^\perp), \Delta} \text{cut}_3}{\Gamma \vdash \Delta} \text{cut}_3 \quad x \notin \text{FV}(\Gamma, \Delta)$$

using the admissibility of instantiation.

$$\frac{\frac{\Gamma \vdash N}{\Gamma \vdash [N]} \quad \frac{\Gamma, N \vdash \Delta}{\Gamma \vdash (N^\perp), \Delta} \text{cut}_3}{\Gamma \vdash \Delta} \text{cut}_3 \quad \text{reduces to} \quad \frac{\Gamma \vdash N \quad \Gamma, N \vdash \Delta}{\Gamma \vdash \Delta} \text{cut}_4$$

We will describe below how  $\text{cut}_4$  is reduced.

$$\frac{\frac{\Gamma, p, p \vdash \Delta}{\Gamma, p \vdash [p]} \quad \frac{\Gamma, p \vdash (p^\perp), \Delta}{\Gamma, p \vdash \Delta} \text{cut}_3}{\Gamma, p \vdash \Delta} \text{cut}_3 \quad \text{reduces to} \quad \frac{\Gamma, p, p \vdash \Delta}{\Gamma, p \vdash \Delta}$$

using the admissibility of contraction.

$$\frac{\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp)}{\Gamma \vdash [p]} \quad \frac{\Gamma, p \vdash \Delta}{\Gamma \vdash (p^\perp), \Delta} \text{cut}_3}{\Gamma \vdash \Delta} \text{cut}_3 \quad \text{reduces to} \quad \frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash \Delta}{\Gamma \vdash \Delta} \text{cut}_1$$

In order to reduce  $\text{cut}_4$ , we analyse which rule is used to prove the right premiss.

$$\frac{\frac{\mathcal{T}(\text{lit}(\Gamma))}{\Gamma \vdash N} \quad \frac{\Gamma, N \vdash \Delta}{\Gamma \vdash (N^\perp), \Delta} \text{cut}_3}{\Gamma \vdash \Delta} \text{cut}_3 \quad \text{reduces to} \quad \frac{\mathcal{T}(\text{lit}(\Gamma))}{\Gamma \vdash \Delta}$$

if  $N$  is not a literal (hence, it is not passed on to the procedure).

$$\frac{\frac{\Gamma, p \vdash \Delta}{\Gamma \vdash p^\perp} \quad \frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp)}{\Gamma, p^\perp \vdash \Delta} \text{cut}_3}{\Gamma \vdash \Delta} \text{cut}_3 \quad \text{reduces to} \quad \frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash \Delta}{\Gamma \vdash \Delta} \text{cut}_1$$

if  $p^\perp$  is a literal passed on to the procedure.

$$\frac{\frac{\Gamma, N \vdash [N^\perp]}{\Gamma \vdash N} \quad \frac{\Gamma, N \vdash \Delta}{\Gamma \vdash (N^\perp), \Delta} \text{cut}_3}{\Gamma \vdash \Delta} \text{cut}_3 \quad \text{reduces to} \quad \frac{\Gamma \vdash N \quad \Gamma, N \vdash [N^\perp]}{\Gamma \vdash [N^\perp]} \text{cut}_5}{\Gamma \vdash \Delta} \text{cut}_3$$

$$\frac{\Gamma, P^\perp \vdash N \quad \frac{\Gamma, P^\perp, N \vdash [P]}{\Gamma, P^\perp, N \vdash \Delta} \text{cut}_3}{\Gamma, P^\perp \vdash \Delta} \text{cut}_3 \quad \text{reduces to} \quad \frac{\Gamma, P^\perp \vdash N \quad \Gamma, P^\perp, N \vdash [P]}{\Gamma, P^\perp \vdash [P]} \text{cut}_5}{\Gamma, P^\perp \vdash \Delta} \text{cut}_5$$



$$\frac{\Gamma, N \vdash B, \Delta \quad \Gamma, N \vdash C, \Delta}{\Gamma \vdash N \quad \Gamma, N \vdash B \wedge^- C, \Delta} \text{cut}_4$$

reduces to

$$\frac{\frac{\Gamma \vdash N \quad \Gamma, N \vdash B, \Delta}{\Gamma \vdash B, \Delta} \text{cut}_4 \quad \frac{\Gamma \vdash N \quad \Gamma, N \vdash C, \Delta}{\Gamma \vdash C, \Delta} \text{cut}_4}{\Gamma \vdash B \wedge^- C, \Delta}$$

$$\frac{\frac{\Gamma, N \vdash B, C, \Delta}{\Gamma \vdash N \quad \Gamma, N \vdash B \vee^- C, \Delta} \text{cut}_4}{\Gamma \vdash B \vee^- C, \Delta} \text{cut}_4 \quad \text{reduces to} \quad \frac{\Gamma \vdash N \quad \Gamma, N \vdash B, C, \Delta}{\Gamma \vdash B, C, \Delta} \text{cut}_4$$

$$\frac{\frac{\Gamma, N \vdash B, \Delta}{\Gamma \vdash N \quad \Gamma, N \vdash \forall x B, \Delta} \text{cut}_4}{\Gamma \vdash \forall x B, \Delta} \text{cut}_4 \quad \text{reduces to} \quad \frac{\Gamma \vdash N \quad \Gamma, N \vdash B, \Delta}{\Gamma \vdash B, \Delta} \text{cut}_4$$

$$\frac{\frac{\Gamma, N, B^\perp \vdash \Delta}{\Gamma \vdash N \quad \Gamma, N \vdash B, \Delta} \text{cut}_4}{\Gamma \vdash B, \Delta} \text{cut}_4 \quad \text{reduces to} \quad \frac{\Gamma, B^\perp \vdash N \quad \Gamma, N, B^\perp \vdash \Delta}{\Gamma, B^\perp \vdash \Delta} \text{cut}_4$$

using weakening, and if  $B$  is positive or a negative literal.

We have reduced all cases of  $\text{cut}_4$ ; we now reduce the cases for  $\text{cut}_5$  (again, by case analysis on the last rule used to prove the right premiss).

$$\frac{\frac{\Gamma, N \vdash [B] \quad \Gamma, N \vdash [C]}{\Gamma \vdash N \quad \Gamma, N \vdash [B \wedge^+ C]} \text{cut}_5}{\Gamma \vdash [B \wedge^+ C]} \text{cut}_5 \quad \text{reduces to} \quad \frac{\Gamma \vdash N \quad \Gamma, N \vdash [B]}{\Gamma \vdash [B]} \text{cut}_5 \quad \frac{\Gamma \vdash N \quad \Gamma, N \vdash [C]}{\Gamma \vdash [C]} \text{cut}_5$$

$$\frac{\frac{\Gamma, N \vdash [B_i]}{\Gamma \vdash N \quad \Gamma, N \vdash [B_1 \vee^+ B_2]} \text{cut}_5}{\Gamma \vdash [B_1 \vee^+ B_2]} \text{cut}_5 \quad \text{reduces to} \quad \frac{\Gamma \vdash N \quad \Gamma, N \vdash [B_i]}{\Gamma \vdash [B_i]} \text{cut}_5$$

$$\frac{\frac{\Gamma, N \vdash [\{x\} B]}{\Gamma \vdash N \quad \Gamma, N \vdash [\exists x B]} \text{cut}_5}{\Gamma \vdash [\exists x B]} \text{cut}_5 \quad \text{reduces to} \quad \frac{\Gamma \vdash N \quad \Gamma, N \vdash [\{x\} B]}{\Gamma \vdash [\{x\} B]} \text{cut}_5$$

$$\frac{\frac{\Gamma, N \vdash N'}{\Gamma \vdash N \quad \Gamma, N \vdash [N']} \text{cut}_5}{\Gamma \vdash [N']} \text{cut}_5 \quad \text{reduces to} \quad \frac{\Gamma \vdash N \quad \Gamma, N \vdash N'}{\Gamma \vdash N'} \text{cut}_4$$

$$\frac{\frac{\Gamma \vdash N \quad \Gamma, N \vdash [p]}{\Gamma \vdash [p]} \text{cut}_5}{\Gamma \vdash [p]} \text{cut}_5 \quad \text{reduces to} \quad \frac{\Gamma \vdash [p]}{\Gamma \vdash [p]}$$

since  $p$  has to be in  $\Gamma$ .

$$\frac{\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp)}{\Gamma \vdash N \quad \Gamma, N \vdash [p]} \text{cut}_5}{\Gamma \vdash [p]} \text{cut}_5 \quad \text{reduces to} \quad \frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp)}{\Gamma \vdash [p]}$$

□

**Theorem 6 ( $\text{cut}_6$ ,  $\text{cut}_7$ , and  $\text{cut}_8$ )** *The following rules are admissible in  $\text{LK}(\mathcal{T})$ .*

$$\frac{\Gamma \vdash N, \Delta \quad \Gamma, N \vdash \Delta}{\Gamma \vdash \Delta} \text{cut}_6 \quad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash A^\perp, \Delta}{\Gamma \vdash \Delta} \text{cut}_7 \quad \frac{\Gamma, l \vdash \Delta \quad \Gamma, l^\perp \vdash \Delta}{\Gamma \vdash \Delta} \text{cut}_8$$

**Proof:**  $\text{cut}_6$  is proved admissible by induction on the multiset  $\Delta$ : the base case is the admissibility of  $\text{cut}_4$ , and the other cases just require the inversion of the connectives in  $\Delta$ .

For  $\text{cut}_7$ , we can assume without loss of generality (swapping  $A$  and  $A^\perp$ ) that  $A$  is negative. Applying inversion on  $\Gamma \vdash A^\perp, \Delta$  gives a proof of  $\Gamma, A \vdash \Delta$ , and  $\text{cut}_7$  is then obtained by  $\text{cut}_6$ :

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta} \text{cut}_6$$

$\text{cut}_8$  is obtained as follows:

$$\frac{\frac{\Gamma, l^\perp \vdash \Delta}{\Gamma \vdash l, \Delta} \quad \frac{\Gamma, l \vdash \Delta}{\Gamma \vdash l^\perp, \Delta}}{\Gamma \vdash \Delta} \text{cut}_7$$

□

## 5 Changing the Polarity of Predicates and Connectives

### 5.1 Changing the polarity of Predicates

In this section we try to show that changing the polarity of the predicates and the connectives that are present in a sequent, does not change the provability of the sequent in  $\text{LK}(\mathcal{T})$ . First, we deal with the polarities of the predicates and then we deal with the polarities of the connectives.

#### Notation 5

Let  $p$  and  $q$  be positive literals, such that  $q$  and  $p^\perp$  have the same meaning for  $\mathcal{T}$ , i.e. for all  $\Gamma$ ,  $\mathcal{T}(\text{lit}(\Gamma), p^\perp)$  if and only if  $\mathcal{T}(\text{lit}(\Gamma), q)$ , and  $\mathcal{T}(\text{lit}(\Gamma), p)$  if and only if  $\mathcal{T}(\text{lit}(\Gamma), q^\perp)$ .

We write  $\Gamma'$  for  $\{p^\perp, p/q, q^\perp\}\Gamma$ ,  $\Delta'$  for  $:= \{p^\perp, p/q, q^\perp\}\Delta$ ,  $A'$ ,  $\{p^\perp, p/q, q^\perp\}A$ , etc.

#### Lemma 7

1. If  $\Gamma \vdash_{\text{LK}(\mathcal{T})} [A]$ , then
  - (a)  $\Gamma', p \vdash_{\text{LK}(\mathcal{T})} [A']$ .
  - (b)  $\Gamma', p^\perp \vdash_{\text{LK}(\mathcal{T})} [A']$  or  $\Gamma', p^\perp \vdash_{\text{LK}(\mathcal{T})} \cdot$
2. If  $\Gamma \vdash_{\text{LK}(\mathcal{T})} \Delta$  then
  - (a)  $\Gamma', p \vdash_{\text{LK}(\mathcal{T})} \Delta'$ .
  - (b)  $\Gamma', p^\perp \vdash_{\text{LK}(\mathcal{T})} \Delta'$ .

**Proof:** By simultaneous induction on the last rule of the derivation.

1. For the first item of the lemma, by case analysis on the last rule of the derivation, we get

•

$$\frac{\Gamma \vdash [A_1] \quad \Gamma \vdash [A_2]}{\Gamma \vdash [A_1 \wedge^+ A_2]}$$

with  $A = A_1 \wedge^+ A_2$ .

- (a) The induction hypothesis on  $\Gamma \vdash_{\text{LK}(\mathcal{T})} [A_1]$  (1(a)) gives  $\Gamma', p \vdash_{\text{LK}(\mathcal{T})} [A'_1]$  and the induction hypothesis on  $\Gamma \vdash_{\text{LK}(\mathcal{T})} [A_2]$  (1(a)) gives  $\Gamma', p \vdash_{\text{LK}(\mathcal{T})} [A'_2]$ . Now, we get:

$$\frac{\Gamma', p \vdash [A'_1] \quad \Gamma', p \vdash [A'_2]}{\Gamma', p \vdash [A'_1 \wedge^+ A'_2]}$$

- (b) The induction hypothesis on  $\Gamma \vdash_{\text{LK}(\mathcal{T})} [A_1]$  (1(b)) gives  $\Gamma', p^\perp \vdash_{\text{LK}(\mathcal{T})} [A'_1]$  or  $\Gamma', p^\perp \vdash_{\text{LK}(\mathcal{T})} \cdot$  and the induction hypothesis on  $\Gamma \vdash_{\text{LK}(\mathcal{T})} [A_2]$  (1(b)) gives  $\Gamma', p^\perp \vdash_{\text{LK}(\mathcal{T})} [A'_2]$  or  $\Gamma', p^\perp \vdash_{\text{LK}(\mathcal{T})} \cdot$

If we get a proof of  $\Gamma', p^\perp \vdash_{\text{LK}(\mathcal{T})}$  from one of the two applications of the induction hypothesis, we are done. If not then we get:

$$\frac{\frac{\Gamma', p^\perp \vdash [A'_1] \quad \Gamma', p \vdash [A'_2]}{\Gamma', p^\perp \vdash [A'_1 \wedge^+ A'_2]}}{\Gamma \vdash [A_i]} \\ \Gamma \vdash [A_1 \vee^+ A_2]$$

with  $A = A_1 \vee^+ A_2$ .

(a) The induction hypothesis on  $\Gamma \vdash_{\text{LK}(\mathcal{T})} [A_i]$  1(a) gives  $\Gamma', p \vdash_{\text{LK}(\mathcal{T})} [A'_i]$ . We get:

$$\frac{\Gamma', p \vdash [A'_i]}{\Gamma', p \vdash [A'_1 \vee^+ A'_2]}$$

(b) The induction hypothesis on  $\Gamma \vdash_{\text{LK}(\mathcal{T})} [A_i]$  (1(b)) gives  $\Gamma', p^\perp \vdash_{\text{LK}(\mathcal{T})} [A'_i]$  or  $\Gamma', p^\perp \vdash_{\text{LK}(\mathcal{T})}$ .

In the latter case we are done. For the former case, we get:

$$\frac{\frac{\Gamma', p^\perp \vdash [A'_i]}{\Gamma', p^\perp \vdash [A'_1 \vee^+ A'_2]}}{\Gamma \vdash [\{\! \{^x_x\} A]} \\ \Gamma \vdash [\exists x A]$$

with  $A = \exists x A$

(a) The induction hypothesis on  $\Gamma \vdash_{\text{LK}(\mathcal{T})} [\{\! \{^x_x\} A]$  1(a) gives  $\Gamma', p \vdash_{\text{LK}(\mathcal{T})} [\{\! \{^x_x\} A']$ . We get:

$$\frac{\Gamma', p \vdash [\{\! \{^x_x\} A']}{\Gamma', p \vdash [\exists x A']}$$

(b) The induction hypothesis on  $\Gamma \vdash_{\text{LK}(\mathcal{T})} [\{\! \{^x_x\} A]$  (1(b)) gives  $\Gamma', p^\perp \vdash_{\text{LK}(\mathcal{T})} [\{\! \{^x_x\} A']$  or  $\Gamma', p^\perp \vdash_{\text{LK}(\mathcal{T})}$ .

In the latter case we are done. For the former case, we get:

$$\frac{\frac{\Gamma', p^\perp \vdash [\{\! \{^x_x\} A']}{\Gamma', p^\perp \vdash [\exists x A']}}{\Gamma \vdash A} \\ \Gamma \vdash [A]$$

where  $A$  is Negative .

– Either  $A = q^\perp$  and therefore:

(a)  $A' = p$  and we get:

$$\frac{\mathcal{T}(\text{lit}(\Gamma'), p, p^\perp)}{\Gamma', p \vdash [A']}$$

by the inconsistency of the theory  $\mathcal{T}$ .

(b)  $\Gamma \vdash A$  can only be proved by

$$\frac{\Gamma, q \vdash}{\Gamma \vdash A}$$

The induction hypothesis on  $\Gamma, q \vdash_{\text{LK}(\mathcal{T})}$  (2(b)) gives  $\Gamma', p^\perp, p^\perp \vdash_{\text{LK}(\mathcal{T})}$  and then we get:

$$\frac{\Gamma', p^\perp, p^\perp \vdash}{\Gamma', p^\perp \vdash}$$

by contraction.

– Or  $A \neq q^\perp$  and therefore

- (a) The induction hypothesis on  $\Gamma \vdash_{\text{LK}(\mathcal{T})} A$  (2(a)) gives  $\Gamma', p \vdash_{\text{LK}(\mathcal{T})} A'$  and we get:

$$\frac{\Gamma', p \vdash A'}{\Gamma', p \vdash [A']}$$

- (b) The induction hypothesis on  $\Gamma \vdash_{\text{LK}(\mathcal{T})} A$  (2(b)) gives  $\Gamma', p^\perp \vdash_{\text{LK}(\mathcal{T})} A'$  and we get:

$$\frac{\Gamma', p^\perp \vdash A'}{\Gamma', p^\perp \vdash [A']}$$

•

$$\frac{\mathcal{T}(\text{lit}(\Gamma), p'^\perp)}{\Gamma \vdash [p']}$$

where  $p'$  is a positive literal.

- Either  $p' = q$  and therefore

- (a) We get:

$$\frac{\mathcal{T}(\text{lit}(\Gamma'), p, p)}{\frac{\Gamma', p, p \vdash}{\Gamma', p \vdash [p^\perp]}}$$

We derive  $\mathcal{T}(\text{lit}(\Gamma'), p, p)$  from  $\mathcal{T}(\text{lit}(\Gamma), p'^\perp)$  since  $p$  has the same meaning as  $q^\perp = p'^\perp$  and  $\mathcal{T}$  satisfies the weakening.

- (b) We get:

$$\frac{\mathcal{T}(\text{lit}(\Gamma'), p^\perp, p)}{\frac{\Gamma', p^\perp, p \vdash}{\Gamma', p^\perp \vdash [p^\perp]}}$$

using the fact that  $\mathcal{T}$  satisfies inconsistency.

- or  $p' \neq q$  and therefore

- (a) We get:

$$\frac{\mathcal{T}(\text{lit}(\Gamma'), p, p'^\perp)}{\Gamma', p \vdash [p']}$$

using the fact that  $\mathcal{T}$  satisfies weakening.

- (b) We get:

$$\frac{\mathcal{T}(\text{lit}(\Gamma'), p^\perp, p'^\perp)}{\Gamma', p^\perp \vdash [p']}$$

using the fact that  $\mathcal{T}$  satisfies weakening.

2. For the second item of the lemma, by case analysis on the last rule of the derivation, we get:

•

$$\frac{\Gamma \vdash A_1, \Delta \quad \Gamma \vdash A_2, \Delta}{\Gamma \vdash A_1 \wedge^- A_2, \Delta}$$

for  $\Delta = A_1 \wedge^- A_2, \Delta$

- (a) The induction hypothesis on  $\Gamma \vdash_{\text{LK}(\mathcal{T})} A_1, \Delta$  (2(a)) gives  $\Gamma', p \vdash_{\text{LK}(\mathcal{T})} A'_1, \Delta'$  and the one on  $\Gamma \vdash_{\text{LK}(\mathcal{T})} A_2, \Delta$  (2(a)) gives  $\Gamma', p \vdash_{\text{LK}(\mathcal{T})} A'_2, \Delta'$ .

We get:

$$\frac{\Gamma', p \vdash A'_1, \Delta' \quad \Gamma', p \vdash A'_2, \Delta'}{\Gamma', p \vdash A'_1 \wedge^- A'_2, \Delta'}$$

- (b) The induction hypothesis on  $\Gamma \vdash_{\text{LK}(\mathcal{T})} A_1, \Delta$  (2(b)) gives  $\Gamma', p^\perp \vdash_{\text{LK}(\mathcal{T})} A'_1, \Delta'$  and induction hypothesis on  $\Gamma \vdash_{\text{LK}(\mathcal{T})} A_2, \Delta$  (2(b)) gives  $\Gamma', p^\perp \vdash_{\text{LK}(\mathcal{T})} A'_2, \Delta'$ .

We get:

$$\frac{\Gamma', p^\perp \vdash A'_1, \Delta' \quad \Gamma', p^\perp \vdash A'_2, \Delta'}{\Gamma', p^\perp \vdash A'_1 \wedge^- A'_2, \Delta'}$$

•

$$\frac{\Gamma \vdash A_1, A_2, \Delta}{\Gamma \vdash A_1 \vee^- A_2, \Delta}$$

for  $\Delta = A_1 \vee^- A_2, \Delta$

- (a) The induction hypothesis on  $\Gamma \vdash_{\text{LK}(\mathcal{T})} A_1, A_2, \Delta$  (2(a)) gives  $\Gamma', p \vdash_{\text{LK}(\mathcal{T})} A'_1, A'_2, \Delta'$ .  
We get:

$$\frac{\Gamma', p \vdash A'_1, A'_2, \Delta'}{\Gamma', p \vdash A'_1 \vee^- A'_2, \Delta'}$$

- (b) The induction hypothesis on  $\Gamma \vdash_{\text{LK}(\mathcal{T})} A_1, A_2, \Delta$  (2(b)) gives  $\Gamma', p^\perp \vdash_{\text{LK}(\mathcal{T})} A'_1, A'_2, \Delta'$ .  
We get:

$$\frac{\Gamma', p^\perp \vdash A'_1, A'_2, \Delta'}{\Gamma', p^\perp \vdash A'_1 \vee^- A'_2, \Delta'}$$

•

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash (\forall x A), \Delta} \quad x \notin \text{FV}(\Gamma, \Delta)$$

with  $\Delta = (\forall x A), \Delta$

- (a) The induction hypothesis on  $\Gamma \vdash_{\text{LK}(\mathcal{T})} A, \Delta$  (2(a)) gives  $\Gamma', p \vdash_{\text{LK}(\mathcal{T})} A', \Delta'$ .  
We get:

$$\frac{\Gamma \vdash A', \Delta'}{\Gamma' \vdash (\forall x A'), \Delta'} \quad x \notin \text{FV}(\Gamma', \Delta')$$

- (b) The induction hypothesis on  $\Gamma \vdash_{\text{LK}(\mathcal{T})} A, \Delta$  (2(b)) gives  $\Gamma', p^\perp \vdash_{\text{LK}(\mathcal{T})} A', \Delta'$ .  
We get:

$$\frac{\Gamma', p^\perp \vdash A', \Delta'}{\Gamma', p^\perp \vdash (\forall x A'), \Delta'} \quad x \notin \text{FV}(\Gamma', \Delta')$$

•

$$\frac{\Gamma, A^\perp \vdash \Delta}{\Gamma \vdash A, \Delta}$$

for  $\Delta = A, \Delta$  where  $A$  is positive or negative literal.

- (a) The induction hypothesis on  $\Gamma, A^\perp \vdash_{\text{LK}(\mathcal{T})} \Delta$  (2(a)) gives  $\Gamma', p, A'^\perp \vdash_{\text{LK}(\mathcal{T})} \Delta'$   
and we get:

$$\frac{\Gamma', p, A'^\perp \vdash \Delta'}{\Gamma', p \vdash A', \Delta'}$$

- (b) The induction hypothesis on  $\Gamma, A \vdash_{\text{LK}(\mathcal{T})} \Delta$  (2(b)) gives  $\Gamma', p^\perp, A'^\perp \vdash_{\text{LK}(\mathcal{T})} \Delta'$   
and we get:

$$\frac{\Gamma', p^\perp, A'^\perp \vdash \Delta'}{\Gamma', p^\perp \vdash A', \Delta'}$$

•

$$\frac{\Gamma \vdash [A]}{\Gamma \vdash}$$

with  $A$  is positive and  $A^\perp \in \Gamma$ .

– Either  $A = q$ , and then  $q^\perp \in \Gamma$  and  $\mathcal{T}(\text{lit}(\Gamma), q^\perp)$

- (a) We get:

$$\frac{\mathcal{T}(\text{lit}(\Gamma'), p)}{\Gamma', p \vdash}$$

since  $p$  and  $q^\perp$  have the same meaning for  $\mathcal{T}$

- (b) We get:

$$\frac{\mathcal{T}(\text{lit}(\Gamma'), p^\perp)}{\Gamma', p^\perp \vdash}$$

since  $p \in \Gamma'$  and by using the fact that  $\mathcal{T}$  is inconsistent.

– or  $A \neq q$

(a) The induction hypothesis on  $\Gamma \vdash_{\text{LK}(\mathcal{T})} [A]$  gives  $\Gamma', p \vdash_{\text{LK}(\mathcal{T})} [A']$  and we get

$$\frac{\Gamma', p \vdash [A']}{\Gamma', p \vdash}$$

(b) The induction hypothesis on  $\Gamma \vdash_{\text{LK}(\mathcal{T})} [A]$  gives either  $\Gamma', p^\perp \vdash_{\text{LK}(\mathcal{T})} [A']$  or  $\Gamma', p^\perp \vdash_{\text{LK}(\mathcal{T})}$ . For the latter case we are done and for the former case we get:

$$\frac{\Gamma', p^\perp \vdash [A']}{\Gamma', p^\perp \vdash}$$

□

**Corollary 8 (Changing the polarity of a predicate)** *If  $\Gamma \vdash_{\text{LK}(\mathcal{T})} \Delta$  then  $\Gamma' \vdash_{\text{LK}(\mathcal{T})} \Delta'$ .*

**Proof:** Lemma 7 (2) provides  $\Gamma', p \vdash_{\text{LK}(\mathcal{T})} \Delta'$  and  $\Gamma', p^\perp \vdash_{\text{LK}(\mathcal{T})} \Delta'$ . Then we can construct:

$$\frac{\Gamma', p \vdash \Delta' \quad \Gamma', p^\perp \vdash \Delta'}{\Gamma' \vdash \Delta'} \text{ cuts}$$

and we use the admissibility of  $\text{cuts}$ . □

We have proven that changing the polarities of the predicate that are present in a sequent, does not change the provability of that sequent in  $\text{LK}(\mathcal{T})$ .

## 5.2 Changing the polarity of connectives

**Definition 6 ( $\text{LK}^+(\mathcal{T})$ )** The sequent calculus  $\text{LK}^+(\mathcal{T})$  manipulates one kind of sequent:

$$\Gamma \vdash [\mathcal{X}] \Delta \text{ where } \mathcal{X} :: \bullet \mid A$$

Here  $\Gamma$  is a multiset of negative formulae and positive literals,  $\Delta$  is a multiset of formulae,  $\mathcal{X}$  is said to be in the *focus* of the (focused) system. By  $\text{lit}(\Gamma)$  we denote the sub-multiset of  $\Gamma$  consisting of its literals and  $\text{lit}(\Delta)$  we denote the sub-multiset of  $\Delta$  consisting of its literals.

The rules of  $\text{LK}^+(\mathcal{T})$ , given in Figure 2, are of three kinds: synchronous rules, asynchronous rules, and structural rules. These correspond to three alternating phases in the proof-search process that is described by the rules.

The  $\text{LK}^+(\mathcal{T})$  system is an extension system of  $\text{LK}(\mathcal{T})$ .

**Remark 9** As in  $\text{LK}(\mathcal{T})$ , weakening and contraction are admissible in  $\text{LK}^+(\mathcal{T})$ .

**Lemma 10 (Identities)**  $\vdash [A^\perp]A$  is provable in  $\text{LK}^+(\mathcal{T})$ .

**Proof:** By induction on  $A$  using an extended but well founded order on formulae: a formula is smaller than another one when either it contains fewer connectives, or the number of connectives is equal the former formula is negative and the latter is positive.

- $A = A_1 \wedge^- A_2$

$$\frac{\frac{\vdash [A_1^\perp]A_1}{\vdash [A_1^\perp \vee^+ A_2^\perp]A_1} \quad \frac{\vdash [A_2^\perp]A_2}{\vdash [A_1^\perp \vee^+ A_2^\perp]A_2}}{\vdash [A_1^\perp \vee^+ A_2^\perp]A_1 \wedge^- A_2}$$

We can complete the proof on the left-hand side by applying the induction hypothesis on  $A_1$  and on the right-hand side by applying the induction hypothesis on  $A_2$ .

- $A = A_1 \vee^- A_2$

$$\frac{\frac{\vdash [A_1^\perp]A_1}{\vdash [A_1^\perp]A_1, A_2} \text{ WK} \quad \frac{\vdash [A_2^\perp]A_2}{\vdash [A_2^\perp]A_1, A_2} \text{ WK}}{\frac{\vdash [A_1^\perp \wedge^+ A_2^\perp]A_1, A_2}{\vdash [A_1^\perp \wedge^+ A_2^\perp]A_1 \vee^- A_2}}$$

We can complete the proof on the left-hand side by applying the induction hypothesis on  $A_1$  and on the right-hand side by applying the induction hypothesis on  $A_2$ .

Synchronous rules		
$\frac{\Gamma \vdash [A]\Delta \quad \Gamma \vdash [B]\Delta}{\Gamma \vdash [A \wedge^+ B]\Delta}$	$\frac{\Gamma \vdash [A_i]\Delta}{\Gamma \vdash [A_1 \vee^+ A_2]\Delta}$	$\frac{\Gamma \vdash [\{t/x\}A]\Delta}{\Gamma \vdash [\exists x A]\Delta}$
$\frac{}{\Gamma, p \vdash [p]\Delta} p \text{ positive literal}$	$\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp, \text{lit}(\Delta))}{\Gamma \vdash [p]\Delta} p \text{ positive literal}$	
$\frac{\Gamma \vdash [\bullet]N}{\Gamma \vdash [N]} N \text{ negative}$		
Asynchronous rules		
$\frac{\Gamma \vdash [\mathcal{X}]A, \Delta \quad \Gamma \vdash [\mathcal{X}]B, \Delta}{\Gamma \vdash [\mathcal{X}]A \wedge^- B, \Delta}$	$\frac{\Gamma \vdash [\mathcal{X}]A_1, A_2, \Delta}{\Gamma \vdash [\mathcal{X}]A_1 \vee^- A_2, \Delta}$	$\frac{\Gamma \vdash [\mathcal{X}]A, \Delta}{\Gamma \vdash [\mathcal{X}](\forall x A), \Delta} x \notin \text{FV}(\Gamma, \mathcal{X}, \Delta)$
$\frac{\Gamma, A^\perp \vdash [\mathcal{X}]\Delta}{\Gamma \vdash [\mathcal{X}]A, \Delta} A \text{ positive or literal}$		
Structural rules		
$\frac{\Gamma, P^\perp \vdash [P]\Delta}{\Gamma, P^\perp \vdash [\bullet]\Delta} P \text{ positive}$	$\frac{\mathcal{T}(\text{lit}(\Gamma), \text{lit}(\Delta))}{\Gamma \vdash [\bullet]\Delta}$	

Figure 2: System  $\text{LK}^+(\mathcal{T})$

- $A = \forall x A$

$$\frac{\frac{\frac{\vdash [A^\perp]A}{\vdash [\{t/x\}A^\perp]A} \text{choosing } t=x}{\vdash [\exists x A^\perp]A}}{\vdash [\exists x A^\perp]\forall x A} x \notin \text{FV}(\exists x A^\perp)$$

We can complete the proof by applying the induction hypothesis on  $A$ .

- $A = p^\perp$

$$\frac{\overline{p \vdash [p]}}{\vdash [p]p^\perp}$$

- $A = P$  where  $P$  is all positive formulae:

$$\frac{\frac{\frac{\vdash [P]P^\perp}{P^\perp \vdash [P]P^\perp} WK}{P^\perp \vdash [\bullet]P^\perp}}{P^\perp \vdash [P^\perp]} \frac{}{\vdash [P^\perp]P}$$

We can complete the proof by applying the induction hypothesis on  $P$ .

□

**Lemma 11 (Invertibility of asynchronous rules)**

*All asynchronous rules are height-preserving invertible in  $\text{LK}^+(\mathcal{T})$ .*

**Proof:** By induction on the derivation proving the conclusion of the asynchronous rule considered.

- Inversion of  $A\wedge^-B$ : by case analysis on the last rule actually used

$$\frac{\Gamma \vdash [\mathcal{X}]A\wedge^-B, C, \Delta' \quad \Gamma \vdash [\mathcal{X}]A\wedge^-B, D, \Delta'}{\Gamma \vdash [\mathcal{X}]A\wedge^-B, C\wedge^-D, \Delta'}$$

By induction hypothesis we get

$$\frac{\Gamma \vdash [\mathcal{X}]A, C, \Delta' \quad \Gamma \vdash [\mathcal{X}]A, D, \Delta'}{\Gamma \vdash [\mathcal{X}]A, C\wedge^-D, \Delta'} \quad \text{and} \quad \frac{\Gamma \vdash [\mathcal{X}]B, C, \Delta' \quad \Gamma \vdash [\mathcal{X}]B, D, \Delta'}{\Gamma \vdash [\mathcal{X}]B, C\wedge^-D, \Delta'}$$

$$\frac{\Gamma \vdash [\mathcal{X}]A\wedge^-B, C, D, \Delta'}{\Gamma \vdash [\mathcal{X}]A\wedge^-B, C\vee^-D, \Delta'}$$

By induction hypothesis we get:  $\frac{\Gamma \vdash [\mathcal{X}]A, C, D, \Delta'}{\Gamma \vdash [\mathcal{X}]A, C\vee^-D, \Delta'}$  and  $\frac{\Gamma \vdash [\mathcal{X}]B, C, D, \Delta'}{\Gamma \vdash [\mathcal{X}]B, C\vee^-D, \Delta'}$

$$\frac{\Gamma \vdash [\mathcal{X}]A\wedge^-B, C, \Delta'}{\Gamma \vdash [\mathcal{X}]A\wedge^-B, (\forall xC), \Delta'} \quad x \notin \text{FV}(\Gamma, \mathcal{X}, \Delta', A\wedge^-B)$$

By induction hypothesis we get:

$$\frac{\Gamma \vdash [\mathcal{X}]A, C, \Delta'}{\Gamma \vdash [\mathcal{X}]A, (\forall xC), \Delta'} \quad x \notin \text{FV}(\Gamma, \mathcal{X}, \Delta', A) \quad \text{and} \quad \frac{\Gamma \vdash [\mathcal{X}]B, C, \Delta'}{\Gamma \vdash [\mathcal{X}]B, (\forall xC), \Delta'} \quad x \notin \text{FV}(\Gamma, \mathcal{X}, \Delta', B)$$

$$\frac{\Gamma, C^\perp \vdash [\mathcal{X}]A\wedge^-B, \Delta'}{\Gamma \vdash [\mathcal{X}]A\wedge^-B, C, \Delta'} \quad C \text{ positive or literal}$$

By induction hypothesis we get:

$$\frac{\Gamma, C^\perp \vdash [\mathcal{X}]A, \Delta'}{\Gamma \vdash [\mathcal{X}]A, C, \Delta'} \quad C \text{ positive or literal} \quad \text{and} \quad \frac{\Gamma, C^\perp \vdash [\mathcal{X}]B, \Delta'}{\Gamma \vdash [\mathcal{X}]B, C, \Delta'} \quad C \text{ positive or literal}$$

$$\frac{\Gamma \vdash [C]A\wedge^-B, \Delta' \quad \Gamma \vdash [D]A\wedge^-B, \Delta'}{\Gamma \vdash [C\wedge^+D, ]A\wedge^-B, \Delta'}$$

By induction hypothesis we get

$$\frac{\Gamma \vdash [C]A, \Delta' \quad \Gamma \vdash [D]A, \Delta'}{\Gamma \vdash [C\wedge^+D]A, \Delta'} \quad \text{and} \quad \frac{\Gamma \vdash [C]B, \Delta' \quad \Gamma \vdash [D]B, \Delta'}{\Gamma \vdash [C\wedge^+D]B, \Delta'}$$

$$\frac{\Gamma \vdash [C_1\vee^+C_2]A\wedge^-B, \Delta'}{\Gamma \vdash [C_1\vee^+C_2]A\wedge^-B, \Delta'}$$

By induction hypothesis we get

$$\frac{\Gamma \vdash [C_i]A, \Delta' \quad \Gamma \vdash [C_i]B, \Delta'}{\Gamma \vdash [C_1\vee^+C_2]A, \Delta' \quad \Gamma \vdash [C_1\vee^+C_2]B, \Delta'} \quad \text{and} \quad \frac{\Gamma \vdash [\{x\}^t]C]A\wedge^-B, \Delta'}{\Gamma \vdash [\exists xC]A\wedge^-B, \Delta'}$$

$$\frac{\Gamma \vdash [\exists xC]A\wedge^-B, \Delta'}{\Gamma \vdash [\exists xC]A\wedge^-B, \Delta'}$$

By induction hypothesis we get

$$\frac{\Gamma \vdash [\{x\}^t]C]A, \Delta' \quad \Gamma \vdash [\{x\}^t]C]B, \Delta'}{\Gamma \vdash [\exists xC]A, \Delta' \quad \Gamma \vdash [\exists xC]B, \Delta'}$$

- Inversion of  $A\vee^-B$

$$\frac{\Gamma \vdash [\mathcal{X}]A\vee^-B, C, \Delta' \quad \Gamma \vdash [\mathcal{X}]A\vee^-B, D, \Delta'}{\Gamma \vdash [\mathcal{X}]A\vee^-B, C\wedge^-D, \Delta'}$$

By induction hypothesis we get  $\frac{\Gamma \vdash [\mathcal{X}]A, B, C, \Delta' \quad \Gamma \vdash [\mathcal{X}]A, B, D, \Delta'}{\Gamma \vdash [\mathcal{X}]A, B, C\wedge^-D, \Delta'}$

$$\frac{\Gamma \vdash [\mathcal{X}]A\vee^-B, C, D, \Delta'}{\Gamma \vdash [\mathcal{X}]A\vee^-B, C\vee^-D, \Delta'}$$

By induction hypothesis we get  $\frac{\Gamma \vdash [\mathcal{X}]A, B, C, D, \Delta'}{\Gamma \vdash [\mathcal{X}]A, B, C\vee^-D, \Delta'}$



$$\begin{array}{l}
- \frac{\Gamma \vdash [\mathcal{X}]AV^-B, C, \Delta'}{\Gamma \vdash [\mathcal{X}]AV^-B, (\forall xC), \Delta'} x \notin \text{FV}(\Gamma, \mathcal{X}, \Delta') \\
\text{By induction hypothesis we get } \frac{\Gamma \vdash [\mathcal{X}]A, B, C, \Delta'}{\Gamma \vdash [\mathcal{X}], A, B, (\forall xC), \Delta'} x \notin \text{FV}(\Gamma, \mathcal{X}, \Delta') \\
- \frac{\Gamma, C^\perp \vdash [\mathcal{X}], AV^-B, \Delta'}{\Gamma \vdash [\mathcal{X}], AV^-B, C, \Delta'} C \text{ positive or literal} \\
\text{By induction hypothesis we get } \frac{\Gamma, C^\perp \vdash [\mathcal{X}], A, B, \Delta'}{\Gamma \vdash [\mathcal{X}], A, B, C, \Delta'} C \text{ positive or literal} \\
- \frac{\Gamma \vdash [C]AV^-B, \Delta' \quad \Gamma \vdash [D]AV^-B, \Delta'}{\Gamma \vdash [C \wedge^+ D]AV^-B, \Delta'} \\
\text{By induction hypothesis we get } \frac{\Gamma \vdash [C]A, B, \Delta' \quad \Gamma \vdash [D]A, B, \Delta'}{\Gamma \vdash [C \wedge^+ D]A, B, C \wedge^- D, \Delta'} \\
- \frac{\Gamma \vdash [C_i]AV^-B, \Delta'}{\Gamma \vdash [C_1 \vee^+ C_2]AV^-B, \Delta'} \\
\text{By induction hypothesis we get } \frac{\Gamma \vdash [C_i]A, B, \Delta'}{\Gamma \vdash [C_1 \vee^+ C_2]A, B, \Delta'} \\
- \frac{\Gamma \vdash [\{x\}_x]C]AV^-B, \Delta'}{\Gamma \vdash [\exists xC]AV^-B, \Delta'} \\
\text{By induction hypothesis we get } \frac{\Gamma \vdash [\{x\}_x]C]A, B, \Delta'}{\Gamma \vdash [\exists xC]A, B, \Delta'} \\
\bullet \text{ Inversion of } \forall xA \\
- \frac{\Gamma \vdash [\mathcal{X}], (\forall xA), C, \Delta' \quad \Gamma \vdash [\mathcal{X}], (\forall xA), D, \Delta'}{\Gamma \vdash [\mathcal{X}], (\forall xA), C \wedge^- D, \Delta'} \\
\text{By induction hypothesis we get } \frac{\Gamma \vdash [\mathcal{X}], A, C, \Delta' \quad \Gamma \vdash [\mathcal{X}], A, D, \Delta'}{\Gamma \vdash [\mathcal{X}], A, C \wedge^- D, \Delta'} x \notin \text{FV}(\Gamma, \mathcal{X}, \Delta') \\
- \frac{\Gamma \vdash [\mathcal{X}], (\forall xA), C, D, \Delta'}{\Gamma \vdash [\mathcal{X}], (\forall xA), C \vee^- D, \Delta'} \\
\text{By induction hypothesis we get } \frac{\Gamma \vdash [\mathcal{X}], A, C, D, \Delta'}{\Gamma \vdash [\mathcal{X}], A, C \vee^- D, \Delta'} \\
- \frac{\Gamma \vdash [\mathcal{X}], (\forall xA), D, \Delta'}{\Gamma \vdash [\mathcal{X}], (\forall xA), (\forall xD), \Delta'} x \notin \text{FV}(\Gamma, \mathcal{X}, \Delta') \\
\text{By induction hypothesis we get } \frac{\Gamma \vdash [\mathcal{X}], A, D, \Delta'}{\Gamma \vdash [\mathcal{X}], A, (\forall xD), \Delta'} x \notin \text{FV}(\Gamma, \mathcal{X}, \Delta') \\
- \frac{\Gamma, C^\perp \vdash [\mathcal{X}], (\forall xA), \Delta'}{\Gamma \vdash [\mathcal{X}], (\forall xA), C, \Delta'} C \text{ positive or literal} \\
\text{By induction hypothesis we get } \frac{\Gamma, C^\perp \vdash [\mathcal{X}], A, \Delta'}{\Gamma \vdash [\mathcal{X}], A, C, \Delta'} C \text{ positive or literal} \\
- \frac{\Gamma \vdash [C], (\forall xA), \Delta' \quad \Gamma \vdash [D], (\forall xA), \Delta'}{\Gamma \vdash [C \wedge^+ D](\forall xA), \Delta'} \\
\text{By induction hypothesis we get } \frac{\Gamma \vdash [C], A, \Delta' \quad \Gamma \vdash [D], A, \Delta'}{\Gamma \vdash [C \wedge^+ D]A, \Delta'} x \notin \text{FV}(\Gamma, C \wedge^+ D, \Delta') \\
- \frac{\Gamma \vdash [C_i], (\forall xA), \Delta'}{\Gamma \vdash [C_1 \vee^+ C_2], (\forall xA), \Delta'} \\
\text{By induction hypothesis we get } \frac{\Gamma \vdash [C_i], A, \Delta'}{\Gamma \vdash [C_1 \vee^+ C_2]A, \Delta'}
\end{array}$$

$$\begin{array}{l}
- \frac{\Gamma \vdash \{\check{x}\}D(\forall xA), \Delta'}{\Gamma \vdash [\exists xD], (\forall xA), \Delta'} x \notin \text{FV}(\Gamma, \exists xD, \Delta') \\
\text{By induction hypothesis we get } \frac{\Gamma \vdash \{\check{x}\}D, A, \Delta'}{\Gamma \vdash [\exists xD], A, \Delta'} x \notin \text{FV}(\Gamma, \exists xD, \Delta') \\
\bullet \text{ Inversion of literals and positive formulae (A)} \\
- \frac{\Gamma \vdash [\mathcal{X}], A, C, \Delta' \quad \Gamma \vdash [\mathcal{X}], A, D, \Delta'}{\Gamma \vdash [\mathcal{X}], A, C \wedge D, \Delta'} \\
\text{By induction hypothesis we get } \frac{\Gamma, A^\perp \vdash [\mathcal{X}], C, \Delta' \quad \Gamma, A^\perp \vdash [\mathcal{X}], D, \Delta'}{\Gamma, A^\perp \vdash [\mathcal{X}], C \wedge D, \Delta'} \\
- \frac{\Gamma \vdash [\mathcal{X}], A, C, D, \Delta'}{\Gamma \vdash [\mathcal{X}], A, C \vee D, \Delta'} \\
\text{By induction hypothesis } \frac{\Gamma, A^\perp \vdash [\mathcal{X}], C, D, \Delta'}{\Gamma, A^\perp \vdash [\mathcal{X}], C \vee D, \Delta'} \\
- \frac{\Gamma \vdash [\mathcal{X}], A, D, \Delta'}{\Gamma \vdash [\mathcal{X}], A, (\forall xD), \Delta'} x \notin \text{FV}(\Gamma, \mathcal{X}, \Delta') \\
\text{By induction hypothesis we get } \frac{\Gamma, A^\perp \vdash [\mathcal{X}], D, \Delta'}{\Gamma, A^\perp \vdash [\mathcal{X}], (\forall xD), \Delta'} x \notin \text{FV}(\Gamma, \mathcal{X}, \Delta') \\
- \frac{\Gamma, B^\perp \vdash [\mathcal{X}], A, \Delta'}{\Gamma \vdash [\mathcal{X}], A, B, \Delta'} B \text{ positive or literal} \\
\text{By induction hypothesis we get } \frac{\Gamma, A^\perp, B^\perp \vdash [\mathcal{X}], \Delta'}{\Gamma, A^\perp \vdash [\mathcal{X}], B, \Delta'} B \text{ positive or literal} \\
- \frac{\Gamma \vdash [C], A, \Delta' \quad \Gamma \vdash [D], A, \Delta'}{\Gamma \vdash [C \wedge D], A, \Delta'} \\
\text{By induction hypothesis we get } \frac{\Gamma, A^\perp \vdash [C], \Delta' \quad \Gamma, A^\perp \vdash [D], \Delta'}{\Gamma, A^\perp \vdash [C \wedge D], \Delta'} \\
- \frac{\Gamma \vdash [C_i], A, \Delta'}{\Gamma \vdash [C_1 \vee^+ C_2], A, \Delta'} \\
\text{By induction hypothesis } \frac{\Gamma, A^\perp \vdash [C_i], \Delta'}{\Gamma, A^\perp \vdash [C_1 \vee^+ C_2], \Delta'} \\
- \frac{\Gamma \vdash \{\check{x}\}D, A, \Delta'}{\Gamma \vdash [\exists xD], A, \Delta'} \\
\text{By induction hypothesis we get } \frac{\Gamma, A^\perp \vdash \{\check{x}\}D, \Delta'}{\Gamma, A^\perp \vdash [\exists xD], \Delta'}
\end{array}$$

□

**Lemma 12 (Encoding  $\text{LK}^+(\mathcal{T})$  in  $\text{LK}(\mathcal{T})$ )**

1. If  $\Gamma \vdash [A]$  is provable in  $\text{LK}^+(\mathcal{T})$ , then  $\Gamma \vdash [A]$  is provable in  $\text{LK}(\mathcal{T})$ .
2. If  $\Gamma \vdash [\bullet]\Delta$  is provable in  $\text{LK}^+(\mathcal{T})$ , then  $\Gamma \vdash \Delta$  is provable in  $\text{LK}(\mathcal{T})$ .

**Proof:** By simultaneous induction on the assumed derivation.

1. For the first item we get, by case analysis on the last rule of the derivation:

•

$$\frac{\Gamma \vdash [A_1] \quad \Gamma \vdash [A_2]}{\Gamma \vdash [A_1 \wedge^+ A_2]}$$

with  $A = A_1 \wedge^+ A_2$ . The induction hypothesis on  $\Gamma \vdash_{\text{LK}^+(\mathcal{T})} [A_1]$  gives  $\Gamma \vdash_{\text{LK}(\mathcal{T})} [A_1]$

and the induction hypothesis on  $\Gamma \vdash_{\text{LK}^+(\mathcal{T})} [A_2]$  gives  $\Gamma \vdash_{\text{LK}(\mathcal{T})} [A_2]$ . We get:

$$\frac{\Gamma \vdash [A_1] \quad \Gamma \vdash [A_2]}{\Gamma \vdash [A_1 \wedge^+ A_2]}$$

•

$$\frac{\Gamma \vdash [A_i]}{\Gamma \vdash [A_1 \vee^+ A_2]}$$

with  $A = A_1 \vee^+ A_2$ .

The induction hypothesis on  $\Gamma \vdash_{\text{LK}^+(\mathcal{T})} [A_i]$  gives  $\Gamma \vdash_{\text{LK}(\mathcal{T})} [A_i]$ . We get:

$$\frac{\Gamma \vdash [A_i]}{\Gamma \vdash [A_1 \vee^+ A_2]}$$

•

$$\frac{\Gamma \vdash [\{t/x\}A]}{\Gamma \vdash [\exists x A]}$$

with  $A = \exists x A$ .

The induction hypothesis on  $\Gamma \vdash_{\text{LK}^+(\mathcal{T})} [\{t/x\}A]$  gives  $\Gamma \vdash_{\text{LK}(\mathcal{T})} [\{t/x\}A]$ . We get:

$$\frac{\Gamma \vdash [\{t/x\}A]}{\Gamma \vdash [\exists x A]}$$

•

$$\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp)}{\Gamma \vdash [p]}$$

with  $A = p$  where  $p$  is a positive literal. We can perform the same step in  $\text{LK}(\mathcal{T})$ :

$$\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp)}{\Gamma \vdash [p]}$$

•

$$\frac{\Gamma \vdash [\bullet]N}{\Gamma \vdash [N]}$$

with  $A = N$  and  $N$  is negative.

The induction hypothesis on  $\Gamma \vdash_{\text{LK}^+(\mathcal{T})} [\bullet]N$  gives  $\Gamma \vdash_{\text{LK}(\mathcal{T})} N$ . We get:

$$\frac{\Gamma \vdash N}{\Gamma \vdash [N]}$$

2. For the second item, we use the height-preserving invertibility of the asynchronous rules, so that we can assume without loss of generality that if  $\Delta$  is not empty then the last rule of the derivation decomposes one of its formulae.

•

$$\frac{\Gamma \vdash [\bullet]A_1, \Delta_1 \quad \Gamma \vdash [\bullet]A_2, \Delta_1}{\Gamma \vdash [\bullet]A_1 \wedge^- A_2, \Delta_1}$$

with  $\Delta = A_1 \wedge^- A_2, \Delta_1$ . The induction hypothesis on  $\Gamma \vdash_{\text{LK}^+(\mathcal{T})} [\bullet]A_1, \Delta_1$  gives  $\Gamma \vdash_{\text{LK}(\mathcal{T})} A_1, \Delta_1$  and the induction hypothesis on  $\Gamma \vdash_{\text{LK}^+(\mathcal{T})} [\bullet]A_2, \Delta_2$  gives  $\Gamma \vdash_{\text{LK}(\mathcal{T})} A_2, \Delta_2$ . We get:

$$\frac{\Gamma \vdash A_1, \Delta_1 \quad \Gamma \vdash A_2, \Delta_1}{\Gamma \vdash A_1 \wedge^- A_2, \Delta_1}$$

•

$$\frac{\Gamma \vdash [\bullet]A_1, A_2, \Delta_1}{\Gamma \vdash [\bullet]A_1 \vee^- A_2, \Delta_1}$$

with  $\Delta = A_1 \vee^- A_2, \Delta_1$ . The induction hypothesis on  $\Gamma \vdash_{\text{LK}^+(\mathcal{T})} [\bullet]A_1, A_2 \Delta_1$  gives  $\Gamma \vdash_{\text{LK}(\mathcal{T})} A_1, A_2 \Delta_1$  and we get:

$$\frac{\Gamma \vdash A_1, A_2, \Delta_1}{\Gamma \vdash A_1 \vee^- A_2, \Delta_1}$$

•

$$\frac{\Gamma \vdash [\bullet]A, \Delta_1}{\Gamma \vdash [\bullet]\forall x A, \Delta_1} \quad x \notin \text{FV}(\Gamma, \Delta_1)$$

with  $\Delta = \forall x A, \Delta_1$ .

The induction hypothesis on  $\Gamma \vdash_{\text{LK}^+(\mathcal{T})} [\bullet]A, \Delta_1$  gives  $\Gamma \vdash_{\text{LK}(\mathcal{T})} A, \Delta_1$ . We get:

$$\frac{\Gamma \vdash A, \Delta_1}{\Gamma \vdash \forall x A, \Delta_1} \quad x \notin \text{FV}(\Gamma, \Delta_1)$$

•

$$\frac{\Gamma, A^\perp \vdash [\bullet]\Delta_1}{\Gamma \vdash [\bullet]A, \Delta_1}$$

with  $\Delta = A, \Delta_1$  and  $A$  is a positive or literal.

The induction hypothesis on  $\Gamma, A^\perp \vdash_{\text{LK}^+(\mathcal{T})} [\bullet]\Delta_1$  gives  $\Gamma, A^\perp \vdash_{\text{LK}(\mathcal{T})} \Delta_1$ . We get:

$$\frac{\Gamma, A^\perp \vdash \Delta_1}{\Gamma \vdash A, \Delta_1}$$

•

$$\frac{\Gamma, P^\perp \vdash [P]\Delta}{\Gamma, P^\perp \vdash [\bullet]\Delta}$$

where  $P$  is positive.

As already mentioned, we can assume without loss of generality that  $\Delta$  is empty. The induction hypothesis on  $\Gamma, P^\perp \vdash_{\text{LK}^+(\mathcal{T})} [P](1)$  gives  $\Gamma, P^\perp \vdash_{\text{LK}(\mathcal{T})} [P]$ . We get:

$$\frac{\Gamma, P^\perp \vdash [P]}{\Gamma, P^\perp \vdash}$$

•

$$\frac{\mathcal{T}(\text{lit}(\Gamma), \text{lit}(\Delta))}{\Gamma \vdash [\bullet]\Delta}$$

As already mentioned, we can assume without loss of generality that  $\Delta$  is empty. We get:

$$\frac{\mathcal{T}(\text{lit}(\Gamma))}{\Gamma \vdash}$$

□

**Lemma 13** *We have:*

$$1. \vdash_{\text{LK}(\mathcal{T})} (A \wedge^+ B)^\perp, (A \wedge^- B)$$

$$2. \vdash_{\text{LK}(\mathcal{T})} (A \wedge^- B)^\perp, (A \wedge^+ B).$$

**Proof:**

1. For the first item we get:

$$\begin{array}{c}
\frac{\frac{\frac{\vdash [A^\perp]A}{A \vdash [A^\perp]A} \text{WK}}{A \vdash [\bullet]A}}{\vdash [\bullet]A^\perp, A} \text{WK} \quad \frac{\frac{\frac{\vdash [B^\perp]B}{B \vdash [B^\perp]B} \text{WK}}{B \vdash [\bullet]B}}{\vdash [\bullet]B^\perp, B} \text{WK}}{\vdash [\bullet](A^\perp \vee^- B^\perp), (A \wedge^- B)} \\
\frac{\vdash [\bullet](A \wedge^+ B)^\perp, (A \wedge^- B)}{\vdash (A \wedge^+ B)^\perp, (A \wedge^- B)} \text{Lemma 12(2)}
\end{array}$$

Both left hand side and right hand side can be closed by Lemma 10.

2. For the second item, we get:

$$\begin{array}{c}
\frac{\frac{\frac{\vdash [A^\perp]A}{A \wedge^- B \vdash [A^\perp]A} \text{WK}}{A \wedge^- B \vdash [\bullet]A} \text{Lemma 12(2)} \quad \frac{\frac{\frac{\frac{\vdash [B^\perp]B}{A \wedge^- B \vdash [A^\perp \vee^+ B^\perp]B} \text{WK}}{A \wedge^- B \vdash [\bullet]B} \text{Lemma 12(2)}}{A \wedge^- B \vdash A^\perp, B} \text{WK}}{(A \wedge^- B), (A^\perp \vee^- B^\perp) \vdash A^\perp, B} \text{WK}}{(A \wedge^- B), (A^\perp \vee^- B^\perp) \vdash A} \text{WK} \quad \frac{\frac{\frac{\frac{\frac{\vdash [A]A^\perp}{\vdash [A]A^\perp, B^\perp} \text{WK}}{\vdash [A \wedge^+ B]A^\perp, B^\perp} \text{WK}}{A^\perp \vee^- B^\perp \vdash [A \wedge^+ B]A^\perp, B^\perp} \text{WK}}{A^\perp \vee^- B^\perp \vdash [\bullet]A^\perp, B^\perp} \text{Lemma 12(2)}}{(A \wedge^- B), (A^\perp \vee^- B^\perp) \vdash A^\perp, B^\perp} \text{WK}}{(A \wedge^- B), (A^\perp \vee^- B^\perp) \vdash A^\perp} \text{cut}_7} \\
\frac{(A \wedge^- B), (A^\perp \vee^- B^\perp) \vdash A^\perp}{(A \wedge^- B), (A \wedge^+ B)^\perp \vdash} \text{cut}_7 \\
\frac{(A \wedge^- B), (A \wedge^+ B)^\perp \vdash}{\vdash (A \wedge^- B)^\perp, (A \wedge^+ B)}
\end{array}$$

All branches are closed by Lemma 10.  $\square$

**Lemma 14** *If  $\Gamma \vdash_{LK(\mathcal{T})} \Delta, C$  and  $\Gamma \vdash_{LK(\mathcal{T})} D, C^\perp$  then  $\Gamma \vdash_{LK(\mathcal{T})} \Delta, D$ .*

**Proof:**

$$\frac{\frac{\frac{\Gamma \vdash \Delta, C}{\Gamma \vdash D, \Delta, C} \text{WK}}{\Gamma \vdash \Delta, D} \text{cut}_7 \quad \frac{\frac{\Gamma \vdash D, C^\perp}{\Gamma \vdash \Delta, D, C^\perp} \text{WK}}{\Gamma \vdash \Delta, D} \text{cut}_7}$$

$\square$

**Corollary 15 (Changing the polarity of connectives)**

1. *If  $\Gamma \vdash A \wedge^+ B, \Delta$  then  $\Gamma \vdash A \wedge^- B, \Delta$ . **Proof:** By Lemma 14 and Lemma 13(1).  $\square$*
2. *If  $\Gamma \vdash A \wedge^- B, \Delta$  then  $\Gamma \vdash A \wedge^+ B, \Delta$ . **Proof:** By Lemma 14 and Lemma 13(2).  $\square$*
3. *If  $\Gamma \vdash A \vee^+ B, \Delta$  then  $\Gamma \vdash A \vee^- B, \Delta$ . **Proof:** By Lemma 14 and Lemma 13(1).  $\square$*
4. *If  $\Gamma \vdash A \vee^- B, \Delta$  then  $\Gamma \vdash A \vee^+ B, \Delta$ . **Proof:** By Lemma 14 and Lemma 13(2).  $\square$*

We have proven that changing the polarities of the connectives that are present in a sequent, does not change the provability of that sequent in  $LK(\mathcal{T})$ .

## 6 Completeness

**Definition 7 (Formulae)**

Let  $\mathcal{A}$  be a set, whose elements will be called *atoms*.

The formulae of first-order logic are given by the following grammar:

Formulae  $A, B, \dots ::= a \mid A \vee B \mid A \wedge B \mid \forall x A \mid \exists x A \mid \neg A$

where  $a$  ranges over atoms.

**Definition 8** ( $\psi$  and  $\bar{\psi}$ ) Let  $\psi$  be a function that maps every atom to a polarised literal that has the same meaning for  $\mathcal{T}$ .

Let  $\bar{\psi}$  be the function that maps every formula of first-order logic to a set of polarised formulae defined as follows:

$$\begin{aligned} \bar{\psi}(a) &:= \{\psi(a)\} \\ \bar{\psi}(A \wedge B) &:= \{A' \wedge^- B', A' \wedge^+ B' \mid A' \in \psi(A), B' \in \psi(B)\} \\ \bar{\psi}(A \vee B) &:= \{A' \vee^- B', A' \vee^+ B' \mid A' \in \psi(A), B' \in \psi(B)\} \\ \bar{\psi}(\exists x A) &:= \{\exists x A' \mid A' \in \psi(A)\} \\ \bar{\psi}(\forall x A) &:= \{\forall x A' \mid A' \in \psi(A)\} \\ \bar{\psi}(\neg A) &:= \{A'^{\perp} \mid A' \in \psi(A)\} \\ \bar{\psi}(\Delta, A) &:= \{\Delta', A' \mid \Delta' \in \psi(\Delta), A' \in \psi(A)\} \\ \bar{\psi}(\emptyset) &:= \emptyset \end{aligned}$$

**Remark 16**  $\bar{\psi}(A) \neq \emptyset$

**Remark 17** If  $A' \in \bar{\psi}(A)$ , then  $\{\check{x}_x\} A' \in \bar{\psi}(\{\check{x}_x\} A')$ .

If  $C' \in \bar{\psi}(\{\check{x}_x\} A)$ , then  $C' = \{\check{x}_x\} A'$  for some  $A' \in \bar{\psi}(A)$ .

**Notation 9** In the rest of this section we will use the notation  $A \wedge^? B$  (resp.  $A \vee^? B$ ) to ambiguously represent either  $A \wedge^+ B$  or  $A \wedge^- B$  (resp.  $A \vee^+ B$  or  $A \vee^- B$ ). This will make the proofs more compact, noticing that Corollary 15(2) and 15(4) respectively imply the admissibility of

$$\frac{\Gamma \vdash \Delta, A \wedge^- B}{\Gamma \vdash \Delta, A \wedge^? B} \quad \frac{\Gamma \vdash \Delta, A \vee^- B}{\Gamma \vdash \Delta, A \vee^? B}$$

**Lemma 18** For all  $A', A'' \in \bar{\psi}(A)$ , we have  $\vdash_{LK(\mathcal{T})} A', A''^{\perp}$

**Proof:** In the proof below, for any formula  $A$ , the notations  $A'$  and  $A''$  will systematically designate elements of  $\bar{\psi}(A)$ .

The proof is by induction on  $A$ :

1.  $A = a$ .

Let  $A', A'' \in \bar{\psi}(a) = \{\psi(a)\}$ . Therefore  $A' = A'' = A = \psi(a)$ .

$$\frac{\frac{\psi(a) \vdash [\psi(a)]}{\psi^{\perp}(a), \psi(a) \vdash}}{\vdash \psi(a), \psi^{\perp}(a)}$$

2.  $A = A_1 \wedge A_2$

Let  $A'_1, A''_1 \in \bar{\psi}(A_1)$ ,  $A'_2, A''_2 \in \bar{\psi}(A_2)$  and  $A' = A'_1 \wedge^? A'_2$ ,  $A'' = A''_1 \wedge^? A''_2$ .

$$\frac{\frac{\vdash A'_1, A''_1^{\perp}}{\vdash A'_1, A''_1^{\perp}, A''_2^{\perp}} \text{WK} \quad \frac{\vdash A'_2, A''_2^{\perp}}{\vdash A'_2, A''_1^{\perp}, A''_2^{\perp}} \text{WK}}{\frac{\vdash A'_1 \wedge^- A'_2, A''_1^{\perp} \vee^- A''_2^{\perp}}{\vdash A', A''_1^{\perp} \vee^- A''_2^{\perp}}}{\vdash A', A''^{\perp}}$$

We can complete the proof on the left-hand side by applying the induction on  $A_1$  and on the right-hand side by applying the induction hypothesis on  $A_2$ .

3.  $A = A_1 \vee A_2$

Let  $A'_1, A''_1 \in \bar{\psi}(A_1)$ ,  $A'_2, A''_2 \in \bar{\psi}(A_2)$  and  $A' = A'_1 \vee^? A'_2$ ,  $A'' = A''_1 \vee^? A''_2$ .

$$\frac{\frac{\frac{\vdash A'_1, A''_1^\perp}{\vdash A'_1, A'_2, A''_1^\perp} \text{WK} \quad \frac{\vdash A'_2, A''_2^\perp}{\vdash A'_1, A'_2, A''_2^\perp} \text{WK}}{\vdash A'_1 \vee^- A'_2, A''_1^\perp \wedge^- A''_2^\perp}}{\vdash A', A''_1^\perp \wedge^- A''_2^\perp}}{\vdash A', A''^\perp}$$

We can complete the proof on the left-hand side by applying the induction on  $A_1$  and on the right-hand side by applying the induction hypothesis on  $A_2$ .

4.  $A = \forall x A_1$

Let  $A' = \forall x A'_1$  and  $A'' = \forall x A''_1$ .

$$\frac{\frac{\frac{\vdash [A''_1^\perp] A''}{\vdash [\exists x A''_1] A''} \text{WK}}{\forall x A''_1 \vdash [\bullet] A''} \text{Lemma 12(2)}}{\vdash A'_1, \exists x A''_1^\perp} \text{Lemma 14}}{\vdash A'_1, \exists x A''_1^\perp} \text{Lemma 14}}{\vdash \forall x A'_1, \exists x A''_1^\perp}$$

We can complete the proof on the left-hand side by Lemma 10 and the right-hand side by applying the induction hypothesis on  $A_1$ .

5.  $A = \exists x A_1$

Let  $A' = \exists x A'_1$  and  $A'' = \exists x A''_1$ .

$$\frac{\frac{\frac{\vdash [A'_1^\perp] A'_1}{\vdash [A'_1] \exists x A'_1}}{A'_1 \vdash [\bullet] \exists x A'_1} \text{Lemma 12(2)}}{\vdash A'_1^\perp, \exists x A'_1} \text{Lemma 14}}{\vdash \exists x A'_1, A''_1^\perp} \text{Lemma 14}}{\vdash \exists x A'_1, \forall x A''_1^\perp}$$

We can complete the proof on the left-hand side by Lemma 10 and the right-hand side by applying the induction hypothesis on  $A_1$ .

6.  $A = \neg A_1$

Let  $A', A'' \in \bar{\psi}(\neg A_1)$ .

Let  $A' = A'_1^\perp$  with  $A'_1 \in \bar{\psi}(A_1)$  and  $A'' = A''_1^\perp$  with  $A''_1 \in \bar{\psi}(A_1)$ .

The induction hypothesis on  $A_1$  we get:  $\vdash_{\text{LK}(\mathcal{T})} A', A''^\perp$  and we are done.  $\square$

### Theorem 19 (Completeness of $\text{LK}(\mathcal{T})$ )

Let  $\Gamma_{\mathcal{T}} = \{l_1 \vee \dots \vee l_n \mid \mathcal{T}(l_1^\perp, \dots, l_n^\perp)\}$ .

If  $\Gamma_{\mathcal{T}}, \Delta \vdash_{\text{FOL}} A$  then for all  $A' \in \bar{\psi}(A)$  and  $\Delta' \in \bar{\psi}(\Delta)$ , we have  $\vdash_{\text{LK}(\mathcal{T})} A', \Delta'^\perp$ .

#### Proof:

For any formula  $A$ , the notation  $A'$  will systematically designate elements of  $\psi(A)$  and for any multiset of formulae  $\Delta$ , the notation  $\Delta'$  will systematically designate elements of  $\psi(\Delta)$ . The proof is by induction of  $\Gamma_{\mathcal{T}}, \Delta \vdash_{\text{FOL}} A$ , and case analysis on the last rule:

- Axiom:

$$\frac{}{\Gamma_{\mathcal{T}}, \Delta \vdash A} A \in \Gamma_{\mathcal{T}}, \Delta$$

By case analysis:

– If  $A \in \Delta$  then:

$$\frac{\vdash A', A''^\perp}{\vdash A', \Delta'^\perp} \text{WK}$$

with  $A', A'' \in \bar{\psi}(A)$ .

We can close the branch by Lemma 18.

– If  $A \in \Gamma_{\mathcal{T}}$  then:

$A$  is of the form  $l_1 \vee \dots \vee l_n$  with  $\mathcal{T}(\psi(l_1)^\perp, \dots, \psi(l_n)^\perp)$ . Let  $C' \in \bar{\psi}(A)$ .  $C'$  is of the form  $\psi(l_1) \vee^? \dots \vee^? \psi(l_n)$ .

$$\frac{\frac{\frac{\mathcal{T}(\psi(l_1)^\perp, \dots, \psi(l_n)^\perp)}{\psi(l_1)^\perp, \dots, \psi(l_n)^\perp \vdash}}{\vdash \psi(l_1), \dots, \psi(l_n)}}{\vdash \psi(l_1) \vee^- \dots \vee^- \psi(l_n)}}{\vdash C'} \text{WK}$$

This is a complete proof since  $\mathcal{T}(\psi(l_1)^\perp, \dots, \psi(l_n)^\perp)$  returns *UNSAT*.

• And Intro:

$$\frac{\Gamma_{\mathcal{T}}, \Delta \vdash A_1 \quad \Gamma_{\mathcal{T}}, \Delta \vdash A_2}{\Gamma_{\mathcal{T}}, \Delta \vdash A_1 \wedge A_2}$$

$A' \in \bar{\psi}(A_1 \wedge A_2)$  is of the form  $A'_1 \wedge^? A'_2$  with  $A'_1 \in \psi(A_1)$  and  $A'_2 \in \psi(A_2)$ .

The induction hypothesis on  $\Gamma_{\mathcal{T}}, \Delta \vdash_{\text{FOL}} A_1$  gives  $\vdash_{\text{LK}(\mathcal{T})} A'_1, \Delta'^\perp$  and the induction hypothesis on  $\Gamma_{\mathcal{T}}, \Delta \vdash_{\text{FOL}} A_2$  gives  $\vdash_{\text{LK}(\mathcal{T})} A'_2, \Delta'^\perp$ . We build:

$$\frac{\frac{\vdash A'_1, \Delta'^\perp \quad \vdash A'_2, \Delta'^\perp}{\vdash A'_1 \wedge^- A'_2, \Delta'^\perp}}{\vdash A'_1 \wedge^? A'_2, \Delta'^\perp}$$

• And Elim

$$\frac{\Gamma_{\mathcal{T}}, \Delta \vdash A \wedge B}{\Gamma_{\mathcal{T}}, \Delta \vdash A}$$

– Since  $\bar{\psi}(B) \neq \emptyset$ , let  $B' \in \bar{\psi}(B)$  and  $C' = A' \wedge^- B'$  ( $C' \in \bar{\psi}(A \wedge B)$ ).

The induction hypothesis on the premise, with  $\Delta'$  and  $C'$ , gives  $\vdash_{\text{LK}(\mathcal{T})} C', \Delta'^\perp$  and we get:

$$\frac{\vdash C', \Delta'^\perp}{\vdash A', \Delta'^\perp}$$

by Lemma 3.

$$\frac{\Gamma_{\mathcal{T}}, \Delta \vdash A \wedge B}{\Gamma_{\mathcal{T}}, \Delta \vdash B}$$

– Since  $\bar{\psi}(A) \neq \emptyset$ , let  $A' \in \bar{\psi}(A)$  and  $C' = A' \wedge^- B'$  ( $C' \in \bar{\psi}(A \wedge B)$ ).

The induction hypothesis on the premise, with  $\Delta'$  and  $C'$ , gives  $\vdash_{\text{LK}(\mathcal{T})} C', \Delta'^\perp$  and we get:

$$\frac{\vdash C', \Delta'^\perp}{\vdash B', \Delta'^\perp}$$

by Lemma 3.

• Or Intro

$$\frac{\Gamma_{\mathcal{T}}, \Delta \vdash A_i}{\Gamma_{\mathcal{T}}, \Delta \vdash A_1 \vee A_2}$$



$A' \in \bar{\psi}(A_1 \vee A_2)$  is of the form  $A'_1 \vee^? A'_2$  with  $A'_1 \in \psi(A_1)$  and  $A'_2 \in \psi(A_2)$ .

The induction hypothesis on  $\Gamma_{\mathcal{T}}, \Delta \vdash_{\text{FOL}} A_i$  gives  $\vdash_{\text{LK}(\mathcal{T})} A'_i, \Delta'^{\perp}$  and we build:

$$\frac{\frac{\vdash A'_i, \Delta'^{\perp}}{\vdash A'_1, A'_2, \Delta'^{\perp}} WK}{\vdash A'_1 \vee^- A'_2, \Delta'^{\perp}}}{\vdash A'_1 \vee^? A'_2, \Delta'^{\perp}}$$

- Or Elim

$$\frac{\Gamma_{\mathcal{T}}, \Delta \vdash A_1 \vee A_2 \quad \Gamma_{\mathcal{T}}, \Delta, A_1 \vdash C \quad \Gamma_{\mathcal{T}}, \Delta, A_2 \vdash C}{\Gamma_{\mathcal{T}}, \Delta \vdash C}$$

Let  $D' = A'_1 \vee^- A'_2$  with  $A'_1 \in \psi(A_1)$  and  $A'_2 \in \psi(A_2)$ .

The induction hypothesis on  $\Gamma_{\mathcal{T}}, \Delta \vdash_{\text{FOL}} A_1 \vee A_2$  gives  $\vdash_{\text{LK}(\mathcal{T})} D', \Delta'^{\perp}$ , the induction hypothesis on  $\Gamma_{\mathcal{T}}, A_1, \Delta \vdash_{\text{FOL}} C$  gives  $\vdash_{\text{LK}(\mathcal{T})} A'_1{}^{\perp}, C', \Delta'^{\perp}$  and the induction hypothesis on  $\Gamma_{\mathcal{T}}, A_2, \Delta \vdash_{\text{FOL}} C$  gives  $\vdash_{\text{LK}(\mathcal{T})} A'_2{}^{\perp}, C', \Delta'^{\perp}$ . We get:

$$\frac{\frac{\vdash A'_1{}^{\perp}, C', \Delta'^{\perp} \quad \vdash A'_2{}^{\perp}, C', \Delta'^{\perp}}{\vdash A'_1{}^{\perp} \wedge^+ A'_2{}^{\perp}, C', \Delta'^{\perp}}}{\vdash D', C', \Delta'^{\perp} \quad \frac{\vdash (A'_1 \vee^- A'_2)^{\perp}, C', \Delta'^{\perp}}{\vdash C', \Delta'^{\perp}} cut_7}}{\vdash C', \Delta'^{\perp}}$$

- Universal quantifier Intro

$$\frac{\Gamma_{\mathcal{T}}, \Delta \vdash A}{\Gamma_{\mathcal{T}}, \Delta \vdash \forall x A} \quad x \notin \Gamma$$

$C' \in \bar{\psi}(\forall x A)$  is of the form  $\forall x A'$  with  $A' \in \bar{\psi}(A)$ .

The induction hypothesis on  $\Gamma_{\mathcal{T}}, \Delta \vdash_{\text{FOL}} A$  gives  $\vdash_{\text{LK}(\mathcal{T})} A', \Delta'^{\perp}$ . We get:

$$\frac{\vdash A', \Delta'^{\perp}}{\vdash \forall x A', \Delta'^{\perp}}$$

- Universal quantifier Elim

$$\frac{\Gamma_{\mathcal{T}}, \Delta \vdash \forall x A}{\Gamma_{\mathcal{T}}, \Delta \vdash \{^t_x\} A}$$

$C' \in \bar{\psi}(\{^t_x\} A)$  is of the form  $\{^t_x\} A'$  with  $A' \in \bar{\psi}(A)$  (by remark 17).

The induction hypothesis on  $\Gamma_{\mathcal{T}}, \Delta \vdash_{\text{FOL}} \forall x A$  gives  $\vdash_{\text{LK}(\mathcal{T})} \forall x A', \Delta'^{\perp}$ . We get:

$$\frac{\frac{\vdash \forall x A', \Delta'^{\perp}}{\vdash A', \Delta'^{\perp}}}{\vdash \{^t_x\} A', \Delta'^{\perp}}$$

by Lemma 3 and Lemma 2.

- Existential quantifier Intro

$$\frac{\Gamma_{\mathcal{T}}, \Delta \vdash \{^t_x\} A}{\Gamma_{\mathcal{T}}, \Delta \vdash \exists x A}$$

$C' \in \bar{\psi}(\exists x A)$  is of the form  $\exists x A'$  with  $A' \in \bar{\psi}(A)$ .

Let  $A'_t = \{^t_x\} A'$  ( $A'_t \in \bar{\psi}(\{^t_x\} A)$ ) by remark 17).

The induction hypothesis on  $\Gamma_{\mathcal{T}}, \Delta \vdash_{\text{FOL}} \{^t_x\} A$  gives  $\vdash_{\text{LK}(\mathcal{T})} A'_t, \Delta'^{\perp}$ .

By Lemma 14 it suffices to prove  $\vdash_{\text{LK}(\mathcal{T})} \exists x A', A'_t{}^{\perp}$  in order to get  $\vdash_{\text{LK}(\mathcal{T})} C', \Delta'^{\perp}$ :

$$\frac{\frac{\frac{\vdash [A'_t]A'_t{}^\perp}{\vdash [\exists x A']A'_t{}^\perp}}{\forall x A'^\perp \vdash [\bullet]A'_t{}^\perp}}{\vdash \exists x A', A'_t{}^\perp} \text{WK}}{\vdash \exists x A', A'_t{}^\perp} \text{Lemma 12(2)}$$

We can complete the proof by applying Lemma 10.

- Existential quantifier Elim

$$\frac{\Gamma_{\mathcal{T}}, \Delta \vdash \exists x A \quad \Gamma, \Delta, A \vdash B}{\Gamma_{\mathcal{T}}, \Delta \vdash B} x \notin \Gamma, B$$

Let  $C' = \exists x A'$  with  $A' \in \bar{\psi}(A)$ .

$$\frac{\frac{\frac{\vdash A'^\perp, B', \Delta'^\perp}{\vdash \forall x A'^\perp, B', \Delta'^\perp}}{\vdash C', \Delta'^\perp} \quad \frac{\vdash A'^\perp, B', \Delta'^\perp}{\vdash \forall x A'^\perp, B', \Delta'^\perp}}{\vdash C', \Delta'^\perp} \quad \frac{\vdash \forall x A'^\perp, B', \Delta'^\perp}{\vdash C'^\perp, B', \Delta'^\perp}}{\vdash B', \Delta'^\perp} \text{cut}_7$$

We can complete the proof on the left-hand side by applying the induction hypothesis on  $\Gamma_{\mathcal{T}}, \Delta \vdash_{\text{FOL}} \exists x A$  and on the right-hand side by applying the induction hypothesis on  $\Gamma_{\mathcal{T}}, \Delta, A \vdash_{\text{FOL}} B$ .

- Negation Intro

$$\frac{\Gamma_{\mathcal{T}}, \Delta, A \vdash B \wedge \neg B}{\Gamma_{\mathcal{T}}, \Delta \vdash \neg A}$$

If  $C' \in \bar{\psi}(\neg A)$  then  $C'^\perp \in \bar{\psi}(A)$ . Let  $D' = D'_1 \wedge^- D'_2$  with  $D'_1 \in \bar{\psi}(B)$  and  $D'_2 \in \bar{\psi}(\neg B)$ . Therefore  $D'_2{}^\perp \in \bar{\psi}(B)$ ,  $D' \in \bar{\psi}(B \wedge \neg B)$  and  $\Delta', C'^\perp \in \bar{\psi}(\Delta, A)$ .

$$\frac{\frac{\frac{\vdash D'_1{}^\perp, D'_2{}^\perp}{\vdash D'_1{}^\perp \vee^- D'_2{}^\perp}}{\vdash D'^\perp} \text{Corollary 15(4)}}{\vdash \Delta'^\perp, C', D'} \quad \frac{\vdash D'_1{}^\perp, D'_2{}^\perp}{\vdash \Delta'^\perp, C', D'^\perp} \text{WK}}{\vdash \Delta'^\perp, C'} \text{cut}_7$$

We can complete the proof on the left-hand side by applying the induction hypothesis on  $\Gamma_{\mathcal{T}}, \Delta, A \vdash B \wedge \neg B$  and on the right-hand side by applying Lemma 18 with  $A''^\perp = D'_1{}^\perp$  and  $A' = D'_2{}^\perp$ .

- Negation Elimination

$$\frac{\Gamma_{\mathcal{T}}, \Delta \vdash \neg \neg A}{\Gamma_{\mathcal{T}}, \Delta \vdash A}$$

$A' \in \bar{\psi}(A)$  is such that  $A' \in \bar{\psi}(\neg \neg A)$ .

The induction hypothesis on  $\Gamma_{\mathcal{T}}, \Delta \vdash \neg \neg A$  gives  $\vdash \Delta'^\perp, A'$  and we are done.  $\square$

## 7 The $\text{LK}^p(\mathcal{T})$ sequent calculus: on-the-fly polarisation of literals

**Definition 10 (Literals)** From now on we distinguish polarised literals and unpolarised literals. The former are those literals used so far in  $\text{LK}(\mathcal{T})$ ; the later are introduced as follows:

Let  $\mathcal{L}$  be a set of elements called *unpolarised literals*, equipped with an involutive function called *negation* from  $\mathcal{L}$  to  $\mathcal{L}$ . In the rest of this paper, a possibly primed or indexed lowercase  $l$  always denotes a literal, and  $l^\perp$  its negation.

From now on, the expression *literals* refers to unpolarised literals.

**Definition 11 (Formulae, negation)** The formulae  $\text{LK}^{\mathcal{P}}(\mathcal{T})$  are given by the following grammar:

$$\text{Formulae } A, B, \dots ::= l \mid A \wedge^+ B \mid A \vee^+ B \mid A \wedge^- B \mid A \vee^- B$$

where  $l$  ranges over literals.

A *polarisation set*  $\mathcal{P}$  is a set of literals ( $\mathcal{P} \subseteq \mathcal{L}$ ) such that if  $l \in \mathcal{P}$ , then  $l^\perp \notin \mathcal{P}$ .

Given such a set, we define  $\mathcal{P}$ -positive formulae and  $\mathcal{P}$ -negative formulae as the formulae generated by the following grammars:

$$\begin{array}{l} \mathcal{P}\text{-positive formulae } P, \dots ::= l \mid A \wedge^+ B \mid A \vee^+ B \\ \mathcal{P}\text{-negative formulae } N, \dots ::= l^\perp \mid A \wedge^- B \mid A \vee^- B \end{array}$$

where  $l$  ranges over  $\mathcal{P}$ .

Negation is extended from literals to all formulae using the following definitions:

$(A \wedge^+ B)^\perp := A^\perp \vee^- B^\perp$	$(A \wedge^- B)^\perp := A^\perp \vee^+ B^\perp$
$(A \vee^+ B)^\perp := A^\perp \wedge^- B^\perp$	$(A \vee^- B)^\perp := A^\perp \wedge^+ B^\perp$

**Definition 12 (System  $\text{LK}^{\mathcal{P}}(\mathcal{T})$ )** The sequent calculus  $\text{LK}^{\mathcal{P}}(\mathcal{T})$  is given by the rules of Figure 3, where  $\Gamma$  and  $\Delta$  are multisets of formulae.

$\frac{\Gamma \vdash^{\mathcal{P}} [A] \quad \Gamma \vdash^{\mathcal{P}} [B]}{\Gamma \vdash^{\mathcal{P}} [A \wedge^+ B]}$	$\frac{\Gamma \vdash^{\mathcal{P}} [A_i]}{\Gamma \vdash^{\mathcal{P}} [A_1 \vee^+ A_2]}$
$\frac{\mathcal{T}(\text{lit}(\Gamma), l^\perp)}{\Gamma \vdash^{\mathcal{P}, l} [l]}$	$\frac{\Gamma \vdash^{\mathcal{P}} N}{\Gamma \vdash^{\mathcal{P}} [N]}$ $N$ is not $\mathcal{P}$ -positive
$\frac{\Gamma \vdash^{\mathcal{P}} A, \Delta \quad \Gamma \vdash^{\mathcal{P}} B, \Delta}{\Gamma \vdash^{\mathcal{P}} A \wedge^- B, \Delta}$	$\frac{\Gamma \vdash^{\mathcal{P}} A_1, A_2, \Delta}{\Gamma \vdash^{\mathcal{P}} A_1 \vee^- A_2, \Delta}$
$\frac{\Gamma \vdash^{\mathcal{P}, l} \Delta}{\Gamma \vdash^{\mathcal{P}} \Delta}$	$\frac{\Gamma, A^\perp \vdash^{\mathcal{P}} \Gamma'}{\Gamma \vdash^{\mathcal{P}} A, \Gamma'}$ $A$ is $\mathcal{P}$ -positive or a $\mathcal{P}$ -negative literal
$\frac{\Gamma, P^\perp \vdash^{\mathcal{P}} [P]}{\Gamma, P^\perp \vdash^{\mathcal{P}}}$ $P$ is $\mathcal{P}$ -positive	$\frac{\mathcal{T}(\text{lit}(\Gamma))}{\Gamma \vdash^{\mathcal{P}}}$

Figure 3: System  $\text{LK}^{\mathcal{P}}(\mathcal{T})$

**Definition 13 ( $\phi$  compatible with  $\mathcal{P}$ )** Let  $\phi$  be a function that maps every unpolarised literal to a polarised literal that has the same meaning for  $\mathcal{T}$  and such that  $\phi(l^\perp) = \phi(l)^\perp$  for all  $l$ .

- $\phi$  is said to be *compatible* with a polarisation set  $\mathcal{P}$  if for all  $l \in \mathcal{P}$ ,  $\phi(l)$  is a positive literal of  $\text{LK}(\mathcal{T})$ .
- $\phi$  is extended into a mapping of formulae, and multisets of formulae, so that we can write  $\phi(A)$ ,  $\phi(B)$ ,  $\phi(\Gamma)$ ,  $\phi(\Delta)$ , etc. .

**Theorem 20 (Encoding  $\text{LK}^{\mathcal{P}}(\mathcal{T})$  in  $\text{LK}(\mathcal{T})$ )**

1. If  $\Gamma \vdash^{\mathcal{P}} \Delta$  in  $\text{LK}^{\mathcal{P}}(\mathcal{T})$ , then for all  $\phi$  compatible with  $\mathcal{P}$ ,  $\phi(\Gamma) \vdash \phi(\Delta)$  in  $\text{LK}(\mathcal{T})$ .
2. If  $\Gamma \vdash^{\mathcal{P}} [A]$  in  $\text{LK}^{\mathcal{P}}(\mathcal{T})$ , then:
  - (a) either for all  $\phi$  compatible with  $\mathcal{P}$ ,  $\phi(\Gamma) \vdash$  in  $\text{LK}(\mathcal{T})$ ;
  - (b) or there exists a polarisation set  $\mathcal{P}' \supseteq \mathcal{P}$  such that:
    - i. for all  $\phi$  compatible with  $\mathcal{P}'$ ,  $\phi(\Gamma) \vdash [\phi(A)]$  in  $\text{LK}(\mathcal{T})$ ;
    - ii. for all  $l \in \mathcal{P}' \setminus \mathcal{P}$ , for all  $\phi$  compatible with  $\mathcal{P}$ ,  $\phi(\Gamma) \vdash \phi(l^\perp)$  in  $\text{LK}(\mathcal{T})$ .

**Proof:** In our proof we use  $\Gamma'$  for  $\phi(\Gamma)$ ,  $A'$  for  $\phi(A)$ ,  $B'$  for  $\phi(B)$  and  $\Delta'$  for  $\phi(\Delta)$ , etc.

1.

$$\frac{\Gamma \vdash^{\mathcal{P}} [A_1] \quad \Gamma \vdash^{\mathcal{P}} [A_2]}{\Gamma \vdash^{\mathcal{P}} [A_1 \wedge^+ A_2]}$$

for  $A = A_1 \wedge^+ A_2$ .

We apply the induction hypothesis (2.) on  $\Gamma \vdash^{\mathcal{P}} [A_1]$  and on  $\Gamma \vdash^{\mathcal{P}} [A_2]$ .

- If we get (2.a) for either of the two sides, then it proves (2.a) for  $\Gamma \vdash^{\mathcal{P}} [A_1 \wedge^+ A_2]$  (same statement).
- If we get (2.b) for both sides, then we get two polarisation sets  $\mathcal{P}_1 \supseteq \mathcal{P}$  and  $\mathcal{P}_2 \supseteq \mathcal{P}$ . Let  $\mathcal{P}' := \mathcal{P}_1 \cup \mathcal{P}_2$ .
  - If  $\mathcal{P}'$  is not a polarisation set, then it means there is a literal  $l \in \mathcal{P}_1 \setminus \mathcal{P}$  and  $l^\perp \in \mathcal{P}_2 \setminus \mathcal{P}$ . The induction hypothesis (2.b.ii) ensures that for any  $\phi$  compatible with  $\mathcal{P}$ ,  $\Gamma' \vdash \phi(l)$  and  $\Gamma' \vdash \phi(l)^\perp$  in  $\text{LK}(\mathcal{T})$ , and using  $\text{cut}_7$  (Theorem 6), we get  $\Gamma' \vdash$  in  $\text{LK}(\mathcal{T})$ . This is exactly (2.a) for  $\Gamma \vdash^{\mathcal{P}} [A_1 \wedge^+ A_2]$ .
  - If on the other hand,  $\mathcal{P}'$  is a polarisation set, then:
    - \* Any  $\phi$  compatible with  $\mathcal{P}'$  is compatible with both  $\mathcal{P}_1$  and  $\mathcal{P}_2$ , so the induction hypothesis (2.b.i) provides  $\Gamma' \vdash [A'_1]$  and  $\Gamma' \vdash [A'_2]$  in  $\text{LK}(\mathcal{T})$ , and therefore  $\Gamma' \vdash [A'_1 \wedge^+ A'_2]$  in  $\text{LK}(\mathcal{T})$ . This is (2.b.i) for  $\Gamma \vdash^{\mathcal{P}} [A_1 \wedge^+ A_2]$ .
    - \* If  $l \in \mathcal{P}' \setminus \mathcal{P}$ , then  $l \in \mathcal{P}_i \setminus \mathcal{P}$  for  $i = 1$  or  $i = 2$ , and induction hypothesis (2.b.ii) on side  $i$  provides  $\Gamma' \vdash \phi(l)^\perp$  for any  $\phi$  compatible with  $\mathcal{P}$ . This is (2.b.ii) for  $\Gamma \vdash^{\mathcal{P}} [A_1 \wedge^+ A_2]$ .

2.

$$\frac{\Gamma \vdash^{\mathcal{P}} [A_i]}{\Gamma \vdash^{\mathcal{P}} [A_1 \vee^+ A_2]}$$

for  $A = A_1 \vee^+ A_2$ .

We apply the induction hypothesis (2.) on  $\Gamma \vdash^{\mathcal{P}} [A_i]$ .

- If we get (2.a), then it proves (2.a) for  $\Gamma \vdash^{\mathcal{P}} [A_1 \vee^+ A_2]$  (same statement).
- If we get (2.b), then we get a polarisation set  $\mathcal{P}_i \supseteq \mathcal{P}$ .

We prove (2.b) for  $\Gamma \vdash^{\mathcal{P}} [A_1 \vee^+ A_2]$ , choosing  $\mathcal{P}' := \mathcal{P}_i$ :

- (2.b.ii) for  $\Gamma \vdash^{\mathcal{P}} [A_1 \vee^+ A_2]$  is the same statement as for  $\Gamma \vdash^{\mathcal{P}} [A_i]$ ;
- in order to prove (2.b.i) for  $\Gamma \vdash^{\mathcal{P}} [A_1 \vee^+ A_2]$ , let  $\phi$  be compatible with  $\mathcal{P}'$ ; the induction hypothesis (2.b.i) provides  $\Gamma \vdash^{\mathcal{P}} [A_i]$  in  $\text{LK}(\mathcal{T})$ , from which we get  $\Gamma \vdash^{\mathcal{P}} [A_1 \vee^+ A_2]$ .

3.

$$\frac{\mathcal{T}(\text{lit}(\Gamma), l^\perp)}{\Gamma \vdash_{\mathcal{T}}^{p, l} [l]}$$

for  $A = l$ .

Let  $p = \phi(l)$  and we can build:

$$\frac{\mathcal{T}(\text{lit}(\Gamma'), p^\perp)}{\Gamma' \vdash [p]}$$

4.

$$\frac{\Gamma \vdash^{\mathcal{P}} N}{\Gamma \vdash^{\mathcal{P}} [N]}$$

where  $A = N$  is not  $\mathcal{P}$ -positive.

We apply the induction hypothesis (1.) on  $\Gamma \vdash^{\mathcal{P}} N$ :

for any  $\phi$  compatible with  $\mathcal{P}$  we have  $\Gamma' \vdash N'$  in  $\text{LK}(\mathcal{T})$ .

- If  $N$  is  $\mathcal{P}$ -negative, then  $N'$  is negative, and therefore for any  $\phi$  compatible with  $\mathcal{P}$  we can apply the same rule in  $\text{LK}(\mathcal{T})$ :

$$\frac{\Gamma' \vdash N'}{\Gamma' \vdash [N']}$$

This proves (2.b) for  $\Gamma \vdash^{\mathcal{P}} [N]$ , choosing  $\mathcal{P}' := \mathcal{P}$ .

- If  $N$  is not  $\mathcal{P}$ -negative, then it is necessarily a literal  $l$  whose polarity is undetermined by  $\mathcal{P}$ , and we then prove (2.b) for  $\Gamma \vdash^{\mathcal{P}} [N]$ , this time choosing  $\mathcal{P}' := \mathcal{P} \cup \{l^\perp\}$ .

(2.b.i) holds because for any  $\phi$  compatible with  $\mathcal{P}'$  we can apply the same rule in  $\text{LK}(\mathcal{T})$ :

$$\frac{\Gamma' \vdash \phi(l)}{\Gamma' \vdash [\phi(l)]}$$

(2.b.ii) holds because  $\mathcal{P}' \setminus \mathcal{P} = \{l^\perp\}$ , and for any  $\phi$  compatible with  $\mathcal{P}$ , induction hypothesis (1.) provides  $\Gamma' \vdash \phi(l)$  in  $\text{LK}(\mathcal{T})$ .

5.

$$\frac{\Gamma \vdash^{\mathcal{P}, l} \Delta}{\Gamma \vdash^{\mathcal{P}} \Delta}$$

Let  $\phi$  be compatible with  $\mathcal{P}$ .

- If  $\phi(l)$  is positive, then the induction hypothesis gives  $\phi(\Gamma) \vdash \phi(\Delta)$ .
- If  $\phi(l)$  is negative, let  $p$  be a positive literal with the same meaning as  $\phi(l)$  for  $\mathcal{T}$ . Then let  $\phi'$  be defined as:

$$\begin{cases} \phi'(l') = \phi(l') & \text{if } l' \neq l \text{ and } l' \neq l^\perp \\ \phi'(l) = p \\ \phi'(l^\perp) = p^\perp \end{cases}$$

$\phi'$  is compatible with  $\mathcal{P}, l$ , so by induction hypothesis we get  $\phi'(\Gamma) \vdash \phi'(\Delta)$ . By corollary 8, we have  $\{\phi(l), \phi(l)^\perp / \phi'(l), \phi'(l)^\perp\} \phi'(\Gamma) \vdash \phi'(\Delta)$ .

6.

$$\frac{\Gamma \vdash^{\mathcal{P}} A_1, \Delta \quad \Gamma \vdash^{\mathcal{P}} A_2, \Delta}{\Gamma \vdash^{\mathcal{P}} A_1 \wedge^- A_2, \Delta}$$

for  $\Delta = A_1 \wedge^- A_2, \Delta$

The induction hypothesis on  $\Gamma \vdash^{\mathcal{P}} A_1, \Delta$  gives  $\Gamma' \vdash A'_1, \Delta'$  in  $\text{LK}(\mathcal{T})$  and the induction hypothesis on  $\Gamma \vdash^{\mathcal{P}} A_2, \Delta$  gives  $\Gamma' \vdash A'_2, \Delta'$  in  $\text{LK}(\mathcal{T})$ . We get:

$$\frac{\Gamma' \vdash A'_1, \Delta' \quad \Gamma' \vdash A'_2, \Delta'}{\Gamma' \vdash A'_1 \wedge^- A'_2, \Delta'}$$

7.

$$\frac{\Gamma \vdash^{\mathcal{P}} A_1, A_2, \Delta}{\Gamma \vdash^{\mathcal{P}} A_1 \vee^- A_2, \Delta}$$

for  $\Delta = A_1 \vee^- A_2, \Delta$ .

The induction hypothesis on  $\Gamma \vdash^{\mathcal{P}} A_1, A_2, \Delta$  gives  $\Gamma' \vdash A'_1, A'_2, \Delta'$  in  $\text{LK}(\mathcal{T})$ . We get:

$$\frac{\Gamma' \vdash A'_1, A'_2, \Delta'}{\Gamma' \vdash A'_1 \vee^- A'_2, \Delta'}$$

8.

$$\frac{\Gamma, A^\perp \vdash_{\mathcal{T}} \Delta}{\Gamma \vdash^{\mathcal{P}} A, \Delta}$$

for  $\Delta = A, \Delta$  where  $A$  is  $\mathcal{P}$ -positive or literal.

The induction hypothesis on  $\Gamma, A^\perp \vdash_{\mathcal{T}} \Delta$  gives  $\Gamma', A'^\perp \vdash \Delta'$  in  $\text{LK}(\mathcal{T})$ . We get:

$$\frac{\Gamma', A'^\perp \vdash \Delta'}{\Gamma' \vdash A', \Delta'}$$

9.

$$\frac{\Gamma, P^\perp \vdash^{\mathcal{P}} [P]}{\Gamma, P^\perp \vdash^{\mathcal{P}}} \quad P \text{ is } \mathcal{P}\text{-positive}$$

where  $P$  is  $\mathcal{P}$ -positive.

We apply the induction hypothesis (2.) on  $\Gamma, P^\perp \vdash^{\mathcal{P}} [P]$ .

- If we get (2.a), then it proves (1) for  $\Gamma, P^\perp \vdash^{\mathcal{P}}$  (same statement).
- If we get (2.b), then there exists a polarisation set  $\hat{\mathcal{P}} \supseteq \mathcal{P}$ , such that for all  $\phi'$  compatible with  $\hat{\mathcal{P}}$  we get :  $\phi'(\Gamma, P^\perp) \vdash [\phi'(P)]$  in  $\text{LK}(\mathcal{T})$ .

Let  $\phi$  be compatible with  $\mathcal{P}$ . Since  $\phi$  is not necessarily compatible with  $\hat{\mathcal{P}}$ , let  $\hat{\phi}$  be defined from  $\phi$  and  $\hat{\mathcal{P}}$  :

$$\begin{aligned} \hat{\phi}(l) &:= p && \text{if } l \in \hat{\mathcal{P}}, \text{ where } p \text{ is a positive literal with the same meaning as } \phi(l) \text{ for } \mathcal{T} \\ \hat{\phi}(l) &:= p^\perp && \text{if } l^\perp \in \hat{\mathcal{P}}, \text{ where } p^\perp \text{ is a negative literal with the same meaning as } \phi(l) \text{ for } \mathcal{T} \\ \hat{\phi}(l) &:= \phi(l) && \text{if } l \notin \hat{\mathcal{P}} \text{ and } l^\perp \notin \hat{\mathcal{P}} \end{aligned}$$

By construction,  $\hat{\phi}$  is compatible with  $\hat{\mathcal{P}}$  and therefore we get:  $\hat{\phi}(\Gamma, P^\perp) \vdash [\hat{\phi}(P)]$ .

Then we construct:

$$\frac{\hat{\phi}(\Gamma, P^\perp) \vdash [\hat{\phi}(P)]}{\hat{\phi}(\Gamma, P^\perp) \vdash}$$

By corollary 8, we get  $\phi(\Gamma, P^\perp) \vdash$  in  $\text{LK}(\mathcal{T})$  from  $\hat{\phi}(\Gamma, P^\perp) \vdash$  in  $\text{LK}(\mathcal{T})$ , since  $\phi$  and  $\hat{\phi}$  differ on a finite number of literals.

10.

$$\frac{\mathcal{T}(\text{lit}(\Gamma))}{\Gamma \vdash}$$

We get:

$$\frac{\mathcal{T}(\text{lit}(\Gamma'))}{\Gamma' \vdash}$$

□

We can also add those two cuts in  $\text{LK}^{\mathcal{P}}(\mathcal{T})$  (the former is a particular case of the latter). Since they are instances of cuts in  $\text{LK}(\mathcal{T})$  (Theorem 6), the previous theorem also holds for  $\text{LK}^{\mathcal{P}}(\mathcal{T})$  with cuts.

- Analytic cut:

$$\frac{\Gamma \vdash^{\mathcal{P}} l \quad \Gamma \vdash^{\mathcal{P}} l^\perp}{\Gamma \vdash^{\mathcal{P}}}$$

- General cut:

$$\frac{\Gamma \vdash^{\mathcal{P}} A \quad \Gamma \vdash^{\mathcal{P}} A^\perp}{\Gamma \vdash^{\mathcal{P}}}$$

**Definition 14 (Removing polarities)** Let  $\mathcal{A}$  be a subset of  $\mathcal{L}$  such that the image of  $\mathcal{A}$  by  $^\perp$  is its complement in  $\mathcal{L}$ :  $\mathcal{L} \setminus \mathcal{A}$ .

For any formula  $A$ , in the grammar of Definition 11, let  $\underline{A}$  be the formula, in the grammar of Definition 7, obtained by removing all polarities on connectives and interpreting  $l \in \mathcal{A}$  as  $l$  and  $l \in \mathcal{L} \setminus \mathcal{A}$  as  $\neg l$ .

**Theorem 21 (Completeness of  $\text{LK}^{\mathcal{P}}(\mathcal{T})$ )** Let  $\Gamma_{\mathcal{T}} = \{l_1 \vee \dots \vee l_n \mid \mathcal{T}(l_1^\perp, \dots, l_n^\perp)\}$ .

If  $\Gamma_{\mathcal{T}} \vdash F$  in propositional logic, then for all  $A$  such that  $\underline{A} = F$ , and all polarisation sets  $\mathcal{P}$ , we have  $\vdash^{\mathcal{P}} A$  in  $\text{LK}^{\mathcal{P}}(\mathcal{T})$ .

**Proof:** The hypothesis is a particular case of  $\Gamma_{\mathcal{T}} \vdash_{\text{FOL}} F$ . Let  $\phi$  be compatible with  $\mathcal{P}$ , and let  $\psi$  be the restriction of  $\phi$  to  $\mathcal{A}$ . By a straightforward induction on  $F$ ,  $\phi(A) \in \bar{\psi}(F)$ , so by Theorem 19 we get  $\vdash_{\text{LK}(\mathcal{T})} \phi(A)$ . Let  $\mathcal{P}' \supseteq \mathcal{P}$  be the (finite) set of all literals that appear

in  $A$  and are mapped by  $\phi$  to positive literals. The proof of  $\vdash_{\text{LK}(\mathcal{T})} \phi(A)$  is isomorphic to a proof of  $\vdash^{\mathcal{P}'} A$  in  $\text{LK}^{\mathcal{P}}(\mathcal{T})$ , and then we can finish the proof with polarisation steps:

$$\frac{\vdash^{\mathcal{P}'} A}{\vdash^{\mathcal{P}} A}$$

□

## 8 Conclusion

It is worth noting that an instance of such a theory is the theory where  $\mathcal{T}(S)$  holds if and only if there is a literal  $p \in S$  such that  $p^\perp \in S$ .

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