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Sequent calculi with procedure calls

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Abstract

In this paper, we study 3 focussed sequent calculi that are based on Miller-Liang's LKF system for polarised classical logic, and integrate the possibility to call a decision procedure.

The main sequent calculus out of the three is $LK(T)$, in which we prove cut-elimination. The second one is less focussed, and is introduced for the proof that changing polarities do not change provability, only the shape of proofs.

The third one features the possibility to polarise literals "on the fly", and is used in other works to simulate the $DPLL(T)$ procedure.

Finally, completeness of the 3 calculi is proved.

Contents

1	The sequent calculus $LK(\mathcal{T})$	2
2	Admissible rules	2
3	Invertibility of the asynchronous phase	3
4	Cut-elimination	5
5	Changing the Polarity of Predicates and Connectives	9
	5.1 Changing the polarity of Predicates	9
	5.2 Changing the polarity of connectives	13
6	The $LK^p(\mathcal{T})$ sequent calculus: on-the-fly polarisation of literals	21
7	Completeness	24
8	Conclusion	29

1 The sequent calculus $LK(\mathcal{T})$

The sequent calculus $LK(\mathcal{T})$ manipulates the formulae of first-order logic, with the specificity that every predicate symbol is classified as either positive or negative, and boolean connectives come in two versions: positive and negative.

Definition 1 (Formulae)

$$\begin{array}{lll} \text{Positive formulae} & P & ::= p \mid A \wedge^+ B \mid A \vee^+ B \mid \exists x A \\ \text{Negative formulae} & N & ::= p^\perp \mid A \wedge^- B \mid A \vee^- B \mid \forall x A \\ \text{Formulae} & A, B & ::= P \mid N \end{array}$$

where p ranges over a set of elements called *positive literals*. Formulae of the form p^\perp are called *negative literals*.

Definition 2 (Negation) Negation is extended from literals to all formulae:

$(p)^\perp$	$:= p^\perp$	$(p^\perp)^\perp$	$:= p$
$(A \wedge^+ B)^\perp$	$:= A^\perp \vee^- B^\perp$	$(A \wedge^- B)^\perp$	$:= A^\perp \vee^+ B^\perp$
$(A \vee^+ B)^\perp$	$:= A^\perp \wedge^- B^\perp$	$(A \vee^- B)^\perp$	$:= A^\perp \wedge^+ B^\perp$
$(\exists x A)^\perp$	$:= \forall x A^\perp$	$(\forall x A)^\perp$	$:= \exists x A^\perp$

Definition 3 ($LK(\mathcal{T})$) The sequent calculus $LK(\mathcal{T})$ manipulates two kinds of sequents:

$$\begin{array}{ll} \text{Focused sequents} & \Gamma \vdash [A] \\ \text{Unfocused sequents} & \Gamma \vdash \Delta \end{array}$$

where Γ is a multiset of negative formulae and positive literals, Δ is a multiset of formulae, and A is said to be in the *focus* of the (focused) sequent. By $\text{lit}(\Gamma)$ we denote the sub-multiset of Γ consisting of its literals.

The rules of $LK(\mathcal{T})$, given in Figure 1, are of three kinds: synchronous rules, asynchronous rules, and structural rules. These correspond to three alternating phases in the proof-search process that is described by the rules.

If S is a set of literals, $\mathcal{T}(S)$ is the call to the decision procedure on the conjunction of all literals of S . It holds if the procedure returns UNSAT.

2 Admissible rules

Definition 4 (Assumptions on the procedure)

We assume that the procedure calls satisfy the following properties:

- Weakening If $\mathcal{T}(S)$ then $\mathcal{T}(S, S')$.
- Contraction If $\mathcal{T}(S, A, A)$ then $\mathcal{T}(S, A)$.
- Instantiation If $\mathcal{T}(S)$ then $\mathcal{T}(\{\forall_x\}S)$.
- Consistency If $\mathcal{T}(S, p)$ and $\mathcal{T}(S, p^\perp)$ then $\mathcal{T}(S)$.
- Inconsistency $\mathcal{T}(S, p, p^\perp)$.

where S is a set of literals.

Lemma 1 (Admissibility of weakening and contraction)

The following rules are admissible in $LK(\mathcal{T})$.

$$\begin{array}{c} \frac{\Gamma \vdash [B]}{\Gamma, A \vdash [B]} \quad \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \\ \\ \frac{\Gamma, A, A \vdash [B]}{\Gamma, A \vdash [B]} \quad \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \end{array}$$

Proof: By induction on the derivation of the premiss. □

Synchronous rules	
$\frac{\Gamma \vdash [A] \quad \Gamma \vdash [B]}{\Gamma \vdash [A \wedge^+ B]}$	$\frac{\Gamma \vdash [A_i]}{\Gamma \vdash [A_1 \vee^+ A_2]}$
$\frac{\Gamma \vdash [\{\not{x}\} A]}{\Gamma \vdash [\exists x A]}$	
$\frac{}{\Gamma, p \vdash [p]} p \text{ positive literal}$	$\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp)}{\Gamma \vdash [p]} p \text{ positive literal}$
$\frac{\Gamma \vdash N}{\Gamma \vdash [N]} N \text{ negative}$	
Asynchronous rules	
$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge^- B, \Delta}$	$\frac{\Gamma \vdash A_1, A_2, \Delta}{\Gamma \vdash A_1 \vee^- A_2, \Delta}$
$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash (\forall x A), \Delta} x \notin \text{FV}(\Gamma, \Delta)$	
$\frac{\Gamma, A^\perp \vdash \Delta}{\Gamma \vdash A, \Delta} A \text{ positive or literal}$	
Structural rules	
$\frac{\Gamma, P^\perp \vdash [P]}{\Gamma, P^\perp \vdash} P \text{ positive}$	$\frac{\mathcal{T}(\text{lit}(\Gamma))}{\Gamma \vdash}$

Figure 1: System $\text{LK}(\mathcal{T})$

Lemma 2 (Admissibility of instantiation) *The following rules are admissible in $\text{LK}(\mathcal{T})$.*

$$\frac{\Gamma \vdash [B]}{\{\not{x}\} \Gamma \vdash [\{\not{x}\} B]} \quad \frac{\Gamma \vdash \Delta}{\{\not{x}\} \Gamma \vdash \{\not{x}\} \Delta}$$

Proof: By induction on the derivation of the premiss. □

3 Invertibility of the asynchronous phase

Lemma 3 (Invertibility of asynchronous rules) *All asynchronous rules are invertible in $\text{LK}(\mathcal{T})$.*

Proof: By induction on the derivation proving the conclusion of the asynchronous rule considered.

- Inversion of $A \wedge^- B$: by case analysis on the last rule actually used

$$\begin{aligned} & \frac{\Gamma \vdash A \wedge^- B, C, \Delta' \quad \Gamma \vdash A \wedge^- B, D, \Delta'}{\Gamma \vdash A \wedge^- B, C \wedge^- D, \Delta'} \\ & \text{By induction hypothesis we get} \\ & \frac{\Gamma \vdash A, C, \Delta' \quad \Gamma \vdash A, D, \Delta'}{\Gamma \vdash A, C \wedge^- D, \Delta'} \quad \text{and} \quad \frac{\Gamma \vdash B, C, \Delta' \quad \Gamma \vdash B, D, \Delta'}{\Gamma \vdash B, C \wedge^- D, \Delta'} \\ & \frac{\Gamma \vdash A \wedge^- B, C, D, \Delta'}{\Gamma \vdash A \wedge^- B, C \vee^- D, \Delta'} \\ & \text{By induction hypothesis we get} \quad \frac{\Gamma \vdash A, C, D, \Delta'}{\Gamma \vdash A, C \vee^- D, \Delta'} \quad \text{and} \quad \frac{\Gamma \vdash B, C, D, \Delta'}{\Gamma \vdash B, C \vee^- D, \Delta'} \\ & \frac{\Gamma \vdash A \wedge^- B, C, \Delta'}{\Gamma \vdash A \wedge^- B, (\forall x C), \Delta'} x \notin \text{FV}(\Gamma, \Delta', A \wedge^- B) \end{aligned}$$

- By induction hypothesis we get
- $$\frac{\Gamma \vdash A, C, \Delta'}{\Gamma \vdash A, (\forall x C), \Delta'} x \notin \text{FV}(\Gamma, \Delta', A) \text{ and } \frac{\Gamma \vdash B, C, \Delta'}{\Gamma \vdash B, (\forall x C), \Delta'} x \notin \text{FV}(\Gamma, \Delta', B)$$
- $\frac{\Gamma, C^\perp \vdash A \wedge^- B, \Delta'}{\Gamma \vdash A \wedge^- B, C, \Delta'}$ C positive or literal
- By induction hypothesis we get
- $$\frac{\Gamma, C^\perp \vdash A, \Delta'}{\Gamma \vdash A, C, \Delta'} C \text{ positive or literal and } \frac{\Gamma, C^\perp \vdash B, \Delta'}{\Gamma \vdash B, C, \Delta'} C \text{ positive or literal}$$
- Inversion of AV^-B

$$\frac{\Gamma \vdash AV^-B, C, \Delta' \quad \Gamma \vdash AV^-B, D, \Delta'}{\Gamma \vdash AV^-B, C \wedge^- D, \Delta'}$$

By induction hypothesis we get $\frac{\Gamma \vdash A, B, C, \Delta' \quad \Gamma \vdash A, B, D, \Delta'}{\Gamma \vdash A, B, C \wedge^- D, \Delta'}$

$$\frac{\Gamma \vdash AV^-B, C, D, \Delta'}{\Gamma \vdash AV^-B, C \vee^- D, \Delta'}$$

By induction hypothesis we get $\frac{\Gamma \vdash A, B, C, D, \Delta'}{\Gamma \vdash A, B, C \vee^- D, \Delta'}$

$$\frac{\Gamma \vdash AV^-B, C, \Delta'}{\Gamma \vdash AV^-B, (\forall x C), \Delta'} x \notin \text{FV}(\Gamma, \Delta')$$

By induction hypothesis we get $\frac{\Gamma \vdash A, B, C, \Delta'}{\Gamma \vdash A, B, (\forall x C), \Delta'} x \notin \text{FV}(\Gamma, \Delta')$

$$\frac{\Gamma, C^\perp \vdash AV^-B, \Delta'}{\Gamma \vdash AV^-B, C, \Delta'} C \text{ positive or literal}$$

By induction hypothesis we get $\frac{\Gamma, C^\perp \vdash A, B, \Delta'}{\Gamma \vdash A, B, C, \Delta'} C \text{ positive or literal}$
 - Inversion of $\forall x A$

$$\frac{\Gamma \vdash (\forall x A), C, \Delta' \quad \Gamma \vdash (\forall x A), D, \Delta'}{\Gamma \vdash (\forall x A), C \wedge^- D, \Delta'}$$

By induction hypothesis we get $\frac{\Gamma \vdash A, C, \Delta' \quad \Gamma \vdash A, D, \Delta'}{\Gamma \vdash A, C \wedge^- D, \Delta'} x \notin \text{FV}(\Gamma, \Delta')$

$$\frac{\Gamma \vdash (\forall x A), C, D, \Delta'}{\Gamma \vdash (\forall x A), C \vee^- D, \Delta'}$$

By induction hypothesis we get $\frac{\Gamma \vdash A, C, D, \Delta'}{\Gamma \vdash A, C \vee^- D, \Delta'}$

$$\frac{\Gamma \vdash (\forall x A), D, \Delta'}{\Gamma \vdash (\forall x A), (\forall x D), \Delta'} x \notin \text{FV}(\Gamma, \Delta')$$

By induction hypothesis we get $\frac{\Gamma \vdash A, D, \Delta'}{\Gamma \vdash A, (\forall x D), \Delta'} x \notin \text{FV}(\Gamma, \Delta')$

$$\frac{\Gamma, C^\perp \vdash (\forall x A), \Delta'}{\Gamma \vdash (\forall x A), C, \Delta'} C \text{ positive or literal}$$

By induction hypothesis we get $\frac{\Gamma, C^\perp \vdash A, \Delta'}{\Gamma \vdash A, C, \Delta'} C \text{ positive or literal}$
 - Inversion of literals and positive formulae (A)
$$\frac{\Gamma \vdash A, C, \Delta' \quad \Gamma \vdash A, D, \Delta'}{\Gamma \vdash A, C \wedge^- D, \Delta'}$$

By induction hypothesis we get $\frac{\Gamma, A^\perp \vdash C, \Delta' \quad \Gamma, A^\perp \vdash D, \Delta'}{\Gamma, A^\perp \vdash C \wedge^- D, \Delta'}$

$$\begin{array}{l}
- \frac{\Gamma \vdash A, C, D, \Delta'}{\Gamma \vdash A, C \vee^- D, \Delta'} \\
\text{By induction hypothesis } \frac{\Gamma, A^\perp \vdash C, D, \Delta'}{\Gamma, A^\perp \vdash C \vee^- D, \Delta'} \\
- \frac{\Gamma \vdash A, D, \Delta'}{\Gamma \vdash A, (\forall x D), \Delta'} \quad x \notin \text{FV}(\Gamma, \Delta') \\
\text{By induction hypothesis we get } \frac{\Gamma, A^\perp \vdash D, \Delta'}{\Gamma, A^\perp \vdash (\forall x D), \Delta'} \quad x \notin \text{FV}(\Gamma, \Delta') \\
- \frac{\Gamma, B^\perp \vdash A, \Delta'}{\Gamma \vdash A, B, \Delta'} \quad B \text{ positive or literal} \\
\text{By induction hypothesis we get } \frac{\Gamma, A^\perp, B^\perp \vdash \Delta'}{\Gamma, A^\perp \vdash B, \Delta'} \quad B \text{ positive or literal}
\end{array}$$

□

4 Cut-elimination

Theorem 4 (cut₁ and cut₂) *The following rules are admissible in LK(T).*

$$\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash \Delta}{\Gamma \vdash \Delta} \text{cut}_1 \quad \frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash [B]}{\Gamma \vdash [B]} \text{cut}_2$$

Proof: By simultaneous induction on the derivation of the right premiss.

We reduce cut₈ by case analysis on the last rule used to prove the right premiss.

$$\begin{array}{l}
\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash B, \Delta \quad \Gamma, p \vdash C, \Delta}{\Gamma \vdash B \wedge^- C, \Delta} \text{cut}_1 \\
\text{reduces to} \\
\frac{\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash B, \Delta}{\Gamma \vdash B, \Delta} \text{cut}_1 \quad \frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash C, \Delta}{\Gamma \vdash C, \Delta} \text{cut}_1}{\Gamma \vdash B \wedge^- C, \Delta} \\
\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \frac{\Gamma, p \vdash B_1, B_2, \Delta}{\Gamma, p \vdash B_1 \vee^- B_2, \Delta} \text{cut}_1}{\Gamma \vdash B_1 \vee^- B_2, \Delta} \text{cut}_1 \quad \text{reduces to} \quad \frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash B_1, B_2, \Delta}{\Gamma \vdash B_1, B_2, \Delta} \text{cut}_1 \\
\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \frac{\Gamma, p \vdash B, \Delta}{\Gamma, p \vdash \forall x B, \Delta} \text{cut}_1}{\Gamma \vdash \forall x B, \Delta} \text{cut}_1 \quad \text{reduces to} \quad \frac{\frac{\Gamma \vdash B, \Delta}{\Gamma \vdash \forall x B, \Delta} \quad \mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash B, \Delta}{\Gamma \vdash B, \Delta} \text{cut}_1 \\
\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \frac{\Gamma, p, B^\perp \vdash \Delta}{\Gamma, p \vdash B, \Delta} \text{cut}_1}{\Gamma \vdash B, \Delta} \text{cut}_1 \quad \text{reduces to} \quad \frac{\mathcal{T}(\text{lit}(\Gamma, B^\perp), p^\perp) \quad \Gamma, p, B^\perp \vdash \Delta}{\Gamma, B^\perp \vdash \Delta} \text{cut}_1 \\
\frac{\Gamma \vdash B, \Delta}{\Gamma \vdash B, \Delta} \\
\text{We have } \mathcal{T}(\text{lit}(\Gamma), p^\perp, B^\perp) \text{ as we assume the procedure to satisfy weakening.} \\
\text{If } P^\perp \in (\Gamma, p), \\
\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \frac{\Gamma, p \vdash [P]}{\Gamma, p \vdash} \text{cut}_1}{\Gamma \vdash} \text{reduces to} \quad \frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \Gamma, p^\perp \vdash [P]}{\Gamma \vdash [P]} \text{cut}_2 \\
\text{as } P^\perp \in (\Gamma).
\end{array}$$

$$\frac{\frac{\mathcal{T}(\text{lit}(\Gamma), p)}{\Gamma, p \vdash} \text{cut}_1}{\Gamma \vdash} \mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \text{reduces to} \quad \frac{\mathcal{T}(\text{lit}(\Gamma))}{\Gamma \vdash}$$

using the assumption of consistency.

We reduce cut_2 again by case analysis on the last rule used to prove the right premiss.

$$\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp)}{\Gamma \vdash [B \wedge^+ C]} \text{cut}_2 \quad \frac{\Gamma, p \vdash [B] \quad \Gamma, p \vdash [C]}{\Gamma, p \vdash [B \wedge^+ C]}$$

reduces to

$$\frac{\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash [B]}{\Gamma \vdash [B]} \text{cut}_2 \quad \frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash [C]}{\Gamma \vdash [C]} \text{cut}_2}{\Gamma \vdash [B \wedge^+ C]}$$

$$\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \frac{\Gamma, p \vdash [B_i]}{\Gamma, p \vdash [B_1 \vee^+ B_2]} \text{cut}_2}{\Gamma \vdash [B_1 \vee^+ B_2]} \text{cut}_2 \quad \text{reduces to} \quad \frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash [B_i]}{\Gamma \vdash [B_i]} \text{cut}_2$$

$$\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \frac{\Gamma, p \vdash [\{\! \int_x \! \} B]}{\Gamma, p \vdash [\exists x B]} \text{cut}_2}{\Gamma \vdash [\exists x B]} \text{cut}_2 \quad \text{reduces to} \quad \frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash [\{\! \int_x \! \} B]}{\Gamma \vdash [\{\! \int_x \! \} B]} \text{cut}_2$$

$$\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \frac{\Gamma, p \vdash N}{\Gamma, p \vdash [N]} \text{cut}_2}{\Gamma \vdash [N]} \text{cut}_2 \quad \text{reduces to} \quad \frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash N}{\Gamma \vdash N} \text{cut}_1$$

If $p' \in \Gamma, p$,

$$\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \overline{\Gamma, p \vdash [p']}}{\Gamma \vdash [p']} \text{cut}_2 \quad \text{reduces to} \quad \overline{\Gamma \vdash [p']} \quad \text{if} \quad p' \in \Gamma$$

$$\text{reduces to} \quad \frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp)}{\Gamma \vdash [p']} \quad \text{if} \quad p' = p$$

Finally,

$$\frac{\mathcal{T}(\text{lit}(\Gamma), p, p'^\perp)}{\Gamma, p \vdash [p']} \text{cut}_2}{\Gamma \vdash [p']} \text{cut}_2 \quad \text{reduces to} \quad \frac{\mathcal{T}(\text{lit}(\Gamma), p'^\perp)}{\Gamma \vdash [p']}$$

since weakening gives $\mathcal{T}(\text{lit}(\Gamma), p^\perp, p'^\perp)$ and consistency then gives $\mathcal{T}(\text{lit}(\Gamma), p'^\perp)$. \square

Theorem 5 (cut_3 , cut_4 and cut_5) *The following rules are admissible in $LK(\mathcal{T})$.*

$$\frac{\Gamma \vdash [A] \quad \Gamma \vdash A^\perp, \Delta}{\Gamma \vdash \Delta} \text{cut}_3$$

$$\frac{\Gamma \vdash N \quad \Gamma, N \vdash \Delta}{\Gamma \vdash \Delta} \text{cut}_4 \quad \frac{\Gamma \vdash N \quad \Gamma, N \vdash [B]}{\Gamma \vdash [B]} \text{cut}_5$$

Proof: By simultaneous induction on the following lexicographical measure:

- the size of the cut-formula (A or N)
- the fact that the cut-formula (A or N) is positive or negative (if of equal size, a positive formula is considered smaller than a negative formula)
- the height of the derivation of the right premiss

Weakenings and contractions (as they are admissible in the system) are implicitly used throughout this proof.

In order to eliminate cut_3 , we analyse which rule is used to prove the left premiss. We then use invertibility of the negative phase so that the last rule used in the right premiss is its dual one.

$$\frac{\frac{\Gamma \vdash [A] \quad \Gamma \vdash [B]}{\Gamma \vdash [A \wedge^+ B]} \quad \frac{\Gamma \vdash A^\perp, B^\perp, \Delta}{\Gamma \vdash A \vee^- B, \Delta} \quad \text{reduces to} \quad \frac{\Gamma \vdash [B] \quad \frac{\Gamma \vdash [A] \quad \Gamma \vdash A^\perp, B^\perp, \Delta}{\Gamma \vdash B^\perp, \Delta} \text{cut}_3}{\Gamma \vdash \Delta} \text{cut}_3$$

$$\frac{\frac{\Gamma \vdash [A_i]}{\Gamma \vdash [A_1 \vee^+ A_2]} \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A_1^\perp, \Delta} \quad \frac{\Gamma \vdash A_2^\perp, \Delta}{\Gamma \vdash A_1 \wedge^- A_2, \Delta} \quad \text{reduces to} \quad \frac{\Gamma \vdash [A_i] \quad \Gamma \vdash A_i^\perp, \Delta}{\Gamma \vdash \Delta} \text{cut}_3}{\Gamma \vdash \Delta} \text{cut}_3$$

$$\frac{\Gamma \vdash [\{\not{x}\}A]}{\Gamma \vdash [\exists x A]} \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A^\perp, \Delta} \quad \text{reduces to} \quad \frac{\Gamma \vdash [\{\not{x}\}A] \quad \frac{\Gamma \vdash A^\perp, \Delta}{\Gamma \vdash (\{\not{x}\}A^\perp), \Delta} \quad x \notin \text{FV}(\Gamma, \Delta)}{\Gamma \vdash \Delta} \text{cut}_3$$

using the admissibility of instantiation.

$$\frac{\frac{\Gamma \vdash N}{\Gamma \vdash [N]} \quad \frac{\Gamma, N \vdash \Delta}{\Gamma \vdash (N^\perp), \Delta} \quad \text{reduces to} \quad \frac{\Gamma \vdash N \quad \Gamma, N \vdash \Delta}{\Gamma \vdash \Delta} \text{cut}_4}{\Gamma \vdash \Delta} \text{cut}_3$$

We will describe below how cut_4 is reduced.

$$\frac{\frac{\Gamma, p \vdash \Delta}{\Gamma, p \vdash [p]} \quad \frac{\Gamma, p \vdash \Delta}{\Gamma, p \vdash (p^\perp), \Delta} \quad \text{reduces to} \quad \frac{\Gamma, p \vdash \Delta}{\Gamma, p \vdash \Delta}}$$

using the admissibility of contraction.

$$\frac{\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp)}{\Gamma \vdash [p]} \quad \frac{\Gamma, p \vdash \Delta}{\Gamma \vdash (p^\perp), \Delta} \quad \text{reduces to} \quad \frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash \Delta}{\Gamma \vdash \Delta} \text{cut}_1}{\Gamma \vdash \Delta} \text{cut}_3$$

In order to reduce cut_4 , we analyse which rule is used to prove the right premiss.

$$\frac{\frac{\Gamma \vdash N \quad \frac{\mathcal{T}(\text{lit}(\Gamma))}{\Gamma, N \vdash}}{\Gamma \vdash} \quad \text{reduces to} \quad \frac{\mathcal{T}(\text{lit}(\Gamma))}{\Gamma \vdash}}{\Gamma \vdash} \text{cut}_4$$

if N is not an literal (hence, it is not passed on to the procedure).

$$\frac{\frac{\Gamma, p \vdash}{\Gamma \vdash p^\perp} \quad \frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp)}{\Gamma, p^\perp \vdash} \quad \text{reduces to} \quad \frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp) \quad \Gamma, p \vdash}{\Gamma \vdash} \text{cut}_1}{\Gamma \vdash} \text{cut}_4$$

if p^\perp is an literal passed on to the procedure.

$$\frac{\frac{\Gamma, N \vdash [N^\perp]}{\Gamma, N \vdash} \quad \text{reduces to} \quad \frac{\Gamma \vdash N \quad \Gamma, N \vdash [N^\perp]}{\Gamma \vdash [N^\perp]} \text{cut}_5}{\Gamma \vdash} \text{cut}_4 \quad \frac{\Gamma \vdash N}{\Gamma \vdash N} \text{cut}_3$$

$$\frac{\frac{\Gamma, P^\perp, N \vdash [P]}{\Gamma, P^\perp, N \vdash} \quad \text{reduces to} \quad \frac{\Gamma, P^\perp \vdash N \quad \Gamma, P^\perp, N \vdash [P]}{\Gamma, P^\perp \vdash [P]} \text{cut}_5}{\Gamma, P^\perp \vdash} \text{cut}_4$$

$$\frac{\frac{\Gamma, N \vdash B, \Delta \quad \Gamma, N \vdash C, \Delta}{\Gamma \vdash N \quad \Gamma, N \vdash B \wedge^- C, \Delta} \quad \text{reduces to} \quad \frac{\Gamma \vdash B \wedge^- C, \Delta}{\Gamma \vdash B \wedge^- C, \Delta}}{\Gamma \vdash B \wedge^- C, \Delta} \text{cut}_4$$

reduces to

$$\frac{\frac{\Gamma \vdash N \quad \Gamma, N \vdash B, \Delta}{\Gamma \vdash B, \Delta} \text{cut}_4 \quad \frac{\Gamma \vdash N \quad \Gamma, N \vdash C, \Delta}{\Gamma \vdash C, \Delta} \text{cut}_4}{\Gamma \vdash B \wedge^- C, \Delta} \text{cut}_4$$

$$\begin{array}{c}
\frac{\Gamma, N \vdash B, C, \Delta}{\Gamma \vdash N \quad \Gamma, N \vdash B \vee^- C, \Delta} \text{cut}_4 \\
\frac{\Gamma \vdash B \vee^- C, \Delta}{\Gamma, N \vdash B, \Delta} \\
\frac{\Gamma \vdash N \quad \Gamma, N \vdash \forall x B, \Delta}{\Gamma \vdash \forall x B, \Delta} \text{cut}_4 \\
\frac{\Gamma \vdash \forall x B, \Delta}{\Gamma, N, B^\perp \vdash \Delta} \\
\frac{\Gamma \vdash N \quad \Gamma, N \vdash B, \Delta}{\Gamma \vdash B, \Delta} \text{cut}_4 \\
\text{reduces to} \\
\frac{\Gamma \vdash N \quad \Gamma, N \vdash B, C, \Delta}{\Gamma \vdash B, C, \Delta} \text{cut}_4 \\
\frac{\Gamma \vdash B \vee^- C, \Delta}{\Gamma \vdash N \quad \Gamma, N \vdash B, \Delta} \text{cut}_4 \\
\frac{\Gamma \vdash B, \Delta}{\Gamma \vdash \forall x B, \Delta} \\
\frac{\Gamma \vdash \forall x B, \Delta}{\Gamma, B^\perp \vdash N \quad \Gamma, N, B^\perp \vdash \Delta} \text{cut}_4 \\
\frac{\Gamma, B^\perp \vdash \Delta}{\Gamma \vdash B, \Delta} \text{cut}_4
\end{array}$$

using weakening, and if B is positive or a negative literal.

We have reduced all cases of cut_4 ; we now reduce the cases for cut_5 (again, by case analysis on the last rule used to prove the right premiss).

$$\begin{array}{c}
\frac{\Gamma, N \vdash [B] \quad \Gamma, N \vdash [C]}{\Gamma \vdash N \quad \Gamma, N \vdash [B \wedge^+ C]} \text{cut}_5 \\
\frac{\Gamma \vdash N \quad \Gamma, N \vdash [B \wedge^+ C]}{\Gamma \vdash [B \wedge^+ C]} \text{cut}_5 \\
\text{reduces to} \\
\frac{\Gamma \vdash N \quad \Gamma, N \vdash [B]}{\Gamma \vdash [B]} \text{cut}_5 \quad \frac{\Gamma \vdash N \quad \Gamma, N \vdash [C]}{\Gamma \vdash [C]} \text{cut}_5 \\
\frac{\Gamma \vdash [B \wedge^+ C]}{\Gamma \vdash [B \wedge^+ C]} \\
\frac{\Gamma, N \vdash [B_i]}{\Gamma \vdash N \quad \Gamma, N \vdash [B_1 \vee^+ B_2]} \text{cut}_5 \\
\frac{\Gamma \vdash N \quad \Gamma, N \vdash [B_1 \vee^+ B_2]}{\Gamma \vdash [B_1 \vee^+ B_2]} \text{cut}_5 \\
\frac{\Gamma \vdash [B_1 \vee^+ B_2]}{\Gamma, N \vdash [\{\checkmark_x\} B]} \\
\frac{\Gamma \vdash N \quad \Gamma, N \vdash [\exists x B]}{\Gamma \vdash [\exists x B]} \text{cut}_5 \\
\frac{\Gamma \vdash [\exists x B]}{\Gamma \vdash [\exists x B]} \\
\frac{\Gamma, N \vdash N'}{\Gamma \vdash N \quad \Gamma, N \vdash [N']} \text{cut}_5 \\
\frac{\Gamma \vdash N \quad \Gamma, N \vdash [N']}{\Gamma \vdash [N']} \text{cut}_5 \\
\text{reduces to} \\
\frac{\Gamma \vdash N \quad \Gamma, N \vdash [B_i]}{\Gamma \vdash [B_i]} \text{cut}_5 \\
\frac{\Gamma \vdash N \quad \Gamma, N \vdash [\{\checkmark_x\} B]}{\Gamma \vdash [\{\checkmark_x\} B]} \text{cut}_5 \\
\frac{\Gamma \vdash [\{\checkmark_x\} B]}{\Gamma \vdash [\{\checkmark_x\} B]} \\
\frac{\Gamma \vdash [\exists x B]}{\Gamma \vdash N \quad \Gamma, N \vdash N'} \text{cut}_4 \\
\frac{\Gamma \vdash N'}{\Gamma \vdash [N']} \text{cut}_4
\end{array}$$

since p has to be in Γ .

$$\begin{array}{c}
\frac{\Gamma \vdash N \quad \Gamma, N \vdash [p]}{\Gamma \vdash [p]} \text{cut}_5 \\
\text{reduces to} \\
\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp)}{\Gamma \vdash N \quad \Gamma, N \vdash [p]} \text{cut}_5 \\
\frac{\Gamma \vdash [p]}{\Gamma \vdash [p]} \\
\text{reduces to} \\
\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp)}{\Gamma \vdash [p]}
\end{array}$$

□

Theorem 6 (cut₆, cut₇, cut₈, and cut₉) *The following rules are admissible in LK(\mathcal{T}).*

$$\frac{\Gamma \vdash N, \Delta \quad \Gamma, N \vdash \Delta}{\Gamma \vdash \Delta} \text{cut}_6 \quad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash A^\perp, \Delta}{\Gamma \vdash \Delta} \text{cut}_7$$

$$\frac{\Gamma, l \vdash \Delta \quad \Gamma, l^\perp \vdash \Delta}{\Gamma \vdash \Delta} \text{cut}_8 \quad \frac{\Gamma, l_1, \dots, l_n \vdash \Delta \quad \Gamma, (l_1^\perp \vee^- \dots \vee^- l_n^\perp) \vdash \Delta}{\Gamma \vdash \Delta} \text{cut}_9$$

Proof: cut₆ is proved admissible by induction on the multiset Δ : the base case is the admissibility of cut₄, and the other cases just require the inversion of the connectives in Δ .

For cut₇, we can assume without loss of generality (swapping A and A^\perp) that A is negative. Applying inversion on $\Gamma \vdash A^\perp, \Delta$ gives a proof of $\Gamma, A \vdash \Delta$, and cut₇ is then obtained by cut₆:

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta} \text{cut}_6$$

cut₈ is obtained as follows:

$$\frac{\frac{\Gamma, l^\perp \vdash \Delta}{\Gamma \vdash l, \Delta} \quad \frac{\Gamma, l \vdash \Delta}{\Gamma \vdash l^\perp, \Delta}}{\Gamma \vdash \Delta} \text{cut}_7$$

cut₉ is obtained as follows:

$$\frac{\frac{\Gamma, (l_1^\perp \vee^- \dots \vee^- l_n^\perp) \vdash \Delta}{\Gamma \vdash l_1 \wedge^+ \dots \wedge^+ l_n, \Delta} \quad \frac{\frac{\Gamma, l_1, \dots, l_n \vdash \Delta}{\Gamma \vdash l_1^\perp, \dots, l_n^\perp, \Delta}}{\Gamma \vdash (l_1^\perp \vee^- \dots \vee^- l_n^\perp), \Delta}}{\Gamma \vdash \Delta} \text{cut}_7$$

□

5 Changing the Polarity of Predicates and Connectives

5.1 Changing the polarity of Predicates

In this section we try to show that changing the polarity of the predicates and the connectives that are present in a sequent, does not change the provability of the sequent in LK(\mathcal{T}). First, we deal with the polarities of the predicates and then we deal with the polarities of the connectives.

Lemma 7 *Let p and q be positive literals, and let us write $\Gamma' = \{p^\perp, p/q, q^\perp\}\Gamma$, $\Delta' = \{p^\perp, p/q, q^\perp\}\Delta$, $A' = \{p^\perp, p/q, q^\perp\}A$. Assume that q and p^\perp have the same meaning for \mathcal{T} , i.e. $\mathcal{T}(\text{lit}(\Gamma))$ if and only if $\mathcal{T}(\text{lit}(\Gamma'))$.*

1. If $\Gamma \vdash_{LK(\mathcal{T})} [A]$, then
 - (a) $\Gamma', p \vdash_{LK(\mathcal{T})} [A']$.
 - (b) $\Gamma', p^\perp \vdash_{LK(\mathcal{T})} [A']$ or $\Gamma', p^\perp \vdash_{LK(\mathcal{T})} \cdot$.
2. If $\Gamma \vdash_{LK(\mathcal{T})} \Delta$ then
 - (a) $\Gamma', p \vdash_{LK(\mathcal{T})} \Delta'$.
 - (b) $\Gamma', p^\perp \vdash_{LK(\mathcal{T})} \Delta'$.

Proof: By simultaneous induction on the last rule of the derivation.

1. For the first item of the lemma, by case analysis on the last rule of the derivation, we get

•

$$\frac{\Gamma \vdash [A_1] \quad \Gamma \vdash [A_2]}{\Gamma \vdash [A_1 \wedge^+ A_2]}$$

with $A = A_1 \wedge^+ A_2$.

- (a) The induction hypothesis on $\Gamma \vdash_{LK(\mathcal{T})} [A_1]$ (1(a)) gives $\Gamma', p \vdash_{LK(\mathcal{T})} [A'_1]$ and the induction hypothesis on $\Gamma \vdash_{LK(\mathcal{T})} [A_2]$ (1(a)) gives $\Gamma', p \vdash_{LK(\mathcal{T})} [A'_2]$. Now, we get:

$$\frac{\Gamma', p \vdash [A'_1] \quad \Gamma', p \vdash [A'_2]}{\Gamma', p \vdash [A'_1 \wedge^+ A'_2]}$$

- (b) The induction hypothesis on $\Gamma \vdash_{LK(\mathcal{T})} [A_1]$ (1(b)) gives $\Gamma', p^\perp \vdash_{LK(\mathcal{T})} [A'_1]$ or $\Gamma', p^\perp \vdash_{LK(\mathcal{T})}$ and the induction hypothesis on $\Gamma \vdash_{LK(\mathcal{T})} [A_2]$ (1(b)) gives $\Gamma', p^\perp \vdash_{LK(\mathcal{T})} [A'_2]$ or $\Gamma', p^\perp \vdash_{LK(\mathcal{T})}$.

If we get a proof of $\Gamma', p^\perp \vdash_{LK(\mathcal{T})}$ from one of the two applications of the induction hypothesis, we are done. If not then we get:

$$\frac{\Gamma', p^\perp \vdash [A'_1] \quad \Gamma', p \vdash [A'_2]}{\Gamma', p^\perp \vdash [A'_1 \wedge^+ A'_2]}$$

$$\frac{\Gamma \vdash [A_i]}{\Gamma \vdash [A_1 \vee^+ A_2]}$$

•

with $A = A_1 \vee^+ A_2$.

- (a) The induction hypothesis on $\Gamma \vdash_{LK(\mathcal{T})} [A_i]$ 1(a) gives $\Gamma', p \vdash_{LK(\mathcal{T})} [A'_i]$. We get:

$$\frac{\Gamma', p \vdash [A'_i]}{\Gamma', p \vdash [A'_i \vee^+ A'_2]}$$

- (b) The induction hypothesis on $\Gamma \vdash_{LK(\mathcal{T})} [A_i]$ (1(b)) gives $\Gamma', p^\perp \vdash_{LK(\mathcal{T})} [A'_i]$ or $\Gamma', p^\perp \vdash_{LK(\mathcal{T})}$.

In the latter case we are done. For the former case, we get:

$$\frac{\Gamma', p^\perp \vdash [A'_i]}{\Gamma', p^\perp \vdash [A'_i \vee^+ A'_2]}$$

$$\frac{\Gamma \vdash [\{\! \{ \! \}_x A]}{\Gamma \vdash [\exists x A]}$$

•

with $A = \exists x A$

- (a) The induction hypothesis on $\Gamma \vdash_{LK(\mathcal{T})} [\{\! \{ \! \}_x A]$ 1(a) gives $\Gamma', p \vdash_{LK(\mathcal{T})} [\{\! \{ \! \}_x A']$. We get:

$$\frac{\Gamma', p \vdash [\{\! \{ \! \}_x A']}{\Gamma', p \vdash [\exists x A']}$$

- (b) The induction hypothesis on $\Gamma \vdash_{LK(\mathcal{T})} [\{\! \{ \! \}_x A]$ (1(b)) gives $\Gamma', p^\perp \vdash_{LK(\mathcal{T})} [\{\! \{ \! \}_x A']$ or $\Gamma', p^\perp \vdash_{LK(\mathcal{T})}$.

In the latter case we are done. For the former case, we get:

$$\frac{\Gamma', p^\perp \vdash [\{\! \{ \! \}_x A']}{\Gamma', p^\perp \vdash [\exists x A']}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash [A]}$$

•

where A is Negative .

– Either $A = q^\perp$ and therefore:

- (a) $A' = p$ and we get:

$$\frac{\mathcal{T}(\text{lit}(\Gamma'), p, p^\perp)}{\Gamma', p \vdash [A']}$$

by the inconsistency of the theory \mathcal{T} .

(b) $\Gamma \vdash A$ can only be proved by

$$\frac{\Gamma, q \vdash}{\Gamma \vdash A}$$

The induction hypothesis on $\Gamma, q \vdash_{LK(\mathcal{T})}$ (2(b)) gives $\Gamma', p^\perp, p^\perp \vdash_{LK(\mathcal{T})}$ and then we get:

$$\frac{\Gamma', p^\perp, p^\perp \vdash}{\Gamma', p^\perp \vdash}$$

by contraction.

– Or $A \neq q^\perp$ and therefore

(a) The induction hypothesis on $\Gamma \vdash_{LK(\mathcal{T})} A$ (2(a)) gives $\Gamma', p \vdash_{LK(\mathcal{T})} A'$ and we get :

$$\frac{\Gamma', p \vdash A'}{\Gamma', p \vdash [A']}$$

(b) The induction hypothesis on $\Gamma \vdash_{LK(\mathcal{T})} A$ (2(b)) gives $\Gamma', p^\perp \vdash_{LK(\mathcal{T})} A'$ and we get:

$$\frac{\Gamma', p^\perp \vdash A'}{\Gamma', p^\perp \vdash [A']}$$

•

$$\frac{\mathcal{T}(\text{lit}(\Gamma), p'^\perp)}{\Gamma \vdash [p']}$$

where p' is a positive literal.

– Either $p' = q$ and therefore

(a) We get:

$$\frac{\mathcal{T}(\text{lit}(\Gamma'), p, p)}{\frac{\Gamma', p, p \vdash}{\Gamma', p \vdash [p^\perp]}}$$

We derive $\mathcal{T}(\text{lit}(\Gamma'), p, p)$ from $\mathcal{T}(\text{lit}(\Gamma), p'^\perp)$ since p has the same meaning as $q^\perp = p'^\perp$ and \mathcal{T} satisfies the weakening.

(b) We get:

$$\frac{\mathcal{T}(\text{lit}(\Gamma'), p^\perp, p)}{\frac{\Gamma', p^\perp, p \vdash}{\Gamma', p^\perp \vdash [p^\perp]}}$$

using the fact that \mathcal{T} satisfies inconsistency.

– or $p' \neq q$ and therefore

(a) We get:

$$\frac{\mathcal{T}(\text{lit}(\Gamma'), p, p'^\perp)}{\Gamma', p \vdash [p']}$$

using the fact that \mathcal{T} satisfies weakening.

(b) We get :

$$\frac{\mathcal{T}(\text{lit}(\Gamma'), p^\perp, p'^\perp)}{\Gamma', p^\perp \vdash [p']}$$

using the fact that \mathcal{T} satisfies weakening.

2. For the second item of the lemma, by case analysis on the last rule of the derivation, we get:

•

$$\frac{\Gamma \vdash A_1, \Delta \quad \Gamma \vdash A_2, \Delta}{\Gamma \vdash A_1 \wedge^\perp A_2, \Delta}$$

for $\Delta = A_1 \wedge^\perp A_2, \Delta$

- (a) The induction hypothesis on $\Gamma \vdash_{LK(\mathcal{T})} A_1, \Delta$ (2(a)) gives $\Gamma', p \vdash_{LK(\mathcal{T})} A'_1, \Delta'$ and the one on $\Gamma \vdash_{LK(\mathcal{T})} A_2, \Delta$ (2(a)) gives $\Gamma', p \vdash_{LK(\mathcal{T})} A'_2, \Delta'$.

We get:

$$\frac{\Gamma', p \vdash A'_1, \Delta' \quad \Gamma', p \vdash A'_2, \Delta'}{\Gamma', p \vdash A'_1 \wedge A'_2, \Delta'}$$

- (b) The induction hypothesis on $\Gamma \vdash_{LK(\mathcal{T})} A_1, \Delta$ (2(b)) gives $\Gamma', p^\perp \vdash_{LK(\mathcal{T})} A'_1, \Delta'$ and induction hypothesis on $\Gamma \vdash_{LK(\mathcal{T})} A_2, \Delta$ (2(b)) gives $\Gamma', p^\perp \vdash_{LK(\mathcal{T})} A'_2, \Delta'$.

We get:

$$\frac{\Gamma', p^\perp \vdash A'_1, \Delta' \quad \Gamma', p^\perp \vdash A'_2, \Delta'}{\Gamma', p^\perp \vdash A'_1 \wedge A'_2, \Delta'}$$

•

$$\frac{\Gamma \vdash A_1, A_2, \Delta}{\Gamma \vdash A_1 \vee A_2, \Delta}$$

for $\Delta = A_1 \vee A_2, \Delta$

- (a) The induction hypothesis on $\Gamma \vdash_{LK(\mathcal{T})} A_1, A_2, \Delta$ (2(a)) gives $\Gamma', p \vdash_{LK(\mathcal{T})} A'_1, A'_2, \Delta'$.

We get:

$$\frac{\Gamma', p \vdash A'_1, A'_2, \Delta'}{\Gamma', p \vdash A'_1 \vee A'_2, \Delta'}$$

- (b) The induction hypothesis on $\Gamma \vdash_{LK(\mathcal{T})} A_1, A_2, \Delta$ (2(b)) gives $\Gamma', p^\perp \vdash_{LK(\mathcal{T})} A'_1, A'_2, \Delta'$.

We get:

$$\frac{\Gamma', p^\perp \vdash A'_1, A'_2, \Delta'}{\Gamma', p^\perp \vdash A'_1 \vee A'_2, \Delta'}$$

•

$$\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash (\forall x A), \Delta} \quad x \notin \text{FV}(\Gamma, \Delta)$$

with $\Delta = (\forall x A), \Delta$

- (a) The induction hypothesis on $\Gamma \vdash_{LK(\mathcal{T})} A, \Delta$ (2(a)) gives $\Gamma', p \vdash_{LK(\mathcal{T})} A', \Delta'$.

We get:

$$\frac{\Gamma \vdash A', \Delta'}{\Gamma' \vdash (\forall x A'), \Delta'} \quad x \notin \text{FV}(\Gamma', \Delta')$$

- (b) The induction hypothesis on $\Gamma \vdash_{LK(\mathcal{T})} A, \Delta$ (2(b)) gives $\Gamma', p^\perp \vdash_{LK(\mathcal{T})} A', \Delta'$.

We get:

$$\frac{\Gamma', p^\perp \vdash A', \Delta'}{\Gamma', p^\perp \vdash (\forall x A'), \Delta'} \quad x \notin \text{FV}(\Gamma', \Delta')$$

•

$$\frac{\Gamma, A^\perp \vdash \Delta}{\Gamma \vdash A, \Delta}$$

for $\Delta = A, \Delta$ where A is positive or negative literal.

- (a) The induction hypothesis on $\Gamma, A^\perp \vdash_{LK(\mathcal{T})} \Delta$ (2(a)) gives $\Gamma', p, A'^\perp \vdash_{LK(\mathcal{T})} \Delta'$ and we get:

$$\frac{\Gamma', p, A'^\perp \vdash \Delta'}{\Gamma', p \vdash A', \Delta'}$$

- (b) The induction hypothesis on $\Gamma, A \vdash_{LK(\mathcal{T})} \Delta$ (2(b)) gives $\Gamma', p^\perp, A'^\perp \vdash_{LK(\mathcal{T})} \Delta'$ and we get:

$$\frac{\Gamma', p^\perp, A'^\perp \vdash \Delta'}{\Gamma', p^\perp \vdash A', \Delta'}$$

•

$$\frac{\Gamma \vdash [A]}{\Gamma \vdash}$$

with A is positive and $A^\perp \in \Gamma$.

– Either $A = q$, and then $q^\perp \in \Gamma$ and $\mathcal{T}(\text{lit}(\Gamma), q^\perp)$

(a) We get :

$$\frac{\mathcal{T}(\text{lit}(\Gamma'), p)}{\Gamma', p \vdash}$$

since p and q^\perp have the same meaning for \mathcal{T}

(b) We get :

$$\frac{\mathcal{T}(\text{lit}(\Gamma'), p^\perp)}{\Gamma', p^\perp \vdash}$$

since $p \in \Gamma'$ and by using the fact that \mathcal{T} is inconsistent.

– or $A \neq q$

(a) The induction hypothesis on $\Gamma \vdash_{LK(\mathcal{T})} [A]$ gives $\Gamma', p \vdash_{LK(\mathcal{T})} [A']$ and we get

$$\frac{\Gamma', p \vdash [A']}{\Gamma', p \vdash}$$

(b) The induction hypothesis on $\Gamma \vdash_{LK(\mathcal{T})} [A]$ gives either $\Gamma', p^\perp \vdash_{LK(\mathcal{T})} [A']$ or $\Gamma', p^\perp \vdash_{LK(\mathcal{T})}$. For the latter case we are done and for the former case we get:

$$\frac{\Gamma', p^\perp \vdash [A']}{\Gamma', p^\perp \vdash}$$

□

Corollary 8 *If $\Gamma \vdash_{LK(\mathcal{T})} \Delta$ then $\Gamma' \vdash_{LK(\mathcal{T})} \Delta'$.*

Proof: Lemma 7 (2) provides $\Gamma', p \vdash_{LK(\mathcal{T})} \Delta'$ and $\Gamma', p^\perp \vdash_{LK(\mathcal{T})} \Delta'$. Then we can construct:

$$\frac{\Gamma', p' \vdash \Delta' \quad \Gamma', p^\perp \vdash \Delta'}{\Gamma' \vdash \Delta'} \text{ cut}_8$$

and we use the admissibility of cut_8 . □

We have proven that changing the polarities of the predicate that are present in a sequent, does not change the provability of that sequent in $LK(\mathcal{T})$.

5.2 Changing the polarity of connectives

Definition 5 ($LK^+(\mathcal{T})$) The sequent calculus $LK^+(\mathcal{T})$ manipulates one kind of sequent:

$$\Gamma \vdash [\mathcal{X}] \Delta \text{ where } \mathcal{X} :: \bullet \mid A$$

Here Γ is a multiset of negative formulae and positive literals, Δ is a multiset of formulae, \mathcal{X} is said to be in the *focus* of the (focused) system. By $\text{lit}(\Gamma)$ we denote the sub-multiset of Γ consisting of its literals and $\text{lit}(\Delta)$ we denote the sub-multiset of Δ consisting of its literals.

The rules of $LK^+(\mathcal{T})$, given in Figure 2, are of three kinds: synchronous rules, asynchronous rules, and structural rules. These correspond to three alternating phases in the proof-search process that is described by the rules.

The $LK^+(\mathcal{T})$ system is an extension system of $LK(\mathcal{T})$.

Remark 9 As in $LK(\mathcal{T})$, weakening and contraction are admissible in $LK^+(\mathcal{T})$.

Lemma 10 $\vdash [A^\perp]A$ is provable in $LK^+(\mathcal{T})$.

Proof: By induction on A using an extended but well founded order on formulae: a formula is smaller than another one when either it contains fewer connectives, or the number of connectives is equal the former formula is negative and the latter is positive.

Synchronous rules	
$\frac{\Gamma \vdash [A]\Delta \quad \Gamma \vdash [B]\Delta}{\Gamma \vdash [A \wedge^+ B]\Delta}$	$\frac{\Gamma \vdash [A_i]\Delta}{\Gamma \vdash [A_1 \vee^+ A_2]\Delta}$
$\frac{\Gamma \vdash [\not{x}A]\Delta}{\Gamma \vdash [\exists x A]\Delta}$	$\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp, \text{lit}(\Delta))}{\Gamma \vdash [p]\Delta}$ p positive literal
$\frac{}{\Gamma, p \vdash [p]\Delta}$ p positive literal	$\frac{\Gamma \vdash [\bullet]N}{\Gamma \vdash [N]}$ N negative
Asynchronous rules	
$\frac{\Gamma \vdash [\mathcal{X}]A, \Delta \quad \Gamma \vdash [\mathcal{X}]B, \Delta}{\Gamma \vdash [\mathcal{X}]A \wedge^- B, \Delta}$	$\frac{\Gamma \vdash [\mathcal{X}]A_1, A_2, \Delta}{\Gamma \vdash [\mathcal{X}]A_1 \vee^- A_2, \Delta}$
$\frac{\Gamma \vdash [\mathcal{X}]A, \Delta}{\Gamma \vdash [\mathcal{X}](\forall x A), \Delta}$ $x \notin \text{FV}(\Gamma, \mathcal{X}, \Delta)$	$\frac{\Gamma, A^\perp \vdash [\mathcal{X}]\Delta}{\Gamma \vdash [\mathcal{X}]A, \Delta}$ A positive or literal
Structural rules	
$\frac{\Gamma, P^\perp \vdash [P]\Delta}{\Gamma, P^\perp \vdash [\bullet]\Delta}$ P positive	$\frac{\mathcal{T}(\text{lit}(\Gamma), \text{lit}(\Delta))}{\Gamma \vdash [\bullet]\Delta}$

Figure 2: System $\text{LK}^+(\mathcal{T})$

- $A = A_1 \wedge^- A_2$

$$\frac{\frac{\vdash [A_1^\perp]A_1}{\vdash [A_1^\perp \vee^+ A_2^\perp]A_1} \quad \frac{\vdash [A_2^\perp]A_2}{\vdash [A_1^\perp \vee^+ A_2^\perp]A_2}}{\vdash [A_1^\perp \vee^+ A_2^\perp]A_1 \wedge^- A_2}$$

We can complete the proof on the left-hand side by applying the induction hypothesis on A_1 and on the right-hand side by applying the induction hypothesis on A_2 .

- $A = A_1 \vee^- A_2$

$$\frac{\frac{\vdash [A_1^\perp]A_1}{\vdash [A_1^\perp]A_1, A_2} \text{WK} \quad \frac{\vdash [A_2^\perp]A_2}{\vdash [A_2^\perp]A_1, A_2} \text{WK}}{\frac{\vdash [A_1^\perp \wedge^+ A_2^\perp]A_1, A_2}{\vdash [A_1^\perp \wedge^+ A_2^\perp]A_1 \vee^- A_2}}$$

We can complete the proof on the left-hand side by applying the induction hypothesis on A_1 and on the right-hand side by applying the induction hypothesis on A_2 .

- $A = \forall x A$

$$\frac{\frac{\vdash [A^\perp]A}{\vdash [\{t/x\}A^\perp]A} \text{choosing } t=x}{\frac{\vdash [\exists x A^\perp]A}{\vdash [\exists x A^\perp]\forall x A} \text{ } x \notin \text{FV}(\exists x A^\perp)}$$

We can complete the proof by applying the induction hypothesis on A .

- $A = p^\perp$

$$\frac{p \vdash [p]}{\vdash [p]p^\perp}$$

- $A = P$ where P is all positive formulae:

$$\frac{\frac{\frac{\frac{\vdash [P]P^\perp}{P^\perp \vdash [P]P^\perp} WK}{P^\perp \vdash [\bullet]P^\perp}}{P^\perp \vdash [P^\perp]}}{\vdash [P^\perp]P}$$

We can complete the proof by applying the induction hypothesis on P .

□

Lemma 11 (Invertibility of asynchronous rules)

All asynchronous rules are height-preserving invertible in $LK^+(\mathcal{T})$.

Proof: By induction on the derivation proving the conclusion of the asynchronous rule considered.

- Inversion of $A\wedge^-B$: by case analysis on the last rule actually used

$$\frac{\Gamma \vdash [\mathcal{X}]A\wedge^-B, C, \Delta' \quad \Gamma \vdash [\mathcal{X}]A\wedge^-B, D, \Delta'}{\Gamma \vdash [\mathcal{X}]A\wedge^-B, C\wedge^-D, \Delta'}$$

By induction hypothesis we get

$$\frac{\Gamma \vdash [\mathcal{X}]A, C, \Delta' \quad \Gamma \vdash [\mathcal{X}]A, D, \Delta'}{\Gamma \vdash [\mathcal{X}]A, C\wedge^-D, \Delta'} \quad \text{and} \quad \frac{\Gamma \vdash [\mathcal{X}]B, C, \Delta' \quad \Gamma \vdash [\mathcal{X}]B, D, \Delta'}{\Gamma \vdash [\mathcal{X}]B, C\wedge^-D, \Delta'}$$

$$\frac{\Gamma \vdash [\mathcal{X}]A\wedge^-B, C, D, \Delta'}{\Gamma \vdash [\mathcal{X}]A\wedge^-B, C\vee^-D, \Delta'}$$

By induction hypothesis we get: $\frac{\Gamma \vdash [\mathcal{X}]A, C, D, \Delta'}{\Gamma \vdash [\mathcal{X}]A, C\vee^-D, \Delta'}$ and $\frac{\Gamma \vdash [\mathcal{X}]B, C, D, \Delta'}{\Gamma \vdash [\mathcal{X}]B, C\vee^-D, \Delta'}$

$$\frac{\Gamma \vdash [\mathcal{X}]A\wedge^-B, C, \Delta'}{\Gamma \vdash [\mathcal{X}]A\wedge^-B, (\forall xC), \Delta'} \quad x \notin \text{FV}(\Gamma, \mathcal{X}, \Delta', A\wedge^-B)$$

By induction hypothesis we get:

$$\frac{\Gamma \vdash [\mathcal{X}]A, C, \Delta'}{\Gamma \vdash [\mathcal{X}]A, (\forall xC), \Delta'} \quad x \notin \text{FV}(\Gamma, \mathcal{X}, \Delta', A) \quad \text{and} \quad \frac{\Gamma \vdash [\mathcal{X}]B, C, \Delta'}{\Gamma \vdash [\mathcal{X}]B, (\forall xC), \Delta'} \quad x \notin \text{FV}(\Gamma, \mathcal{X}, \Delta', B)$$

$$\frac{\Gamma, C^\perp \vdash [\mathcal{X}]A\wedge^-B, \Delta'}{\Gamma \vdash [\mathcal{X}]A\wedge^-B, C, \Delta'} \quad C \text{ positive or literal}$$

By induction hypothesis we get:

$$\frac{\Gamma, C^\perp \vdash [\mathcal{X}]A, \Delta'}{\Gamma \vdash [\mathcal{X}]A, C, \Delta'} \quad C \text{ positive or literal} \quad \text{and} \quad \frac{\Gamma, C^\perp \vdash [\mathcal{X}]B, \Delta'}{\Gamma \vdash [\mathcal{X}]B, C, \Delta'} \quad C \text{ positive or literal}$$

$$\frac{\Gamma \vdash [C]A\wedge^-B, \Delta' \quad \Gamma \vdash [D]A\wedge^-B, \Delta'}{\Gamma \vdash [C\wedge^+D,]A\wedge^-B, \Delta'}$$

By induction hypothesis we get

$$\frac{\Gamma \vdash [C]A, \Delta' \quad \Gamma \vdash [D]A, \Delta'}{\Gamma \vdash [C\wedge^+D]A, \Delta'} \quad \text{and} \quad \frac{\Gamma \vdash [C]B, \Delta' \quad \Gamma \vdash [D]B, \Delta'}{\Gamma \vdash [C\wedge^+D]B, \Delta'}$$

$$\frac{\Gamma \vdash [C_i]A\wedge^-B, \Delta'}{\Gamma \vdash [C_1\vee^+C_2]A\wedge^-B, \Delta'}$$

By induction hypothesis we get

$$\frac{\Gamma \vdash [C_i]A, \Delta'}{\Gamma \vdash [C_1\vee^+C_2]A, \Delta'} \quad \text{and} \quad \frac{\Gamma \vdash [C_i]B, \Delta'}{\Gamma \vdash [C_1\vee^+C_2]B, \Delta'}$$

$$\frac{\Gamma \vdash \{\{t_x\}C\}A \wedge^- B, \Delta'}{\Gamma \vdash [\exists x C]A \wedge^- B, \Delta'}$$

By induction hypothesis we get

$$\frac{\Gamma \vdash \{\{t_x\}C\}A, \Delta'}{\Gamma \vdash [\exists x C]A, \Delta'} \quad \text{and} \quad \frac{\Gamma \vdash \{\{t_x\}C\}B, \Delta'}{\Gamma \vdash [\exists x C]B, \Delta'}$$

- Inversion of AV^-B

$$\frac{\Gamma \vdash [\mathcal{X}]AV^-B, C, \Delta' \quad \Gamma \vdash [\mathcal{X}]AV^-B, D, \Delta'}{\Gamma \vdash [\mathcal{X}]AV^-B, C \wedge^- D, \Delta'}$$

By induction hypothesis we get $\frac{\Gamma \vdash [\mathcal{X}]A, B, C, \Delta' \quad \Gamma \vdash [\mathcal{X}]A, B, D, \Delta'}{\Gamma \vdash [\mathcal{X}]A, B, C \wedge^- D, \Delta'}$

$$\frac{\Gamma \vdash [\mathcal{X}]AV^-B, C, D, \Delta'}{\Gamma \vdash [\mathcal{X}]AV^-B, C \vee^- D, \Delta'}$$

By induction hypothesis we get $\frac{\Gamma \vdash [\mathcal{X}]A, B, C, D, \Delta'}{\Gamma \vdash [\mathcal{X}]A, B, C \vee^- D, \Delta'}$

$$\frac{\Gamma \vdash [\mathcal{X}]AV^-B, C, \Delta'}{\Gamma \vdash [\mathcal{X}]AV^-B, (\forall x C), \Delta'} \quad x \notin \text{FV}(\Gamma, \mathcal{X}, \Delta')$$

By induction hypothesis we get $\frac{\Gamma \vdash [\mathcal{X}]A, B, C, \Delta'}{\Gamma \vdash [\mathcal{X}], A, B, (\forall x C), \Delta'} \quad x \notin \text{FV}(\Gamma, \mathcal{X}, \Delta')$

$$\frac{\Gamma, C^\perp \vdash [\mathcal{X}], AV^-B, \Delta'}{\Gamma \vdash [\mathcal{X}], AV^-B, C, \Delta'} \quad C \text{ positive or literal}$$

By induction hypothesis we get $\frac{\Gamma, C^\perp \vdash [\mathcal{X}], A, B, \Delta'}{\Gamma \vdash [\mathcal{X}], A, B, C, \Delta'} \quad C \text{ positive or literal}$

$$\frac{\Gamma \vdash [C]AV^-B, \Delta' \quad \Gamma \vdash [D]AV^-B, \Delta'}{\Gamma \vdash [C \wedge^+ D]AV^-B, \Delta'}$$

By induction hypothesis we get $\frac{\Gamma \vdash [C]A, B, \Delta' \quad \Gamma \vdash [D]A, B, \Delta'}{\Gamma \vdash [C \wedge^+ D]A, B, C \wedge^- D, \Delta'}$

$$\frac{\Gamma \vdash [C_i]AV^-B, \Delta'}{\Gamma \vdash [C_1 \vee^+ C_2]AV^-B, \Delta'}$$

By induction hypothesis we get $\frac{\Gamma \vdash [C_i]A, B, \Delta'}{\Gamma \vdash [C_1 \vee^+ C_2]A, B, \Delta'}$

$$\frac{\Gamma \vdash \{\{t_x\}C\}AV^-B, \Delta'}{\Gamma \vdash [\exists x C]AV^-B, \Delta'}$$

By induction hypothesis we get $\frac{\Gamma \vdash \{\{t_x\}C\}A, B, \Delta'}{\Gamma \vdash [\exists x C]A, B, \Delta'}$

- Inversion of $\forall x A$

$$\frac{\Gamma \vdash [\mathcal{X}], (\forall x A), C, \Delta' \quad \Gamma \vdash [\mathcal{X}], (\forall x A), D, \Delta'}{\Gamma \vdash [\mathcal{X}], (\forall x A), C \wedge^- D, \Delta'}$$

By induction hypothesis we get $\frac{\Gamma \vdash [\mathcal{X}], A, C, \Delta' \quad \Gamma \vdash [\mathcal{X}], A, D, \Delta'}{\Gamma \vdash [\mathcal{X}], A, C \wedge^- D, \Delta'} \quad x \notin \text{FV}(\Gamma, \mathcal{X}, \Delta')$

$$\frac{\Gamma \vdash [\mathcal{X}], (\forall x A), C, D, \Delta'}{\Gamma \vdash [\mathcal{X}], (\forall x A), C \vee^- D, \Delta'}$$

By induction hypothesis we get $\frac{\Gamma \vdash [\mathcal{X}], A, C, D, \Delta'}{\Gamma \vdash [\mathcal{X}], A, C \vee^- D, \Delta'}$

$$\frac{\Gamma \vdash [\mathcal{X}], (\forall x A), D, \Delta'}{\Gamma \vdash [\mathcal{X}], (\forall x A), (\forall x D), \Delta'} \quad x \notin \text{FV}(\Gamma, \mathcal{X}, \Delta')$$

$$\begin{array}{l}
\text{By induction hypothesis we get } \frac{\Gamma \vdash [\mathcal{X}], A, D, \Delta'}{\Gamma \vdash [\mathcal{X}], A, (\forall x D), \Delta'} x \notin \text{FV}(\Gamma, \mathcal{X}, \Delta') \\
- \frac{\Gamma, C^\perp \vdash [\mathcal{X}], (\forall x A), \Delta'}{\Gamma \vdash [\mathcal{X}], (\forall x A), C, \Delta'} C \text{ positive or literal} \\
\text{By induction hypothesis we get } \frac{\Gamma, C^\perp \vdash [\mathcal{X}], A, \Delta'}{\Gamma \vdash [\mathcal{X}], A, C, \Delta'} C \text{ positive or literal} \\
- \frac{\Gamma \vdash [C], (\forall x A), \Delta' \quad \Gamma \vdash [D], (\forall x A), \Delta'}{\Gamma \vdash [C \wedge^+ D], (\forall x A), \Delta'} \\
\text{By induction hypothesis we get } \frac{\Gamma \vdash [C], A, \Delta' \quad \Gamma \vdash [D], A, \Delta'}{\Gamma \vdash [C \wedge^+ D], A, \Delta'} x \notin \text{FV}(\Gamma, C \wedge^+ D, \Delta') \\
- \frac{\Gamma \vdash [C_i], (\forall x A), \Delta'}{\Gamma \vdash [C_1 \vee^+ C_2], (\forall x A), \Delta'} \\
\text{By induction hypothesis we get } \frac{\Gamma \vdash [C_i], A, \Delta'}{\Gamma \vdash [C_1 \vee^+ C_2], A, \Delta'} \\
- \frac{\Gamma \vdash [\{x\}^t D], (\forall x A), \Delta'}{\Gamma \vdash [\exists x D], (\forall x A), \Delta'} x \notin \text{FV}(\Gamma, \exists x D, \Delta') \\
\text{By induction hypothesis we get } \frac{\Gamma \vdash [\{x\}^t D], A, \Delta'}{\Gamma \vdash [\exists x D], A, \Delta'} x \notin \text{FV}(\Gamma, \exists x D, \Delta') \\
\bullet \text{ Inversion of literals and positive formulae (A)} \\
- \frac{\Gamma \vdash [\mathcal{X}], A, C, \Delta' \quad \Gamma \vdash [\mathcal{X}], A, D, \Delta'}{\Gamma \vdash [\mathcal{X}], A, C \wedge^- D, \Delta'} \\
\text{By induction hypothesis we get } \frac{\Gamma, A^\perp \vdash [\mathcal{X}], C, \Delta' \quad \Gamma, A^\perp \vdash [\mathcal{X}], D, \Delta'}{\Gamma, A^\perp \vdash [\mathcal{X}], C \wedge^- D, \Delta'} \\
- \frac{\Gamma \vdash [\mathcal{X}], A, C, D, \Delta'}{\Gamma \vdash [\mathcal{X}], A, C \vee^- D, \Delta'} \\
\text{By induction hypothesis } \frac{\Gamma, A^\perp \vdash [\mathcal{X}], C, D, \Delta'}{\Gamma, A^\perp \vdash [\mathcal{X}], C \vee^- D, \Delta'} \\
- \frac{\Gamma \vdash [\mathcal{X}], A, D, \Delta'}{\Gamma \vdash [\mathcal{X}], A, (\forall x D), \Delta'} x \notin \text{FV}(\Gamma, \mathcal{X}, \Delta') \\
\text{By induction hypothesis we get } \frac{\Gamma, A^\perp \vdash [\mathcal{X}], D, \Delta'}{\Gamma, A^\perp \vdash [\mathcal{X}], (\forall x D), \Delta'} x \notin \text{FV}(\Gamma, \mathcal{X}, \Delta') \\
- \frac{\Gamma, B^\perp \vdash [\mathcal{X}], A, \Delta'}{\Gamma \vdash [\mathcal{X}], A, B, \Delta'} B \text{ positive or literal} \\
\text{By induction hypothesis we get } \frac{\Gamma, A^\perp, B^\perp \vdash [\mathcal{X}], \Delta'}{\Gamma, A^\perp \vdash [\mathcal{X}], B, \Delta'} B \text{ positive or literal} \\
- \frac{\Gamma \vdash [C], A, \Delta' \quad \Gamma \vdash [D], A, \Delta'}{\Gamma \vdash [C \wedge^+ D], A, \Delta'} \\
\text{By induction hypothesis we get } \frac{\Gamma, A^\perp \vdash [C], \Delta' \quad \Gamma, A^\perp \vdash [D], \Delta'}{\Gamma, A^\perp \vdash [C \wedge^+ D], \Delta'} \\
- \frac{\Gamma \vdash [C_i], A, \Delta'}{\Gamma \vdash [C_1 \vee^+ C_2], A, \Delta'} \\
\text{By induction hypothesis } \frac{\Gamma, A^\perp \vdash [C_i], \Delta'}{\Gamma, A^\perp \vdash [C_1 \vee^+ C_2], \Delta'} \\
- \frac{\Gamma \vdash [\{x\}^t D], A, \Delta'}{\Gamma \vdash [\exists x D], A, \Delta'}
\end{array}$$

By induction hypothesis we get $\frac{\Gamma, A^\perp \vdash [\{t/x\}D]\Delta'}{\Gamma, A^\perp \vdash [\exists xD], \Delta'}$

□

Lemma 12

1. If $\Gamma \vdash [A]$ is provable in $LK^+(\mathcal{T})$, then $\Gamma \vdash [A]$ is provable in $LK(\mathcal{T})$.
2. If $\Gamma \vdash [\bullet]\Delta$ is provable in $LK^+(\mathcal{T})$, then $\Gamma \vdash \Delta$ is provable in $LK(\mathcal{T})$.

Proof: By simultaneous induction on the assumed derivation.

1. For the first item we get, by case analysis on the last rule of the derivation:

•

$$\frac{\Gamma \vdash [A_1] \quad \Gamma \vdash [A_2]}{\Gamma \vdash [A_1 \wedge^+ A_2]}$$

with $A = A_1 \wedge^+ A_2$. The induction hypothesis on $\Gamma \vdash_{LK^+(\mathcal{T})} [A_1]$ gives $\Gamma \vdash_{LK(\mathcal{T})} [A_1]$ and the induction hypothesis on $\Gamma \vdash_{LK^+(\mathcal{T})} [A_2]$ gives $\Gamma \vdash_{LK(\mathcal{T})} [A_2]$. We get:

$$\frac{\Gamma \vdash [A_1] \quad \Gamma \vdash [A_2]}{\Gamma \vdash [A_1 \wedge^+ A_2]}$$

•

$$\frac{\Gamma \vdash [A_i]}{\Gamma \vdash [A_1 \vee^+ A_2]}$$

with $A = A_1 \vee^+ A_2$.

The induction hypothesis on $\Gamma \vdash_{LK^+(\mathcal{T})} [A_i]$ gives $\Gamma \vdash_{LK(\mathcal{T})} [A_i]$. We get:

$$\frac{\Gamma \vdash [A_i]}{\Gamma \vdash [A_1 \vee^+ A_2]}$$

•

$$\frac{\Gamma \vdash [\{t/x\}A]}{\Gamma \vdash [\exists xA]}$$

with $A = \exists xA$.

The induction hypothesis on $\Gamma \vdash_{LK^+(\mathcal{T})} [\{t/x\}A]$ gives $\Gamma \vdash_{LK(\mathcal{T})} [\{t/x\}A]$. We get:

$$\frac{\Gamma \vdash [\{t/x\}A]}{\Gamma \vdash [\exists xA]}$$

•

$$\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp)}{\Gamma \vdash [p]}$$

with $A = p$ where p is a positive literal.

The induction hypothesis on $\mathcal{T}(\text{lit}(\Gamma), p^\perp)$ gives $\mathcal{T}(\text{lit}(\Gamma), p^\perp)$ in $LK(\mathcal{T})$ and we get:

$$\frac{\mathcal{T}(\text{lit}(\Gamma), p^\perp)}{\Gamma \vdash [p]}$$

•

$$\frac{\Gamma \vdash [\bullet]N}{\Gamma \vdash [N]}$$

with $A = N$ and N is negative.

The induction hypothesis on $\Gamma \vdash_{LK^+(\mathcal{T})} [\bullet]N$ gives $\Gamma \vdash_{LK(\mathcal{T})} N$. We get:

$$\frac{\Gamma \vdash N}{\Gamma \vdash [N]}$$

2. For the second item, we use the height-preserving invertibility of the asynchronous rules, so that we can assume without loss of generality that if Δ is not empty then the last rule of the derivation decomposes one of its formulae.

•

$$\frac{\Gamma \vdash [\bullet]A_1, \Delta_1 \quad \Gamma \vdash [\bullet]A_2, \Delta_1}{\Gamma \vdash [\bullet]A_1 \wedge^- A_2, \Delta_1}$$

with $\Delta = A_1 \wedge^- A_2, \Delta_1$. The induction hypothesis on $\Gamma \vdash_{LK+(\mathcal{T})} [\bullet]A_1, \Delta_1$ gives $\Gamma \vdash_{LK(\mathcal{T})} A_1, \Delta_1$ and the induction hypothesis on $\Gamma \vdash_{LK+(\mathcal{T})} [\bullet]A_2, \Delta_2$ gives $\Gamma \vdash_{LK(\mathcal{T})} A_2, \Delta_2$. We get :

$$\frac{\Gamma \vdash A_1, \Delta_1 \quad \Gamma \vdash A_2, \Delta_1}{\Gamma \vdash A_1 \wedge^- A_2, \Delta_1}$$

•

$$\frac{\Gamma \vdash [\bullet]A_1, A_2, \Delta_1}{\Gamma \vdash [\bullet]A_1 \vee^- A_2, \Delta_1}$$

with $\Delta = A_1 \vee^- A_2, \Delta_1$. The induction hypothesis on $\Gamma \vdash_{LK+(\mathcal{T})} [\bullet]A_1, A_2, \Delta_1$ gives $\Gamma \vdash_{LK(\mathcal{T})} A_1, A_2, \Delta_1$ and we get :

$$\frac{\Gamma \vdash A_1, A_2, \Delta_1}{\Gamma \vdash A_1 \vee^- A_2, \Delta_1}$$

•

$$\frac{\Gamma \vdash [\bullet]A, \Delta_1}{\Gamma \vdash [\bullet]\forall xA, \Delta_1} \quad x \notin \text{FV}(\Gamma, \Delta_1)$$

with $\Delta = \forall xA, \Delta_1$.

The induction hypothesis on $\Gamma \vdash_{LK+(\mathcal{T})} [\bullet]A, \Delta_1$ gives $\Gamma \vdash_{LK(\mathcal{T})} A, \Delta_1$. We get:

$$\frac{\Gamma \vdash A, \Delta_1}{\Gamma \vdash \forall xA, \Delta_1} \quad x \notin \text{FV}(\Gamma, \Delta_1)$$

•

$$\frac{\Gamma, A^\perp \vdash [\bullet]\Delta_1}{\Gamma \vdash [\bullet]A, \Delta_1}$$

with $\Delta = A, \Delta_1$ and A is a positive or literal.

The induction hypothesis on $\Gamma, A^\perp \vdash_{LK+(\mathcal{T})} [\bullet]\Delta_1$ gives $\Gamma, A^\perp \vdash_{LK(\mathcal{T})} \Delta_1$. We get :

$$\frac{\Gamma, A^\perp \vdash \Delta_1}{\Gamma \vdash A, \Delta_1}$$

•

$$\frac{\Gamma, P^\perp \vdash [P]\Delta}{\Gamma, P^\perp \vdash [\bullet]\Delta}$$

where P is positive.

As already mentioned, we can assume without loss of generality that Δ is empty. The induction hypothesis on $\Gamma, P^\perp \vdash_{LK+(\mathcal{T})} [P] (1)$ gives $\Gamma, P^\perp \vdash_{LK(\mathcal{T})} [P]$. We get:

$$\frac{\Gamma, P^\perp \vdash [P]}{\Gamma, P^\perp \vdash}$$

•

$$\frac{\mathcal{T}(\text{lit}(\Gamma), \text{lit}(\Delta))}{\Gamma \vdash [\bullet]\Delta}$$

As already mentioned, we can assume without loss of generality that Δ is empty. We get:

$$\frac{\mathcal{T}(\text{lit}(\Gamma))}{\Gamma \vdash}$$

□

Lemma 13 *We have:*

1. $\vdash_{LK(\mathcal{T})} (A\wedge^+B)^\perp, (A\wedge^-B)$ and
2. $\vdash_{LK(\mathcal{T})} (A\wedge^-B)^\perp, (A\wedge^+B)$.

Proof:

1. For the first item we get:

$$\begin{array}{c}
\frac{\frac{\vdash [A^\perp]A}{A \vdash [A^\perp]A} \text{WK} \quad \frac{\vdash [B^\perp]B}{B \vdash [B^\perp]B} \text{WK}}{A \vdash [\bullet]A \quad B \vdash [\bullet]B} \\
\frac{\frac{\vdash [\bullet]A^\perp, A}{\vdash [\bullet]A^\perp, B^\perp, A} \text{WK} \quad \frac{\vdash [\bullet]B^\perp, B}{\vdash [\bullet]A^\perp, B^\perp, B} \text{WK}}{\vdash [\bullet](A^\perp \vee^- B^\perp), (A\wedge^- B)} \\
\frac{\vdash [\bullet](A\wedge^+ B)^\perp, (A\wedge^- B)}{\vdash (A\wedge^+ B)^\perp, (A\wedge^- B)} \text{Lemma 12(2)}
\end{array}$$

Both left hand side and right hand side can be closed by Lemma 10.

2. For the second item, we get:

$$\begin{array}{c}
\frac{\frac{\vdash [A^\perp]A}{\vdash [A^\perp \vee^+ B^\perp]A} \text{WK} \quad \frac{\frac{\vdash [B^\perp]B}{\vdash [A^\perp \vee^+ B^\perp]B} \text{WK} \quad \frac{\frac{\vdash [A]A^\perp}{\vdash [A]A^\perp, B^\perp} \text{WK} \quad \frac{\vdash [B]B^\perp}{\vdash [B]A^\perp, B^\perp} \text{WK}}{\vdash [A\wedge^+ B]A^\perp, B^\perp} \text{WK}}{A\wedge^- B \vdash [A^\perp \vee^+ B^\perp]A} \text{WK}}{A\wedge^- B \vdash [\bullet]A} \text{Lemma 12(2)} \\
\frac{\frac{\frac{\vdash [\bullet]B}{A\wedge^- B \vdash [\bullet]B} \text{Lemma 12(2)} \quad \frac{\vdash [\bullet]A^\perp, B}{A\wedge^- B \vdash A^\perp, B} \text{WK}}{(A\wedge^- B), (A^\perp \vee^- B^\perp) \vdash A^\perp, B} \text{WK} \quad \frac{\frac{\frac{\vdash [A\wedge^+ B]A^\perp, B^\perp}{A^\perp \vee^- B^\perp \vdash [A\wedge^+ B]A^\perp, B^\perp} \text{WK}}{A^\perp \vee^- B^\perp \vdash [\bullet]A^\perp, B^\perp} \text{Lemma 12(2)}}{(A\wedge^- B), (A^\perp \vee^- B^\perp) \vdash A^\perp, B^\perp} \text{WK}}{(A\wedge^- B), (A^\perp \vee^- B^\perp) \vdash A^\perp} \text{cut}_7 \\
\frac{\vdash (A\wedge^- B), (A\wedge^+ B)^\perp \vdash}{\vdash (A\wedge^- B)^\perp, (A\wedge^+ B)}
\end{array}$$

All branches are closed by Lemma 10. \square

Lemma 14 *If $\Gamma \vdash_{LK(\mathcal{T})} \Delta, C$ and $\Gamma \vdash_{LK(\mathcal{T})} D, C^\perp$ then $\Gamma \vdash_{LK(\mathcal{T})} \Delta, D$.*

Proof:

$$\frac{\frac{\Gamma \vdash \Delta, C}{\Gamma \vdash D, \Delta, C} \text{WK} \quad \frac{\Gamma \vdash D, C^\perp}{\Gamma \vdash \Delta, D, C^\perp} \text{WK}}{\Gamma \vdash \Delta, D} \text{cut}_7$$

\square

Corollary 15

1. *If $\Gamma \vdash A\wedge^+ B, \Delta$ then $\Gamma \vdash A\wedge^- B, \Delta$. **Proof:** By Lemma 14 and Lemma 13(1). \square*
2. *If $\Gamma \vdash A\wedge^- B, \Delta$ then $\Gamma \vdash A\wedge^+ B, \Delta$. **Proof:** By Lemma 14 and Lemma 13(2). \square*
3. *If $\Gamma \vdash A\vee^+ B, \Delta$ then $\Gamma \vdash A\vee^- B, \Delta$. **Proof:** By Lemma 14 and Lemma 13(1). \square*
4. *If $\Gamma \vdash A\vee^- B, \Delta$ then $\Gamma \vdash A\vee^+ B, \Delta$. **Proof:** By Lemma 14 and Lemma 13(2). \square*

We have proven that changing the polarities of the connectives that are present in a sequent, does not change the provability of that sequent in $\text{LK}(\mathcal{T})$.

6 The $\text{LK}^p(\mathcal{T})$ sequent calculus: on-the-fly polarisation of literals

Definition 6 (Literals) From now on we distinguish polarised literals and unpolarised literals. The former are those literals used so far in $\text{LK}(\mathcal{T})$; the later are introduced as follows:

Let \mathcal{L} be a set of elements called *unpolarised literals*, equipped with an involutive function called *negation* from \mathcal{L} to \mathcal{L} . In the rest of this paper, a possibly primed or indexed lowercase l always denotes a literal, and l^\perp its negation.

From now on, the expression literals refers to unpolarised literals.

Definition 7 (Formulae, negation) The formulae $\text{LK}^p(\mathcal{T})$ are given by the following grammar:

$$\text{Formulae } A, B, \dots ::= l \mid A \wedge^+ B \mid A \vee^+ B \mid A \wedge^- B \mid A \vee^- B$$

where l ranges over literals.

Let $\mathcal{P} \subseteq \mathcal{L}$ such that if $l \in \mathcal{P}$, $l^\perp \notin \mathcal{P}$.

We define \mathcal{P} -positive formulae and \mathcal{P} -negative formulae as the formulae generated by the following grammars:

$$\mathcal{P}\text{-positive formulae } P, \dots ::= l \mid A \wedge^+ B \mid A \vee^+ B$$

$$\mathcal{P}\text{-negative formulae } N, \dots ::= l^\perp \mid A \wedge^- B \mid A \vee^- B$$

where l ranges over \mathcal{P} .

Negation is extended from literals to all formulae using the following definitions:

$(A \wedge^+ B)^\perp$	$:= A^\perp \vee^- B^\perp$	$(A \wedge^- B)^\perp$	$:= A^\perp \vee^+ B^\perp$
$(A \vee^+ B)^\perp$	$:= A^\perp \wedge^- B^\perp$	$(A \vee^- B)^\perp$	$:= A^\perp \wedge^+ B^\perp$
$(\exists x A)^\perp$	$:= \forall x A^\perp$	$(\forall x A)^\perp$	$:= \exists x A^\perp$

Definition 8 (System $\text{LK}^p(\mathcal{T})$) The sequent calculus $\text{LK}^p(\mathcal{T})$ is given by the rules of Figure 3, where Γ and Δ are multisets of formulae.

$\frac{\Gamma \vdash_{\mathcal{T}}^{\mathcal{P}} [A] \quad \Gamma \vdash_{\mathcal{T}}^{\mathcal{P}} [B]}{\Gamma \vdash_{\mathcal{T}}^{\mathcal{P}} [A \wedge^+ B]} \quad \frac{\Gamma \vdash_{\mathcal{T}}^{\mathcal{P}} [A_i]}{\Gamma \vdash_{\mathcal{T}}^{\mathcal{P}} [A_1 \vee^+ A_2]}$
$\frac{\mathcal{T}(\text{lit}(\Gamma), l^\perp)}{\Gamma \vdash_{\mathcal{T}}^{\mathcal{P}, l} [l]} \quad l \text{ is } \mathcal{P}\text{-positive} \quad \frac{\Gamma \vdash_{\mathcal{T}}^{\mathcal{P}} N}{\Gamma \vdash_{\mathcal{T}}^{\mathcal{P}} [N]} \quad N \text{ is } \mathcal{P}\text{-negative}$
$\frac{\Gamma \vdash_{\mathcal{T}}^{\mathcal{P}} A, \Delta \quad \Gamma \vdash_{\mathcal{T}}^{\mathcal{P}} B, \Delta}{\Gamma \vdash_{\mathcal{T}}^{\mathcal{P}} A \wedge^- B, \Delta} \quad \frac{\Gamma \vdash_{\mathcal{T}}^{\mathcal{P}} A_1, A_2, \Delta}{\Gamma \vdash_{\mathcal{T}}^{\mathcal{P}} A_1 \vee^- A_2, \Delta} \quad \frac{\Gamma, A^\perp \vdash_{\mathcal{T}}^{\mathcal{P}} \Gamma'}{\Gamma \vdash_{\mathcal{T}}^{\mathcal{P}} A, \Gamma'} \quad A \text{ is } \mathcal{P}\text{-positive or literal}$
$\frac{\Gamma \vdash_{\mathcal{T}}^{\mathcal{P}, l}}{\Gamma \vdash_{\mathcal{T}}^{\mathcal{P}}} \quad \frac{\Gamma, P^\perp \vdash_{\mathcal{T}}^{\mathcal{P}} [P]}{\Gamma, P^\perp \vdash_{\mathcal{T}}^{\mathcal{P}}} \quad P \text{ is } \mathcal{P}\text{-positive} \quad \frac{\mathcal{T}(\text{lit}(\Gamma))}{\Gamma \vdash_{\mathcal{T}}^{\mathcal{P}}}$

Figure 3: System $\text{LK}^p(\mathcal{T})$

There are two cuts that we use in $\text{LK}^p(\mathcal{T})$.

- Analytic cut:

$$\frac{\Gamma, l \vdash_{\mathcal{T}}^{\mathcal{P}} \quad \Gamma, l^\perp \vdash_{\mathcal{T}}^{\mathcal{P}}}{\Gamma \vdash_{\mathcal{T}}^{\mathcal{P}}}$$

- General cut:

$$\frac{\Gamma, l_1, \dots, l_n \vdash_{\mathcal{T}}^{\mathcal{P}} \quad \Gamma, (l_1^\perp \vee^- \dots \vee^- l_n^\perp) \vdash_{\mathcal{T}}^{\mathcal{P}}}{\Gamma \vdash_{\mathcal{T}}^{\mathcal{P}}}$$

Definition 9 (ϕ compatible with \mathcal{P}) Let ϕ be a function that maps every unpolarised literal to a polarised literal that has the same meaning for \mathcal{T} and such that $\phi(l^\perp) = \phi(l)^\perp$ for all l . Let \mathcal{P} be a set of literals of $\text{LK}^p(\mathcal{T})$.

- ϕ is said to be compatible with \mathcal{P} if $\forall l \in \mathcal{P}$, $\phi(l)$ is a positive literal of $\text{LK}(\mathcal{T})$.
- ϕ is extended into a mapping of formulae, and multisets of formulae, so that we can write $\phi(A)$, $\phi(B)$, $\phi(\Gamma)$, $\phi(\Delta)$, etc. .

Corollary 16

- If $\Gamma \vdash^{\mathcal{P}} \Delta$ in $\text{LK}^p(\mathcal{T})$ then for all ϕ compatible with \mathcal{P} , $\phi(\Gamma) \vdash \phi(\Delta)$ in $\text{LK}(\mathcal{T})$.
- If $\Gamma \vdash^{\mathcal{P}} [A]$ in $\text{LK}^p(\mathcal{T})$ then for all ϕ compatible with \mathcal{P} , $\phi(\Gamma) \vdash [\phi(A)]$ in $\text{LK}(\mathcal{T})$.

Proof: In our proof we use Γ' for $\phi(\Gamma)$, A' for $\phi(A)$, B' for $\phi(B)$ and Δ' for $\phi(\Delta)$, etc.

1.

$$\frac{\Gamma \vdash^{\mathcal{P}, l}}{\Gamma \vdash^{\mathcal{P}}}$$

since ϕ is compatible with \mathcal{P}, l .

- If $\phi(l)$ is positive, then the induction hypothesis gives $\phi(\Gamma) \vdash$.
- If $\phi(l)$ is negative, let p be a positive literal with the same meaning as $\phi(l)$ for \mathcal{T} . Then let ϕ' be defined as:

$$\begin{cases} \phi'(l') = \phi(l') & \text{if } l' \neq l \text{ and } l' \neq l^\perp \\ \phi'(l) = p \\ \phi'(l^\perp) = p^\perp \end{cases}$$

ϕ' is compatible with \mathcal{P}, l , so by induction hypothesis we get $\phi'(\Gamma) \vdash$. By corollary 8, we have $\{\phi(l), \phi(l)^\perp / \phi'(l), \phi'(l)^\perp\} \phi' \Gamma \vdash$.

2.

$$\frac{\Gamma \vdash^{\mathcal{P}} [A_1] \quad \Gamma \vdash^{\mathcal{P}} [A_2]}{\Gamma \vdash^{\mathcal{P}} [A_1 \wedge^+ A_2]}$$

for $A = A_1 \wedge^+ A_2$.

The induction hypothesis on $\Gamma \vdash^{\mathcal{P}} [A_1]$ gives $\Gamma' \vdash [A'_1]$ in $\text{LK}(\mathcal{T})$ and the induction hypothesis on $\Gamma \vdash^{\mathcal{P}} [A_2]$ gives $\Gamma' \vdash [A'_2]$ in $\text{LK}(\mathcal{T})$. We build:

$$\frac{\Gamma' \vdash [A'_1] \quad \Gamma' \vdash [A'_2]}{\Gamma' \vdash [A'_1 \wedge^+ A'_2]}$$

3.

$$\frac{\Gamma \vdash^{\mathcal{P}} [A_i]}{\Gamma \vdash^{\mathcal{P}} [A_1 \vee^+ A_2]}$$

for $A = A_1 \vee^+ A_2$.

The induction hypothesis on $\Gamma \vdash^{\mathcal{P}} [A_i]$ gives $\Gamma' \vdash [A'_i]$ in $\text{LK}(\mathcal{T})$ and then we build:

$$\frac{\Gamma' \vdash [A'_i]}{\Gamma' \vdash [A'_1 \vee^+ A'_2]}$$

4.

$$\frac{\mathcal{T}(\text{lit}(\Gamma), l^\perp)}{\Gamma \vdash_{\mathcal{T}}^{\mathcal{P}, l} [l]}$$

for $A = l$.

Let $p = \phi(l)$ and we can build:

$$\frac{\mathcal{T}(\text{lit}(\Gamma'), p^\perp)}{\Gamma' \vdash [p]}$$

5.

$$\frac{\Gamma \vdash^{\mathcal{P}} N}{\Gamma \vdash^{\mathcal{P}} [N]}$$

for $A = N$ where N is \mathcal{P} -negative.

The induction hypothesis on $\Gamma \vdash^{\mathcal{P}} N$ gives $\Gamma' \vdash N'$ in $\text{LK}(\mathcal{T})$ then we build:

$$\frac{\Gamma' \vdash N'}{\Gamma' \vdash [N']}$$

6.

$$\frac{\Gamma \vdash^{\mathcal{P}} A_1, \Delta \quad \Gamma \vdash^{\mathcal{P}} A_2, \Delta}{\Gamma \vdash^{\mathcal{P}} A_1 \wedge^- A_2, \Delta}$$

for $\Delta = A_1 \wedge^- A_2, \Delta$

The induction hypothesis on $\Gamma \vdash^{\mathcal{P}} A_1, \Delta$ gives $\Gamma' \vdash A'_1, \Delta'$ in $\text{LK}(\mathcal{T})$ and the induction hypothesis on $\Gamma \vdash^{\mathcal{P}} A_2, \Delta$ gives $\Gamma' \vdash A'_2, \Delta'$ in $\text{LK}(\mathcal{T})$. We get:

$$\frac{\Gamma' \vdash A', \Delta' \quad \Gamma' \vdash B', \Delta'}{\Gamma' \vdash A' \wedge^- B', \Delta'}$$

7.

$$\frac{\Gamma \vdash^{\mathcal{P}} A_1, A_2, \Delta}{\Gamma \vdash^{\mathcal{P}} A_1 \vee^- A_2, \Delta}$$

for $\Delta = A_1 \vee^- A_2, \Delta$.

The induction hypothesis on $\Gamma \vdash^{\mathcal{P}} A_1, A_2, \Delta$ gives $\Gamma' \vdash A'_1, A'_2, \Delta'$ in $\text{LK}(\mathcal{T})$. We get:

$$\frac{\Gamma' \vdash A'_1, A'_2, \Delta'}{\Gamma' \vdash A'_1 \vee^- A'_2, \Delta'}$$

8.

$$\frac{\Gamma, A^\perp \vdash^{\mathcal{P}} \Delta}{\Gamma \vdash^{\mathcal{P}} A, \Delta}$$

for $\Delta = A, \Delta$ where A is \mathcal{P} -positive or literal.

The induction hypothesis on $\Gamma, A^\perp \vdash^{\mathcal{P}} \Delta$ gives $\Gamma', A'^\perp \vdash \Delta'$ in $\text{LK}(\mathcal{T})$. We get:

$$\frac{\Gamma', A'^\perp \vdash \Delta'}{\Gamma' \vdash A', \Delta'}$$

9.

$$\frac{\Gamma, P^\perp \vdash^{\mathcal{P}} [P]}{\Gamma, P^\perp \vdash^{\mathcal{P}}} \quad P \text{ is } \mathcal{P}\text{-positive}$$

where P is \mathcal{P} -positive.

The induction hypothesis on $\Gamma, P^\perp \vdash^{\mathcal{P}} [P]$ gives $\Gamma, P'^\perp \vdash [P']$ in $\text{LK}(\mathcal{T})$. We get:

$$\frac{\Gamma, P'^\perp \vdash [P']}{\Gamma', P'^\perp \vdash}$$

10.

$$\frac{\mathcal{T}(\text{lit}(\Gamma))}{\Gamma \vdash}$$

We get:

$$\frac{\mathcal{T}(\text{lit}(\Gamma'))}{\Gamma' \vdash}$$

11.

$$\frac{\Gamma, l \vdash^{\mathcal{P}} \quad \Gamma, l^{\perp} \vdash^{\mathcal{P}}}{\Gamma \vdash^{\mathcal{P}}}$$

For analytic cut in $\text{LK}^{\mathcal{P}}(\mathcal{T})$.

The induction hypothesis on $\Gamma, l \vdash^{\mathcal{P}}$ gives $\Gamma', l' \vdash$ in $\text{LK}(\mathcal{T})$ and the induction hypothesis on $\Gamma, l^{\perp} \vdash^{\mathcal{P}}$ gives $\Gamma', l'^{\perp} \vdash$ in $\text{LK}(\mathcal{T})$. Then we build:

$$\frac{\Gamma', l' \vdash \quad \Gamma', l'^{\perp} \vdash}{\Gamma' \vdash} \text{cut}_8$$

and we eliminate the cut_8 by Theorem 6.

12.

$$\frac{\Gamma, l_1, \dots, l_n \vdash^{\mathcal{P}} \quad \Gamma, (l_1^{\perp} \vee^{-} \dots \vee^{-} l_n^{\perp}) \vdash^{\mathcal{P}}}{\Gamma \vdash^{\mathcal{P}}}$$

The induction hypothesis on $\Gamma, l_1, \dots, l_n \vdash^{\mathcal{P}}$ gives $\Gamma', l'_1, \dots, l'_n \vdash$ and the induction hypothesis on $\Gamma, (l_1^{\perp} \vee^{-} \dots \vee^{-} l_n^{\perp}) \vdash^{\mathcal{P}}$ gives $\Gamma', (l'_1{}^{\perp} \vee^{-} \dots \vee^{-} l'_n{}^{\perp}) \vdash$. Then we build:

$$\frac{\Gamma', l'_1, \dots, l'_n \vdash \quad \Gamma', (l'_1{}^{\perp} \vee^{-} \dots \vee^{-} l'_n{}^{\perp}) \vdash}{\Gamma' \vdash} \text{cut}_9$$

and we eliminate cut_9 by Theorem 6. □

7 Completeness

Definition 10 (Formulae) Let \mathcal{L} be the set of unpolarised literals as defined in Section 6.

Let \mathcal{A} be a subset of \mathcal{L} , whose elements will be called *atoms*, and such that the image of \mathcal{A} by $^{\perp}$ is its complement in \mathcal{L} : $\mathcal{L} \setminus \mathcal{A}$.

The formulae of first-order logic are given by the following grammar:

$$\text{Formulae } A, B, \dots ::= a \mid A \vee B \mid A \wedge B \mid \forall x A \mid \exists x A \mid \neg A$$

where a ranges over atoms.

Definition 11 (ϕ and $\bar{\phi}$) Let ϕ be a function that maps every atom to a polarised literal that has the same meaning for \mathcal{T} .

Let $\bar{\phi}$ be the function that maps every formula of first-order logic to a set of polarised formulae defined as follows:

$$\begin{aligned} \bar{\phi}(a) &= \{\phi(a)\} \\ \bar{\phi}(A \wedge B) &= \{A' \wedge^{-} B', A' \wedge^{+} B' \mid A' \in \phi(A), B' \in \phi(B)\} \\ \bar{\phi}(A \vee B) &= \{A' \vee^{-} B', A' \vee^{+} B' \mid A' \in \phi(A), B' \in \phi(B)\} \\ \bar{\phi}(\exists x A) &= \{\exists x A' \mid A' \in \phi(A)\} \\ \bar{\phi}(\forall x A) &= \{\forall x A' \mid A' \in \phi(A)\} \\ \bar{\phi}(\neg A) &= \{A'^{\perp} \mid A' \in \phi(A)\} \\ \bar{\phi}(\Delta, A) &= \{\Delta', A' \mid \Delta' \in \phi(\Delta), A' \in \phi(A)\} \\ \bar{\phi}(\emptyset) &= \emptyset \end{aligned}$$

Remark 17 $\bar{\phi}(A) \neq \emptyset$

Remark 18 If $A' \in \bar{\phi}(A)$, then $\{\check{v}_x\} A' \in \bar{\phi}(\{\check{v}_x\} A)$.

If $C' \in \bar{\phi}(\{\check{v}_x\} A)$, then $C' = \{\check{v}_x\} A'$ for some $A' \in \bar{\phi}(A)$.

Notation 19 In the rest of this section we will use the notation $A \wedge^? B$ (resp. $A \vee^? B$) to ambiguously represent either $A \wedge^{+} B$ or $A \wedge^{-} B$ (resp. $A \vee^{+} B$ or $A \vee^{-} B$). This will make the proofs more compact, noticing that Corollary 15(2) and 15(4) respectively imply the admissibility of

$$\frac{\Gamma \vdash \Delta, A \wedge^{-} B}{\Gamma \vdash \Delta, A \wedge^? B} \quad \frac{\Gamma \vdash \Delta, A \vee^{-} B}{\Gamma \vdash \Delta, A \vee^? B}$$

Lemma 20 For all $A', A'' \in \bar{\phi}(A)$, we have $\vdash_{LK(\mathcal{T})} A', A''^\perp$

Proof: In the proof below, for any formula A , the notations A' and A'' will systematically designate elements of $\phi(A)$.

The proof is by induction on A :

1. $A = a$.

Let $A', A'' \in \bar{\phi}(a) = \{\phi(a)\}$. Therefore $A' = A'' = A = \phi(a)$.

$$\frac{\frac{\phi(a) \vdash [\phi(a)]}{\phi^\perp(a), \phi(a) \vdash}}{\vdash \phi(a), \phi^\perp(a)}$$

2. $A = A_1 \wedge A_2$

Let $A'_1, A''_1 \in \bar{\phi}(A_1)$, $A'_2, A''_2 \in \bar{\phi}(A_2)$ and $A' = A'_1 \wedge^? A'_2$, $A'' = A''_1 \wedge^? A''_2$.

$$\frac{\frac{\frac{\vdash A'_1, A''_1^\perp}{\vdash A'_1, A''_1^\perp, A''_2^\perp} \text{WK} \quad \frac{\vdash A'_2, A''_2^\perp}{\vdash A'_2, A''_1^\perp, A''_2^\perp} \text{WK}}{\vdash A'_1 \wedge^- A'_2, A''_1^\perp \vee^- A''_2^\perp}}{\vdash A', A''_1^\perp \vee^- A''_2^\perp}}{\vdash A', A''^\perp}$$

We can complete the proof on the left-hand side by applying the induction on A_1 and on the right-hand side by applying the induction hypothesis on A_2 .

3. $A = A_1 \vee A_2$

Let $A'_1, A''_1 \in \bar{\phi}(A_1)$, $A'_2, A''_2 \in \bar{\phi}(A_2)$ and $A' = A'_1 \vee^? A'_2$, $A'' = A''_1 \vee^? A''_2$.

$$\frac{\frac{\frac{\vdash A'_1, A''_1^\perp}{\vdash A'_1, A'_2, A''_1^\perp} \text{WK} \quad \frac{\vdash A'_2, A''_2^\perp}{\vdash A'_1, A'_2, A''_2^\perp} \text{WK}}{\vdash A'_1 \vee^- A'_2, A''_1^\perp \wedge^- A''_2^\perp}}{\vdash A', A''_1^\perp \wedge^- A''_2^\perp}}{\vdash A', A''^\perp}$$

We can complete the proof on the left-hand side by applying the induction on A_1 and on the right-hand side by applying the induction hypothesis on A_2 .

4. $A = \forall x A_1$

Let $A' = \forall x A'_1$ and $A'' = \forall x A''_1$.

$$\frac{\frac{\frac{\vdash [A''_1^\perp] A''_1}{\vdash [\exists x A''_1] A''_1} \text{WK}}{\forall x A''_1 \vdash [\bullet] A''_1} \text{WK}}{\vdash A''_1, \exists x A''_1^\perp} \text{Lemma 12(2)} \quad \vdash A'_1, A''_1^\perp}{\vdash A'_1, \exists x A''_1^\perp} \text{Lemma 14}}{\vdash \forall x A'_1, \exists x A''_1^\perp}$$

We can complete the proof on the left-hand side by Lemma 10 and the right-hand side by applying the induction hypothesis on A_1 .

5. $A = \exists x A_1$

Let $A' = \exists x A'_1$ and $A'' = \exists x A''_1$.

$$\begin{array}{c}
\frac{}{\vdash [A_1^\perp]A_1'} \\
\frac{}{\vdash [A_1']\exists xA_1'} \\
\frac{}{A_1' \vdash [\bullet]\exists xA_1'} \\
\frac{}{\vdash A_1'^\perp, \exists xA_1'} \text{Lemma 12(2)} \quad \vdash A_1', A_1''^\perp \\
\hline
\vdash \exists xA_1', A_1''^\perp \text{Lemma 14} \\
\hline
\vdash \exists xA_1', \forall xA_1''^\perp
\end{array}$$

We can complete the proof on the left-hand side by Lemma 10 and the right-hand side by applying the induction hypothesis on A_1 .

6. $A = \neg A_1$

Let $A', A'' \in \bar{\phi}(\neg A_1)$.

Let $A' = A_1'^\perp$ with $A_1' \in \bar{\phi}(A_1)$ and $A'' = A_1''^\perp$ with $A_1'' \in \bar{\phi}(A_1)$.

The induction hypothesis on A_1 we get: $\vdash_{LK(\mathcal{T})} A', A''^\perp$ and we are done. □

Theorem 21 Let $\Gamma_{\mathcal{T}} = \{l_1 \vee \dots \vee l_n \mid \mathcal{T}(l_1^\perp, \dots, l_n^\perp) \text{ returns UNSAT}\}$.

If $\Gamma_{\mathcal{T}}, \Delta \vdash_{FOL} A$ then for all $A' \in \bar{\phi}(A)$ and $\Delta' \in \bar{\phi}(\Delta)$, $\vdash_{LK(\mathcal{T})} A', \Delta'^\perp$.

Proof:

For any formula A , the notation A' will systematically designate elements of $\phi(A)$ and for any multiset of formulae Δ , the notation Δ' will systematically designate elements of $\phi(\Delta)$. The proof is by induction of $\Gamma_{\mathcal{T}}, \Delta \vdash_{FOL} A$, and case analysis on the last rule:

• Axiom:

$$\frac{}{\Gamma_{\mathcal{T}}, \Delta \vdash A} \quad A \in \Gamma_{\mathcal{T}}, \Delta$$

By case analysis:

– If $A \in \Delta$ then:

$$\frac{\vdash A', A''^\perp}{\vdash A', \Delta'^\perp} \text{WK}$$

with $A', A'' \in \bar{\phi}(A)$.

We can close the branch by Lemma 20.

– If $A \in \Gamma_{\mathcal{T}}$ then:

A is of the form $l_1 \vee \dots \vee l_n$ with $\mathcal{T}(\phi(l_1)^\perp, \dots, \phi(l_n)^\perp)$. Let $C' \in \bar{\phi}(A)$. C' is of the form $\phi(l_1) \vee^? \dots \vee^? \phi(l_n)$.

$$\begin{array}{c}
\frac{\mathcal{T}(\phi(l_1)^\perp, \dots, \phi(l_n)^\perp)}{\phi(l_1)^\perp, \dots, \phi(l_n)^\perp \vdash} \\
\frac{}{\vdash \phi(l_1), \dots, \phi(l_n)} \\
\hline
\vdash \phi(l_1) \vee^- \dots \vee^- \phi(l_n) \\
\hline
\vdash C' \\
\frac{}{\vdash \Delta'^\perp, C'} \text{WK}
\end{array}$$

This is a complete proof since $\mathcal{T}(\phi(l_1)^\perp, \dots, \phi(l_n)^\perp)$ returns *UNSAT*.

• And Intro:

$$\frac{\Gamma_{\mathcal{T}}, \Delta \vdash A_1 \quad \Gamma_{\mathcal{T}}, \Delta \vdash A_2}{\Gamma_{\mathcal{T}}, \Delta \vdash A_1 \wedge A_2}$$

$A' \in \bar{\phi}(A_1 \wedge A_2)$ is of the form $A_1' \wedge^? A_2'$ with $A_1' \in \phi(A_1)$ and $A_2' \in \phi(A_2)$.

The induction hypothesis on $\Gamma_{\mathcal{T}}, \Delta \vdash_{FOL} A_1$ gives $\vdash_{LK(\mathcal{T})} A_1', \Delta_1'^\perp$ and the induction hypothesis on $\Gamma_{\mathcal{T}}, \Delta \vdash_{FOL} A_2$ gives $\vdash_{LK(\mathcal{T})} A_2', \Delta_2'^\perp$. We build:

$$\frac{\frac{\frac{\vdash A'_1, \Delta'^\perp \quad \vdash A'_2, \Delta'^\perp}{\vdash A'_1 \wedge^- A'_2, \Delta'^\perp}}{\vdash A'_1 \wedge^? A'_2, \Delta'^\perp}}$$

- And Elim

$$\frac{\Gamma_{\mathcal{T}}, \Delta \vdash A \wedge B}{\Gamma_{\mathcal{T}}, \Delta \vdash A}$$

- Since $\bar{\phi}(B) \neq \emptyset$, let $B' \in \bar{\phi}(B)$ and $C' = A' \wedge^- B'$ ($C' \in \bar{\phi}(A \wedge B)$).

The induction hypothesis on the premise, with Δ' and C' , gives $\vdash_{LK(\mathcal{T})} C', \Delta'^\perp$ and we get:

$$\frac{\vdash C', \Delta'^\perp}{\vdash A', \Delta'^\perp}$$

by Lemma 3.

$$\frac{\Gamma_{\mathcal{T}}, \Delta \vdash A \wedge B}{\Gamma_{\mathcal{T}}, \Delta \vdash B}$$

- Since $\bar{\phi}(A) \neq \emptyset$, let $A' \in \bar{\phi}(A)$ and $C' = A' \wedge^- B'$ ($C' \in \bar{\phi}(A \wedge B)$).

The induction hypothesis on the premise, with Δ' and C' , gives $\vdash_{LK(\mathcal{T})} C', \Delta'^\perp$ and we get:

$$\frac{\vdash C', \Delta'^\perp}{\vdash B', \Delta'^\perp}$$

by Lemma 3.

- Or Intro

$$\frac{\Gamma_{\mathcal{T}}, \Delta \vdash A_i}{\Gamma_{\mathcal{T}}, \Delta \vdash A_1 \vee A_2}$$

$A' \in \bar{\phi}(A_1 \vee A_2)$ is of the form $A'_1 \vee^? A'_2$ with $A'_1 \in \phi(A_1)$ and $A'_2 \in \phi(A_2)$.

The induction hypothesis on $\Gamma_{\mathcal{T}}, \Delta \vdash_{FOL} A_i$ gives $\vdash_{LK(\mathcal{T})} A'_i, \Delta'^\perp$ and we build:

$$\frac{\frac{\frac{\vdash A'_i, \Delta'^\perp}{\vdash A'_1, A'_2, \Delta'^\perp} WK}{\vdash A'_1 \vee^- A'_2, \Delta'^\perp}}{\vdash A'_1 \vee^? A'_2, \Delta'^\perp}$$

- Or Elim

$$\frac{\Gamma_{\mathcal{T}}, \Delta \vdash A_1 \vee A_2 \quad \Gamma_{\mathcal{T}}, \Delta, A_1 \vdash C \quad \Gamma_{\mathcal{T}}, \Delta, A_2 \vdash C}{\Gamma_{\mathcal{T}}, \Delta \vdash C}$$

Let $D' = A'_1 \vee^- A'_2$ with $A'_1 \in \phi(A_1)$ and $A'_2 \in \phi(A_2)$.

The induction hypothesis on $\Gamma_{\mathcal{T}}, \Delta \vdash_{FOL} A_1 \vee A_2$ gives $\vdash_{LK(\mathcal{T})} D', \Delta'^\perp$, the induction hypothesis on $\Gamma_{\mathcal{T}}, A_1, \Delta \vdash_{FOL} C$ gives $\vdash_{LK(\mathcal{T})} A_1^\perp, C', \Delta'^\perp$ and the induction hypothesis on $\Gamma_{\mathcal{T}}, A_2, \Delta \vdash_{FOL} C$ gives $\vdash_{LK(\mathcal{T})} A_2^\perp, C', \Delta'^\perp$. We get:

$$\frac{\frac{\frac{\vdash A_1^\perp, C', \Delta'^\perp \quad \vdash A_2^\perp, C', \Delta'^\perp}{\vdash A_1^\perp \wedge^+ A_2^\perp, C', \Delta'^\perp}}{\vdash (A'_1 \vee^- A'_2)^\perp, C', \Delta'^\perp}}{\vdash D', C', \Delta'^\perp} cut_7$$

- Universal quantifier Intro

$$\frac{\Gamma_{\mathcal{T}}, \Delta \vdash A}{\Gamma_{\mathcal{T}}, \Delta \vdash \forall x A} \quad x \notin \Gamma$$

$C' \in \bar{\phi}(\forall x A)$ is of the form $\forall x A'$ with $A' \in \bar{\phi}(A)$.

The induction hypothesis on $\Gamma_{\mathcal{T}}, \Delta \vdash_{FOL} A$ gives $\vdash_{LK(\mathcal{T})} A', \Delta'^\perp$. We get:

$$\frac{\vdash A', \Delta'^\perp}{\vdash \forall x A', \Delta'^\perp}$$

- Universal quantifier Elim

$$\frac{\Gamma_{\mathcal{T}}, \Delta \vdash \forall x A}{\Gamma_{\mathcal{T}}, \Delta \vdash \{\forall_x\} A}$$

$C' \in \bar{\phi}(\{\forall_x\} A)$ is of the form $\{\forall_x\} A'$ with $A' \in \phi(A)$ (by remark 18).

The induction hypothesis on $\Gamma_{\mathcal{T}}, \Delta \vdash_{FOL} \forall x A$ gives $\vdash_{LK(\mathcal{T})} \forall x A', \Delta'^{\perp}$. We get:

$$\frac{\vdash \forall x A', \Delta'^{\perp}}{\vdash A', \Delta'^{\perp}} \\ \frac{}{\vdash \{\forall_x\} A', \Delta'^{\perp}}$$

by Lemma 3 and Lemma 2.

- Existential quantifier Intro

$$\frac{\Gamma_{\mathcal{T}}, \Delta \vdash \{\exists_x\} A}{\Gamma_{\mathcal{T}}, \Delta \vdash \exists x A}$$

$C' \in \bar{\phi}(\exists x A)$ is of the form $\exists x A'$ with $A' \in \bar{\phi}(A)$.

Let $A'_t = \{\exists_x\} A'$ ($A'_t \in \bar{\phi}(\{\exists_x\} A)$) by remark 18).

The induction hypothesis on $\Gamma_{\mathcal{T}}, \Delta \vdash_{FOL} \{\exists_x\} A$ gives $\vdash_{LK(\mathcal{T})} A'_t, \Delta'^{\perp}$.

By Lemma 14 it suffices to prove $\vdash_{LK(\mathcal{T})} \exists x A', A'_t^{\perp}$ in order to get $\vdash_{LK(\mathcal{T})} C', \Delta'^{\perp}$:

$$\frac{\vdash [A'_t] A'_t^{\perp}}{\vdash [\exists x A'] A'_t^{\perp}} \\ \frac{}{\forall x A'^{\perp} \vdash [\bullet] A'_t^{\perp}} \text{WK} \\ \frac{}{\vdash \exists x A', A'_t^{\perp}} \text{Lemma 12(2)}$$

We can complete the proof by applying Lemma 10.

- Existential quantifier Elim

$$\frac{\Gamma_{\mathcal{T}}, \Delta \vdash \exists x A \quad \Gamma, \Delta, A \vdash B}{\Gamma_{\mathcal{T}}, \Delta \vdash B} \quad x \notin \Gamma, B$$

Let $C' = \exists x A'$ with $A' \in \bar{\phi}(A)$.

$$\frac{\vdash A'^{\perp}, B', \Delta'^{\perp}}{\vdash \forall x A'^{\perp}, B', \Delta'^{\perp}} \\ \frac{\vdash C', \Delta'^{\perp} \quad \vdash C'^{\perp}, B', \Delta'^{\perp}}{\vdash B', \Delta'^{\perp}} \text{cut}_7$$

We can complete the proof on the left-hand side by applying the induction hypothesis on $\Gamma_{\mathcal{T}}, \Delta \vdash_{FOL} \exists x A$ and on the right-hand side by applying the induction hypothesis on $\Gamma_{\mathcal{T}}, \Delta, A \vdash_{FOL} B$.

- Negation Intro

$$\frac{\Gamma_{\mathcal{T}}, \Delta, A \vdash B \wedge \neg B}{\Gamma_{\mathcal{T}}, \Delta \vdash \neg A}$$

If $C' \in \bar{\phi}(\neg A)$ then $C'^{\perp} \in \bar{\phi}(A)$. Let $D' = D'_1 \wedge D'_2$ with $D'_1 \in \bar{\phi}(B)$ and $D'_2 \in \bar{\phi}(\neg B)$. Therefore $D'_2^{\perp} \in \bar{\phi}(B)$, $D' \in \bar{\phi}(B \wedge \neg B)$ and $\Delta', C'^{\perp} \in \bar{\phi}(\Delta, A)$.

$$\frac{\vdash D'_1^{\perp}, D'_2^{\perp}}{\vdash D'_1^{\perp} \vee D'_2^{\perp}} \text{Corollary 15(4)} \\ \frac{}{\vdash D'^{\perp}} \text{WK} \\ \frac{\vdash \Delta'^{\perp}, C', D' \quad \vdash \Delta'^{\perp}, C', D'^{\perp}}{\vdash \Delta'^{\perp}, C'} \text{cut}_7$$

We can complete the proof on the left-hand side by applying the induction hypothesis on $\Gamma_{\mathcal{T}}, \Delta, A \vdash B \wedge \neg B$ and on the right-hand side by applying Lemma 20 with $A''^{\perp} = D'_1^{\perp}$ and $A' = D'_2^{\perp}$.

- Negation Elimination

$$\frac{\Gamma_{\mathcal{T}}, \Delta \vdash \neg \neg A}{\Gamma_{\mathcal{T}}, \Delta \vdash A}$$

$A' \in \bar{\phi}(A)$ is such that $A' \in \bar{\phi}(\neg\neg A)$.

The induction hypothesis on $\Gamma_{\mathcal{T}}, \Delta \vdash \neg\neg A$ gives $\vdash \Delta'^{\perp}, A'$ and we are done.

□

8 Conclusion

It is worth noting that an instance of such a theory is the theory where $\mathcal{T}(S)$ holds if and only if there is a literal $p \in S$ such that $p^{\perp} \in S$.

We proved the admissibility of cut_8 and cut_9 as they are used to simulate the $\text{DPLL}(\mathcal{T})$ procedure [NOT06] as the proof-search mechanism of $\text{LK}(\mathcal{T})$.

References

- [LM09] C. Liang and D. Miller. Focusing and polarization in linear, intuitionistic, and classical logics. *Theoret. Comput. Sci.*, 410(46):4747–4768, 2009.
- [NOT06] R. Nieuwenhuis, A. Oliveras, and C. Tinelli. Solving SAT and SAT Modulo Theories: From an abstract Davis–Putnam–Logemann–Loveland procedure to $\text{DPLL}(T)$. *J. of the ACM Press*, 53(6):937–977, 2006.