



HAL
open science

Development of one-point statistical closure from a spectral description of the turbulence

Anne Cadiou, Kemal Hanjalic

► **To cite this version:**

Anne Cadiou, Kemal Hanjalic. Development of one-point statistical closure from a spectral description of the turbulence. 1998. hal-00779039

HAL Id: hal-00779039

<https://hal.science/hal-00779039>

Submitted on 21 Jan 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Development of one-point statistical closure from a spectral description of the turbulence

Anne Cadiou, Kemal Hanjalić

Report APTF-R / 98-07

Department of Applied Physics
Delft University of Technology
The Netherlands

Contents

Introduction	1
1 Two-point correlations and spatial scales	5
1.1 The two-point description and notations	5
1.2 Dissipation- and length-scale	9
2 Treatment of the equations in the spectral space	11
2.1 Application of the Fourier transform to the Reynolds stress equations . . .	13
2.2 Transformation of the terms which do not explicitly involve the mean velocity	13
2.2.1 Unsteady term	14
2.2.2 Turbulent diffusion terms	14
2.2.3 Pressure-velocity correlations	14
2.2.4 Viscous diffusion	14
2.2.5 Viscous dissipation	15
2.3 Transformation of the terms which explicitly involve the mean velocity . .	15
2.3.1 Convective terms	16
2.3.2 Production terms	17
2.4 Calculation of the pressure terms	17
2.4.1 Contribution from the triple correlations	18
2.4.2 Contribution involving mean velocity gradients	19
2.5 Second order expansion of the Reynolds stress equation in the spectral space	26
2.6 Closure of the Reynolds stress equation in the spectral space	30
2.7 Spherical integration	31
2.8 The spectral closure	32
2.8.1 Production	33
2.8.2 Dissipation	33
2.8.3 Redistribution by the pressure-strain correlations	34
2.8.4 Linear spectral transfer	37
2.8.5 Turbulent diffusion	37
2.8.6 Synthesis	38

3	Derivation of a multiple-scale model	41
3.1	Closure on the spherically integrated spectrum tensor	41
3.2	Choice of the variables	44
3.3	Derivation of the equations	46
3.3.1	The transport equation for the variable $\varepsilon^{(1)2/3} k_L^{(m-2/3)} T_{ij}^{(m)}$	46
3.3.2	Transport equation for the variable $\varepsilon^{(1)2/3} k_\eta^{(m-2/3)} T_{ij}^{(m)}$	53
3.3.3	Summary of the derived transport equations	59
3.3.4	Comments	64
	Appendix	72
A	Governing equations for one-point statistical closure	73
A.1	Statistical description of the motion	73
A.1.1	Reynolds averaged equations	74
A.1.2	The mean equations	74
A.1.3	Equations for fluctuating motion	75
A.2	Transport equation for one-point statistic quantities	75
A.2.1	Governing equation for the second moment	75
A.2.2	Governing equation for the dissipation rate tensor	76
A.2.3	Governing equation for the kinetic energy	77
A.2.4	Governing equation for the energy dissipation rate	78
B	Governing equations for the two-point statistical closure	79
B.1	Two - point description with centered variables	79
B.1.1	Rules of the centered transformation	79
B.1.2	Transport equations	84
B.1.3	Evolution equation for each part of the dissipation tensor using the centered transformation	95
B.2	Two - point description with decentered variables	98
B.2.1	Rules of the decentered transformation	98
B.2.2	Transport equations	102
B.2.3	Evolution equation for each part of the dissipation tensor in the decentered transformation	108
C	Application of the Fourier transform	111
C.1	Basic notations	111
C.2	Expansion of the velocity components	112
C.3	Expression for the pressure-velocity correlations	112
C.3.1	Expression in the physical space	113
C.3.2	Solution of the Poisson equation in the spectral space	113
C.3.3	Solution of the Poisson equation in the physical space	115
C.4	Expansion of the operators involved in the Poisson equations	116

D	Two-point equations and one-point equations	119
D.1	Reynolds stress equation	119
D.1.1	Expression in the one-point limit	119
D.1.2	Remarks	121
D.2	Dissipation equation	121
D.2.1	Expression in the one-point limit	121
D.2.2	Remarks	123
E	Spectral equations and the one-point equations	125
E.1	Reynolds stress equation	125
E.1.1	Expression in the one-point limit	126
E.1.2	Remarks	128
F	Classical one point closure	129
G	Choice of the multiple-scale variables	133
G.1	Closures assumptions	133
G.2	Integration	133
G.2.1	Large scales contribution	135
G.2.2	Small scales contribution	135
G.2.3	Synthesis	137
G.2.4	Approximations	138
	Bibliography	141

Introduction

The turbulence in a fluid flow can be defined as an ensemble of eddy motion where the velocity and other flow properties evolve in an apparently irregular, disorderly and chaotic manner. Because of the inherent non-linearity, an infinite number of realization is possible so that details of the fluid particles trajectory seem to be unpredictable. These, seemingly random motions in time and space, occur even if ensemble-averaged properties exhibit a unique steady and unidirectional character, as in the case of a channel flow : the turbulence is by nature a three-dimensional and unsteady phenomenon. Associated with fluctuations of the velocity are the fluctuations of pressure in accord with the law of fluid motion expressed in form of Navier-Stokes equations, which are assumed to be satisfied at each time instant. The same applies for fluctuations of density, temperature and other scalars, governed by energy and mass (species concentration) conservation equations in non-isothermal and multi-component systems.

Turbulent motion is not only random but possess a rotational character : the eddy motion is indeed present in a broad spectrum in space and time, corresponding to a wide range of eddies of various size and life-time. In reality, the smallest scales present in a flow are limited by the action of molecular viscosity, whereas the largest scales are of the order of size of the flow domain (Launder, 1991).

A direct access to the details of the turbulent motion is still difficult and restricted to very specific simple flows. In most cases of practical interest, the temporal variations of the flow properties appear to be of secondary importance, the main interest being in the time or ensemble averaged velocity and the pressure field (also the temperature, concentration, density), evaluated over a time long enough compared to the characteristic time scale of the fluctuating field, or over a large number of samples. Therefore, for practical applications, the Navier-Stokes equations for the instantaneous motion can be averaged and treated by a statistical approach, as proposed a century ago by Reynolds (1895).

The application of any statistical operator to the Navier-Stokes equation results in unknown statistical moments as a consequence of the non-linear nature of the convective terms. Of primary importance are the second order moments of the fluctuating velocity in the mean momentum equation, which constitute the Reynolds stress tensor when evaluated in the same point. The transport equation for any statistical moment involves the moments of immediate higher order which are unknown and remain to be closed.

An one-point statistical description of the turbulence involves furthermore other unknowns in addition to those arising directly from averaging the convection process. These are the terms involving pressure and molecular dissipation. The pressure terms are characterized by nonlocal effects because of the elliptic nature of the pressure in an incompressible flow, whereas the dissipation terms reflect the character of turbulent spatial scales. These are properties which are not explicitly taken into account by a single point statistics and, therefore, are unknown.

The transport equations for the Reynolds stresses describe in essence the dynamics of large scales, energy-containing motion, because they are of the same order of magnitude as the kinetic energy, providing thus far an information on the characteristic turbulent velocity scale. In order to close the unknown processes, it is necessary to introduce another scale, in length or time. One of the most common choices is the scalar dissipation rate of kinetic energy, because this quantity appears as a sink term in the kinetic energy equation, and its physical interpretation is straightforward. The entire turbulent field is then described by a unique set of characteristic time and space scales.

This is one of the major assumption underlining the one-point statistical closure hierarchy. The turbulent spectrum is supposed to be in an equilibrium state. This implicitly assumes that all spatial information can be given by a unique length scale for the whole spectrum and that various eddies and the major processes associated with them are linked together by a transfer of energy at a constant rate. Therefore, the dissipation process, acting on the small eddies, is assumed to respond with the same characteristic time to any change occurring on the energy containing eddies, located at the large scales.

This assumption seems to be satisfactory for a wide range of flows of practical interest. That is why the single-scale modelling approach has been so successful since the early development. There are, however, many flows where such a spectral equilibrium is not attained. A case in point is the flow in reciprocating engines, where a moving piston generates periodic perturbations. To capture the salient features of such turbulent flows, it may be necessary to introduce different scales by which to describe eddies of different sizes, their evolution and mutual interaction.

A first attempt to distinguish different scales associated with various processes in the dynamics of turbulence within the one-point methodology is the multiple-scale approach. This type of closure, which is the focus in this study, has conceptually been proposed by Launder and Schiestel (1978) and formulated for both the eddy-viscosity and second-moment closure levels by Hanjalić *et al.* (1979). The main idea is based on a decomposition of kinetic energy or Reynolds stress spectrum into several wave-number regions to which major turbulence interactions are associated, and to propose a closure for each spectrum slice. This led to the development of the split spectrum technique, which has been widely applied in most of the multiple-scales models derived thereafter.

This approach can be regarded as an intermediate route between the classical one-point approach and the two-point statistical description of turbulence.

The split-spectrum method follows the one-point methodology, and therefore, can ac-

cess neither the non-linear nor the non-local character of the energy distribution between the eddies of various sizes and their interactions. The full information can only be recovered by the full resolution of the turbulence spectra. However, some information can be provided by formulating and solving the transport equations for additional turbulence scales. The closures of these equations can be performed with the help of a two-point description of turbulence in which the non-local and non-linear effect are naturally taken into account.

It is, therefore, useful to revisit the two-point approach, which can serve as a basis for deriving at least one additional scale equation. Far from an exhaustive review, the short outline that follows recollects some of the past ideas as well as some novelties, and also, introduces the notations used in this report.

The development of any multiple-scale model is strongly linked to the treatment of various scale processes. The Fourier transformation of the two-point equation permits a better description and interpretation of the processes and governing equations. This is presented in the chapter 1 where the Reynolds stress equations are derived. This also leads to the hierarchy of spectral models which are defined by different sets of the transformed equations.

The spectral approach is presented in order to illustrate the derivation of the spectral equations used for multiple-scale derivation. The brief review, however, does not pretend to be exhaustive, nor to give a detailed view of the spectral models. It is only intended to point out at some interesting features of the multiple-scale modelling and to provide a better understanding of the processes and the underlying assumptions of the well-known closures.

Chapter 1

Two-point correlations and spatial scales

The statistical two-point description of turbulence has been regarded as a convenient mathematical framework for better understanding and modelling of the one-point quantities. Kármán and Howarth (1938) used this technique to study the evolution of the turbulent length scales in decaying isotropic turbulence. It was also used later by Crow (1968) to express the rapid distortion of isotropic turbulence and by Naot *et al.* (1973) to derive a transport equation for the Reynolds stress tensor. The two-point description of turbulence provides indeed a convenient framework to study the length scales equations. Rotta (1951) applied the integral operator of the two-point correlation equations to derive the equation for the product of the Reynolds stresses and a scalar integral length-scale. The two-point description served also to derive transport equations for the length-scale tensor (Lin and Wolfshtein (1980), Donaldson and Sandri (1981) or Oberlack and Peters (1993)), and to provide a better insight in the closure of the classical one-point dissipation equation (Jovanović *et al.* (1995), Oberlack (1994)). However, most of these studies were confined to homogeneous turbulent flows. The two-point spatial approach has also been used to develop multiple-scales models (Schiestel, 1987), though this can also be achieved by starting from a description of the two-point equations in the spectral space through a Fourier transformation.

The latter approach is adopted in this study. The spectral equations are presented in the next chapter. As a starting point for the Fourier transformation it is interesting first to look at the two-point equations in the physical space.

1.1 The two-point description and notations

The evolution equation of the two-point second order tensor is derived in this chapter. The expression for the source term is written in a form that permits to isolate the contribution specific to a two-point description of turbulent flow. This is achieved by a suitable transformation of the independent variable, currently used in classical two-point

representation of turbulence (Hinze, 1975). Details of this transformation are given in Appendix B and only the interpretation of each term is recalled in this chapter.

The fluctuating quantities are evaluated at points M' and M'' defined by coordinates (\underline{x}', t) and (\underline{x}'', t) respectively. In the following text each quantity defined at M' (resp. M'') is denoted by $'$ (resp. $''$). The differential operators written in the Einstein notation are also indexed by $'$ (resp. $''$) when applied to variables \underline{x}' (resp. \underline{x}''). This notation is also valid for two-point quantities. However, when the two-point tensor corresponds to the one-point quantity, the indices related to \underline{x}' and those related to \underline{x}'' are only separated by a dot.

The derivation of the two-point equation is obtained by the combination of the two fluctuating equations evaluated at each point (see Appendix A) :

$$u'_{i,t} + L_i(\underline{u}') = 0 \quad (1.1a)$$

$$u''_{j,t} + L_j(\underline{u}'') = 0 \quad (1.1b)$$

in order to form :

$$u''_j [u'_{i,t} + L_i(\underline{u}')] + u'_i [u''_{j,t} + L_j(\underline{u}'')] = 0 \quad (1.2)$$

The evolution equation of the fluctuating velocity written at points M' and M'' are :

$$u'_{i,t} + \overline{V}'_n u'_{i,n'} = -\frac{1}{\rho} p'_{,i'} - u'_n \overline{V}'_{i,n'} - [u'_i u'_n - \overline{u'_i u'_n}]_{,n'} + \nu u'_{i,n'n'} + f'_i \quad (1.3a)$$

$$u''_{j,t} + \overline{V}''_m u''_{j,m''} = -\frac{1}{\rho} p''_{,j''} - u''_m \overline{V}''_{j,m''} - [u''_j u''_m - \overline{u''_j u''_m}]_{,m''} + \nu u''_{j,m''m''} + f''_j \quad (1.3b)$$

Therefore, the two-point Reynolds-stress tensor equation may be written as :

$$\begin{aligned} (\overline{u'_i u''_j})_{,t} + \overline{V}'_k (\overline{u'_i u''_j})_{,k'} + \overline{V}''_k (\overline{u'_i u''_j})_{,k''} &= -\overline{u'_i u''_k} \overline{V}''_{j,k''} - \overline{u''_j u'_k} \overline{V}'_{i,k'} \\ &+ \overline{f'_i u''_j} + \overline{f''_j u'_i} \\ &- (\overline{u'_i u''_j u''_k})_{,k''} - (\overline{u'_i u'_k u''_j})_{,k'} \\ &- \frac{1}{\rho} (\overline{u'_i p''_{,j''}} + \overline{u''_j p'_{,i'}}) \\ &+ \nu [(\overline{u'_i u''_j})_{,k'k'} + (\overline{u'_i u''_j})_{,k''k''}] \end{aligned} \quad (1.4)$$

Each source term can be written in the form used in the one-point equation and with the above notations, leading to :

$$C_{i'.j''} = \overline{V}'_k R_{i,j,k'} + \overline{V}''_k R_{i,j,k''} \quad (1.5a)$$

$$P_{i'.j''} = -R_{i,k} \overline{V}''_{j,k''} - R_{k,j} \overline{V}'_{i,k'} \quad (1.5b)$$

$$G_{i'.j''} = \overline{f'_i u''_j} + \overline{f''_j u'_i} \quad (1.5c)$$

$$d_{i'.j''}^u = T_{i,jk,k''}^u + T_{ik,j,k'}^u \quad (1.5d)$$

$$\Pi_{i'.j''} = -\frac{1}{\rho} (\overline{u'_i p''_{,j''}} + \overline{u''_j p'_{,i'}}) \quad (1.5e)$$

$$d_{i'.j''}^\nu = \nu [(\overline{u'_i u''_j})_{,k'k'} + (\overline{u'_i u''_j})_{,k''k''}] \quad (1.5f)$$

with

$$R_{i,j} = \overline{u'_i u''_j} \quad (1.6a)$$

$$T_{i,jk}^u = -\overline{u'_i u''_j u''_k} \quad (1.6b)$$

$$T_{ik,j}^u = -\overline{u'_i u''_k u''_j} \quad (1.6c)$$

The viscous tensor above includes both the viscous diffusion and viscous dissipation. Their corresponding forms in the two-point description will be obtained by adequate splitting, exactly as it is done for one-point description.

In order to identify clearly the meaning of each process and to separate specific two-point contributions from the expressions that degenerate into the corresponding one point terms when the two points collapse, it is necessary to adopt a local coordinate system and to rewrite the equation in terms of the adopted two-point independent variables. The most immediate choice is the well-known mid-point description, i.e. to place the local coordinate system in the middle between the two points M' and M'' . The two-point properties are then expressed in terms of the mid-point position vector and the distance vector between M' and M'' . Such transformation is used by Chou (1945), Hinze (1975) or again Jovanović *et al.* (1995). The kinematic characteristics of this centered transformation are given in Appendix B. This, of course, is not the only possible choice and a popular alternative is the decentered local coordinate system with its origin at one of the points. This is used by Rotta (1972) and Oberlack and Peters (1993). Although the Jacobian of the latter transformation is simpler, the first choice leads to a clearer description of different processes in the equations, because it allows to separate in an obvious manner each of the local or spatial contributions. The illustration of this is shown in Appendix B.

The adopted independent variables are :

$$X_k = \frac{1}{2} (x'_k + x''_k) \quad (1.7a)$$

$$r_k = (x''_k - x'_k) \quad (1.7b)$$

Any derivative with respect to X_k will be denoted by \cdot_k and the derivative with respect to r_k by $\cdot|_k$. The details of the application of this transformation are provided in Appendix B. The evolution equation of the two-point Reynolds stress tensor can now be written as :

$$\frac{\overline{d}R_{i,j}}{dt} = P_{i,j} + d_{i,j}^u + d_{i,j}^{u\cdot} + d_{i,j}^p + d_{i,j}^{p\cdot} + d_{i,j}^\nu - \varepsilon_{i,j} - C_{i,j} \quad (1.8)$$

where the material derivative is defined as

$$\frac{\overline{d}\cdot}{dt} = \frac{\partial\cdot}{\partial t} + \frac{1}{2} [\overline{V}'_k + \overline{V}''_k] \frac{\partial\cdot}{\partial X_k} \quad (1.9)$$

The part of the convective term included in this derivation is denoted as

$$C_{i,j} = \frac{1}{2} [\overline{V}'_k + \overline{V}''_k] R_{i,j,k} \quad (1.10a)$$

whereas the second contribution of the two-point convective tensor is defined as

$$C_{i,j}^{\cdot\cdot} = [\bar{V}_k'' - \bar{V}_k'] R_{i,j|k} \quad (1.10b)$$

The production terms are then defined by

$$P_{i,j} = -R_{i,k} \bar{V}_{j,k}'' - R_{k,j} \bar{V}_{i,k}' \quad (1.11)$$

The turbulent diffusion involves the triple correlations :

$$d_{i,j}^u = \frac{1}{2} [T_{i,jk}^u + T_{ik,j}^u]_{,k} \quad (1.12a)$$

$$d_{i,j}^{u\cdot\cdot} = [T_{i,jk}^u - T_{ik,j}^u]_{|k} \quad (1.12b)$$

where

$$T_{i,jk}^u = -\overline{u_i' u_j'' u_k''} \quad (1.13a)$$

$$T_{ik,j}^u = -\overline{u_i' u_k' u_j''} \quad (1.13b)$$

The same notation can be used for the pressure-velocity tensor, so that :

$$d_{i,j}^p = \frac{1}{2} [T_{i,jk}^p + T_{ik,j}^p]_{,k} \quad (1.14a)$$

$$d_{i,j}^{p\cdot\cdot} = [T_{i,jk}^p - T_{ik,j}^p]_{|k} \quad (1.14b)$$

where

$$T_{i,jk}^p = -\frac{1}{\rho} \overline{u_i' p''} \delta_{jk} \quad (1.15a)$$

$$T_{ik,j}^p = -\frac{1}{\rho} \overline{u_j'' p'} \delta_{ik} \quad (1.15b)$$

The viscous diffusion can be written as

$$d_{i,j}^{\nu} = \nu R_{i,j,kk} \quad (1.16)$$

The dissipation tensor can be directly expressed in terms of second derivatives of the two-point Reynolds stress tensor such as

$$\varepsilon_{i,j} = 2\nu \left[\frac{1}{4} R_{i,j,kk} - R_{i,j|kk} \right] \quad (1.17)$$

This particular splitting of the dissipation rate tensor has already been applied by several authors working in the physical space (Jovanović *et al.*, 1995) or in the spectral space (Laporta, 1995). It enables to write a part of the dissipation tensor with the same operator as the viscous diffusion process, and is denoted as the inhomogeneous part. It is noted that this formulation is entirely due to the choice of a centered description of the two

independent variables, as it can be seen by comparison with the decentered description, preferred by some researchers (Rotta (1972), Oberlack and Peters (1993)), which leads to a different outcome, (see Appendix B). The splitting presented here with the centered transformation has been shown to be convenient to describe the wall limit behaviour of the scalar dissipation in the case of a low-Reynolds number channel flow, as shown by comparison with direct numerical simulation (Kim *et al.*, 1987). It allows to close the homogeneous part of the viscous dissipation with an equation of the same form as the classical closure usually adopted for the entire term, but also to capture some feature of the wall effect through the inhomogeneous part.

The two-point equations presented here can serve as a basis for the development of spectral equations, and this will be briefly presented in the following chapter.

The approach adopted for the two-point Reynolds stress tensor can also be used to derive an equation for the dissipation rate tensor or its trace (see Appendix B). A direct application of the system of two-point equations to the one-point modelling using the closure hypothesis based on local homogeneity approximation and dimensional analysis leads to a form very similar to the core of the usual one-point models. Such a two-point description does not provide much insight into the two-point characteristics, but can be used to provide other parameters describing the turbulent scales. Some of the proposal are presented in the next section.

1.2 Dissipation- and length-scale

In order to close the equation set, the Reynolds stress equation needs to be associated with at least one characteristic turbulence scale. The dissipation rate ε is widely used as the additional scale-providing parameter, although other proposals have also been studied. An alternative choice, frequently adopted for practical applications is based on ω , interpreted as the rate of dissipation per unit turbulence kinetic energy (e.g. Wilcox (1988)). Such a closure is, however, mainly linked to two-equations models. In one of the earliest Reynolds stress models, developed by Rotta (1951), the closure was achieved using a scalar integral length scale equation. This equation was derived by applying an integral operator to the two-point correlation equation in the physical space. More recently Besnard *et al.* (1992) derived a similar equation starting the analysis in the spectral space. In order to account for some features of the turbulence structures, additional tensorial equations have been studied, such as the tensor length scale equation. Such an equation was earlier proposed by Donaldson and Sandri (1981), but some important processes have been neglected in their formulation. This is also the case in ? (?) approach as mentioned recently by ? (?). The latter authors use a volume integral of the two-point correlation as well as an homogeneity approximation. The same restriction can also be found in Oberlack (1997) equation, who retains higher order terms in the extension of the functional, but did not provide suggestions for all of the coefficients arising in the derivation. The expression for some of these closures will be used in the next chapter as a basis for comparison with the model derived in this report.

Chapter 2

Treatment of the equations in the spectral space

The two-point statistical description of turbulence allows in principle to account for interaction and energy transfer between eddies of different sizes. It involves only the unknowns arising from the averaging process. The potential of the method for a refined description of turbulence has long been recognized and various routes to deriving a closed set of equations have been investigated and reported in the literature. Three major approaches can be distinguished, each offering some advantages, but also shortcomings.

The most straightforward method is to derive equations for statistical moments up to an adopted order and expressing the higher moments in terms of the lower ones. For instance, at the third order level the fourth order correlation can be closed by the quasi-normal approximation which assumes a quasi gaussianity (Millionschtchikov, 1941). The quadruple correlations can then be written in terms of second moments. The numerical calculations performed for homogeneous isotropic turbulence by O'Brien and Francis (1962) and Ogura (1963) showed some inconsistencies of this assumption resulting in negative kinetic energy spectrum. Orszag (1970) noted that the relaxation time introduced by the QN closure on the triple correlations does not take into account the non-linear processes and proposed an eddy damping approximation. The original spectral closure of this type, known as the Eddy Damped Quasi Normal (EDQN) approximation, ensures the widely accepted $K^{-5/3}$ energy spectrum decay in the inertial zone in accord with the theory of Kolmogorov (1941). The EDQN closure does not, however, satisfy the realisability of the kinetic energy spectrum. Therefore, Orszag added a modification to the original closure called Markovianisation, which led to the EDQNM closure. The one-point closure for the triple correlation proposed by Hanjalić and Launder (1972) has some similarities with this procedure, even though it was derived directly from the single-point transport equation and for different reason and purpose. The EDQNM closure has been refined by several authors (Leith (1971), Pouquet *et al.* (1975) or André and Lesieur (1977)) and it is basically aimed at retaining the non-locality of the triadic interactions captured by the spectral description, while proposing a simpler formulation of the interactions themselves.

The approach developed by Kraichnan (1958) has also been targeted at deriving the

spectral equations, but using the perturbation expansion (Direct Interaction Approximation). This can be seen from the exact solution of the hierarchy of the equations of moments associated with a particular stochastic model equation. This equation stands instead of the Navier-Stokes equation, though it keeps the same structural properties. The realisability of the energy spectrum is, therefore, assured by this equation. However the first derivation of the DIA did not satisfy the Galilean invariance of the statistics and led to a $K^{-3/2}$ variation of the kinetic energy spectrum in the inertial range (Kraichnan, 1959). This deficiency was later modified by adopting a Lagrangian formulation (LHDIA) of the DIA (Kraichnan, 1965), but doing so the existence of a stochastic model could not be proved anymore. This technique is rather cumbersome and so far it has not been proved viable for numerical computation of turbulent flows.

The Test Field Model (TFM) method has also been initially proposed by Kraichnan (1972) to restore the Galilean invariance in its DIA formulation. This technique leads to a closure of the triple correlations similar to the EDQNM approximation, where the relaxation time is determined by the advection of a test field by mean velocity. Its formulation is simpler than the LHDIA, but still more complex than the EDQNM approach.

All studies mentioned above have considered only simple flows, in most cases the homogeneous turbulence. However, they lead to a better understanding of the evolution of turbulence in presence of very specific physical effects such as, for instance, the rotation (Cambon *et al.*, 1991). In the inhomogeneous case it is essential to simplify the closure and to restrict the spectral description to a manageable level if the method is to be used for practical purpose. For this reason several authors have integrated the spectral equations over a shell of the wave-number radius and proposed closure for this directional information (Jeandel *et al.* (1978), Cambon *et al.* (1981) or Besnard *et al.* (1992)). Such an integration introduces additional unknowns and again three different approaches can be found in the literature to deal with the problem.

In the first approach the unknown terms are closed in a similar way as in the one-point method, which means that the nonlocal interactions are modelled by a local formulation (Leith (1967) or Kovasznay (1948)). This approach abandons any directional information but still allows to capture the cascade of energy in the Fourier space. The model, adopted in this report as a basis for the derivation of the multiple scale model, belongs to this category. It will be presented at the end of this chapter.

The second approach adopts a three-dimensional spectral closure of the EDQNM type and introduces some directional information in order to integrate the spectral equations over the shells of the wave-number radius (Cambon (1979) or Laporta (1995)).

The third route is based more on the DIA approximation while using two-scales (Yoshizawa, 1987). In contrast to the other two approaches, it has not been used for practical calculations, but rather to explain some features of Large Eddy Simulation or one-point two-equations closures.

2.1 Application of the Fourier transform to the Reynolds stress equations

The Fourier transform of the Reynolds stress transport equation can be performed following the definitions recalled in Appendix C. However the treatment of the terms involving the mean velocity components necessitates an approximation. The direct transformation of such terms introduces not only the Fourier transform of the mean velocity but also convolution products. In order to avoid these products, the mean velocity components are usually expanded around an averaged position (Cambon *et al.* (1981), Besnard *et al.* (1992) or Laporta (1995)). The mean velocity components are then defined with a differential operator (Laporta, 1995)

$$\bar{V}'_n = \exp\left(-\frac{1}{2} r_k \frac{\partial \bar{V}'}{\partial X_k}\right) \bar{V}_n \quad (2.1a)$$

$$\bar{V}''_n = \exp\left(\frac{1}{2} r_k \frac{\partial \bar{V}''}{\partial X_k}\right) \bar{V}_n \quad (2.1b)$$

that can be transformed into the Fourier space as

$$\bar{V}'_n = \exp\left(-\frac{I}{2} \frac{\partial}{\partial k_m} \frac{\partial \bar{V}'}{\partial X_m}\right) \bar{V}_n \quad (2.2a)$$

$$\bar{V}''_n = \exp\left(\frac{I}{2} \frac{\partial}{\partial k_m} \frac{\partial \bar{V}''}{\partial X_m}\right) \bar{V}_n \quad (2.2b)$$

where the superscript \bar{V} over the partial derivative indicates that this operator only applies to the mean velocity component.

The truncation applied in this expansion retains the major characteristics of an inhomogeneous turbulent flow, what is assumed to be sufficient to expand the spectral Reynolds stress tensor equation up to the second order in the spatial derivative (Laporta, 1995). This assumption is not too restrictive as long as the terms present in the one-point description of turbulence come from no more than the second order spatial derivatives in the spectral space. Higher contributions are entirely spectral processes. This implies that a truncation of the spectral equation at the second order level is justified as a first approximation in the sense of one-point models.

2.2 Transformation of the terms which do not explicitly involve the mean velocity

The Fourier transformation of these terms is straightforward and is listed below.

2.2.1 Unsteady term

This transformation is :

$$\text{TF} [R_{i,j,t}] = \hat{R}_{i,j,t} \equiv \Phi_{ij,t} \quad (2.3)$$

where TF stands for Fourier transformation of the quantity in square brackets. In order to simplify further the notations, the transformed quantities are denoted by a hat $\{\hat{\cdot}\}$.

2.2.2 Turbulent diffusion terms

In order to simplify the notations, the Fourier transform of the triple correlations are here denoted without a superscript, i.e. :

$$T_{i,jk} = \hat{T}_{i,jk}^u \quad (2.4)$$

so that the expression for the turbulent diffusion can be written as :

$$\text{TF} [d_{i,j}^u + d_{i,j}^{u\cdot}] = \frac{1}{2} [T_{i,jk} + T_{ik,j}]_{,k} + I k_k [T_{i,jk} - T_{ik,j}] \quad (2.5)$$

2.2.3 Pressure-velocity correlations

These terms can be expressed in the spectral space as functions of the second order tensor and the triple correlations using Poisson equations. They can be denoted as :

$$\text{TF} [d_{i,j}^p + d_{i,j}^{p\cdot}] = \frac{1}{2} [\hat{T}_{i,jk}^p + \hat{T}_{ik,j}^p]_{,k} + I k_k [\hat{T}_{i,jk}^p - \hat{T}_{ik,j}^p] \quad (2.6)$$

or again

$$\text{TF} [d_{i,j}^p + d_{i,j}^{p\cdot}] = \frac{1}{2} [\Pi_{j,i} + \Pi_{i,j}^*] + I [k_j \Pi_i^* - k_i \Pi_j] \quad (2.7)$$

with

$$\Pi_i^* = \text{TF} \left[-\frac{1}{\rho} \overline{u_i' p''} \right] \quad (2.8a)$$

$$\Pi_j = \text{TF} \left[-\frac{1}{\rho} \overline{u_j'' p'} \right] \quad (2.8b)$$

The treatment of the Poisson equation is elaborated later.

2.2.4 Viscous diffusion

The transformation of the viscous diffusion is direct :

$$\text{TF} [\nu R_{i,j,kk}] = \nu \Phi_{ij,kk} \quad (2.9)$$

2.2.5 Viscous dissipation

This term is exact in the spectral space and can be written as :

$$\text{TF} [\varepsilon_{i,j}] = 2\nu \left[\frac{1}{4} \Phi_{ij,kk} + k^2 \Phi_{ij} \right] \quad (2.10)$$

2.3 Transformation of the terms which explicitly involve the mean velocity

It can be noted that the terms which explicitly involve the mean velocity gradient might be written in a condensed form. They appear in the convection and production terms, which can be considered jointly :

$$S_{i,j}^{\bar{V}} = -\frac{1}{2} [\bar{V}'_k + \bar{V}''_k] R_{i,j,k} - [\bar{V}''_k - \bar{V}'_k] R_{i,j|k} - R_{i,k} \bar{V}''_{j,k} - R_{k,j} \bar{V}'_{i,k} \quad (2.11)$$

Using the incompressibility condition this term can be reduced to :

$$\begin{aligned} S_{i,j}^{\bar{V}} = & -\left[\left(\frac{1}{2} \cdot_{,k} + \cdot_{|k} \right) [\bar{V}''_k R_{i,j} + \bar{V}'_j R_{i,k}] \right. \\ & \left. - \left[\left(\frac{1}{2} \cdot_{,k} - \cdot_{|k} \right) [\bar{V}'_k R_{i,j} + \bar{V}'_i R_{k,j}] \right] \right] \end{aligned} \quad (2.12)$$

or, in the form :

$$\begin{aligned} S_{i,j}^{\bar{V}} = & - \left(\frac{1}{2} \cdot_{,k} + \cdot_{|k} \right) [\delta_{nk} \delta_{mj} + \delta_{mk} \delta_{nj}] \bar{V}''_n R_{i,m} \\ & - \left(\frac{1}{2} \cdot_{,k} - \cdot_{|k} \right) [\delta_{nk} \delta_{mi} + \delta_{mk} \delta_{ni}] \bar{V}'_n R_{m,j} \end{aligned} \quad (2.13)$$

It is clear that any expansion of the mean velocities needs to be calculated only once. As already mentioned, for this reason the velocity components are expanded in Taylor series where :

$$\bar{V}'_n = \bar{V}(\underline{X} - \frac{1}{2} \underline{r}) \quad (2.14a)$$

$$\bar{V}''_n = \bar{V}(\underline{X} + \frac{1}{2} \underline{r}) \quad (2.14b)$$

is expanded as :

$$\bar{V}'_n = \exp\left(-\frac{1}{2} r_k \frac{\partial \bar{V}}{\partial X_k}\right) \bar{V}_n \quad (2.15a)$$

$$\bar{V}''_n = \exp\left(\frac{1}{2} r_k \frac{\partial \bar{V}}{\partial X_k}\right) \bar{V}_n \quad (2.15b)$$

where the superscript $\bar{\cdot}$ over the partial derivative indicates that this operator only applies to the mean velocity component. The Fourier transform of those quantities leads to :

$$\bar{V}'_n = \exp\left(-\frac{I}{2} \frac{\partial}{\partial k_m} \frac{\partial \bar{\cdot}}{\partial X_m}\right) \bar{V}_n \quad (2.16a)$$

$$\bar{V}''_n = \exp\left(\frac{I}{2} \frac{\partial}{\partial k_m} \frac{\partial \bar{\cdot}}{\partial X_m}\right) \bar{V}_n \quad (2.16b)$$

It is convenient to introduce here the notation proposed by Besnard *et al.* (1992) for this operator :

$$\Sigma(\underline{X}, \underline{k}) = \sum_{m=0}^{\infty} \left(\frac{I}{2} \frac{\partial}{\partial k_n} \frac{\partial \bar{\cdot}}{\partial X_n}\right)^m \frac{1}{m!} \quad (2.17)$$

The transformation of the derivative operators with respect to the first and second point is denoted by the following notation :

$$\nabla_n(\underline{k}) = \frac{1}{2} \frac{\partial}{\partial X_n} + I k_n \quad (2.18)$$

This quantity is precisely the derivative with respect to the second point in the spectral space. Their conjugate forms are applied to the first point and will be denoted with a star, as :

$$\nabla_n^*(\underline{k}) = \nabla_n(-\underline{k}) \quad (2.19)$$

Using these notations the transformation of the group of terms involving explicitly the mean velocity is straightforward :

$$\begin{aligned} \hat{S}_{i,j}^{\bar{V}} = & - \nabla_k [\delta_{nk} \delta_{mj} + \delta_{mk} \delta_{nj}] \Sigma \bar{V}_n \Phi_{im} \\ & - \nabla_k^* [\delta_{nk} \delta_{mi} + \delta_{mk} \delta_{ni}] \Sigma^* \bar{V}_n \Phi_{mj} \end{aligned} \quad (2.20)$$

Again it has to be pointed out that the combinations of the expansion of ∇ and Σ operators have to be used with care since the spatial derivative in the Σ operator only concern the mean velocity components.

The compact form of the terms explicitly dependent on mean velocity may seem convenient for a practical calculation, but this notation has also some drawbacks. It does not allow to clearly separate every physical effect. Therefore, the Fourier transform is applied to each process separately, using the earlier adopted notations.

2.3.1 Convective terms

$C_{i,j}$ contribution

The expression involved in the particular derivative can be written as :

$$\hat{C}_{ij} = \frac{1}{2} [\Sigma + \Sigma^*] \bar{V}_k \frac{\partial \Phi_{ij}}{\partial X_k} \quad (2.21)$$

and its expansion up to the second order in spatial derivatives yields :

$$\hat{C}_{ij} = \bar{V}_k \frac{\partial \Phi_{ij}}{\partial X_k} + O(^3) \quad (2.22)$$

$C_{ij}^{\ddot{}}$ contribution

The expression involved in the particular derivative can be written as :

$$\hat{C}_{ij}^{\ddot{}} = -[\Sigma - \Sigma^*] \bar{V}_k (I k_k \Phi_{ij}) \quad (2.23)$$

and its expansion up to the second order leads to :

$$\hat{C}_{ij}^{\ddot{}} = -\left[I \frac{\partial}{\partial k_p} \frac{\partial \bar{V}}{\partial X_p} \right] \bar{V}_k (I k_k \Phi_{ij}) + O(^3) \quad (2.24)$$

which can be simplified as :

$$\hat{C}_{ij}^{\ddot{}} = \bar{V}_{m,n} \frac{\partial}{\partial k_n} \left[k_m \Phi_{ij} \right] + O(^3) \quad (2.25)$$

This term is usually named as the linear transfer term.

2.3.2 Production terms

The Fourier transformation of the production terms can be written as :

$$\hat{P}_{ij} = -[\Sigma \bar{V}_j]_{,k} \Phi_{ik} - [\Sigma^* \bar{V}_i]_{,k} \Phi_{kj} \quad (2.26)$$

so that the expansion yields :

$$\begin{aligned} \hat{P}_{ij} &= -\bar{V}_{j,k} \Phi_{ik} - \bar{V}_{i,k} \Phi_{kj} \\ &+ \frac{I}{2} \left[\bar{V}_{i,kp} \frac{\partial \Phi_{kj}}{\partial k_p} - \bar{V}_{j,kp} \frac{\partial \Phi_{ik}}{\partial k_p} \right] \\ &+ O(^3) \end{aligned} \quad (2.27)$$

The fact that the mean velocity gradients are expanded up to the second order can be traced only in the production terms which explicitly involve the mean velocity.

2.4 Calculation of the pressure terms

The pressure-velocity terms can be expressed using the earlier introduced notations as :

$$\begin{aligned} \hat{T}_{ij}^p &= \text{TF} [d_{i,j}^p + d_{i,j}^{p\cdot}] \\ &= \nabla_i^* \Pi_j + \nabla_j \Pi_i^* \end{aligned} \quad (2.28)$$

They can be calculated using the Poisson equations obtained by taking the divergence of the transport equations for the Reynolds stresses. These expressions is recalled in Appendix C.

When the Poisson equations are integrated in the physical space and transformed into the spectral space, the resulting contribution of the pressure-velocity correlation can be written as :

$$\nabla_j \Pi_i^*(\underline{X}, \underline{k}) = -\nabla_j \nabla_m \nabla_n G [2 \Sigma \bar{V}_n \Phi_{im} - T_{i.mn}] \quad (2.29)$$

where the spectral transform of the Green function can be chosen to be :

$$G = \left[- \left(\frac{1}{2} \frac{\partial.}{\partial X_n} + I k_n \right)^2 \right]^{-1} \quad (2.30)$$

Hence, using the Taylor expansion of the Poisson equation operators the expanded expression of the pressure-velocity correlation can be obtained.

For a shorter description it may be convenient to split these correlations into two contributions, the first one involving the spectral Reynolds stress tensor and the second one the triple correlations.

2.4.1 Contribution from the triple correlations

The contribution involving explicitly the triple correlations can be written as :

$$\hat{T}_{ij}^{p(u)} = \nabla_i^* \nabla_m^* \nabla_n^* G^* T_{mn.j} + \nabla_j \nabla_m \nabla_n G T_{i.mn} \quad (2.31)$$

and it immediately leads to :

$$\begin{aligned} \hat{T}_{ij}^{p(u)} &= I \frac{k_m k_n}{k^2} \left[k_i T_{mn.j} - k_j T_{i.mn} \right] \\ &+ \frac{k_m k_n k_l}{k^4} \frac{\partial.}{\partial X_l} \left[k_i T_{mn.j} + k_j T_{i.mn} \right] \\ &- \frac{1}{2} \frac{k_m k_n}{k^2} \left[\frac{\partial.}{\partial X_i} T_{mn.j} + \frac{\partial.}{\partial X_j} T_{i.mn} \right] \\ &- \left[\frac{k_i k_m}{k^2} \frac{\partial.}{\partial X_n} T_{mn.j} + \frac{k_j k_m}{k^2} \frac{\partial.}{\partial X_n} T_{i.mn} \right] \\ &- \frac{I}{4} \frac{1}{k^2} \frac{\partial^2.}{\partial X_n \partial X_m} \left[k_i T_{mn.j} - k_j T_{i.mn} \right] \\ &- \frac{I}{2} \frac{k_n}{k^2} \left[\frac{\partial^2.}{\partial X_i \partial X_m} T_{mn.j} - \frac{\partial^2.}{\partial X_j \partial X_m} T_{i.mn} \right] \\ &+ \frac{I}{2} \frac{k_m k_n k_l}{k^4} \left[\frac{\partial^2.}{\partial X_i \partial X_l} T_{mn.j} - \frac{\partial^2.}{\partial X_j \partial X_l} T_{i.mn} \right] \\ &+ I \frac{k_l k_n}{k^4} \frac{\partial^2.}{\partial X_m \partial X_l} \left[k_i T_{mn.j} - k_j T_{i.mn} \right] \end{aligned} \quad (2.32a)$$

$$\begin{aligned}
 & + \frac{I}{4} \left(\delta_{ls} - 4 \frac{k_l k_s}{k^2} \right) \frac{k_m k_n}{k^4} \frac{\partial^2}{\partial X_l \partial X_s} \left[k_i T_{mn,j} - k_j T_{i,mm} = \right] \\
 & + O(3)
 \end{aligned}$$

2.4.2 Contribution involving mean velocity gradients

The contribution involving explicitly the mean velocity can be written as :

$$\hat{T}_{ij}^{p(\bar{V})} = -\nabla_i^* \nabla_m^* \nabla_n^* G^* [2 \Sigma^* \bar{V}_n \Phi_{mj}] - \nabla_j \nabla_m \nabla_n G [2 \Sigma \bar{V}_n \Phi_{im}] \quad (2.33)$$

The operator involved can be conveniently split into a first order and a second order contribution (in terms of mean velocity derivatives).

First order expansion

A direct application of the first order terms of the developed operators leads to the following expression :

$$\hat{T}_{ij,1}^{p(\bar{V})} = -2I \frac{k_m k_n}{k^2} \left[k_i \bar{V}_n \Phi_{mj} - k_j \bar{V}_n \Phi_{im} \right] \quad (2.34a)$$

$$\begin{aligned}
 & + \frac{k_m k_n}{k^2} \left[\bar{V}_{n,i} \Phi_{mj} + \bar{V}_{n,j} \Phi_{im} \right. \\
 & \quad \left. + \bar{V}_n \Phi_{mj,i} + \bar{V}_n \Phi_{im,j} \right] \quad (2.34b)
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{k_m}{k^2} \left[k_i \bar{V}_{n,n} \Phi_{mj} + k_j \bar{V}_{n,n} \Phi_{im} \right. \\
 & \quad \left. + k_i \bar{V}_n \Phi_{mj,n} + k_j \bar{V}_n \Phi_{im,n} \right] \quad (2.34c)
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{k_n}{k^2} \left[k_i \bar{V}_{n,m} \Phi_{mj} + k_j \bar{V}_{n,m} \Phi_{im} \right. \\
 & \quad \left. + k_i \bar{V}_n \Phi_{mj,m} + k_j \bar{V}_n \Phi_{im,m} \right] \quad (2.34d)
 \end{aligned}$$

$$\begin{aligned}
 & - 2 \frac{k_m k_n k_l}{k^4} \left[k_i \bar{V}_{n,l} \Phi_{mj} + k_j \bar{V}_{n,l} \Phi_{im} \right. \\
 & \quad \left. + k_i \bar{V}_n \Phi_{mj,l} + k_j \bar{V}_n \Phi_{im,l} \right] \quad (2.34e)
 \end{aligned}$$

$$- \bar{V}_{n,l} \frac{k_m k_n}{k^2} \left[k_i \frac{\partial \Phi_{mj}}{\partial k_l} + k_j \frac{\partial \Phi_{im}}{\partial k_l} \right] \quad (2.34f)$$

$$+ O(2)$$

In order to simplify these expressions and to display the order of expansion in a more transparent way, the incompressibility condition is applied allowing to replace the following terms :

$$k_m \Phi_{mj} = -\frac{I}{2} \frac{\partial \Phi_{mj}}{\partial X_m} \quad (2.35a)$$

$$k_m \Phi_{im} = \frac{I}{2} \frac{\partial \Phi_{im}}{\partial X_m} \quad (2.35b)$$

Each term has been labeled in order to make the simplification more obvious.

Simplification of (2.34a)

$$^{[2.34a]} \hat{T}_{ij.1}^{p(\bar{V})} = -\frac{k_n}{k^2} \bar{V}_n \left[k_i \Phi_{mj,m} + k_j \Phi_{im,m} \right] \quad (2.36a)$$

Simplification of (2.34b)

$$\begin{aligned} ^{[2.34b]} \hat{T}_{ij.1}^{p(\bar{V})} &= -\frac{I}{2} \frac{k_n}{k^2} \left[\bar{V}_{n,i} \Phi_{mj,m} - \bar{V}_{n,j} \Phi_{im,m} \right] \\ &\quad -\frac{I}{2} \frac{k_n}{k^2} \bar{V}_n \left[\Phi_{mj,im} - \Phi_{im,jm} \right] \end{aligned} \quad (2.36b)$$

This shows explicitly a contribution of second order terms.

Simplification of (2.34c)

$$\begin{aligned} ^{[2.34c]} \hat{T}_{ij.1}^{p(\bar{V})} &= -\frac{I}{2} \frac{1}{k^2} \bar{V}_{n,n} \left[k_i \Phi_{mj,m} - k_j \Phi_{im,m} \right] \\ &\quad -\frac{I}{2} \frac{1}{k^2} \bar{V}_n \left[k_i \Phi_{mj,nm} - k_j \Phi_{im,nm} \right] \end{aligned} \quad (2.36c)$$

Simplification of (2.34d)

This term needs is in fact not to be transformed.

$$\begin{aligned} ^{[2.34d]} \hat{T}_{ij.1}^{p(\bar{V})} &= \frac{k_n}{k^2} \bar{V}_{n,m} \left[k_i \Phi_{mj} + k_j \Phi_{im} \right] \\ &\quad + \frac{k_n}{k^2} \bar{V}_n \left[k_i \Phi_{mj,m} + k_j \Phi_{im,m} \right] \end{aligned} \quad (2.36d)$$

The second part of this term can be simplified using $^{[2.34a]} \hat{T}_{ij.1}^{p(\bar{V})}$.

Simplification of (2.34e)

$$\begin{aligned} ^{[2.34e]} \hat{T}_{ij.1}^{p(\bar{V})} &= I \frac{k_n k_l}{k^4} \bar{V}_{n,l} \left[k_i \Phi_{mj,m} - k_j \Phi_{im,m} \right] \\ &\quad + I \frac{k_n k_l}{k^4} \bar{V}_n \left[k_i \Phi_{mj,lm} - k_j \Phi_{im,lm} \right] \end{aligned} \quad (2.36e)$$

Simplification of (2.34f)

$$\begin{aligned}
 [2.34f] \hat{T}_{ij.1}^{p(\bar{V})} &= \bar{V}_{n,m} \frac{k_n}{k^2} \left[k_i \Phi_{mj} + k_j \Phi_{im} \right] \\
 &+ \frac{I}{2} \bar{V}_{n,l} \frac{k_n}{k^2} \left[k_i \Phi_{mj,m|l} - k_j \Phi_{im,m|l} \right]
 \end{aligned} \tag{2.36f}$$

where the $\cdot_{,n}$ denotes the derivative with respect to X_n and $\cdot_{|n}$ denotes the derivative with respect to k_n . The first part of this term can be grouped with the first part of (2.34d).

Second order expansions

A direct application of the second order terms of the expanded operators leads to the following expression :

$$\hat{T}_{ij.2}^{p(\bar{V})} = \frac{I}{4} \frac{k_m k_n}{k^2} \left[k_i \bar{V}_{n,pq} \Phi_{mj|pq} - k_j \bar{V}_{n,pq} \Phi_{im|pq} \right] \tag{2.37a}$$

$$+ I \frac{k_m k_n k_l}{k^4} \frac{\partial}{\partial X_l} \left[k_i \bar{V}_{n,p} \Phi_{mj|p} - k_j \bar{V}_{n,p} \Phi_{im|p} \right] \tag{2.37b}$$

$$- \frac{I}{2} \frac{k_m k_n}{k^2} \left[\frac{\partial}{\partial X_i} \left(\bar{V}_{n,p} \Phi_{mj|p} \right) - \frac{\partial}{\partial X_j} \left(\bar{V}_{n,p} \Phi_{im|p} \right) \right] \tag{2.37c}$$

$$- \frac{I}{2} \frac{k_m}{k^2} \frac{\partial}{\partial X_n} \left[k_i \bar{V}_{n,p} \Phi_{mj|p} - k_j \bar{V}_{n,p} \Phi_{im|p} \right] \tag{2.37d}$$

$$- \frac{I}{2} \frac{k_n}{k^2} \frac{\partial}{\partial X_m} \left[k_i \bar{V}_{n,p} \Phi_{mj|p} - k_j \bar{V}_{n,p} \Phi_{im|p} \right] \tag{2.37e}$$

$$+ \frac{I}{2} \frac{1}{k^2} \frac{\partial^2}{\partial X_m \partial X_n} \left[k_i \bar{V}_n \Phi_{mj} - k_j \bar{V}_n \Phi_{im} \right] \tag{2.37f}$$

$$+ \frac{I}{2} \frac{k_n}{k^2} \frac{\partial}{\partial X_m} \left[\frac{\partial}{\partial X_i} \left(\bar{V}_n \Phi_{mj} \right) - \frac{\partial}{\partial X_j} \left(\bar{V}_n \Phi_{im} \right) \right] \tag{2.37g}$$

$$+ \frac{I}{2} \frac{1}{k^2} \frac{\partial}{\partial X_n} \left[\frac{\partial}{\partial X_i} \left(\bar{V}_n k_m \Phi_{mj} \right) - \frac{\partial}{\partial X_j} \left(\bar{V}_n k_m \Phi_{im} \right) \right] \tag{2.37h}$$

$$- I \frac{k_l k_n}{k^4} \frac{\partial}{\partial X_l} \left[\frac{\partial}{\partial X_i} \left(\bar{V}_n k_m \Phi_{mj} \right) - \frac{\partial}{\partial X_j} \left(\bar{V}_n k_m \Phi_{im} \right) \right] \tag{2.37i}$$

$$- I \frac{k_l}{k^4} \frac{\partial^2}{\partial X_l \partial X_n} \left[k_i \bar{V}_n k_m \Phi_{mj} - k_j \bar{V}_n k_m \Phi_{im} \right] \tag{2.37j}$$

$$- I \frac{k_l k_n}{k^4} \frac{\partial^2}{\partial X_l \partial X_m} \left[k_i \bar{V}_n \Phi_{mj} - k_j \bar{V}_n \Phi_{im} \right] \tag{2.37k}$$

$$- \frac{I}{2} \frac{k_n}{k^4} \left(\delta_{ls} - 4 \frac{k_l k_s}{k^2} \right) \frac{\partial^2}{\partial X_l \partial X_s} \left[k_i k_m \bar{V}_n \Phi_{mj} - k_j k_m \bar{V}_n \Phi_{im} \right] \tag{2.37l}$$

$$+ O(3)$$

It is again illustrative to write explicitly the derivative with respect to \underline{X} .

Simplification of (2.37a)

This term contains the contributions from higher order terms and therefore can be simplified. It is written as :

$${}^{[2.37a]} \hat{T}_{ij.2}^p(\bar{V}) = \frac{I}{4} \frac{k_n}{k^2} \bar{V}_{n,pq} \left[k_i k_m \Phi_{mj|pq} - k_j k_m \Phi_{im|pq} \right]$$

The second derivatives with respect to \underline{k} can be transformed as :

$$\begin{aligned} k_m \Phi_{mj|pq} &= \frac{\partial.}{\partial k_q} \left[k_m \Phi_{mj|p} \right] - \Phi_{qj|p} \\ k_m \Phi_{im|pq} &= \frac{\partial.}{\partial k_q} \left[k_m \Phi_{im|p} \right] - \Phi_{iq|p} \end{aligned}$$

and again :

$$\begin{aligned} k_m \Phi_{mj|pq} &= \frac{\partial.}{\partial k_q} \left[\left(k_m \Phi_{mj} \right)_{|p} - \Phi_{pj} \right] - \Phi_{qj|p} \\ k_m \Phi_{im|pq} &= \frac{\partial.}{\partial k_q} \left[\left(k_m \Phi_{im} \right)_{|p} - \Phi_{ip} \right] - \Phi_{iq|p} \end{aligned}$$

As for the first order terms, the incompressibility condition allows to express the derivatives with respect to \underline{X} explicitly, so that :

$$\begin{aligned} k_m \Phi_{mj|pq} &= \frac{\partial.}{\partial k_q} \left[\left(-\frac{I}{2} \Phi_{mj,m} \right)_{|p} - \Phi_{pj} \right] - \Phi_{qj|p} \\ k_m \Phi_{im|pq} &= \frac{\partial.}{\partial k_q} \left[\left(\frac{I}{2} \Phi_{im,m} \right)_{|p} - \Phi_{ip} \right] - \Phi_{iq|p} \end{aligned}$$

It is clear that the contributions involving the derivatives of the spectral tensor with respect to \underline{X} makes it a third order expansion term and therefore can be truncated. Furthermore, because the second derivatives are commutative, the term can be written as :

$${}^{[2.37a]} \hat{T}_{ij.2}^p(\bar{V}) = -\frac{I}{2} \frac{k_n}{k^2} \bar{V}_{n,pq} \left[k_i \Phi_{pj|q} - k_j \Phi_{ip|q} \right] \quad (2.38a)$$

Simplification of (2.37b)

The original term can be written as :

$${}^{[2.37b]} \hat{T}_{ij.2}^p(\bar{V}) = I \frac{k_m k_n k_l}{k^4} \frac{\partial.}{\partial X_l} \left[k_i \bar{V}_{n,p} \Phi_{mj|p} - k_j \bar{V}_{n,p} \Phi_{im|p} \right]$$

However it should be noted that :

$$\begin{aligned} k_m \Phi_{mj|p} &= \frac{\partial}{\partial k_p} \left[k_m \Phi_{mj} \right] - \frac{\partial k_m}{\partial k_p} \Phi_{mj} \\ k_m \Phi_{im|p} &= \frac{\partial}{\partial k_p} \left[k_m \Phi_{im} \right] - \frac{\partial k_m}{\partial k_p} \Phi_{im} \end{aligned}$$

which, using the incompressibility conditions, becomes :

$$\begin{aligned} k_m \Phi_{mj|p} &= -\frac{I}{2} \frac{\partial}{\partial k_p} \left[\Phi_{mj,m} \right] - \Phi_{pj} \\ k_m \Phi_{im|p} &= \frac{I}{2} \frac{\partial}{\partial k_p} \left[\Phi_{im,m} \right] - \Phi_{ip} \end{aligned}$$

The term can then be written as :

$$\begin{aligned} [2.37b] \hat{T}_{ij,2}^{p(\bar{V})} &= -I \frac{k_n k_l}{k^4} \frac{\partial}{\partial X_l} \left[k_i \bar{V}_{n,p} \Phi_{pj} - k_j \bar{V}_{n,p} \Phi_{ip} \right] \\ &\quad + \frac{1}{2} \frac{k_n k_l}{k^4} \frac{\partial}{\partial X_l} \left[k_i \bar{V}_{n,p} \Phi_{mj,m|p} - k_j \bar{V}_{n,p} \Phi_{im,m|p} \right] \end{aligned}$$

It is quite obvious that the second part of this term is of the third order with respect to the \underline{X} derivatives. Therefore, the contribution of this term to the second order expansion can be written as :

$$\begin{aligned} [2.37b] \hat{T}_{ij,2}^{p(\bar{V})} &= -I \frac{k_n k_l}{k^4} \bar{V}_{n,pl} \left[k_i \Phi_{pj} - k_j \Phi_{ip} \right] \\ &\quad - I \frac{k_n k_l}{k^4} \bar{V}_{n,p} \left[k_i \Phi_{pj,l} - k_j \Phi_{ip,l} \right] \\ &\quad + O(3) \end{aligned} \tag{2.38b}$$

Simplification of (2.37c)

The same kind of simplification can be applied to the next two terms, (2.37c) and (2.37d).

$$\begin{aligned} [2.37c] \hat{T}_{ij,2}^{p(\bar{V})} &= \frac{I}{2} \frac{k_n}{k^2} \bar{V}_{n,p} \left[\Phi_{pj,i} - \Phi_{ip,j} \right] \\ &\quad + \frac{I}{2} \frac{k_n}{k^2} \left[\bar{V}_{n,pi} \Phi_{pj} - \bar{V}_{n,pj} \Phi_{ip} \right] \\ &\quad + O(3) \end{aligned} \tag{2.38c}$$

Simplification of (2.37d)

$$\begin{aligned}
 [2.37d] \hat{T}_{ij,2}^{p(\bar{V})} &= \frac{I}{2} \frac{1}{k^2} \bar{V}_{n,pn} \left[k_i \Phi_{pj} - k_j \Phi_{ip} \right] \\
 &+ \frac{I}{2} \frac{1}{k^2} \bar{V}_{n,p} \left[k_i \Phi_{pj,n} - k_j \Phi_{ip,n} \right] \\
 &+ O(3)
 \end{aligned} \tag{2.38d}$$

Simplification of (2.37e)

The derivation leads to two terms :

$$\begin{aligned}
 [2.37e] \hat{T}_{ij,2}^{p(\bar{V})} &= -\frac{I}{2} \frac{k_n}{k^2} \bar{V}_{n,p} \left[k_i \Phi_{mj|p,m} - k_j \Phi_{im|p,m} \right] \\
 &- \frac{I}{2} \frac{k_n}{k^2} \bar{V}_{n,pm} \left[k_i \Phi_{mj|p} - k_j \Phi_{im|p} \right]
 \end{aligned} \tag{2.38e}$$

The first term is exactly the opposite of (2.34f), whereas the second term can be added to the second part of (2.37a), since it has exactly the same form.

Simplification of (2.37f)

$$\begin{aligned}
 [2.37f] \hat{T}_{ij,2}^{p(\bar{V})} &= \frac{I}{2} \frac{1}{k^2} \bar{V}_{n,nm} \left[k_i \Phi_{mj} - k_j \Phi_{im} \right] \\
 &+ \frac{I}{2} \frac{1}{k^2} \bar{V}_{n,m} \left[k_i \Phi_{mj,n} - k_j \Phi_{im,n} \right] \\
 &+ \frac{I}{2} \frac{1}{k^2} \bar{V}_n \left[k_i \Phi_{mj,nm} - k_j \Phi_{im,nm} \right] \\
 &+ \frac{I}{2} \frac{1}{k^2} \bar{V}_{n,n} \left[k_i \Phi_{mj,m} - k_j \Phi_{im,m} \right]
 \end{aligned} \tag{2.38f}$$

The last two terms cancel with the (2.34c) term while the first two terms can be lumped with the (2.37d) term.

Simplification of (2.37g)

$$\begin{aligned}
 [2.37g] \hat{T}_{ij,2}^{p(\bar{V})} &= \frac{I}{2} \frac{k_n}{k^2} \bar{V}_{n,m} \left[\Phi_{mj,i} - \Phi_{im,j} \right] \\
 &+ \frac{I}{2} \frac{k_n}{k^2} \left[\bar{V}_{n,im} \Phi_{mj} - \bar{V}_{n,jm} \Phi_{im} \right] \\
 &+ \frac{I}{2} \frac{k_n}{k^2} \bar{V}_n \left[\Phi_{mj,mi} - \Phi_{im,mj} \right] \\
 &+ \frac{I}{2} \frac{k_n}{k^2} \left[\bar{V}_{n,i} \Phi_{mj,m} - \bar{V}_{n,j} \Phi_{im,m} \right]
 \end{aligned} \tag{2.38g}$$

The last two terms cancel with (2.34b) and the first two terms can be added to the (2.37c) term.

Simplification of (2.37h)

This term is in fact of a higher order than two, so that :

$$^{[2.37h]} \hat{T}_{ij.2}^{p(\bar{V})} = O(3) \quad (2.38h)$$

Simplification of (2.37i)

This is also the case for the next term :

$$^{[2.37i]} \hat{T}_{ij.2}^{p(\bar{V})} = O(3) \quad (2.38i)$$

Simplification of (2.37j)

This simplification can again be applied to the next term :

$$^{[2.37j]} \hat{T}_{ij.2}^{p(\bar{V})} = O(3) \quad (2.38j)$$

Simplification of (2.37k)

No change has to be introduced in the next term.

$$\begin{aligned} ^{[2.37k]} \hat{T}_{ij.2}^{p(\bar{V})} = & -I \frac{k_l k_n}{k^4} \bar{V}_{n,lm} \left[k_i \Phi_{mj} - k_j \Phi_{im} \right] \\ & -I \frac{k_l k_n}{k^4} \bar{V}_{n,m} \left[k_i \Phi_{mj,l} - k_j \Phi_{im,l} \right] \\ & -I \frac{k_l k_n}{k^4} \bar{V}_n \left[k_i \Phi_{mj,lm} - k_j \Phi_{im,lm} \right] \\ & -I \frac{k_l k_n}{k^4} \bar{V}_{n,l} \left[k_i \Phi_{mj,m} - k_j \Phi_{im,m} \right] \end{aligned} \quad (2.38k)$$

Here again the last two terms cancel with (2.34e) while the first two terms can be associated with the (2.37b) term.

Simplification of (2.37l)

The incompressibility solution again makes this contribution a third order one, so that :

$$^{[2.37l]} \hat{T}_{ij.2}^{p(\bar{V})} = O(3) \quad (2.38l)$$

Regrouping the first and second order expansions

As already shown, the contribution of the pressure terms involving the Reynolds stress tensor can be simplified as :

$$\begin{aligned}
 \hat{T}_{ij}^{p(\bar{V})} = & 2 \frac{k_n}{k^2} \bar{V}_{n,m} \left[k_i \Phi_{mj} + k_j \Phi_{im} \right] \\
 & - I \frac{k_n}{k^2} \bar{V}_{n,pm} \left[k_i \Phi_{mj|p} - k_j \Phi_{im|p} \right] \\
 & + I \frac{1}{k^2} \bar{V}_{n,nm} \left[k_i \Phi_{mj} - k_j \Phi_{im} \right] \\
 & + I \frac{1}{k^2} \bar{V}_{n,m} \left[k_i \Phi_{mj,n} - k_j \Phi_{im,n} \right] \\
 & + I \frac{k_n}{k^2} \bar{V}_{n,m} \left[\Phi_{mj,i} - \Phi_{im,j} \right] \\
 & + I \frac{k_n}{k^2} \left[\bar{V}_{n,im} \Phi_{mj} - \bar{V}_{n,jm} \Phi_{im} \right] \\
 & - 2 I \frac{k_l k_n}{k^4} \bar{V}_{n,lm} \left[k_i \Phi_{mj} - k_j \Phi_{im} \right] \\
 & - 2 I \frac{k_l k_n}{k^4} \bar{V}_{n,m} \left[k_i \Phi_{mj,l} - k_j \Phi_{im,l} \right]
 \end{aligned} \tag{2.39a}$$

The first terms can be easily recognized to be the zero order rapid term in the spectral space.

2.5 Second order expansion of the Reynolds stress equation in the spectral space

Now that each process has been identified, we can summarize the expression for the second order expansion of the Reynolds stress transport equation in the spectral space. Various contributions are written in the increasing order of the expansion with respect to the spatial derivatives. The physical meaning of each term and the processes which they represent are also indicated. Hence, the transport equation for the spectrum tensor can be summarized as :

$$\begin{aligned}
 \Phi_{ij,t} & + \bar{V}_k \Phi_{ij,k} = \nu \Phi_{ij,kk} - 2 \nu \left[\frac{1}{4} \Phi_{ij,kk} + k^2 \Phi_{ij} \right] \\
 \text{unsteadiness} & \quad \text{convection} & \quad \text{viscous diffusion} & \quad \text{viscous dissipation} \\
 (1a) & \quad (1b) & \quad (1c) & \quad (1d) \\
 & & & + \bar{V}_{m,n} \frac{\partial}{\partial k_n} \left[k_m \Phi_{ij} \right] \\
 & & & \text{linear spectral transfer}
 \end{aligned}$$

(2)

$$- \bar{V}_{j,k} \Phi_{ik} - \bar{V}_{i,k} \Phi_{kj}$$

production by mean velocity gradients (first order terms)

(3)

$$+ 2 \frac{k_n}{k^2} \bar{V}_{n,m} \left[k_i \Phi_{mj} + k_j \Phi_{im} \right]$$

rapid pressure strain term (first order term)

(4)

$$+ \frac{1}{2} \left[T_{i,jm} + T_{im,j} \right]_{,m} + I k_m \left[T_{i,jm} - T_{im,j} \right]$$

inhomogeneous turbulent diffusion *spectral turbulent diffusion*

(5a) (5b)

$$+ I \frac{k_m k_n k_l}{k^2} \left[\delta_{il} T_{mn,j} - \delta_{jl} T_{i,mn} \right]$$

*contribution to the spectral cascade by the triple correlations
arising from the pressure-velocity correlations
(zero order terms)*

(6)

$$+ \left[\frac{k_m k_n k_l k_p}{k^4} - \frac{1}{2} \frac{k_m k_n}{k^2} \delta_{lp} - \frac{k_m k_l}{k^2} \delta_{np} \right] \left[\delta_{il} T_{mn,j} + \delta_{jl} T_{i,mn} \right]_{,p}$$

*triple correlations term arising from pressure-velocity correlations
(first order terms)*

(7)

$$+ \left[-\frac{I}{4} \frac{k_l}{k^2} \delta_{np} \delta_{mq} - \frac{I}{2} \frac{k_n}{k^2} \delta_{lp} \delta_{mq} + \frac{I}{2} \frac{k_m k_n k_p}{k^4} \delta_{ql} + I \frac{k_p k_n k_l}{k^4} \delta_{mq} \right. \\ \left. + \frac{I}{4} \left(\delta_{pq} - 4 \frac{k_p k_q}{k^2} \right) \frac{k_m k_n k_l}{k^4} \right] \left[\delta_{il} T_{mn,j} - \delta_{jl} T_{i,mn} \right]_{,pq}$$

*triple correlations term arising from pressure-velocity correlations
(second order terms)*

(8)

$$+ \frac{I}{2} \delta_{nl} \bar{V}_{n,pm} \left[\delta_{il} \Phi_{mj} - \delta_{jl} \Phi_{im} \right]_{|p}$$

production by mean velocity gradients (second order terms)
(9)

$$- I \frac{k_n k_l}{k^2} \overline{V}_{n,pm} \left[\delta_{il} \Phi_{mj} - \delta_{jl} \Phi_{im} \right]_{,p}$$

terms arising from pressure-velocity correlations
(first part of the second order terms)

(10)

$$+ \left[-2 I \frac{k_l k_n k_p}{k^4} + I \frac{k_n}{k^2} \delta_{pl} + I \frac{k_l}{k^2} \delta_{pn} \right] \left(\overline{V}_{n,m} \left[\delta_{il} \Phi_{mj} - \delta_{jl} \Phi_{im} \right] \right)_{,p}$$

terms arising from pressure-velocity correlations
(end of the second order terms)

(11)

The molecular effects are here written in the first line of the right hand side of this equation. They are respectively the viscous transport and the viscous dissipation terms. Although they are here separately written, it is obvious (as it has been already noted for the two-point equations in the physical space), that the viscous diffusion can be grouped with the non-homogeneous part of the viscous dissipation in order to obtain a more compact form.

The two-point contribution from the convective terms, usually named as the linear spectral transfer, is given in the second line of the equation (term (2)).

The first order contribution of the production by mean velocity gradient is the next term (3), representing the interactions between mean and fluctuating field. Its second order contribution appears later in the equation and can be written in a similar form to the part of the second order terms involving the mean velocity gradient and arising from the pressure-velocity terms.

The next term (4) is the only remaining contribution of the pressure-velocity correlations involving the mean velocity gradient for the case of a homogeneous turbulence. This is the first order term in the expansion of the rapid pressure-strain term.

The terms (5a) and (5b) are derived from the turbulent diffusion. The first one is an inhomogeneous contribution depending on the spatial derivatives. The second one takes into account the influence of the distance between the two-points in the physical space, and is therefore a spectral contribution to the turbulent diffusion.

The zero order expansion of the pressure-velocity correlations involves only the triple correlations (6). It can be noted that for homogeneous turbulence, only the terms up to (6) remain. In addition, the viscous diffusion (1c), the diffusion-like term of the viscous dissipation (1d) as well as the inhomogeneous turbulent diffusion (5a) vanish. Therefore, in the case of a homogeneous turbulence, the cascade of energy is described via the linear spectral transfer (2) and the non linear spectral transfer decomposed in the spectral turbulent diffusion (5b) and the zero order pressure-velocity correlation term (6). The

2.5. SECOND ORDER EXPANSION OF THE REYNOLDS STRESS EQUATION IN
THE SPECTRAL SPACE

integral of each of these terms over all wave numbers is indeed zero in a homogeneous flow. In the case of an inhomogeneous flow, the contribution of the spectral turbulent diffusion (5b) in the one-point physical space remains zero. This is not the case with the integration of the zero order pressure-velocity correlations term (6) which has to be integrated together with the higher order term ((7), (8), (10) and (11) to give a zero contribution. Based on the above interpretation it could be more suitable to name the entire redistribution term in the spectral space as the nonlinear transfer term. However, for convenience and a more transparent comparison with relevant literature, we follow here the nomenclature used by Bertoglio and Jeandel (1993) and Laporta (1995). The name *spectral transfer* is then reserved for the processes which do not come from any spatial inhomogeneity, while the name *diffusion* is associated with typically inhomogeneous terms.

The summarized equation allows to recognize the various terms to be retained for a homogeneous flow, as well as those representing the first order spatial inhomogeneity or the contributions from spatial second derivatives. In a way, it simplifies also the physical interpretation of each term.

However, the non linear spectral transfer term can be written in a more compact form, as shown below.

First, we can rewrite the spectral turbulent diffusion (5b) as :

$$I k_m \left[T_{i.jm} - T_{im.j} \right] = -I k_m \delta_{nl} \left[\delta_{il} T_{mn.j} - \delta_{jl} T_{i.mn} \right]$$

so that it has a form similar to the zero order term (6) :

$$I k_m \frac{k_n k_l}{k^2} \left[\delta_{il} T_{mn.j} - \delta_{jl} T_{i.mn} \right]$$

allowing to write the non linear spectral transfer as

$$\begin{aligned} & (5b) + (6) \\ & = \\ & I k_m \left[\frac{k_n k_l}{k^2} - \delta_{nl} \right] \left[\delta_{il} T_{mn.j} - \delta_{jl} T_{i.mn} \right] \end{aligned}$$

The inhomogeneous turbulent diffusion (5a) can be written together with the first order term from the pressure-velocity correlations involving the triple correlations (7).

The inhomogeneous turbulent diffusion terms are first rewritten as :

$$\frac{1}{2} \left[T_{i.jm} + T_{im.j} \right]_{,m} = \frac{1}{2} \delta_{pm} \delta_{nl} \left[\delta_{il} T_{mn.j} + \delta_{jl} T_{i.mn} \right]_{,p}$$

whereas, using the symmetry of the indices m and n in this expression, the terms of the first order are expanded as :

$$(\gamma) = \left[\frac{k_m k_n k_l k_p}{k^4} - \frac{1}{2} \frac{k_m k_n}{k^2} \delta_{lp} - \frac{1}{2} \frac{k_m k_l}{k^2} \delta_{np} - \frac{1}{2} \frac{k_n k_l}{k^2} \delta_{mp} \right] \left[\delta_{il} T_{mn.j} + \delta_{jl} T_{i.mn} \right]_{,p}$$

allowing to express together all first order term depending on the triple correlations as :

$$(5a) + (7) = \left[k_m \left(\frac{k_n k_l k_p}{k^4} - \frac{1}{2} \frac{k_n}{k^2} \delta_{lp} - \frac{1}{2} \frac{k_l}{k^2} \delta_{np} \right) - \frac{1}{2} \left(\frac{k_n k_l}{k^2} - \delta_{nl} \right) \delta_{mp} \right] \left[\delta_{il} T_{mn,j} + \delta_{jl} T_{i,mn} \right]_{,p}$$

All second order terms depending on the triple correlations are already grouped in one term (8). The second order contributions depending on the spectral tensor can be written as :

$$(9) + (10) + (11) = -2 I \left[\left(\frac{k_n k_l k_p}{k^4} - \frac{1}{2} \frac{k_n}{k^2} \delta_{lp} - \frac{1}{2} \frac{k_l}{k^2} \delta_{np} \right) \right] \left(\bar{\nabla}_{n,m} \left[\delta_{il} \Phi_{mj} - \delta_{jl} \Phi_{im} \right] \right)_{,p} - I \left[- \frac{1}{2} \left(2 \frac{k_n k_l}{k^2} - \delta_{nl} \right) \delta_{mp} \right] \bar{\nabla}_{n,pm} \left[\delta_{il} \Phi_{mj} - \delta_{jl} \Phi_{im} \right]_{|p}$$

It can also be seen that a more compact form of the equation might be obtained for the higher order terms. Since the derivation of one-point equations is the main target of this study, it is convenient to decompose the term arising from the pressure-velocity contribution in the same way as it is done with the correlation between the pressure-gradient and velocity in the physical space. This decomposition leads to a diffusive and a redistributive term, the latter called the pressure-strain term. The correspondence is elaborated in Appendix C.

2.6 Closure of the Reynolds stress equation in the spectral space

The description of turbulence in the spectral space dispenses with the necessity for closing the dissipation tensor because this equation can be treated in the exact manner in the spectral space. The pressure velocity correlation can be directly obtained by solving the Poisson equations that are directly expressed in terms of spectral tensor Φ_{ij} and triple correlations $T_{ij,k}$. Hence, only the third order correlations have to be specified. The information contained in the spectral equation enables to describe the interactions between the different scales and thus far accounts for the non-local processes in turbulence. The non-local processes are mainly associated with the third order correlations so that a full spectral treatment can retain these information. However, since the one-point equations are the final goal of this study, a local treatment can be considered still as a useful approximation, leading to a much simpler model in the spectral space (Cambon (1982), Besnard *et al.* (1992) and Steinkamp *et al.* (1995)). A local closure of the third order moments will, of course, conceal a part of the non-local interactions between the scales.

However, it enables still to introduce several scales. Taking this advantage of a spectral description of turbulence and incorporating the scale information in the one-point closure is basically the aim of this study.

The technique chosen here follows the work of Laporta (1995). It enables to dispense with the definition of a split spectrum. Multiple time scale turbulence models which use a decomposition of the spectrum, as described by Schiestel (1983), lead to a large number of equations, in fact separate equation sets corresponding to each spectrum slice. Particularly at the second moment closure level, this approach generates a formidable number of equations which are unsuitable for practical computations. Even more serious problem is the number of empirical coefficients which increases progressively with the number of spectrum slices considered. Here, we try to identify and incorporate into one-point model the basic information from a very simple spectral approach.

2.7 Spherical integration

The triple correlations are linked to triadic interactions of the wave numbers. In order to reduce the dimensions involved we adopt the classical method which is based on the integration of the equations over shells of radius equal to the norm of the wave number. This operation does not allow anymore to account for the anisotropy of the directions of the wave number. However this information is also totally inaccessible to the one-point approach.

Introducing such integration leads to unknown quantities in almost every term of the Reynolds stress transport equation, terms that, therefore, need to be closed.

Using the standard notation of the spherically integrated tensor is :

$$\varphi_{ij} = \int_A \Phi_{ij} dA \quad (2.44)$$

the exact equation for φ_{ij} can now be written as :

$$\begin{aligned} \varphi_{ij,t} &+ \bar{V}_k \varphi_{ij,k} = \nu \varphi_{ij,kk} - 2\nu \left[\frac{1}{4} \varphi_{ij,kk} + k^2 \varphi_{ij} \right] \\ \text{unsteadiness} & \quad \text{convection} & \quad \text{viscous diffusion} & \quad \text{viscous dissipation} \\ & + \bar{V}_{m,n} \int \frac{\partial}{\partial k_n} [k_m \Phi_{ij}] \\ & \quad \text{linear spectral transfer} \\ & - \bar{V}_{j,k} \varphi_{ik} - \bar{V}_{i,k} \varphi_{kj} \\ & \quad \text{production by mean velocity gradients (first order terms)} \\ & + 2\bar{V}_{n,m} \int \frac{k_n}{k^2} [k_i \Phi_{mj} + k_j \Phi_{im}] \\ & \quad \text{rapid pressure strain term (first order term)} \\ & + \frac{1}{2} \left[\int (T_{i,jm} + T_{im,j}) \right]_{,m} + I \int k_m [T_{i,jm} - T_{im,j}] \end{aligned}$$

$$\begin{aligned}
 & \text{inhomogeneous turbulent diffusion} \quad \text{spectral turbulent diffusion} \\
 + & I \int \frac{k_m k_n k_l}{k^2} \left[\delta_{il} T_{mn,j} - \delta_{jl} T_{i,mn} \right] \\
 & \text{contribution to the spectral cascade by triple correlations arising from} \\
 & \text{pressure-velocity correlations (zero order terms)} \\
 + & \int \left[\frac{k_m k_n k_l k_p}{k^4} - \frac{1}{2} \frac{k_m k_n}{k^2} \delta_{lp} - \frac{k_m k_l}{k^2} \delta_{np} \right] \left[\delta_{il} T_{mn,j} + \delta_{jl} T_{i,mn} \right]_{,p} \\
 & \text{triple correlations effect arising from pressure-velocity correlations} \\
 & \text{(first order terms)} \\
 + & \int \left[-\frac{I}{4} \frac{k_l}{k^2} \delta_{np} \delta_{mq} - \frac{I}{2} \frac{k_n}{k^2} \delta_{lp} \delta_{mq} + \frac{I}{2} \frac{k_m k_n k_p}{k^4} \delta_{ql} + I \frac{k_p k_m k_l}{k^4} \delta_{mq} \right. \\
 & \quad \left. + \frac{I}{4} \left(\delta_{pq} - 4 \frac{k_p k_q}{k^2} \right) \frac{k_m k_n k_l}{k^2} \right] \left[\delta_{il} T_{mn,j} - \delta_{jl} T_{i,mn} \right]_{,pq} \\
 & \text{triple correlations effect arising from pressure-velocity correlations} \\
 & \text{(second order terms)} \\
 + & \int \frac{I}{2} \delta_{nl} \bar{\nabla}_{n,pm} \left[\delta_{il} \Phi_{mj} - \delta_{jl} \Phi_{im} \right]_{|p} \\
 & \text{production by mean velocity gradients (second order terms)} \\
 - & \int I \frac{k_n k_l}{k^2} \bar{\nabla}_{n,pm} \left[\delta_{il} \Phi_{mj} - \delta_{jl} \Phi_{im} \right]_{|p} \\
 & \text{terms arising from pressure-velocity correlations} \\
 & \text{(first part of the second order terms)} \\
 + & \int \left[-2 I \frac{k_l k_n k_p}{k^4} + I \frac{k_n}{k^2} \delta_{pl} + I \frac{k_l}{k^2} \delta_{pm} \right] \left(\bar{\nabla}_{n,m} \left[\delta_{il} \Phi_{mj} - \delta_{jl} \Phi_{im} \right] \right)_{,p} \\
 & \text{terms arising from pressure-velocity correlations} \\
 & \text{(end of the second order terms)}
 \end{aligned}$$

The aim of the derivation that follows is to close the above equation in a form which will lead to the one-point formulation. Two approaches can be seen at this level. The first one consist in closing the unknown terms in function of spherically integrated quantities (Besnard *et al.*, 1992) and then choosing similar expressions as practiced usually in one-point closures. The other approach consist in closing the third order correlations with a spectral model. This leads to a systematic closure provided the expressions can be integrated analytically. It also ensures that each term is closed at the same level. However such derivations are much more sophisticated than the previous ones (Cambon, 1982).

2.8 The spectral closure

The spectral model adopted for further consideration is the one proposed by Clark and Zemach (1995). As a first approximation each contribution coming from the second

order expansion is neglected. The exact equation can, therefore, be written as :

$$\begin{aligned}
 \varphi_{ij,t} + \bar{V}_k \varphi_{ij,k} &= \nu \varphi_{ij,kk} - 2\nu \left[\frac{1}{4} \varphi_{ij,kk} + k^2 \varphi_{ij} \right] \\
 \text{unsteadiness} \quad \text{convection} & \quad \text{viscous diffusion} \quad \quad \text{viscous dissipation} \\
 &+ \bar{V}_{m,n} \int \frac{\partial}{\partial k_n} \left[k_m \Phi_{ij} \right] \\
 & \quad \text{linear spectral transfer} \\
 &- \bar{V}_{j,k} \varphi_{ik} - \bar{V}_{i,k} \varphi_{kj} \\
 & \quad \text{production by mean velocity gradients (first order terms)} \\
 &+ 2\bar{V}_{n,m} \int \frac{k_n}{k^2} \left[k_i \Phi_{mj} + k_j \Phi_{im} \right] \\
 & \quad \text{rapid pressure strain term (first order term)} \\
 &+ \frac{1}{2} \left[\int \left(T_{i,jm} + T_{im,j} \right) \right]_{,m} + I \int k_m \left[T_{i,jm} - T_{im,j} \right] \\
 & \quad \text{inhomogeneous turbulent diffusion} \quad \text{spectral turbulent diffusion} \\
 &+ I \int \frac{k_m k_n k_l}{k^2} \left[\delta_{il} T_{mn,j} - \delta_{jl} T_{i,mn} \right] \\
 & \quad \text{contribution to the spectral cascade by triple correlations arising} \\
 & \quad \text{from pressure-velocity correlations (zero order terms).} \\
 &+ \int \left[\frac{k_m k_n k_l k_p}{k^4} - \frac{1}{2} \frac{k_m k_n}{k^2} \delta_{lp} - \frac{k_m k_l}{k^2} \delta_{np} \right] \left[\delta_{il} T_{mn,j} + \delta_{jl} T_{i,mn} \right]_{,p} \\
 & \quad \text{triple correlations effect arising from pressure-velocity correlations} \\
 & \quad \text{(first order terms)}
 \end{aligned}$$

In this equation only the unsteady term, the advection in physical space, the production by mean velocity gradients and the viscous processes do not require any closure.

2.8.1 Production

As in every second moment closure, these mechanisms are treated exactly and, therefore, do not need any modeling.

2.8.2 Dissipation

In contrast to the treatment in physical space, the spectral description of turbulence allows to describe exactly the viscous dissipation process. It does not require any closure assumption.

2.8.3 Redistribution by the pressure-strain correlations

The rapid term

The only remaining term arising from the pressure-velocity correlation and involving the double correlations is the redistribution term corresponding to the first part of the rapid term. It can be written as :

$$\phi_{ij}^R = 2 \bar{V}_{n,m} \int \frac{k_n}{k^2} \left[k_i \Phi_{mj} + k_j \Phi_{im} \right] \quad (2.47)$$

Introducing the fourth order tensor

$$X_{ijpq} = \int \frac{k_p k_q}{k^2} \Phi_{ij} \quad (2.48)$$

this term can be rewritten as :

$$\phi_{ij}^R = 2 \bar{V}_{m,n} \left[X_{imnj} + X_{mjni} \right] \quad (2.49)$$

Every classical treatment of this term assumes that X_{ijpq} can be developed as an isotropic functional which only depends on the deviatoric of the Reynolds stresses tensor, which represents the stress anisotropy, defined by :

$$b_{ij} = \frac{1}{2K} R_{ij} - \frac{1}{3} \delta_{ij} \quad (2.50)$$

The complete form of this functional can be found using the Cayley-Hamilton theorem and involves *a priori* fifteen unknowns. The corresponding form to that closure in the spherical space can be easily found using the same definition of the anisotropy tensor for φ_{ij} , that can be denoted as (Cambon, 1982) :

$$h_{ij} = \frac{1}{2\varphi} \varphi_{ij} - \frac{1}{3} \delta_{ij} \quad (2.51)$$

where φ is the trace of φ_{ij} and, therefore, identical to the spectrum of kinetic energy denoted by E , so that

$$\varphi = E \quad (2.52)$$

The functional can then be written as :

$$\begin{aligned} \frac{1}{2E} X_{ijpq} &= C_1 \delta_{ij} \delta_{pq} + C_2 (\delta_{ip} \delta_{jq} + \delta_{jp} \delta_{iq}) \\ &+ C_3 \delta_{ij} h_{pq} + C_4 h_{ij} \delta_{pq} \\ &+ C_5 (\delta_{ip} h_{jq} + \delta_{jp} h_{iq} + h_{ip} \delta_{jq} + h_{jp} \delta_{iq}) \\ &+ C_6 \delta_{ij} h_{.pq}^2 + C_7 h_{.ij}^2 \delta_{pq} \\ &+ C_8 (\delta_{ip} h_{.jq}^2 + \delta_{jp} h_{.iq}^2 + h_{.ip}^2 \delta_{jq} + h_{.jp}^2 \delta_{iq}) \\ &+ C_9 h_{ij} h_{.pq} + C_{10} (h_{ip} h_{jq} + h_{jp} h_{iq}) \\ &+ C_{11} h_{ij} h_{.pq}^2 + C_{12} h_{.ij}^2 h_{pq} \\ &+ C_{13} (h_{ip} h_{.jq}^2 + h_{jp} h_{.iq}^2 + h_{.ip}^2 h_{jq} + h_{.jp}^2 h_{iq}) \\ &+ C_{14} h_{.ij}^2 h_{.pq}^2 + C_{15} (h_{.ip}^2 h_{.jq}^2 + h_{.jp}^2 h_{.iq}^2) \end{aligned} \quad (2.53)$$

where $h_{.ij}^2$ means $h_{ik} h_{kj}$. Following the same approach as used within the one-point description of the turbulence, the coefficients of this functionals are scalars that might depend on the invariants of the anisotropy tensor $II_h = -h_{.nn}^2/2$ and $III_h = h_{.nn}^3/3$. Thus, with $\bar{V}_{m,n}$ decomposed into the deformation tensor S_{mn} and the rate of rotation tensor W_{mn} , the rapid pressure-strain term can be written as :

$$\begin{aligned}
 \frac{1}{2E} \phi_{ij}^R &= \beta_1 h_{ij} \\
 &+ \beta_2 [h_{.ij}^2 + 2/3 II_h \delta_{ij}] \\
 &+ \beta_3 S_{ij} \\
 &+ \beta_4 [h_{ik} S_{kj} + h_{jk} S_{ik} - 2/3 h_{mn} S_{mn} \delta_{ij}] \\
 &+ \beta_5 [h_{.ik}^2 S_{kj} + h_{.jk}^2 S_{ik} - 2/3 h_{.mn}^2 S_{mn} \delta_{ij}] \\
 &+ \beta_6 [h_{ik} W_{jk} + h_{jk} W_{ik}] \\
 &+ \beta_7 [h_{.ik}^2 W_{jk} + h_{.jk}^2 W_{ik}] \\
 &+ \beta_8 [h_{ik} W_{kp} h_{.pj}^2 + h_{.ik}^2 W_{pk} h_{jp}]
 \end{aligned} \tag{2.54}$$

The relations between the coefficients as well as the constraints are given in Appendix F. The model chosen here is the linear one with respect to the anisotropy tensor so that the expression for X_{ijpq} reduces to :

$$\begin{aligned}
 \frac{1}{2E} X_{ijpq} &= C_1 \delta_{ij} \delta_{pq} + C_2 (\delta_{ip} \delta_{jq} + \delta_{jp} \delta_{iq}) \\
 &+ C_3 \delta_{ij} h_{pq} + C_4 h_{ij} \delta_{pq} \\
 &+ C_5 (\delta_{ip} h_{jq} + \delta_{jp} h_{iq} + h_{ip} \delta_{jq} + h_{jp} \delta_{iq})
 \end{aligned} \tag{2.55}$$

The application of the incompressibility and normalization constraints respectively yields the following relationship between the coefficients :

$$C_1 + 4C_2 = 0 \tag{2.56a}$$

$$C_3 + C_4 + 5C_5 = 0 \tag{2.56b}$$

and

$$3C_1 + 2C_2 = \frac{1}{3} \tag{2.57a}$$

$$3C_4 + 4C_5 = 1 \tag{2.57b}$$

The above equations can be rearranged to yield all coefficients in terms of only one, which remains to be determined :

$$C_1 = \frac{2}{15} \tag{2.58a}$$

$$C_2 = -\frac{1}{30} \tag{2.58b}$$

$$C_3 = -\frac{11}{3} C_5 - \frac{1}{3} \tag{2.58c}$$

$$C_4 = -\frac{4}{3} C_5 + \frac{1}{3} \tag{2.58d}$$

These relations were obtained in the physical space by Launder *et al.* (1975). The expression for the pressure-strain term now reduces to

$$\begin{aligned} \frac{1}{2E} \phi_{ij}^R &= \beta_3 S_{ij} \\ &+ \beta_4 [h_{ik} S_{kj} + h_{jk} S_{ik} - 2/3 h_{mn} S_{mn} \delta_{ij}] \\ &+ \beta_6 [h_{ik} W_{jk} + h_{jk} W_{ik}] \end{aligned} \quad (2.59)$$

with

$$\beta_3 = \frac{2}{5} \quad (2.60a)$$

$$\beta_4 = -6 C_5 \quad (2.60b)$$

$$\beta_6 = \frac{14}{3} (C_5 + \frac{2}{7}) \quad (2.60c)$$

Besnard *et al.* (1992) retained this model also within the spectral approach and the same expression were also obtained by Jeandel *et al.* (1978). The free coefficient used in the physical model of Launder *et al.* (1975) is

$$C_5 = -\frac{8}{55} \quad (2.61)$$

and this is also the choice made in the spectral space.

The slow term

The other terms arising from the pressure-velocity correlations and involving triple correlations can be identified as the so called slow part of the pressure strain in the physical space. To follow again the same route to closure as in the one-point methodology, all these contributions, which constitute the last two lines of the transport equation, can be closed again by a trace free functional depending only on the anisotropy tensor b_{ij} . The complete functional determined by the Cayley-Hamilton theorem is quadratic and can be written as :

$$\frac{1}{(k^{3/2} E^{3/2})} \phi_{ij}^S = \beta_1^S h_{ij} + \beta_2^S (h_{ij}^2 + \frac{2}{3} II_h \delta_{ij}) \quad (2.62)$$

Here we retain only the first linear part corresponding to the simple linear return to isotropy hypothesis similar to Rotta (1951) model. The value of the coefficient is taken from the proposal of Besnard *et al.* (1992) :

$$\beta_1^S = -2.89 \quad (2.63a)$$

$$\beta_2^S = 0 \quad (2.63b)$$

The value of β_1^S is slightly smaller than the corresponding coefficient in the slow term associated with the the Launder *et al.* (1975) slow term in the physical space, where it is generally adopted to be $\beta_1^S = -3.0$.

2.8.4 Linear spectral transfer

The pure spectral convection term can be closed with the same hypothesis as the one chosen for the pressure-strain term since it can also be expressed in terms of the fourth order tensor X_{ijpq} . The linear spectral transfer is defined as :

$$T_{ij}^L = \bar{V}_{m,n} \int \frac{\partial}{\partial k_n} \left[k_m \Phi_{ij} \right] \quad (2.64)$$

which can be rewritten as :

$$\begin{aligned} T_{ij}^L &= \bar{V}_{m,n} \int \frac{k_m k_n}{k} \frac{\partial \Phi_{ij}}{\partial k} \\ &= \bar{V}_{m,n} \frac{\partial}{\partial k} \int k \frac{k_m k_n}{k^2} \Phi_{ij} \\ &= \bar{V}_{m,n} \frac{\partial}{\partial k} \left[k X_{ijmn} \right] \end{aligned} \quad (2.65)$$

Using the linear closure discussed earlier (similar to Launder *et al.* (1975)) the term can be written as :

$$\begin{aligned} T_{ij}^L &= \frac{\partial}{\partial k} \left[k 2 E \left(2 C_2 S_{ij} \right. \right. \\ &\quad \left. \left. + C_3 S_{mn} h_{mn} \delta_{ij} \right. \right. \\ &\quad \left. \left. + C_5 [h_{ik} \bar{V}_{j,k} + h_{ik} \bar{V}_{k,j} + h_{jk} \bar{V}_{i,k} + h_{jk} \bar{V}_{k,i}] \right) \right] \end{aligned} \quad (2.66)$$

This expression was chosen by Besnard *et al.* (1992) and is also the closure obtained by Jeandel *et al.* (1978) with their own integration.

2.8.5 Turbulent diffusion

The approaches of Besnard *et al.* (1992) and Jeandel *et al.* (1978) differ mainly in the treatment on the terms which do not involve the mean velocity gradient. This difference is especially noticeable in the treatment of the turbulent diffusion terms. As already mentioned, for the sake of simplicity, we adopted here the approach of Besnard *et al.* (1992).

Inhomogeneous transport

The inhomogeneous turbulent transport term is closed using the same assumptions as in the one-point modelling. The turbulent transport is, therefore, represented by the local gradients of the variable in question. The simplest assumption (similar to that of Davidov (1961)) leads to :

$$d_{ij}^u = C_D \left[\nu_T \delta_{mn} \varphi_{ij,m} \right]_{,n} \quad (2.67)$$

with

$$\nu_T = \int k^{-3/2} E^{1/2} dk \quad (2.68)$$

The diffusion coefficient is supposed to be constant and it is proposed here to take the value $C_D = 0.1$.

Non linear transfer

The spectral transfer represented by the triple correlation can be written as :

$$-I \int k_m [T_{i.jm} - T_{im.j}] \quad (2.69)$$

The physical transfer of energy between the scales can be modelled by a combination of diffusion-like and wave-like mechanism as proposed by Leith (1967). This closure is used for the entire non-linear spectral process, a part of which arises from the pressure gradient-velocity correlation through the Poisson equations :

$$I \int \frac{k_m k_n}{k^2} [k_i T_{mn.j} - k_j T_{i.mn}] \quad (2.70)$$

The closure of those two term then can be written as :

$$T_{ij}^{NL} = -c_1 \frac{\partial}{\partial k} \left(k^{5/2} E^{1/2} \varphi_{ij} \right) + c_2 \frac{\partial}{\partial k} \left(k^{7/2} E^{1/2} \frac{\partial \varphi_{ij}}{\partial k} \right) \quad (2.71)$$

The coefficients chosen by Leith (1967) are supposed to satisfy the Kolmogorov's cascade law for an isotropic homogeneous turbulence, and are respectively (Clark and Zemach, 1995)

$$c_1 = 0.297 \quad (2.72a)$$

$$c_2 = 0.148 \quad (2.72b)$$

It is noted that choosing $c_2 = 0$ leads to the flux expression of Kovasznay (1948).

2.8.6 Synthesis

To summarize, the spectral model of $\varphi_{ij,t}$ adopted here is given by the following model equation :

$$\begin{aligned} \varphi_{ij,t} + \overline{V}_k \varphi_{ij,k} &= \nu \varphi_{ij,kk} - 2\nu \left[\frac{1}{4} \varphi_{ij,kk} + k^2 \varphi_{ij} \right] \\ \text{unsteadiness} \quad \text{convection} & \quad \text{viscous diffusion} \quad \quad \text{viscous dissipation} \\ & + \frac{\partial}{\partial k} \left[k^2 E \left(2C_2 S_{ij} \right. \right. \\ & \quad \quad \quad \left. \left. + C_3 S_{mn} h_{mn} \delta_{ij} \right. \right. \\ & \quad \quad \quad \left. \left. + C_5 [h_{ik} \overline{V}_{j,k} + h_{ik} \overline{V}_{k,j} + h_{jk} \overline{V}_{i,k} + h_{jk} \overline{V}_{k,i}] \right) \right] \end{aligned}$$

$$\begin{aligned}
& \text{linear spectral transfer} \\
- & \overline{V}_{j,k} \varphi_{ik} - \overline{V}_{i,k} \varphi_{kj} \\
& \text{production by mean velocity gradients (first order terms)} \\
+ & \beta_3 S_{ij} \\
+ & \beta_4 [h_{ik} S_{kj} + h_{jk} S_{ik} - 2/3 h_{mn} S_{mn} \delta_{ij}] \\
+ & \beta_6 [h_{ik} W_{jk} + h_{jk} W_{ik}] \\
& \text{rapid pressure strain term (first order term)} \\
+ & C_D \left[\nu_T \delta_{mn} \varphi_{ij,m} \right]_{,n} \\
& \text{inhomogeneous turbulent diffusion} \\
- & \beta_1^S k^{3/2} E^{3/2} h_{ij} \\
& \text{slow term} \\
- & c_1 \frac{\partial}{\partial k} \left(k^{5/2} E^{1/2} \varphi_{ij} \right) + c_2 \frac{\partial}{\partial k} \left(k^{7/2} E^{1/2} \frac{\partial \varphi_{ij}}{\partial k} \right) \\
& \text{non linear spectral transfer}
\end{aligned}$$

where the coefficients are supposed to be constant. The turbulent viscosity is closed by

$$\nu_T = \int k^{-3/2} E^{1/2} dk \quad (2.74)$$

and

$$h_{ij} = \frac{1}{2\varphi} \varphi_{ij} - \frac{1}{3} \delta_{ij} \quad (2.75a)$$

$$\varphi = E \quad (2.75b)$$

This is the simplest expression that can be derived with adopted assumptions.

Chapter 3

Derivation of a multiple-scale model

3.1 Closure on the spherically integrated spectrum tensor

Instead of considering the closure for the the spherically integrated spectral tensor function $\varphi_{ij}(k)$, it is convenient to introduce a simplified model of this variable to make the closure easier. Based on dimensional analysis, one of the possibility would be to define a model in the form :

$$\varphi_{ij}(k) = K L g_{ij}(k L) \quad (3.1)$$

where K is the kinetic energy, L is the length scale of energy-containing eddies and g_{ij} a tensorial shape function. However, such a model introduces another tensor, g_{ij} which also needs to be modelled. For practical purposes, and in order to enable analytical integration which would recover the main variables in the $R_{ij} - \varepsilon$ closure framework, i.e. R_{ij} or K and ε , it is more convenient to adopt even simpler model for $\varphi_{ij}(k)$, with a scalar shape function, e.g.

$$\varphi_{ij}(k) = R_{ij} L g(k L) \quad (3.2)$$

This was the route adopted by (Schiestel, 1993), who proposed the following spectrum tensor approximation :

$$\varphi_{ij}(k) = f(k) E(k) \frac{R_{ij}}{K} + (1 - f(k)) \frac{2}{3} E(k) \delta_{ij} \quad (3.3)$$

with the kinetic energy spectrum defined as shown in figure below : Hence, the energy transfer down the spectrum is denoted by $\varepsilon^{(1)}$, whereas the dissipation ratio is directly defined by $\varepsilon^{(2)} = \varepsilon$. The function $f(k)$, to be defined later, is introduced in order to model a diminution of anisotropy with a decrease in turbulence scales (increase in wave number).

The continuity of the energy spectrum at the intersection of the two curves at $k = k_L$, yields the relation between α and β :

$$\alpha = \beta \varepsilon^{(1)2/3} k_L^{-17/3} \quad (3.4)$$

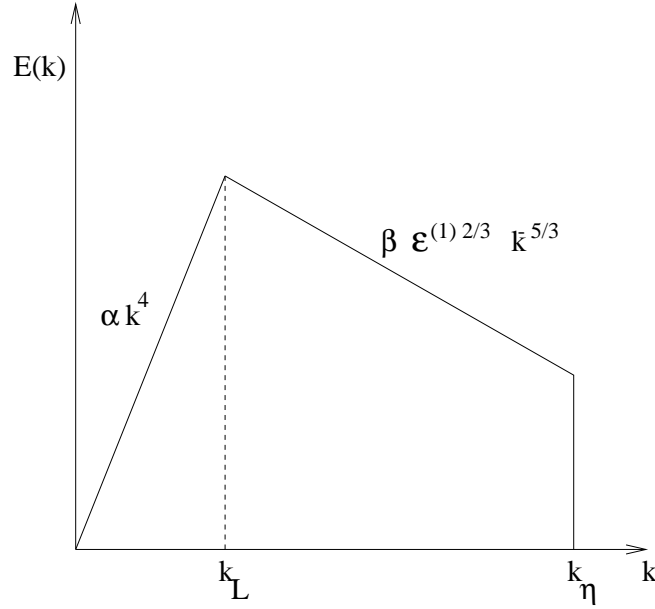


Figure 3.1 – Schematic representation of a high-Reynolds scalar kinetic energy spectrum.

The production region of the kinetic energy spectrum is limited by the large scale wave number, defined in terms of the length scale of the energy containing eddies :

$$k_L = \frac{1}{L} = \frac{\varepsilon^{(1)}}{K^{3/2}} \quad (3.5)$$

whereas the viscous cut-off is located at the Kolmogorov length scale :

$$k_\eta = \varepsilon^{(1)-1/2} \left(\frac{\varepsilon^{(2)}}{\nu} \right)^{3/4} \quad (3.5a)$$

The turbulent kinetic energy is defined by :

$$K = \int_0^{+\infty} E(k) dk \quad (3.6a)$$

so that the analytical integration leads to :

$$K = \frac{1}{5} \beta \varepsilon^{(1)2/3} k_L^{-2/3} + \frac{3}{2} \beta \varepsilon^{(1)2/3} k_L^{-2/3} \left[1 - \left(\frac{k_L}{k_\eta} \right)^{2/3} \right] \quad (3.6b)$$

In the high-Reynolds number limit $(k_L/k_\eta)^{2/3}$ goes to zero so that the kinetic energy can be approximated as :

$$K \sim \varepsilon^{(1)2/3} k_L^{-2/3} \quad (3.6c)$$

The turbulent Reynolds number is defined in terms of the kinetic energy, the viscosity and the dissipation rate :

$$R_t = \frac{K^2}{\nu \varepsilon^{(2)}} \quad (3.7)$$

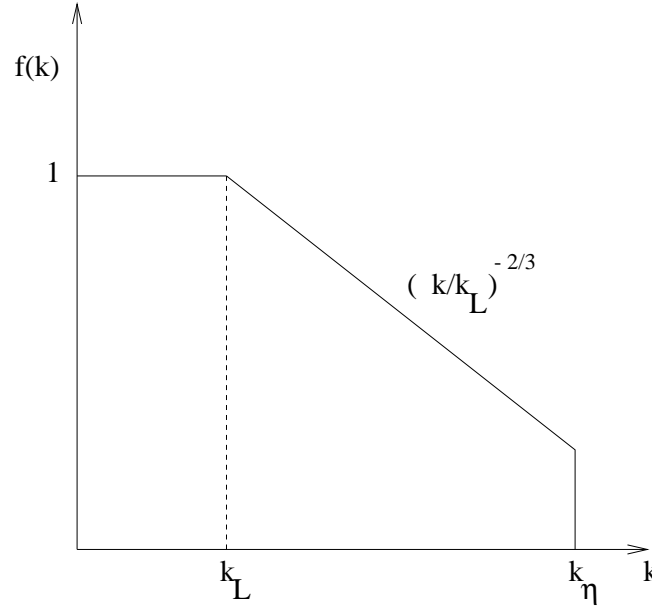


Figure 3.2 – Schematic damping of the anisotropy of small scales.

The shape function $f(k)$ should take into account the decrease of the anisotropic character of the small scales. Various shapes can be considered and in the present work we use the simple form shown in figure below :

The dissipation of the kinetic energy is defined as :

$$\varepsilon = \int_0^{+\infty} 2\nu \left(k^2 + \frac{1}{4} \frac{\partial^2}{\partial X_n^2} \right) E(k) dk \quad (3.8a)$$

and can be decomposed into homogeneous and inhomogeneous contributions (where X_n are the coordinates at the midpoint in the physical space) :

$$\varepsilon^h = \int_0^{+\infty} 2\nu k^2 E(k) dk \quad (3.8b)$$

$$\varepsilon^i = \int_0^{+\infty} 2\nu \frac{1}{4} \frac{\partial^2}{\partial X_n^2} E(k) dk \quad (3.8c)$$

Their integration leads respectively to :

$$\varepsilon^h \sim 2\nu \varepsilon^{(1)2/3} k_\eta^{4/3} \quad (3.8d)$$

$$\varepsilon^i \sim \nu \frac{\partial^2 K}{\partial X_n^2} \quad (3.8e)$$

The inhomogeneous contribution ε^i can be directly computed from the kinetic energy so that the problem of obtaining the dissipation rate is reduced to the evaluation of the homogeneous part ε^h . this quantity will be identified with the dissipation itself in the following sections.

3.2 Choice of the variables

From the definitions introduced above it is now possible to define various variables that can serve for closing the one-point model equations by analytically integrating the following basic integral :

$$I^{(m)} = \int_0^{+\infty} k^m \varphi_{ij} dk \quad (3.9)$$

By choosing the parameter m , desired quantity can be recovered. For the Reynolds stress and homogeneous dissipation rate tensor this is obvious from their definition, i.e. for $m = 0$

$$R_{ij} = \int_0^{+\infty} \varphi_{ij} dk \quad (3.10a)$$

and for $m = 2$

$$\varepsilon_{ij}^h = 2\nu \int_0^{+\infty} k^2 \varphi_{ij} dk \quad (3.11a)$$

It is interesting to note that the integral converges when $m > -5$, the latter limit emerges from the adopted k^4 law in the vicinity of $k = 0$.

Another interesting quantity emerges for $m = -1$ which can be interpreted as a product of the kinetic energy and the length scale tensor :

$$\int_0^{+\infty} k^{-1} \varphi_{ij} dk \sim K \Lambda_{ij} \quad (3.12a)$$

A further simplification can be made to eliminate the tensorial length scale (Donaldson and Sandri, 1982) by

$$\Lambda_{ij} = L \frac{R_{ij}}{K} \quad (3.13a)$$

or, in a more general form :

$$\Lambda_{ij} = L \left[\frac{2}{3} \delta_{ij} + \sigma \left(\frac{R_{ij}}{K} - \frac{2}{3} \delta_{ij} \right) \right] \quad (3.14a)$$

where σ is a parameter to be specified, accounting for scale anisotropy. Here $L = K^{3/2}/\varepsilon^{(1)}$ defines the integral length scale that can be assumed to be close to the scale of the energy containing eddies. Hence, this expression leads in an indirect manner to the definition of the spectral energy transfer rate $\varepsilon^{(1)}$, which, in turn, can be used to define the length scale L .

The above three cases are obtained for specific values of m (0, 2 and -1). We can write, however, a general form of the expression for $I^{(m)}$ by integrating the product $k^m \varphi_{ij}$ over the whole spectrum :

$$\begin{aligned} \int_0^{+\infty} k^m \varphi_{ij} dk &= \frac{\beta}{(m+5)} \frac{R_{ij}}{K} \varepsilon^{(1)2/3} k_\eta^{(m-2/3)} \left(\frac{k_L}{k_\eta} \right)^{(m-2/3)} \\ &+ \frac{\beta}{(m-4/3)} \frac{R_{ij}}{K} \varepsilon^{(1)2/3} k_\eta^{(m-2/3)} \left(\frac{k_L}{k_\eta} \right)^{2/3} \left[1 - \left(\frac{k_L}{k_\eta} \right)^{(m-4/3)} \right] \end{aligned} \quad (3.15)$$

$$+ \frac{\beta}{(m-4/3)} \frac{2}{3} \delta_{ij} \varepsilon^{(1)2/3} k_\eta^{(m-2/3)} \left[\frac{(m-4/3)}{(m-2/3)} - \left(\frac{k_L}{k_\eta}\right)^{2/3} + \frac{2}{3} \frac{1}{(m-2/3)} \left(\frac{k_L}{k_\eta}\right)^{(m-2/3)} \right]$$

which is valid for all values of m greater than -5 , and different from $2/3$ and $4/3$ (see appendix G). The above three values of m separate different intervals in which the above integral expressions for $I^{(m)}$ reduces to specific forms. Of a particular interest are the two cases : the first one corresponds to the interval of $-5 < m < 2/3$ and the second one to the interval $4/3 < m$, i.e. :

$$\int_0^{+\infty} k^m \varphi_{ij} dk \sim \varepsilon^{(1)2/3} k_L^{(m-2/3)} \frac{R_{ij}}{K} \quad (3.16a)$$

when $-5 < m < 2/3$

and

$$\int_0^{+\infty} k^m \varphi_{ij} dk \sim \varepsilon^{(1)2/3} k_\eta^{(m-2/3)} \left[f_{\varepsilon^{(m)}} \frac{R_{ij}}{K} + (1 - f_{\varepsilon^{(m)}}) \frac{2}{3} \delta_{ij} \right] \quad (3.16b)$$

when $4/3 < m$

where $f_{\varepsilon^{(m)}} = \left(\frac{k_L}{k_\eta}\right)^{2/3}$

In order to simplify the notations, the above two expressions are rewritten in an abbreviated form :

$$\int_0^{+\infty} k^m \varphi_{ij} dk \sim \varepsilon^{(1)2/3} k_L^{(m-2/3)} T_{ij}^{(m)} \quad (3.17a)$$

when $-5 < m < 2/3$

and

$$\int_0^{+\infty} k^m \varphi_{ij} dk \sim \varepsilon^{(1)2/3} k_\eta^{(m-2/3)} T_{ij}^{(m)} \quad (3.17b)$$

when $4/3 < m$

Note that T_{ij}^m has different meanings in the two ranges of m considered.

From these expressions a set of three variables, introduced earlier, can be easily be extracted,

$$\int_0^{+\infty} \varphi_{ij} dk \sim R_{ij} \quad (3.18a)$$

$$\int_0^{+\infty} k^{-1} \varphi_{ij} dk \sim \frac{K^{3/2}}{\varepsilon^{(1)}} R_{ij} \quad (3.18b)$$

$$\int_0^{+\infty} \nu k^2 \varphi_{ij} dk \sim \varepsilon^{(2)} \left[f_{\varepsilon^{(2)}} \frac{R_{ij}}{K} + (1 - f_{\varepsilon^{(2)}}) \frac{2}{3} \delta_{ij} \right] \quad (3.18c)$$

where $f_{\varepsilon^{(2)}} = R_t^{-1/2} \left(\frac{\varepsilon^{(1)}}{\varepsilon^{(2)}}\right)$

The last two variables show the closures for the length scale tensor Λ_{ij} and for the dissipation tensor ε_{ij} . Their transport equation can be derived from the spherical spectral closure.

3.3 Derivation of the equations

3.3.1 The transport equation for the variable $\varepsilon^{(1)2/3} k_L^{(m-2/3)} T_{ij}^{(m)}$

As shown above, this variable corresponds to the interval where $-5 < m < 2/3$. The integral $I^{(m)}$ can be used to derive the equations for the Reynolds stress tensor R_{ij} and for the length scale tensor through the variable $K^{3/2}/\varepsilon^{(1)} R_{ij}$.

Unsteady and convective terms

Their integration is straightforward as it comes from the very same assumption as for the choice of the variables themselves :

$$\frac{\bar{d}}{dt} \int_0^{+\infty} k^m \varphi_{ij} dk \sim \frac{\bar{d}}{dt} \left[\varepsilon^{(1)2/3} k_L^{(m-2/3)} T_{ij}^{(m)} \right] \quad (3.19)$$

when $-5 < m < 2/3$

Viscous diffusion term

The viscous diffusion term is defined as :

$$d_{ij}^\nu = \int_0^{+\infty} \nu k^m \varphi_{ij,mm} dk \quad (3.20)$$

so that its integration is also straightforward :

$$d_{ij}^\nu = \nu \left[\varepsilon^{(1)2/3} k_L^{(m-2/3)} T_{ij}^{(m)} \right]_{,mm} \quad (3.21)$$

Viscous dissipation term

The homogeneous part of the viscous dissipation process is written as :

$$\varepsilon_{ij}^h = \int_0^{+\infty} -2\nu k^{(m+2)} \varphi_{ij} dk \quad (3.22)$$

Its form is the same as the variable itself, but with a different power. This allows to evaluate this term in the same manner as the variable itself. The form of the term varies depending whether $-5 < m + 2 < 2/3$ or $4/3 < m + 2$. The integration leads to :

$$\int_0^{+\infty} -2\nu k^{(m+2)} \varphi_{ij} dk \sim -2\nu \varepsilon^{(1)2/3} k_L^{(m+4/3)} \frac{R_{ij}}{K} \quad (3.23a)$$

when $-7 < m < -4/3$

$$\int_0^{+\infty} -2\nu k^{(m+2)} \varphi_{ij} dk \sim -2\nu \varepsilon^{(1)2/3} k_\eta^{(m+4/3)} \frac{2}{3} \delta_{ij} \quad (3.23b)$$

when $-4/3 < m < -2/3$

$$\int_0^{+\infty} -2\nu k^{(m+2)} \varphi_{ij} dk \sim -2\nu \varepsilon^{(1)2/3} k_\eta^{(m+4/3)} \left[f_{\varepsilon^{(m+2)}} \frac{R_{ij}}{K} + (1 - f_{\varepsilon^{(m+2)}}) \frac{2}{3} \delta_{ij} \right] \quad (3.23c)$$

when $-2/3 < m$

$$\text{where } f_{\varepsilon^{(m+2)}} = \left(\frac{k_L}{k_\eta} \right)^{2/3}$$

The first expression is outside the range available here, $-5 < m < 2/3$. It can be noted that for $m = 0$, the integration leads to the exact dissipation tensor.

Linear spectral transfer

The linear spectral transfer is represented by :

$$T_{ij}^L = \int_0^{+\infty} k^m \bar{V}_{p,q} \frac{\partial}{\partial k} \left[k X_{ijpq} \right] dk \quad (3.24)$$

where the fourth order tensor is a linear functional in terms of the deviatoric part of φ_{ij} . It should be noted that this integral gives no contribution in the physical space when $m = 0$. The integration is then to be performed for m different from 0. The adopted closure of the linear spectral transfer can be written as :

$$T_{ij}^L = \int_0^{+\infty} k^m \frac{\partial}{\partial k} \left[k 2 E \left(2 c_2 S_{ij} + c_3 S_{mn} h_{mn} \delta_{ij} + c_5 [h_{ik} \bar{V}_{j,k} + h_{ik} \bar{V}_{k,j} + h_{jk} \bar{V}_{i,k} + h_{jk} \bar{V}_{k,i}] \right) \right] dk \quad (3.25)$$

Its integration requires, therefore, the evaluation of the following integrals :

$$J'_{ij} = \int_0^{+\infty} k^m \frac{\partial}{\partial k} \left[k 2 \varphi h_{ij} \right] dk \quad (3.26a)$$

$$J' = \int_0^{+\infty} k^m \frac{\partial}{\partial k} \left[k 2 \varphi \right] dk \quad (3.26b)$$

They can be written as

$$J'_{ij} = - \int_0^{+\infty} (m+1) k^m 2 \varphi h_{ij} dk \quad (3.27a)$$

$$J' = - \int_0^{+\infty} (m+1) k^m 2 \varphi dk \quad (3.27b)$$

which, in turn, involve the following integrals :

$$J_{ij} = \int_0^{+\infty} k^m 2 \varphi h_{ij} dk \quad (3.28a)$$

$$J = - \int_0^{+\infty} k^m 2 \varphi dk \quad (3.28b)$$

that can be rewritten as :

$$J_{ij} = b_{ij} \int_0^{+\infty} k^m 2 E f dk \quad (3.29a)$$

$$J = \int_0^{+\infty} k^m 2 E dk \quad (3.29b)$$

In the case, when $m < 2/3$, they can be approximated by the leading term of the integrals

$$J_{ij} \sim b_{ij} \varepsilon^{(1)(2/3)} k_L^{(m-2/3)} \quad (3.30a)$$

$$J \sim \varepsilon^{(1)(2/3)} k_L^{(m-2/3)} \quad (3.30b)$$

The expression for the linear transfer term becomes then :

$$\begin{aligned} T_{ij}^L &\sim \varepsilon^{(1)(2/3)} k_L^{(m-2/3)} \left(2 c_2^{(m)} S_{ij} \right. \\ &+ c_3^{(m)} b_{pq} \bar{V}_{p,q} \delta_{ij} \\ &+ c_5^{(m)} [b_{in} \bar{V}_{j,n} + b_{jn} \bar{V}_{i,n} + b_{in} \bar{V}_{n,j} + b_{jn} \bar{V}_{n,i}] \end{aligned} \quad (3.31)$$

when $m \neq 0$, and

$$T_{ij}^L = 0 \quad (3.32)$$

when $m = 0$.

Production

The production by mean velocity gradient term is defined by :

$$P_{ij} = \int_0^{+\infty} k^m \left[-\varphi_{nj} \bar{V}_{i,n} - \varphi_{in} \bar{V}_{j,n} \right] dk \quad (3.33)$$

and therefore can be evaluated as :

$$P_{ij} \sim -\varepsilon^{(1)2/3} k_L^{(m-2/3)} \left[T_{ik}^{(m)} \bar{V}_{j,k} + T_{jk}^{(m)} \bar{V}_{i,k} \right] \quad (3.34)$$

Rapid term

The expression for the rapid term is closed as :

$$\begin{aligned} \Phi_{ij}^R = \int_0^{+\infty} k^m 2 \varphi \left[\right. & \beta_3 S_{ij} \\ & + \beta_4 (h_{in} S_{jn} + h_{jn} S_{in} - \frac{2}{3} h_{pn} S_{pn} \delta_{ij}) \\ & \left. + \beta_6 (W_{in} h_{nj} - h_{in} W_{nj}) \right] dk \end{aligned} \quad (3.35)$$

In order to integrate this expression, the following integrals are needed

$$J_{ij} = \int_0^{+\infty} k^m 2 \varphi h_{ij} dk \quad (3.36a)$$

$$J = - \int_0^{+\infty} k^m 2 \varphi dk \quad (3.36b)$$

These are the same integrals as the one presented in the previous paragraph. Consequently the same type of terms are obtained for the linear transfer term and for the rapid term ; the latter can be written as :

$$\begin{aligned} \Phi_{ij}^R \sim \varepsilon^{(1)2/3} k_L^{(m-2/3)} \left[\right. & \beta_3^{(m)} S_{ij} \\ & + \beta_4^{(m)} (b_{in} S_{jn} + b_{jn} S_{in} - \frac{2}{3} b_{pn} S_{pn} \delta_{ij}) \\ & \left. + \beta_6^{(m)} (W_{in} b_{nj} - b_{in} W_{nj}) \right] \end{aligned} \quad (3.37)$$

Inhomogeneous turbulent diffusion

The inhomogeneous turbulent diffusion term is closed by the following model :

$$d_{ij}^u = \int_0^{+\infty} C_D \left[\nu_T \nu^{(3/4 m - 1/2)} k^m \varphi_{ij,n} \right]_n \quad (3.38)$$

where the turbulent viscosity is defined by

$$\nu_T = \int_0^{+\infty} \frac{\sqrt{k E}}{k^2} dk \quad (3.39)$$

Its integration leads to

$$\nu_T = -\frac{1}{2} \beta^{1/2} \varepsilon^{(1)1/3} \left[\frac{3}{2} k_\eta^{-4/3} - \frac{17}{6} k_L^{-4/3} \right] \quad (3.40)$$

that can be rewritten as

$$\nu_T = \beta^{1/2} \varepsilon^{(1)1/3} k_l^{-4/3} \left[\frac{17}{12} - \frac{3}{4} \left(\frac{k_L}{k_\eta} \right)^{4/3} \right] \quad (3.41)$$

and, for

$$k_L = \frac{\varepsilon^{(1)}}{K^{3/2}} \quad (3.42)$$

the turbulent viscosity can be expressed as :

$$\nu_T = \beta^{1/2} \frac{K^2}{\varepsilon^{(1)}} \left[\frac{17}{12} - \frac{3}{4} \left(\frac{k_L}{k_\eta} \right)^{4/3} \right] \quad (3.43)$$

The ratio of the large scales and the Kolmogorov scale is a function of the turbulent Reynolds number

$$\frac{k_L}{k_\eta} = R_t^{-3/4} \left(\frac{\varepsilon^{(1)}}{\varepsilon^{(2)}} \right)^{3/2} \quad (3.44)$$

and, therefore, can be neglected for high Reynolds number approximation. The turbulent diffusive transport can then be written as :

$$d_{ij}^u \sim C_D^{(m)} \left[\frac{K^2}{\varepsilon^{(1)}} \left(\varepsilon^{(1)2/3} k_L^{(m-2/3)} T_{ij}^{(m)} \right) \right]_{,n},n \quad (3.45)$$

The slow term

The slow term is defined by

$$\Phi_{ij}^S = \int_0^{+\infty} -k^{(m+3/2)} C_M 2 \varphi^{3/2} h_{ij} dk \quad (3.46)$$

which can be rewritten as :

$$\Phi_{ij}^S = -2 C_M b_{ij} \int_0^{+\infty} k^{(m+3/2)} \varphi^{3/2} f dk \quad (3.47)$$

In the case of $m < 2/3$, this integral can be approximated by :

$$\Phi_{ij}^S \sim -\beta_1^{S(m)} b_{ij} \varepsilon^{(1)} k_L^m \quad (3.48)$$

The nonlinear spectral transfer

The nonlinear spectral transfer term is here defined by

$$T_{ij}^{NL} = \int_0^{+\infty} k^m \left[-c_1 \frac{\partial}{\partial k} \left(k^{5/2} \varphi^{1/2} \varphi_{ij} \right) + c_2 \frac{\partial}{\partial k} \left(k^{7/2} \varphi^{1/2} \frac{\partial \varphi_{ij}}{\partial k} \right) \right] dk \quad (3.49)$$

This term again does not, by definition, give any contribution in the physical space when $m = 0$ and its integration is given for the other values within the $-5 < m < 2/3$ interval. The major contribution is

$$T_{ij}^{NL} \sim c_1^{(m)} \varepsilon^{(1)} k_L^m \frac{R_{ij}}{K} \quad (3.50)$$

Synthesis

We can now summarize the integration of the source term in the transport equation for the variable $\varepsilon^{(1)2/3} k_L^{(m-2/3)} T_{ij}^{(m)}$ in the interval $-5 < m < 2/3$. Depending on the value of m , four different expressions can be distinguished, corresponding to the four ranges or values of m .

When $-5 < m < -4/3$,

$$\begin{aligned}
 \frac{\bar{d}}{dt} \left[\varepsilon^{(1)2/3} k_L^{(m-2/3)} T_{ij}^{(m)} \right] &= -2\nu \varepsilon^{(1)2/3} k_\eta^{(m+4/3)} \frac{2}{3} \delta_{ij} \\
 &+ \varepsilon^{(1)2/3} k_L^{(m-2/3)} \left(2 c_2^{(m)} S_{ij} \right. \\
 &\quad \left. + c_3^{(m)} b_{pq} \bar{V}_{p,q} \delta_{ij} \right. \\
 &\quad \left. + c_5^{(m)} [b_{in} \bar{V}_{j,n} + b_{jn} \bar{V}_{i,n} + b_{in} \bar{V}_{n,j} + b_{jn} \bar{V}_{n,i}] \right) \\
 &- \varepsilon^{(1)2/3} k_L^{(m-2/3)} \left[T_{ik}^{(m)} \bar{V}_{j,k} + T_{jk}^{(m)} \bar{V}_{i,k} \right] \\
 &+ \varepsilon^{(1)2/3} k_L^{(m-2/3)} \left[\beta_3^{(m)} S_{ij} \right. \\
 &\quad \left. + \beta_4^{(m)} (b_{in} S_{jn} + b_{jn} S_{in} - \frac{2}{3} b_{pn} S_{pn} \delta_{ij}) \right. \\
 &\quad \left. + \beta_6^{(m)} (W_{in} b_{nj} - b_{in} W_{nj}) \right] \\
 &- \beta_1^{S(m)} \varepsilon^{(1)} k_L^m b_{ij} \\
 &+ c_1^{(m)} \varepsilon^{(1)} k_L^m \frac{R_{ij}}{K} \\
 &+ \left[\left(\nu \delta_{pq} + C_D^{(m)} \frac{K^2}{\varepsilon^{(1)}} \delta_{pq} \right) \left(\varepsilon^{(1)2/3} k_L^{(m-2/3)} T_{ij}^{(m)} \right) \right]_{,p},_q
 \end{aligned}$$

When $-4/3 < m < -2/3$,

$$\begin{aligned}
 \frac{\bar{d}}{dt} \left[\varepsilon^{(1)2/3} k_L^{(m-2/3)} T_{ij}^{(m)} \right] &= -2\nu \varepsilon^{(1)2/3} k_\eta^{(m+4/3)} \frac{2}{3} \delta_{ij} \\
 &+ \varepsilon^{(1)2/3} k_L^{(m-2/3)} \left(2 c_2^{(m)} S_{ij} \right. \\
 &\quad \left. + c_3^{(m)} b_{pq} \bar{V}_{p,q} \delta_{ij} \right. \\
 &\quad \left. + c_5^{(m)} [b_{in} \bar{V}_{j,n} + b_{jn} \bar{V}_{i,n} + b_{in} \bar{V}_{n,j} + b_{jn} \bar{V}_{n,i}] \right) \\
 &- \varepsilon^{(1)2/3} k_L^{(m-2/3)} \left[T_{ik}^{(m)} \bar{V}_{j,k} + T_{jk}^{(m)} \bar{V}_{i,k} \right] \\
 &+ \varepsilon^{(1)2/3} k_L^{(m-2/3)} \left[\beta_3^{(m)} S_{ij} \right.
 \end{aligned}$$

$$\begin{aligned}
 & + \beta_4^{(m)} (b_{in} S_{jn} + b_{jn} S_{in} - \frac{2}{3} b_{pn} S_{pn} \delta_{ij}) \\
 & + \beta_6^{(m)} (W_{in} b_{nj} - b_{in} W_{nj}) \Big] \\
 & - \beta_1^{S(m)} \varepsilon^{(1)} k_L^m b_{ij} \\
 & + c_1^{(m)} \varepsilon^{(1)} k_L^m \frac{R_{ij}}{K} \\
 & + \left[\left(\nu \delta_{pq} + C_D^{(m)} \frac{K^2}{\varepsilon^{(1)}} \delta_{pq} \right) \left(\varepsilon^{(1)2/3} k_L^{(m-2/3)} T_{ij}^{(m)} \right) \right]_{,p},_q
 \end{aligned}$$

When $-2/3 < m$ and $m \neq 0$,

$$\begin{aligned}
 \frac{\bar{d}}{dt} \left[\varepsilon^{(1)2/3} k_L^{(m-2/3)} T_{ij}^{(m)} \right] & = -2\nu \varepsilon^{(1)2/3} k_\eta^{(m+4/3)} \left[f_{\varepsilon^{(m+2)}} \frac{R_{ij}}{K} + (1 - f_{\varepsilon^{(m+2)}}) \frac{2}{3} \delta_{ij} \right] \\
 & + \varepsilon^{(1)2/3} k_L^{(m-2/3)} \left(2 c_2^{(m)} S_{ij} \right. \\
 & \quad + c_3^{(m)} b_{pq} \bar{V}_{p,q} \delta_{ij} \\
 & \quad \left. + c_5^{(m)} [b_{in} \bar{V}_{j,n} + b_{jn} \bar{V}_{i,n} + b_{in} \bar{V}_{n,j} + b_{jn} \bar{V}_{n,i}] \right) \\
 & - \varepsilon^{(1)2/3} k_L^{(m-2/3)} \left[T_{ik}^{(m)} \bar{V}_{j,k} + T_{jk}^{(m)} \bar{V}_{i,k} \right] \\
 & + \varepsilon^{(1)2/3} k_L^{(m-2/3)} \left[\beta_3^{(m)} S_{ij} \right. \\
 & \quad + \beta_4^{(m)} (b_{in} S_{jn} + b_{jn} S_{in} - \frac{2}{3} b_{pn} S_{pn} \delta_{ij}) \\
 & \quad \left. + \beta_6^{(m)} (W_{in} b_{nj} - b_{in} W_{nj}) \right] \\
 & - \beta_1^{S(m)} \varepsilon^{(1)} k_L^m b_{ij} \\
 & + c_1^{(m)} \varepsilon^{(1)} k_L^m \frac{R_{ij}}{K} \\
 & + \left[\left(\nu \delta_{pq} + C_D^{(m)} \frac{K^2}{\varepsilon^{(1)}} \delta_{pq} \right) \left(\varepsilon^{(1)2/3} k_L^{(m-2/3)} T_{ij}^{(m)} \right) \right]_{,p},_q
 \end{aligned}$$

where

$$f_{\varepsilon^{(m+2)}} = \left(\frac{k_L}{k_\eta} \right)^{2/3}$$

When $m = 0$,

$$\begin{aligned}
 \frac{\bar{d}}{dt} \left[\varepsilon^{(1)2/3} k_L^{(m-2/3)} T_{ij}^{(m)} \right] & = -2\nu \varepsilon^{(1)2/3} k_\eta^{(m+4/3)} \left[f_{\varepsilon^{(m+2)}} \frac{R_{ij}}{K} + (1 - f_{\varepsilon^{(m+2)}}) \frac{2}{3} \delta_{ij} \right] \\
 & - \varepsilon^{(1)2/3} k_L^{(m-2/3)} \left[T_{ik}^{(m)} \bar{V}_{j,k} + T_{jk}^{(m)} \bar{V}_{i,k} \right]
 \end{aligned}$$

$$\begin{aligned}
 & + \varepsilon^{(1)2/3} k_L^{(m-2/3)} \left[\beta_3^{(m)} S_{ij} \right. \\
 & \quad + \beta_4^{(m)} (b_{in} S_{jn} + b_{jn} S_{in} - \frac{2}{3} b_{pn} S_{pn} \delta_{ij}) \\
 & \quad \left. + \beta_6^{(m)} (W_{in} b_{nj} - b_{in} W_{nj}) \right] \\
 & - \beta_1^{S(m)} \varepsilon^{(1)} k_L^m b_{ij} \\
 & + \left[\left(\nu \delta_{pq} + C_D^{(m)} \frac{K^2}{\varepsilon^{(1)}} \delta_{pq} \right) \left(\varepsilon^{(1)2/3} k_L^{(m-2/3)} T_{ij}^{(m)} \right) \right]_{,p},q
 \end{aligned}$$

where

$$f_{\varepsilon^{(m+2)}} = \left(\frac{k_L}{k_\eta} \right)^{2/3}$$

3.3.2 Transport equation for the variable $\varepsilon^{(1)2/3} k_\eta^{(m-2/3)} T_{ij}^{(m)}$

We consider now the integration of $I^{(m)}$ in the interval $4/3 < m$, which leads to the derivation of the transport equation for the dissipation, when $m = 2$. For this purpose, as pointed out in the appendix G, it is convenient to express the Kolmogorov scale in terms of viscosity so that

$$\varepsilon^{(1)2/3} k_\eta^{(m-2/3)} = \nu^{-(3/4m-1/2)} \varepsilon^{(2)(3/4m-1/2)} \varepsilon^{(1)(1-m/2)} \quad (3.55)$$

so that the we can focus on the integration of the equation for the variable

$$\varepsilon^{(2)(3/4m-1/2)} \varepsilon^{(1)(1-m/2)} T_{ij}^{(m)} \quad (3.56)$$

which corresponds to the integration of

$$\int_0^{+\infty} \nu^{(3/4m-1/2)} k^m \varphi_{ij} dk \quad (3.57)$$

instead of

$$\int_0^{+\infty} k^m \varphi_{ij} dk \quad (3.58)$$

We consider again the separately each term in the equation.

Unsteady and convective terms

This integration is again straightforward :

$$\begin{aligned}
 \frac{\bar{d}}{dt} \left[\nu^{(3/4m-1/2)} \int_0^{+\infty} k^m \varphi_{ij} dk \right] & \sim \frac{\bar{d}}{dt} \left[\varepsilon^{(2)(3/4m-1/2)} \varepsilon^{(1)(1-m/2)} T_{ij}^{(m)} \right] \\
 & \text{when } 4/3 < m
 \end{aligned} \quad (3.59)$$

Viscous diffusion term

The viscous diffusion term is written as :

$$\begin{aligned} d_{ij}^\nu &= \int_0^{+\infty} \nu \left[\nu^{(3/4 m - 1/2)} k^m \varphi_{ij,nn} \right] dk \\ &= \int_0^{+\infty} \nu^{(3/4 m + 1/2)} k^m \varphi_{ij} dk \end{aligned} \quad (3.60)$$

so that

$$d_{ij}^\nu = \nu \left[\varepsilon^{(2) (3/4 m - 1/2)} \varepsilon^{(1) (1 - m/2)} T_{ij}^{(m)} \right]_{,nn} \quad (3.61)$$

Viscous dissipation term

The homogeneous part of the viscous dissipation process is here defined as :

$$\begin{aligned} \varepsilon_{ij}^h &= \int_0^{+\infty} -2 \nu k^2 \left[\nu^{(3/4 m - 1/2)} k^m \varphi_{ij} \right] dk \\ &= \int_0^{+\infty} -2 \nu^{(3/4 m + 1/2)} k^{(m+2)} \varphi_{ij} dk \end{aligned} \quad (3.62)$$

The evaluation of the principal term of this integral is done here for the case where $4/3 < m$. This is rather straightforward since the term can be treated as the variable itself. Expressing the molecular viscosity and the Kolmogorov length scale in terms of the turbulent Reynolds number and the transfer rates leads to :

$$\varepsilon_{ij}^h \sim R_t^{1/2} \varepsilon^{(2) (3/4 m + 3/2)} \varepsilon^{(1) (-m/2)} K^{-1} T_{ij}^{(m+2)} \quad (3.63)$$

where

$$T_{ij}^{(m+2)} = \left[f_{\varepsilon^{(m+2)}} \frac{R_{ij}}{K} + (1 - f_{\varepsilon^{(m+2)}}) \frac{2}{3} \delta_{ij} \right] \quad (3.64a)$$

and

$$f_{\varepsilon^{(m+2)}} = R_t^{-1/2} \frac{\varepsilon^{(1)}}{\varepsilon^{(2)}} \quad (3.64b)$$

It should be noted that the dissipation is here expressed in terms of a positive exponent of the turbulent Reynolds number. This form of the homogeneous dissipation contribution can *a priori* lead to an infinite value when the turbulent Reynolds number is high, what is in contradiction to the physical process of the dissipation. However the process of the dissipation is counterbalanced by the transfer term and, together with it, it will lead to a closed finite valued model. This is shown below after the derivation of the expression for the non linear transfer itself.

The linear spectral transfer term

This term is defined by

$$T_{ij}^L = \int_0^{+\infty} \nu^{(3/4)m-1/2} k^m \bar{V}_{p,q} \frac{\partial}{\partial k} \left[k X_{ijpq} \right] dk \quad (3.65)$$

Its integration involves the evaluation of integrals such as

$$J'_{ij} = \int_0^{+\infty} k^m \frac{\partial}{\partial k} \left[k 2 \varphi h_{ij} \right] dk \quad (3.66a)$$

$$J' = \int_0^{+\infty} k^m \frac{\partial}{\partial k} \left[k 2 \varphi \right] dk \quad (3.66b)$$

which, again, can be expressed in terms of

$$J_{ij} = \int_0^{+\infty} k^m 2 \varphi h_{ij} dk \quad (3.67a)$$

$$J = \int_0^{+\infty} k^m 2 \varphi dk \quad (3.67b)$$

For $4/3 < m$ the above two integrals can be approximated by

$$J_{ij} \sim b_{ij} \varepsilon^{(1)(2/3)} k_L^{2/3} k_\eta^{(m-4/3)} \quad (3.68a)$$

$$J \sim \varepsilon^{(1)(2/3)} k_\eta^{(m-2/3)} \quad (3.68b)$$

The linear spectral transfer can then be written as

$$\begin{aligned} T_{ij}^L &\sim \varepsilon^{(1)(1-m/2)} \varepsilon^{(2)(3/4m-1/2)} K^{-1} \left(2 c_2^{(m)} S_{ij} \right. \\ &+ c_3^{(m)} f_{\varepsilon^{(m)}} b_{pq} \bar{V}_{p,q} \delta_{ij} \\ &+ \left. c_5^{(m)} f_{\varepsilon^{(m)}} [b_{in} \bar{V}_{j,n} + b_{jn} \bar{V}_{i,n} + b_{in} \bar{V}_{n,j} + b_{jn} \bar{V}_{n,i}] \right) \end{aligned} \quad (3.69)$$

where

$$f_{\varepsilon^{(m)}} = \left(\frac{k_l}{k_\eta} \right)^{2/3}$$

Production terms

The integration of the production terms consists in the evaluation of

$$P_{ij} = \int_0^{+\infty} \nu^{(3/4)m-1/2} k^m \left[-\varphi_{nj} \bar{V}_{i,n} - \varphi_{in} \bar{V}_{j,n} \right] dk \quad (3.70)$$

yielding simply :

$$P_{ij} \sim -\varepsilon^{(2)(3/4m-1/2)} \varepsilon^{(1)(1-m/2)} \left[T_{ik}^{(m)} \bar{V}_{j,k} + T_{jk}^{(m)} \bar{V}_{i,k} \right] \quad (3.71)$$

or,

$$\begin{aligned} P_{ij} &\sim -\varepsilon^{(2)(3/4m-1/2)} \varepsilon^{(1)(1-m/2)} \\ &\left[(1 - f_{\varepsilon^{(m)}}) \frac{2}{3} \left(\bar{V}_{i,j} + \bar{V}_{j,i} \right) + f_{\varepsilon^{(m)}} \left(R_{nj} \bar{V}_{i,n} + R_{in} \bar{V}_{j,n} \right) \right] \end{aligned} \quad (3.72)$$

The rapid term

By definition the rapid term contribution is :

$$\begin{aligned} \Phi_{ij}^R = \int_0^{+\infty} \nu^{(3/4m-1/2)} k^m 2\varphi \left[\right. & \beta_3 S_{ij} \\ & + \beta_4 (h_{in} S_{jn} + h_{jn} S_{in} - \frac{2}{3} h_{pn} S_{pn} \delta_{ij}) \\ & \left. + \beta_6 (W_{in} h_{nj} - h_{in} W_{nj}) \right] dk \end{aligned} \quad (3.73)$$

Its integration involves the following integrals :

$$J_{ij} = \int_0^{+\infty} k^m 2\varphi h_{ij} dk \quad (3.74a)$$

$$J = \int_0^{+\infty} k^m 2\varphi dk \quad (3.74b)$$

As long as $4/3 < m$, they can be approximated by :

$$J_{ij} \sim b_{ij} \varepsilon^{(1)(2/3)} k_L^{2/3} k_\eta^{(m-4/3)} \quad (3.75a)$$

$$J \sim \varepsilon^{(1)(2/3)} k_\eta^{(m-2/3)} \quad (3.75b)$$

Therefore, the rapid term can be written as

$$\begin{aligned} \Phi_{ij}^R \sim \varepsilon^{(1)(1-m/2)} \varepsilon^{(2)(3/4m-1/2)} \left[\right. & \beta_3^{(m)} S_{ij} \\ & + \beta_4^{(m)} f_{\varepsilon^{(m)}} (b_{in} S_{jn} + b_{jn} S_{in} - \frac{2}{3} b_{pn} S_{pn} \delta_{ij}) \\ & \left. + \beta_6^{(m)} f_{\varepsilon^{(m)}} (W_{in} b_{nj} - b_{in} W_{nj}) \right] \end{aligned} \quad (3.76)$$

with again

$$f_{\varepsilon^{(m)}} = \left(\frac{k_l}{k_\eta} \right)^{2/3}$$

where the scales have been expressed in a more convenient way.

Turbulent diffusion term

The turbulent diffusion term is closed by the following model :

$$d_{ij}^u = \int_0^{+\infty} C_D \left[\nu_T \nu^{(3/4m-1/2)} k^m \varphi_{ij,n} \right]_{,n} \quad (3.77)$$

where the turbulent viscosity is defined by

$$\nu_T = \int_0^{+\infty} \frac{\sqrt{k E}}{k^2} dk \quad (3.78)$$

Its integration leads to

$$\nu_T = -\frac{1}{2} \beta^{1/2} \varepsilon^{(1)1/3} \left[\frac{3}{2} k_\eta^{-4/3} - \frac{17}{6} k_L^{-4/3} \right] \quad (3.79)$$

This integral can be approximated by

$$\nu_T \sim \beta^{1/2} \frac{K^2}{\varepsilon^{(1)}} \quad (3.80)$$

If the diffusion coefficient C_D is assumed to be independent of the wave number, the diffusion term can be rewritten as :

$$d_{ij}^u = C_D \left[\int_0^{+\infty} \nu_T \left(\nu^{(3/4 m - 1/2)} k^m \varphi_{ij} \right)_{,n},_n \right] \quad (3.81)$$

For $4/3 < m$, this integral can be approximated by

$$d_{ij}^u \sim C_D^{(m)} \left[\frac{K^2}{\varepsilon^{(1)}} \left(\varepsilon^{(2)(3/4 m - 1/2)} T_{ij}^{(m)} \right)_{,n},_n \right] \quad (3.82)$$

The simplicity of the expression for the diffusion term, compared to that in one-point Reynolds stress model, is due to the scalar character of the turbulent eddy viscosity adopted in the spectral diffusion term.

The slow term

The slow term is defined by

$$\Phi_{ij}^S = \int_0^{+\infty} -\nu^{(3/4 m - 1/2)} k^{(m+3/2)} C_M 2 \varphi^{3/2} h_{ij} dk \quad (3.83)$$

which can be rewritten as :

$$\Phi_{ij}^S = -2 C_M b_{ij} \int_0^{+\infty} \nu^{(3/4 m - 1/2)} k^{(m+3/2)} \varphi^{3/2} f dk \quad (3.84)$$

In the case of $4/3 < m$ this integral can be approximated as :

$$\Phi_{ij}^S \sim -\beta_1^{S(m)} b_{ij} \nu^{(3/4 m - 1/2)} \varepsilon^{(1)} k_L^{2/3} k_\eta^{(m-2/3)} \quad (3.85)$$

that can be rewritten as

$$\Phi_{ij}^S \sim -\beta_1^{S(m)} \varepsilon^{(2)(3/4 m - 1/2)} \varepsilon^{(1)(-1/2 m + 2)} K^{-1} b_{ij} \quad (3.86)$$

The nonlinear spectral transfer

The nonlinear spectral transfer term is here defined by

$$T_{ij}^{NL} = \int_0^{+\infty} \nu^{(3/4 m - 1/2)} k^m \left[-c_1 \frac{\partial}{\partial k} \left(k^{5/2} \varphi^{1/2} \varphi_{ij} \right) + c_2 \frac{\partial}{\partial k} \left(k^{7/2} \varphi^{1/2} \frac{\partial \varphi_{ij}}{\partial k} \right) \right] dk \quad (3.87)$$

The major contribution for $m > 4/3$ is here

$$T_{ij}^{NL} \sim \nu^{(3/4 m - 1/2)} \varepsilon^{(1)} k_\eta^m \left[\left(\frac{k_L}{k_\eta} \right)^{2/3} \frac{R_{ij}}{K} + \left(1 - \left(\frac{k_L}{k_\eta} \right)^{2/3} \right) \frac{2}{3} \delta_{ij} \right] \quad (3.88)$$

which can be rewritten as :

$$T_{ij}^{NL} \sim \nu^{(3/4 m - 1/2)} \varepsilon^{(1)} k_\eta^m T_{ij}^{(m)} \quad (3.89)$$

where

$$T_{ij}^{(m)} = \left[f_{\varepsilon^{(m)}} \frac{R_{ij}}{K} + \left(1 - f_{\varepsilon^{(m)}} \right) \frac{2}{3} \delta_{ij} \right]$$

and

$$f_{\varepsilon^{(m)}} = \left(\frac{k_L}{k_\eta} \right)^{2/3}$$

Expressing the viscosity in terms of the Reynolds number yields :

$$T_{ij}^{NL} \sim R_t^{1/2} K^{-1} \varepsilon^{(1-1/2 m)} \varepsilon^{(2)(3/4 m + 1/2)} T_{ij}^{(m)} \quad (3.90)$$

This term involves again a positive exponent of the turbulent Reynolds number. This is expected to be balanced by the dissipation action. Rewriting the term in terms of the dissipation, as $T_{ij}^{(m)} = T_{ij}^{(m+2)}$, leads to :

$$T_{ij}^{NL} \sim R_t^{1/2} K^{-1} \frac{\varepsilon^{(1)}}{\varepsilon^{(2)}} \varepsilon^{(-1/2 m)} \varepsilon^{(2)(3/4 m + 3/2)} T_{ij}^{(m+2)} \quad (3.91)$$

Hence, writing these terms together yields

$$T_{ij}^{NL} - \varepsilon_{ij}^h \sim c_1(m) R_t^{1/2} K^{-1} \left(1 - \frac{\varepsilon^{(1)}}{\varepsilon^{(2)}} \right) \varepsilon^{(-1/2 m)} \varepsilon^{(2)(3/4 m + 3/2)} T_{ij}^{(m+2)} \quad (3.92)$$

It is seen now that the right hand side has a finite value, as long as

$$R_t^{1/2} \left(1 - \frac{\varepsilon^{(1)}}{\varepsilon^{(2)}} \right)$$

is always finite. This means that the transfer of energy by self stretching of the vortex filaments and the dissipation by viscous effect balance each other. When the turbulent Reynolds number tends to be infinite, the turbulence evolves to an equilibrium state, where $\varepsilon^{(1)} = \varepsilon^{(2)}$. Hence, although the Reynolds number can be very large, the group $(1 - \varepsilon^{(1)}/\varepsilon^{(2)})$ is expected to approach zero in the same time, resulting in a bounded value.

It should be recognized that the presence of this term can be inconvenient for practical prediction of complex non-equilibrium flows. For non-equilibrium flows at very high turbulent Reynolds number R_t where ε_1 and ε_2 may still be different, a delicate imbalance between these two quantities needs to be predicted from the model equations to keep this term finite, yet sufficiently small. This issue and an alternative form of modelling the term in question will be discussed later in this report.

Synthesis

The transport equation for the quantity $\nu^{(3/4 m - 1/2)} \varepsilon^{(1) 2/3} k_\eta^{(m-2/3)} T_{ij}^{(m)}$ for $4/3 < m$ can be summarized as :

$$\begin{aligned} \frac{\bar{d}}{dt} \left[\varepsilon^{(2) (3/4 m - 1/2)} \varepsilon^{(1) (1-m/2)} T_{ij}^{(m)} \right] = & \\ & -c_1(m) R_t^{1/2} \left(1 - \frac{\varepsilon^{(1)}}{\varepsilon^{(2)}} \right) \varepsilon^{(2) (3/4 m + 3/2)} \varepsilon^{(1) (-m/2)} K^{-1} T_{ij}^{(m+2)} \\ & + \varepsilon^{(1) (1-m/2)} \varepsilon^{(2) (3/4 m - 1/2)} \left(2 c_2^{(m)} S_{ij} \right. \\ & \quad + c_3^{(m)} f_{\varepsilon^{(m)}} b_{pq} \bar{V}_{p,q} \delta_{ij} \\ & \quad \left. + c_5^{(m)} f_{\varepsilon^{(m)}} [b_{in} \bar{V}_{j,n} + b_{jn} \bar{V}_{i,n} + b_{in} \bar{V}_{n,j} + b_{jn} \bar{V}_{n,i}] \right) \\ & - \varepsilon^{(2) (3/4 m - 1/2)} \varepsilon^{(1) (1-m/2)} \\ & \quad \left((1 - f_{\varepsilon^{(m)}}) \frac{2}{3} [\bar{V}_{i,j} + \bar{V}_{j,i}] + f_{\varepsilon^{(m)}} [R_{nj} \bar{V}_{i,n} + R_{in} \bar{V}_{j,n}] \right) \\ & + \varepsilon^{(1) (1-m/2)} \varepsilon^{(2) (3/4 m - 1/2)} \left(\beta_3^{(m)} S_{ij} \right. \\ & \quad + \beta_4^{(m)} f_{\varepsilon^{(m)}} [b_{in} S_{jn} + b_{jn} S_{in} - \frac{2}{3} b_{pn} S_{pn} \delta_{ij}] \\ & \quad \left. + \beta_6^{(m)} f_{\varepsilon^{(m)}} [W_{in} b_{nj} - b_{in} W_{nj}] \right) \\ & - \beta_1^{S(m)} \varepsilon^{(2) (3/4 m - 1/2)} \varepsilon^{(1) (-1/2 m + 2)} K^{-1} b_{ij} \\ & + \left[\left(\nu \delta_{pq} + C_D^{(m)} \frac{K^2}{\varepsilon^{(1)}} \delta_{pq} \right) \left(\varepsilon^{(2) (3/4 m - 1/2)} \varepsilon^{(1) (1-m/2)} T_{ij}^{(m)} \right) \right]_{,p,q} \end{aligned}$$

3.3.3 Summary of the derived transport equations

We can now summarize the outcome of the above analysis and the resulting three transport equations obtained for three specific values of the parameter m :

Reynolds stress equation ($m = 0$)

$$\frac{\bar{d}}{dt} \left[\varepsilon^{(1) 2/3} k_L^{-2/3} T_{ij}^{(0)} \right] = -2 \nu \varepsilon^{(1) 2/3} k_\eta^{4/3} \left[f_{\varepsilon^{(2)}} \frac{R_{ij}}{K} + (1 - f_{\varepsilon^{(2)}}) \frac{2}{3} \delta_{ij} \right]$$

$$\begin{aligned}
 & - \varepsilon^{(1)2/3} k_L^{-2/3} \left[T_{ik}^{(0)} \bar{V}_{j,k} + T_{jk}^{(0)} \bar{V}_{i,k} \right] \\
 & + \varepsilon^{(1)2/3} k_L^{-2/3} \left[\beta_3^{(0)} S_{ij} \right. \\
 & \quad \left. + \beta_4^{(0)} (b_{in} S_{jn} + b_{jn} S_{in} - \frac{2}{3} b_{pn} S_{pn} \delta_{ij}) \right. \\
 & \quad \left. + \beta_6^{(0)} (W_{in} b_{nj} - b_{in} W_{nj}) \right] \\
 & - \beta_1^{S(0)} \varepsilon^{(1)} b_{ij} \\
 & + \left[\left(\nu \delta_{pq} + C_D^{(0)} \frac{K^2}{\varepsilon^{(1)}} \delta_{pq} \right) \left(\varepsilon^{(1)2/3} k_L^{-2/3} T_{ij}^{(0)} \right) \right]_{,p},_q
 \end{aligned}$$

where

$$\begin{aligned}
 T_{ij}^{(0)} &= \frac{R_{ij}}{K} \\
 T_{ij}^{(2)} &= f_{\varepsilon^{(2)}} \frac{R_{ij}}{K} + (1 - f_{\varepsilon^{(2)}}) \frac{2}{3} \delta_{ij} \\
 f_{\varepsilon^{(2)}} &= \left(\frac{k_L}{k_\eta} \right)^{2/3}
 \end{aligned}$$

Knowing that

$$\begin{aligned}
 k_L &= \frac{\varepsilon^{(1)}}{K^{3/2}} \\
 \nu \varepsilon^{(1)2/3} k_\eta^{4/3} &= \varepsilon^{(2)}
 \end{aligned}$$

the equation becomes :

$$\begin{aligned}
 \frac{\bar{d}}{dt} \left[R_{ij} \right] &= - \varepsilon^{(2)} \left[f_{\varepsilon^{(2)}} \frac{R_{ij}}{K} + (1 - f_{\varepsilon^{(2)}}) \frac{2}{3} \delta_{ij} \right] \\
 & - \left[R_{ik} \bar{V}_{j,k} + R_{jk} \bar{V}_{i,k} \right] \\
 & + K \left[\beta_3^{(0)} S_{ij} \right. \\
 & \quad \left. + \beta_4^{(0)} (b_{in} S_{jn} + b_{jn} S_{in} - \frac{2}{3} b_{pn} S_{pn} \delta_{ij}) \right. \\
 & \quad \left. + \beta_6^{(0)} (W_{in} b_{nj} - b_{in} W_{nj}) \right] \\
 & - \beta_1^{S(0)} \varepsilon^{(1)} b_{ij} \\
 & + \left[\left(\nu \delta_{pq} + \frac{C_K^{(0)}}{\sigma_K^{(0)}} \frac{K^2}{\varepsilon^{(1)}} \delta_{pq} \right) \left(R_{ij} \right) \right]_{,p},_q
 \end{aligned}$$

It should be noted that the model for the homogeneous part of the dissipation tensor is :

$$\varepsilon_{ij}^h = \varepsilon^{(2)} \left[f_{\varepsilon^{(2)}} \frac{R_{ij}}{K} + (1 - f_{\varepsilon^{(2)}}) \frac{2}{3} \delta_{ij} \right] \quad (3.98)$$

where the damping function is defined in terms of the turbulent Reynolds number :

$$f_{\varepsilon^{(2)}} = R_t^{-1/2} \left(\frac{\varepsilon^{(1)}}{\varepsilon^{(2)}} \right)$$

For high Reynolds numbers this expression reduces to the isotropic model

$$\varepsilon_{ij}^h = \frac{2}{3} \varepsilon^{(2)} \delta_{ij} \quad (3.99)$$

The large scale equation ($m = -1$)

As noted earlier, the length scale tensor is defined by the following model :

$$K \Lambda_{ij} = \frac{K^{3/2}}{\varepsilon^{(1)}} R_{ij} \quad (3.100)$$

and its anisotropy tensor $\lambda_{ij} = \frac{\Lambda_{ij}}{2L} - \frac{1}{3} \delta_{ij}$ (where $L = K^{3/2}/\varepsilon^{(1)}$) is directly related to the stress anisotropy, i.e.

$$\lambda_{ij} = b_{ij} \quad (3.101)$$

The transport equation for the large scale variable is :

$$\begin{aligned} \frac{\bar{d}}{dt} \left[\varepsilon^{(1)2/3} k_L^{-5/3} T_{ij}^{(-1)} \right] = & -2 \nu \varepsilon^{(1)2/3} k_L^{1/3} \frac{2}{3} \delta_{ij} \\ & + \varepsilon^{(1)2/3} k_L^{-5/3} \left(2 c_2^{(-1)} S_{ij} \right. \\ & \quad \left. + c_3^{(-1)} b_{pq} \bar{V}_{p,q} \delta_{ij} \right. \\ & \quad \left. + c_5^{(-1)} [b_{in} \bar{V}_{j,n} + b_{jn} \bar{V}_{i,n} + b_{in} \bar{V}_{n,j} + b_{jn} \bar{V}_{n,i}] \right) \\ & - \varepsilon^{(1)2/3} k_L^{-5/3} \left[T_{ik}^{(-1)} \bar{V}_{j,k} + T_{jk}^{(-1)} \bar{V}_{i,k} \right] \\ & + \varepsilon^{(1)2/3} k_L^{-5/3} \left[\beta_3^{(-1)} S_{ij} \right. \\ & \quad \left. + \beta_4^{(-1)} (b_{in} S_{jn} + b_{jn} S_{in} - \frac{2}{3} b_{pn} S_{pn} \delta_{ij}) \right. \\ & \quad \left. + \beta_6^{(-1)} (W_{in} b_{nj} - b_{in} W_{nj}) \right] \\ & - \beta_1^{S(-1)} \varepsilon^{(1)} k_L^{-1} b_{ij} \\ & + c_1^{(-1)} \varepsilon^{(1)} k_L^{-1} \frac{R_{ij}}{K} \\ & + \left[\left(\nu \delta_{pq} + C_D^{(-1)} \frac{K^2}{\varepsilon^{(1)}} \delta_{pq} \right) \left(\varepsilon^{(1)2/3} k_L^{-5/3} T_{ij}^{(-1)} \right) \right]_{,p,q} \end{aligned}$$

where

$$T_{ij}^{(-1)} = \frac{R_{ij}}{K}$$

Knowing that

$$\begin{aligned} k_L &= \frac{\varepsilon^{(1)}}{K^{3/2}} \\ \nu \varepsilon^{(1)2/3} k_\eta^{4/3} &= \varepsilon^{(2)} \end{aligned}$$

the equation can be rewritten as :

$$\begin{aligned} \frac{\bar{d}}{dt} \left[K \Lambda_{ij} \right] &= -2 \left(\frac{\varepsilon^{(1)}}{\varepsilon^{(2)}} \right)^{-1/2} K^{3/2} R_t^{-3/4} \frac{2}{3} \delta_{ij} \\ &+ \frac{K^{5/2}}{\varepsilon^{(1)}} \left(2 c_2^{(-1)} S_{ij} \right. \\ &\quad + c_3^{(-1)} \lambda_{pq} \bar{V}_{p,q} \delta_{ij} \\ &\quad \left. + c_5^{(-1)} [\lambda_{in} \bar{V}_{j,n} + \lambda_{jn} \bar{V}_{i,n} + \lambda_{in} \bar{V}_{n,j} + \lambda_{jn} \bar{V}_{n,i}] \right) \\ &- K \left[\Lambda_{ik} \bar{V}_{j,k} + \Lambda_{jk} \bar{V}_{i,k} \right] \\ &+ \frac{K^{5/2}}{\varepsilon^{(1)}} \left[\beta_3^{(-1)} S_{ij} \right. \\ &\quad + \beta_4^{(-1)} (\lambda_{in} S_{jn} + \lambda_{jn} S_{in} - \frac{2}{3} \lambda_{pn} S_{pn} \delta_{ij}) \\ &\quad \left. + \beta_6^{(-1)} (W_{in} \lambda_{nj} - \lambda_{in} W_{nj}) \right] \\ &- \beta_1^{S(-1)} K^{3/2} \lambda_{ij} \\ &+ c_1^{(-1)} \varepsilon^{(1)} \Lambda_{ij} \\ &+ \left[\left(\nu \delta_{pq} + C_D^{(-1)} \frac{K^2}{\varepsilon^{(1)}} \delta_{pq} \right) \left(K \Lambda_{ij} \right)_{,p},_q \right] \end{aligned}$$

Taking the trace of this expression leads to the following scalar length scale equation :

$$\begin{aligned} \frac{\bar{d}}{dt} \left[\frac{K^{5/2}}{\varepsilon^{(1)}} \right] &= -C_{KL1}^{(1)} \frac{K^{3/2}}{\varepsilon^{(1)}} R_{pq} \bar{V}_{p,q} \\ &- C_{KL2}^{(1)} K^{3/2} \\ &- C_{KL3}^{(1)} \left(\frac{\varepsilon^{(1)}}{\varepsilon^{(2)}} \right)^{-1/2} K^{3/2} R_t^{-3/4} \\ &+ \left[\left(\nu \delta_{pq} + \frac{C_{KL}^{(1)}}{\sigma_{KL}^{(1)}} \frac{K^2}{\varepsilon^{(1)}} \delta_{pq} \right) \left(\frac{K^{5/2}}{\varepsilon^{(1)}} \right)_{,p},_q \right] \end{aligned}$$

The small scale equation

Recalling that the homogeneous dissipation rate tensor is defined as :

$$\varepsilon_{ij}^h = \varepsilon^{(2)} \left[f_{\varepsilon^{(2)}} \frac{R_{ij}}{K} + (1 - f_{\varepsilon^{(2)}}) \frac{2}{3} \delta_{ij} \right] \quad (3.107)$$

where

$$f_{\varepsilon^{(2)}} = R_t^{-1/2} \left(\frac{\varepsilon^{(1)}}{\varepsilon^{(2)}} \right)$$

Again its anisotropy tensor can be directly expressed in terms of the Reynolds' stress anisotropy tensor, as :

$$e_{ij}^h = f_{\varepsilon^{(2)}} b_{ij} \quad (3.108)$$

The transport equation for the small-scale variable is :

$$\begin{aligned} \frac{\bar{d}}{dt} \left[\varepsilon^{(2)} T_{ij}^{(2)} \right] = & -c_1^{(2)} R_t^{1/2} \left(1 - \frac{\varepsilon^{(1)}}{\varepsilon^{(2)}} \right) \frac{\varepsilon^{(2)3}}{\varepsilon^{(1)}} K^{-1} T_{ij}^{(4)} \\ & + \varepsilon^{(2)} \left(2c_2^{(2)} S_{ij} \right. \\ & \quad + c_3^{(2)} f_{\varepsilon^{(2)}} b_{pq} \bar{V}_{p,q} \delta_{ij} \\ & \quad \left. + c_5^{(2)} f_{\varepsilon^{(2)}} [b_{in} \bar{V}_{j,n} + b_{jn} \bar{V}_{i,n} + b_{in} \bar{V}_{n,j} + b_{jn} \bar{V}_{n,i}] \right) \\ & - \varepsilon^{(2)} \left((1 - f_{\varepsilon^{(2)}}) \frac{2}{3} [\bar{V}_{i,j} + \bar{V}_{j,i}] + f_{\varepsilon^{(2)}} [R_{nj} \bar{V}_{i,n} + R_{in} \bar{V}_{j,n}] \right) \\ & + \varepsilon^{(2)} \left(\beta_3^{(2)} S_{ij} \right. \\ & \quad + \beta_4^{(2)} f_{\varepsilon^{(2)}} [b_{in} S_{jn} + b_{jn} S_{in} - \frac{2}{3} b_{pn} S_{pn} \delta_{ij}] \\ & \quad \left. + \beta_6^{(2)} f_{\varepsilon^{(2)}} [W_{in} b_{nj} - b_{in} W_{nj}] \right) \\ & - \beta_1^{S(2)} \varepsilon^{(2)} \varepsilon^{(1)} K^{-1} b_{ij} \\ & + \left[\left(\nu \delta_{pq} + C_D^{(2)} \frac{K^2}{\varepsilon^{(1)}} \delta_{pq} \right) \left(\varepsilon^{(2)} T_{ij}^{(2)} \right) \right]_{,p},q \end{aligned}$$

Substituting

$$\begin{aligned} T_{ij}^{(4)} &= f_{\varepsilon^{(2)}} \frac{R_{ij}}{K} + (1 - f_{\varepsilon^{(2)}}) \frac{2}{3} \delta_{ij} \\ T_{ij}^{(2)} &= f_{\varepsilon^{(2)}} \frac{R_{ij}}{K} + (1 - f_{\varepsilon^{(2)}}) \frac{2}{3} \delta_{ij} \\ f_{\varepsilon^{(2)}} &= \left(\frac{k_L}{k_\eta} \right)^{2/3} \end{aligned}$$

the equation can be rewritten as :

$$\begin{aligned} \frac{\bar{d}}{dt} \left[\varepsilon_{ij}^h \right] = & c_1^{(2)} R_t^{1/2} \frac{\varepsilon^{(2)}}{K} \left(1 - \frac{\varepsilon^{(2)}}{\varepsilon^{(1)}} \right) \varepsilon_{ij}^h \\ & + \varepsilon^{(2)} \left(2c_2^{(2)} S_{ij} \right. \\ & \quad \left. + c_3^{(2)} e_{pq}^h \bar{V}_{p,q} \delta_{ij} \right) \end{aligned}$$

$$\begin{aligned}
 & + e_5^{(2)} [e_{in}^h \bar{V}_{j,n} + e_{jn}^h \bar{V}_{i,n} + e_{in}^h \bar{V}_{n,j} + e_{jn}^h \bar{V}_{n,i}] \\
 & - [\varepsilon_{ik}^h \bar{V}_{j,k} + \varepsilon_{jk}^h \bar{V}_{i,k}] \\
 & + \varepsilon^{(2)} \left(\beta_3^{(2)} S_{ij} \right. \\
 & \quad + \beta_4^{(2)} [e_{in}^h S_{jn} + e_{jn}^h S_{in} - \frac{2}{3} e_{pn}^h S_{pn} \delta_{ij}] \\
 & \quad \left. + \beta_6^{(2)} [W_{in} e_{nj}^h - e_{in}^h W_{nj}] \right) \\
 & - \beta_1^{S(2)} \varepsilon^{(2)} \varepsilon^{(1)} K^{-1} b_{ij} \\
 & + \left[\left(\nu \delta_{pq} + C_D^{(2)} \frac{K^2}{\varepsilon^{(1)}} \delta_{pq} \right) \left(\varepsilon_{ij}^h \right)_{,p},q \right]
 \end{aligned}$$

The transport equation of the scalar homogeneous dissipation rate is the trace of the former equation :

$$\begin{aligned}
 \frac{\bar{d}}{dt} [\varepsilon^{(2)}] & = -C_{\varepsilon_1}^{(2)} \varepsilon^{(2)} e_{pq}^h \bar{V}_{p,q} \\
 & - C_{\varepsilon_2}^{(2)} R_t^{1/2} \frac{\varepsilon^{(2)2}}{K} \left(1 - \frac{\varepsilon^{(2)}}{\varepsilon^{(1)}} \right) \\
 & + \left[\left(\nu \delta_{pq} + C_D^{(2)} \frac{K^2}{\varepsilon^{(1)}} \delta_{pq} \right) \left(\varepsilon^{(2)} \right)_{,p},q \right]
 \end{aligned}$$

that can again be rewritten as :

$$\begin{aligned}
 \frac{\bar{d}}{dt} [\varepsilon^{(2)}] & = -C_{\varepsilon_1}^{(2)} R_t^{-1/2} \varepsilon^{(1)} K^{-1} R_{pq} \bar{V}_{p,q} \\
 & - C_{\varepsilon_2}^{(2)} R_t^{1/2} \frac{\varepsilon^{(2)2}}{K} \left(1 - \frac{\varepsilon^{(2)}}{\varepsilon^{(1)}} \right) \\
 & + \left[\left(\nu \delta_{pq} + \frac{C_{\varepsilon}^{(2)}}{\sigma_{\varepsilon}^{(2)}} \frac{K^2}{\varepsilon^{(1)}} \delta_{pq} \right) \left(\varepsilon^{(2)} \right)_{,p},q \right]
 \end{aligned}$$

3.3.4 Comments

The variables adopted here are basically the Reynolds stress tensor R_{ij} , the characteristic length scale of the energy containing eddies L from which the length scale tensor Λ_{ij} is obtained, and the dissipation tensor ε_{ij}^h . Some comments, summarizing major assumptions used in deriving the equation and their interpretation are given below.

Homogeneous and nonhomogeneous contribution to dissipation

It can be noted that, as a first approximation, the homogeneous part of the dissipation tensor has been identified as the dissipation tensor, and therefore the parts including a

viscous dissipation-like term has been neglected in the equations. According to the study of the low-Reynolds number fully developed channel flow of Jovanovic *et al.* (1992), this term may be of importance for treating the near-wall region and satisfying the wall limits of the dissipation rate tensor. Such an approach was also proposed by Shima (1995) and ? (?), though the performances of such modifications have not been tested in detail. This splitting of the dissipation into a homogeneous and nonhomogeneous part can be later reintegrated into the previous equations.

Dissipation rate tensor

The expression for the dissipation rate tensor arising from the spectral hypothesis has here the same form as the algebraic expression of Hanjalić and Launder (1976)

$$\varepsilon_{ij} = (1 - f_\varepsilon^{(2)}) \frac{2}{3} \varepsilon^{(2)} \delta_{ij} + f_\varepsilon^{(2)} \varepsilon^{(2)} \frac{R_{ij}}{K} \quad (3.114)$$

where the damping function also depends on the turbulent Reynolds number, though in a somewhat different manner :

$$f_\varepsilon^{(2)} = \frac{1}{1 + R_t/10} \quad (3.115)$$

In the present study the closure is formulated using only the homogeneous part :

$$\varepsilon_{ij}^h = (1 - f_\varepsilon^{(2)}) \frac{2}{3} \varepsilon^{(2)} \delta_{ij} + f_\varepsilon^{(2)} \varepsilon^{(2)} \frac{R_{ij}}{K} \quad (3.116)$$

where the damping function in terms of turbulent Reynolds number, is defined by :

$$f_\varepsilon^{(2)} = R_t^{-1/2} \left(\frac{\varepsilon^{(1)}}{\varepsilon^{(2)}} \right) \quad (3.117)$$

Transport equation of the dissipation rate tensor

The transport equation for the homogeneous dissipation rate tensor can now be written in the following form :

$$\begin{aligned} \frac{\bar{d}}{dt} [\varepsilon_{ij}^h] &= c_1^{(2)} R_t^{1/2} \frac{\varepsilon^{(2)}}{K} \left(1 - \frac{\varepsilon^{(2)}}{\varepsilon^{(1)}} \right) \varepsilon_{ij}^h \\ &\quad - \left[\varepsilon_{ik}^h \bar{V}_{j,k} + \varepsilon_{jk}^h \bar{V}_{i,k} \right] \\ &\quad + \varepsilon^{(2)} \left((\beta_3^{(2)} + 2 c_2^{(2)}) S_{ij} \right. \\ &\quad \quad \left. + (\beta_4^{(2)} + 2 c_5^{(2)}) [e_{in}^h S_{jn} + e_{jn}^h S_{in} - \frac{2}{3} e_{pn}^h S_{pn} \delta_{ij}] \right. \\ &\quad \quad \left. + \beta_6^{(2)} [W_{in}^h e_{nj}^h - e_{in}^h W_{nj}^h] \right) \end{aligned}$$

$$\begin{aligned}
 & + (c_3^{(2)} + \frac{1}{3} c_5^{(2)}) e_{pn}^h S_{pn} \delta_{ij} \\
 & - \beta_1^{S(2)} R_t^{1/2} \frac{\varepsilon^{(2)2}}{K} e_{ij}^h \\
 & + \left[\left(\nu \delta_{pq} + C_D^{(2)} \frac{K^2}{\varepsilon^{(1)}} \delta_{pq} \right) \left(\varepsilon_{ij}^h \right)_{,p},_q \right]
 \end{aligned}$$

This form resembles the equation obtained recently by Speziale and Gatski (1997) using a functional analysis, which for homogeneous turbulence reads :

$$\begin{aligned}
 \frac{\bar{d}}{dt} [\varepsilon_{ij}] &= \frac{\varepsilon^{(2)}}{K} \left(-C_{\varepsilon_1} \frac{R_{mn} \bar{V}_{m,n}}{\varepsilon^{(2)}} - C_{\varepsilon_2} \right) \frac{2}{3} \varepsilon^{(2)} \delta_{ij} \\
 & - \left[\varepsilon_{ik} \bar{V}_{j,k} + \varepsilon_{jk} \bar{V}_{i,k} \right] \\
 & + \varepsilon^{(2)} \left(\beta_3 S_{ij} \right. \\
 & \quad + \beta_4 [e_{in} S_{jn} + e_{jn} S_{in} - \frac{2}{3} e_{pn} S_{pn} \delta_{ij}] \\
 & \quad + \beta_6 [W_{in} e_{nj} - e_{in} W_{nj}] \\
 & \quad \left. + c_R e_{pn} S_{pn} \delta_{ij} \right) \\
 & - \beta_1^S \frac{\varepsilon^{(2)2}}{K} e_{ij}
 \end{aligned}$$

where

$$\begin{aligned}
 C_{\varepsilon_1} &= 1.0 & C_{\varepsilon_2} &= 1.83 \\
 \beta_3 &= \frac{16}{15} & \beta_4 &= \frac{(30\alpha + 20)}{11} & \beta_6 &= -\frac{(14\alpha - 20)}{11} & c_R &= -\frac{(42\alpha - 16)}{33} \\
 \alpha &= 0.6 \\
 \beta_1^S &= 5.80
 \end{aligned}$$

or, by inserting the values of $\alpha = 0.6$:

$$\begin{aligned}
 C_{\varepsilon_1} &= 1.00 & C_{\varepsilon_2} &= 1.83 \\
 \beta_3 &= 1.067 & \beta_4 &= 3.454 & \beta_6 &= 1.054 & c_R &= -0.278 \\
 \beta_1^S &= 5.800
 \end{aligned}$$

It can be noted that the coefficient in the return to isotropy term here is constant, whereas in the equation derived from the spectral model a dependency on the turbulent Reynolds number appears naturally. This term can be rewritten in terms of the Kolmogorov time scale defined as :

$$T = \sqrt{\frac{\nu}{\varepsilon^{(2)}}} = R_t^{-1/2} \frac{K}{\varepsilon^{(2)}}$$

Therefore, it follows that the Kolmogorov time scale is the characteristic time of the return to isotropy of the dissipative motion, i.e. :

$$\beta_1^{S(2)} R_t^{1/2} \frac{\varepsilon^{(2)}}{K} e_{ij}^h = \beta_1^{S(2)} \frac{1}{T} e_{ij}^h$$

As noted by Speziale and Gatski (1997), this would suggest that at relatively high turbulent Reynolds number in a near-homogeneous flow the dissipative structure evolves rapidly towards an isotropic state, although recent experiments and DNS indicate that small scales anisotropies can persist and can not be disregarded. They therefore suggested that a most suitable characteristic time scale should be a combined scale based on both the Kolmogorov time scale and the energy-containing turbulence time scale, a proposition already introduced by Durbin (1993), though in a context of near-wall modifications, who suggested

$$T = \max \left[\frac{K}{\varepsilon}, C_K \sqrt{\frac{\nu}{\varepsilon}} \right]$$

where $C_K \approx 6$ as chosen by Durbin (1993). The same hybrid time scale T can be used in the return to isotropy term in the model equation derived here for ε_{ij}^h .

The second remark concerns the way in which the nonequilibrium is taken into account. The balance between the production by non-linear vortex stretching and the viscous destruction mechanism is in the Speziale and Gatski (1997) model represented in the classical way, used in almost every $K - \varepsilon$ or $R_{ij} - \varepsilon$ closures. The difference between these two terms represents the total source which governs the dissipation equation at high-Reynolds number in an homogeneous flow

$$S_{ij}^\varepsilon = \frac{\varepsilon^{(2)}}{K} \left(-C_{\varepsilon_1} \frac{R_{mn} \bar{V}_{m,n}}{\varepsilon^{(2)}} - C_{\varepsilon_2} \right) \frac{2}{3} \varepsilon^{(2)} \delta_{ij} \quad (3.122)$$

whereas in the present formulation the total source in the limit of high Reynolds numbers is given by :

$$S_{ij}^\varepsilon = c_1^{(2)} R_t^{1/2} \frac{\varepsilon^{(2)}}{K} \left(1 - \frac{\varepsilon^{(2)}}{\varepsilon^{(1)}} \right) \varepsilon_{ij}^h \quad (3.123)$$

In order to explain this difference, we will consider the form of the model in the equilibrium limit. In the case when a homogeneous turbulence reaches an equilibrium state, the rate at which the small scales are dissipated into heat is the same as the rate at which the kinetic energy is transferred through the spectrum from larger to smaller eddies. This is the Kolmogorov (1941) hypothesis, based also on the assumption that the turbulence Reynolds number is high. Therefore, in such a situation, both of the following conditions are expected to be satisfied :

$$\varepsilon^{(1)} = \varepsilon^{(2)} \quad (3.124a)$$

$$R_t \rightarrow \infty \quad (3.124b)$$

From the Reynolds stress equation one can derive the kinetic energy equation by taking the trace :

$$\begin{aligned} \frac{\bar{d}}{dt} [K] &= -\varepsilon^{(2)} \\ &\quad - R_{pq} \bar{V}_{p,q} \\ &\quad + \left[\left(\nu \delta_{pq} + \frac{C_K}{\sigma_K} \frac{K^2}{\varepsilon^{(1)}} \delta_{pq} \right) (K) \right]_{,p],q} \end{aligned}$$

The length scale equation is recalled again

$$\begin{aligned} \frac{\bar{d}}{dt} \left[\frac{K^{5/2}}{\varepsilon^{(1)}} \right] &= -C_{KL_1}^{(1)} \frac{K^{3/2}}{\varepsilon^{(1)}} R_{pq} \bar{V}_{p,q} \\ &\quad - C_{KL_2}^{(1)} K^{3/2} \\ &\quad - C_{KL_3}^{(1)} \left(\frac{\varepsilon^{(1)}}{\varepsilon^{(2)}} \right)^{-1/2} K^{3/2} R_t^{-3/4} \\ &\quad + \left[\left(\nu \delta_{pq} + \frac{C_{KL}^{(1)}}{\sigma_{KL}^{(1)}} \frac{K^2}{\varepsilon^{(1)}} \delta_{pq} \right) \left(\frac{K^{5/2}}{\varepsilon^{(1)}} \right) \right]_{,p],q} \end{aligned}$$

When the flow is homogeneous, the diffusion and the convection of the turbulent quantities can be neglected so that

$$\frac{d\varepsilon^{(1)}}{dt} = \frac{5}{2} \frac{\varepsilon^{(1)}}{K} \frac{dK}{dt} - \frac{\varepsilon^{(1)2}}{K^{5/2}} \frac{d}{dt} \left(\frac{K^{5/2}}{\varepsilon^{(1)}} \right) \quad (3.127)$$

The $\varepsilon^{(1)}$ equation reduces in the case of a homogeneous flow to :

$$\begin{aligned} \frac{d\varepsilon^{(1)}}{dt} &= \left(C_{KL_1}^{(1)} - \frac{5}{2} \right) \frac{\varepsilon^{(1)}}{K} R_{mn} \bar{V}_{m,n} \\ &\quad - \frac{5}{2} \frac{\varepsilon^{(1)} \varepsilon^{(2)}}{K} \\ &\quad + C_{KL_2}^{(1)} \frac{\varepsilon^{(1)2}}{K} \\ &\quad + C_{KL_3}^{(1)} \frac{\varepsilon^{(1)2}}{K} \left(\frac{\varepsilon^{(1)}}{\varepsilon^{(2)}} \right)^{-1/2} R_t^{-3/4} \end{aligned}$$

The homogeneous approximation of the equation for the viscous dissipation rate is :

$$\begin{aligned} \frac{d\varepsilon^{(2)}}{dt} &= -C_{\varepsilon_1}^{(2)} R_t^{-1/2} \varepsilon^{(1)} K^{-1} R_{pq} \bar{V}_{p,q} \\ &\quad - C_{\varepsilon_2}^{(2)} R_t^{1/2} \frac{\varepsilon^{(2)2}}{K} \left(1 - \frac{\varepsilon^{(2)}}{\varepsilon^{(1)}} \right) \end{aligned}$$

Then, in the limit of an equilibrium state, this equation becomes

$$\begin{aligned} \frac{d\varepsilon^{(2)}}{dt} &= \left(C_{KL_1}^{(1)} - \frac{5}{2} \right) \frac{\varepsilon^{(2)}}{K} R_{mn} \bar{V}_{m,n} \\ &\quad + \left(C_{KL_2}^{(1)} - \frac{5}{2} \right) \frac{\varepsilon^{(2)2}}{K} \end{aligned}$$

where $\varepsilon^{(1)} = \varepsilon^{(2)}$ and the turbulent Reynolds number is very large. Hence, the comparison between the two equations yields

$$\begin{aligned} \lim_{\varepsilon^{(1)} \rightarrow \varepsilon^{(2)}} \left[R_t^{1/2} \frac{\varepsilon^{(2)2}}{K} \left(1 - \frac{\varepsilon^{(2)}}{\varepsilon^{(1)}} \right) \right] &= \left(C_{KL_1}^{(1)} - \frac{5}{2} \right) \frac{\varepsilon^{(2)}}{K} R_{mn} \bar{V}_{m,n} \\ &\quad + \left(C_{KL_2}^{(1)} - \frac{5}{2} \right) \frac{\varepsilon^{(2)2}}{K} \end{aligned}$$

This suggests that the vortex stretching and the viscous dissipation mechanisms depend on the production of kinetic energy by the mean velocity gradients in the limit of the equilibrium assumption. This is not unrealistic, for this is precisely the major consequence of the assumption for the spectrum equilibrium. This also gives a link between the coefficients in the classical $K - \varepsilon$ and $R_{ij} - \varepsilon$ closure for the equilibrium assumption.

An equation for the dissipation rate tensor for inhomogeneous turbulence has been proposed by Tagawa *et al.* (1991). Their equation is also derived from functional expansion, but includes an approximate expression for the diffusive terms. It was aimed at improving the boundary layer and channel flow predictions. The derivation of that equation was directly inspired by the classical form of Launder *et al.* (1975) for the Reynolds stress equation. A better basis for comparison for inhomogeneous flow is the more recent proposal by Oberlack (1997). This equation has been derived from two-point correlations and extended to inhomogeneous flows.

$$\begin{aligned} \frac{\bar{d}}{dt} [\varepsilon_{ij}] &= \frac{\varepsilon^{(2)}}{K} \left(-C_{\varepsilon_1} \frac{R_{mn} \bar{V}_{m,n}}{\varepsilon^{(2)}} - C_{\varepsilon_2} \right) \frac{2}{3} \varepsilon^{(2)} \delta_{ij} \\ &\quad - \left[\varepsilon_{ik} \bar{V}_{j,k} + \varepsilon_{jk} \bar{V}_{i,k} \right] \\ &\quad + \varepsilon^{(2)} \left(\beta_3 S_{ij} \right. \\ &\quad \quad + \beta_4 [e_{in} S_{jn} + e_{jn} S_{in} - \frac{2}{3} e_{pn} S_{pn} \delta_{ij}] \\ &\quad \quad + \beta_6 [W_{in} e_{nj} - e_{in} W_{nj}] \\ &\quad \quad \left. + c_R e_{pn} S_{pn} \delta_{ij} \right) \\ &\quad - \beta_1^S \frac{\varepsilon^{(2)2}}{K} e_{ij} \\ &\quad + \left[\left(\nu \delta_{pq} + \frac{C_\varepsilon^{(2)}}{\sigma_\varepsilon^{(2)}} \frac{K}{\varepsilon^{(1)}} R_{pq} \right) \left(\varepsilon_{ij}^{(2)} \right)_{,p},_q \right] \end{aligned}$$

where

$$\begin{aligned}
 C_{\varepsilon_1} &= 2.5 & C_{\varepsilon_2} &= 1.92 \\
 \beta_3 &= \frac{4}{15} - 2a_0 + \frac{64}{15}a_1 & \beta_4 &= \frac{8}{5} - \frac{32}{5}a_1 & \beta_6 &= \frac{4}{5} + \frac{112}{15}a_1 & c_R &= \frac{2}{45} \frac{1}{a_0} (15a_0 + 4 + 14a_1) \\
 a_0 &= 0.05 & a_1 &= 0.00 \\
 \beta_1^S &= (2c_R^\varepsilon + 2C_{\varepsilon_2}) \frac{\varepsilon^{(2)}}{K} - \frac{4}{15} \frac{1}{a_0} (4 + 14a_1) e_{mn} S_{mn} + 10 b_{mn} S_{mn} \\
 c_R^\varepsilon &= 0.8
 \end{aligned}$$

which yield approximately :

$$\begin{aligned}
 C_{\varepsilon_1} &= 2.5 & C_{\varepsilon_2} &= 1.92 \\
 \beta_3 &= 0.167 & \beta_4 &= 1.600 & \beta_6 &= 0.800 & c_R &= 4.222 \\
 \beta_1^S &= 5.440 \frac{\varepsilon^{(2)}}{K} - 21.333 e_{mn} S_{mn} + 10.000 b_{mn} S_{mn}
 \end{aligned}$$

The equation for the length scale tensor

Following the same pattern as for the dissipation rate equation, the tensorial length scale equation can be rewritten as :

$$\begin{aligned}
 \frac{\bar{d}}{dt} [K \Lambda_{ij}] &= -2 \left(\frac{\varepsilon^{(1)}}{\varepsilon^{(2)}} \right)^{-1/2} K^{3/2} R_t^{-3/4} \frac{2}{3} \delta_{ij} \\
 &\quad - K [\Lambda_{ik} \bar{V}_{j,k} + \Lambda_{jk} \bar{V}_{i,k}] \\
 &\quad + \frac{K^{5/2}}{\varepsilon^{(1)}} \left((\beta_3^{(-1)} + 2c_2^{(-1)}) S_{ij} \right. \\
 &\quad \quad + (\beta_4^{(-1)} + 2c_5^{(-1)}) [\lambda_{in} S_{jn} + \lambda_{jn} S_{in} - \frac{2}{3} \lambda_{pn} S_{pn} \delta_{ij}] \\
 &\quad \quad + \beta_6^{(-1)} [W_{in} \lambda_{nj} - \lambda_{in} W_{nj}] \\
 &\quad \quad \left. + (c_3^{(-1)} + \frac{1}{3} c_5^{(-1)}) \lambda_{pn} S_{pn} \delta_{ij} \right) \\
 &\quad - \beta_1^{S(-1)} K^{3/2} \lambda_{ij} \\
 &\quad + c_1^{(-1)} \varepsilon^{(1)} \Lambda_{ij} \\
 &\quad + \left[\left(\nu \delta_{pq} + C_D^{(-1)} \frac{K^2}{\varepsilon^{(1)}} \delta_{pq} \right) \left(K \Lambda_{ij} \right)_{,p} \right]_{,q}
 \end{aligned}$$

Using the definition of the large scale parameter, which is proportional to the trace of the length scale tensor Λ_{ij} , it has been set that

$$\Lambda = L \tag{3.136}$$

Eliminating the $\varepsilon^{(1)}$ contribution from the previous equation leads to

$$\begin{aligned}
 \frac{\bar{d}}{dt} \left[K \Lambda_{ij} \right] &= -2 \left(\frac{\varepsilon^{(1)}}{\varepsilon^{(2)}} \right)^{-1/2} K^{3/2} R_t^{-3/4} \frac{2}{3} \delta_{ij} \\
 &\quad - K \left[\Lambda_{ik} \bar{V}_{j,k} + \Lambda_{jk} \bar{V}_{i,k} \right] \\
 &\quad + K \Lambda \left((\beta_3^{(-1)} + 2 c_2^{(-1)}) S_{ij} \right. \\
 &\quad \quad + (\beta_4^{(-1)} + 2 c_5^{(-1)}) [\lambda_{in} S_{jn} + \lambda_{jn} S_{in} - \frac{2}{3} \lambda_{pn} S_{pn} \delta_{ij}] \\
 &\quad \quad + \beta_6^{(-1)} [W_{in} \lambda_{nj} - \lambda_{in} W_{nj}] \\
 &\quad \quad \left. + (c_3^{(-1)} + \frac{1}{3} c_5^{(-1)}) \lambda_{pn} S_{pn} \delta_{ij} \right) \\
 &\quad - \beta_1^{S(-1)} K^{3/2} \lambda_{ij} \\
 &\quad + c_1^{(-1)} \frac{K^{3/2}}{\Lambda} \Lambda_{ij} \\
 &\quad + \left[\left(\nu \delta_{pq} + C_D^{(-1)} (K^{1/2} \Lambda) \delta_{pq} \right) \left(K \Lambda_{ij} \right)_{,p},q \right]
 \end{aligned}$$

This closure shows some similarity with the expression proposed by Donaldson and Sandri (1981) where

$$\begin{aligned}
 \frac{\bar{d}}{dt} \left[K \Lambda_{ij} \right] &= -K \left[\Lambda_{ik} \bar{V}_{j,k} + \Lambda_{jk} \bar{V}_{i,k} \right] \\
 &\quad - \beta_1^{S(-1)} K^{3/2} \lambda_{ij} \\
 &\quad + c_1^{(-1)} \frac{K^{3/2}}{\Lambda} \Lambda_{ij} \\
 &\quad + \left[\left(\nu \delta_{pq} + C_D^{(-1)} (K^{1/2} \Lambda) \delta_{pq} \right) \left(K \Lambda_{ij} \right)_{,p},q \right]
 \end{aligned}$$

with

$$\beta_1^{S(-1)} = 1.25 \tag{3.139a}$$

$$c_1^{-1} = -1.45 \tag{3.139b}$$

These authors also proposed from their derivation an expression for the diffusive transport which is of the same kind as the one here, but they neglect this term when considering the homogeneous approximation of the equation, so that the diffusive coefficient remained undetermined.

Appendix A

Governing equations for one-point statistical closure

This appendix contains details and notations of the governing equations for an incompressible flow used in the one-point statistical closure.

A.1 Statistical description of the motion

In the Eulerian description of the fluid motion the dynamical variables are expressed as functions depending on space and time. The flow is supposed to be incompressible and the conservation of mass and momentum for an infinitesimal control volume is governed by the following equations :

$$u_{i,i} = 0 \quad (\text{A.1})$$

$$\frac{du_i}{dt} = -\frac{1}{\rho}\sigma_{ij,j} + f_i \quad (\text{A.2})$$

where f_i designs volume forces applied on the fluid and u_i is the velocity component in the i^{th} direction. The index notations are adopted with the convention of summation on repeated indices except when especially mentioned. The material derivative is defined as :

$$\frac{d.}{dt} = \frac{\partial.}{\partial t} + u_k \frac{\partial.}{\partial x_k}$$

For a Newtonian fluid, the stress tensor σ_{ij} can be written as :

$$\sigma_{ij} = -p\delta_{ij} + 2\mu(u_{i,j} + u_{j,i}) \quad (\text{A.3})$$

where p is the static pressure, μ is the dynamic viscosity and ρ is the density of the fluid (here μ and ρ are assumed constant). δ_{ij} is the Kronecker symbol. The Newtonian fluid

hypothesis leads to the Navier-Stokes equations which, for an incompressible fluid, are :

$$u_{i,i} = 0 \quad (\text{A.4})$$

$$\frac{du_i}{dt} = -\frac{1}{\rho}p_{,i} + \nu u_{i,jj} + f_i \quad (\text{A.5})$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid.

A.1.1 Reynolds averaged equations

The instantaneous turbulent motion is assumed to be fully described by the Navier-Stokes equations that can, in principle, be solved numerically (Direct Numerical Simulations). However, the presence of eddy structures of a wide range of sizes and characteristic frequencies makes direct numerical simulation of the flow in complex geometries and at higher Reynolds numbers still inaccessible to the currently available computers. For the applications of practical interest, it is therefore still necessary to follow a statistical approach by considering a form of the Navier-Stokes equation modified by an average operation. The non-linearity of the convective terms of these equations leads to unknown correlations between turbulent fluctuations, resulting in an open system of equations. The closure of this system is achieved with the help of turbulence models that give algebraic or differential equations to simulate the behaviour of the turbulent correlations.

A.1.2 The mean equations

The application of the Reynolds averaging to the Navier-Stokes equations leads to the following transport equations for the mean velocity components :

$$\overline{V}_{i,i} = 0 \quad (\text{A.6})$$

$$\frac{d\overline{V}_i}{dt} = -\frac{1}{\rho}\overline{P}_{,i} - \left[\overline{u'_i u'_j}\right]_{,j} + \nu \overline{V}_{i,jj} + \overline{F}_i \quad (\text{A.7})$$

where the fluctuating quantities are denoted by prime and the mean ones are overlined. These equations differ from the instantaneous equations by the presence of the double correlations (second moments) of the fluctuating velocities on the left hand side. These correlations are interpreted as turbulent or Reynolds stress tensor R_{ij} , half trace of which defines the kinetic energy K . It should also be noted that the material derivative of the mean flow now can be written as :

$$\frac{d.}{dt} = \frac{\partial.}{\partial t} + \overline{V}_k \frac{\partial.}{\partial x_k} \quad (\text{A.8})$$

A.1.3 Equations for fluctuating motion

Subtracting the governing equation for the mean motion from the equation for the instantaneous flow yields the transport equation of the fluctuating motion :

$$u'_{i,i} = 0 \quad (\text{A.9})$$

$$\frac{\overline{d}u'_i}{dt} = -\frac{1}{\rho}p'_{,i} - u'_j \overline{V}_{i,j} - (u'_i u'_j - \overline{u'_i u'_j})_{,j} + \nu u'_{i,jj} + f'_i \quad (\text{A.10})$$

In the following text the prime notations for fluctuating quantities ' will be omitted for more clarity. The above equation can be also written using an operator, i.e. :

$$u_{i,t} + L_i(\underline{u}) = 0 \quad (\text{A.11})$$

A.2 Transport equation for one-point statistic quantities

A.2.1 Governing equation for the second moment

The transport equations for the Reynolds stress tensor can be obtained from the evolution equation for the fluctuating velocity and, for an incompressible flow, it can be written as :

$$\frac{\overline{d}R_{ij}}{dt} = P_{ij} + G_{ij} + \phi_{ij} - \varepsilon_{ij} + d_{ij}^p + d_{ij}^u + d_{ij}^\nu \quad (\text{A.12})$$

where P_{ij} , G_{ij} , ϕ_{ij} , d_{ij}^p , d_{ij}^u et d_{ij}^ν represent respectively the production by the mean velocity gradients, the production by the body forces, the redistribution, the transport by fluctuating pressure and by fluctuating velocity, and the diffusion by the molecular viscosity. These terms are defined as :

$$P_{ij} = -R_{ik} \overline{V}_{j,k} - R_{jk} \overline{V}_{i,k} \quad (\text{A.13a})$$

$$G_{ij} = (\overline{f_i u_j} + \overline{f_j u_i}) \quad (\text{A.13b})$$

$$\varepsilon_{ij} = 2\nu \overline{u_{i,k} u_{j,k}} \quad (\text{A.13c})$$

$$\phi_{ij} = \frac{1}{\rho} \overline{p(u_{i,j} + u_{j,i})} \quad (\text{A.13d})$$

$$d_{ij}^p = -\frac{1}{\rho} [\overline{p u_i} \delta_{jk} + \overline{p u_j} \delta_{ik}]_{,k} \quad (\text{A.13e})$$

$$d_{ij}^u = -[\overline{u_i u_j u_k}]_{,k} \quad (\text{A.13f})$$

$$d_{ij}^\nu = \nu [\overline{u_i u_j}]_{,kk} \quad (\text{A.13g})$$

The ε_{ij} tensor is known as the rate of dissipation tensor, even if it does not exactly represent this action. If the deformation (rate-of-strain) tensor of the fluctuating flow is written as :

$$s_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

the exact dissipation rate tensor is :

$$\varepsilon_{ij}^r = \nu \overline{[u_{i,k} s_{jk} + u_{j,k} s_{ik}]} \quad (\text{A.14})$$

For incompressible flows ε_{ij}^r can be expressed as a function of ε_{ij} :

$$\varepsilon_{ij}^r = \varepsilon_{ij} + \nu \overline{[u_i s_{jk} + u_j s_{ik}]_{,k}} \quad (\text{A.15})$$

In this case, if the correlations between the fluctuating velocity and the fluctuating rate of strain are supposed to be independent of their spatial location, both tensors are rigorously identical. This is the case when statistical quantities are invariant to the spatial translation (homogeneous turbulence), or at high Reynolds number when the large-scale motion represented by velocity fluctuations are expected to be uncorrelated with the small scale motion represented by the fluctuating velocity derivatives, due to a large separation of scales in the wave number or frequency domain.

The organization of different terms in the equation is, of course, not unique. The notation adopted here is convenient since it shows explicitly the influence of the viscous diffusion term. The classical decomposition of the original term involving the fluctuating pressure into the pressure-strain and the pressure transport terms is also adopted :

$$\Pi_{ij} = \phi_{ij} + d_{ij}^p \quad (\text{A.16})$$

where Π_{ij} is the velocity–pressure–gradient correlation tensor defined as :

$$\Pi_{ij} = -\frac{1}{\rho} \overline{[u_i p_{,j} + u_j p_{,i}]} \quad (\text{A.17})$$

It should be noted that the transport terms are usually denoted as triple order tensor :

$$T_{ijk,k}^u = d_{ij}^u \quad (\text{A.18a})$$

$$T_{ijk,k}^p = d_{ij}^p \quad (\text{A.18b})$$

These splitting can also be applied to the two-point equation as adopted in this report, see below.

A.2.2 Governing equation for the dissipation rate tensor

The transport equation for the dissipation rate tensor can also be derived from the evolution equation for the fluctuating velocity. Following the same organization of the source term leads to :

$$\frac{\overline{d\varepsilon_{ij}}}{dt} = P_{ij}^{\varepsilon 1} + P_{ij}^{\varepsilon 2} + P_{ij}^{\varepsilon 3} + P_{ij}^{\varepsilon 4} + G_{ij}^{\varepsilon} - E_{ij}^{\varepsilon} + \phi_{ij}^{\varepsilon} + d_{ij}^{\varepsilon u} + d_{ij}^{\varepsilon p} + d_{ij}^{\varepsilon \nu} \quad (\text{A.19})$$

where the production term may be decomposed into four contributions :

$$P_{ij}^{\varepsilon 1} = -2\nu (\overline{u_{i,m}u_{k,m}} \overline{V}_{j,k} + \overline{u_{j,m}u_{k,m}} \overline{V}_{i,k}) \quad (\text{A.20a})$$

$$P_{ij}^{\varepsilon 2} = -2\nu (\overline{u_{i,m}u_{j,k}} + \overline{u_{j,m}u_{i,k}}) \overline{V}_{k,m} \quad (\text{A.20b})$$

$$P_{ij}^{\varepsilon 3} = -2\nu (\overline{u_k u_{i,m}} \overline{V}_{j,mk} + \overline{u_k u_{j,m}} \overline{V}_{i,mk}) \quad (\text{A.20c})$$

$$P_{ij}^{\varepsilon 4} = -2\nu (\overline{u_{i,k}u_{j,m}u_{k,m}} + \overline{u_{j,k}u_{i,m}u_{k,m}}) \quad (\text{A.20d})$$

The first two terms, $P_{ij}^{\varepsilon 1}$ and $P_{ij}^{\varepsilon 2}$ represents the production by the mean velocity gradients, the third term is the production due to inhomogeneity of the mean velocity gradients (by the mean vorticity gradients), and the last term represents the production by the self stretching of fluctuating vortex filaments (interactions between eddies of different scales). The explicit effect of body forces is defined by the G_{ij}^{ε} tensor :

$$G_{ij}^{\varepsilon} = 2\nu [\overline{f_{i,k}u_{j,k}} + \overline{f_{j,k}u_{i,k}}] \quad (\text{A.21})$$

The rate of the viscous destruction of the dissipation tensor can be written as :

$$E_{ij}^{\varepsilon} = 4\nu^2 \overline{u_{i,km}u_{j,km}} \quad (\text{A.22})$$

The redistribution tensor is :

$$\phi_{ij}^{\varepsilon} = 2\nu \frac{1}{\rho} \overline{p_{,m}(u_{i,jm} + u_{j,im})} \quad (\text{A.23})$$

and the three transport contributions, by the turbulent motion, by the fluctuating pressure and by the viscosity can respectively be written as :

$$d_{ij}^{\varepsilon u} = -2\nu (\overline{u_k u_{i,m}u_{j,m}})_{,k} \quad (\text{A.24a})$$

$$d_{ij}^{\varepsilon p} = -2\nu \frac{1}{\rho} \overline{[p_{,m}(u_{i,m}\delta_{jk} + u_{j,m}\delta_{ik})]}_{,k} \quad (\text{A.24b})$$

$$d_{ij}^{\varepsilon \nu} = 2\nu^2 (\overline{u_{i,m}u_{j,m}})_{,kk} \quad (\text{A.24c})$$

A.2.3 Governing equation for the kinetic energy

Taking the trace of the Reynolds stress transport equation leads to the evolution equation for the kinetic energy :

$$\frac{dK}{dt} = P + G - \varepsilon + d^{uK} + d^{pK} + d^{\nu K} \quad (\text{A.25})$$

with the corresponding interpretation of each terms as in the equation for R_{ij} . P denotes the production by the mean motion, G the explicit effect of the body forces, ε the dissipation rate and d^{uK} , d^{pK} and $d^{\nu K}$ the transport terms respectively by the fluctuating

velocity, the fluctuating pressure and the viscous diffusion.

$$P = -R_{mn} \overline{V}_{m,n} \quad (\text{A.26a})$$

$$G = \overline{u_i f_i} \quad (\text{A.26b})$$

$$\varepsilon = \nu \overline{u_{i,j} u_{i,j}} \quad (\text{A.26c})$$

$$d^{uK} = -\frac{1}{2} (\overline{u_n u_n u_k})_{,k} \quad (\text{A.26d})$$

$$d^{pK} = -\frac{1}{\rho} \overline{u_n p_{,n}} \quad (\text{A.26e})$$

$$d^{\nu K} = \nu K_{,jj} \quad (\text{A.26f})$$

There is no contribution of the pressure-strain term as this redistribution tensor has a zero trace for incompressible turbulence.

A.2.4 Governing equation for the energy dissipation rate

Taking the trace of the dissipation rate tensor transport equation also leads to the evolution equation for the energy dissipation rate :

$$\frac{\overline{d\varepsilon}}{dt} = P^\varepsilon + G^\varepsilon - E^\varepsilon + \phi^\varepsilon + d^{\varepsilon u} + d^{\varepsilon p} + d^{\varepsilon \nu} \quad (\text{A.27})$$

Here P^ε can be decomposed into four terms with different physical meanings, as outlined earlier

$$P^\varepsilon = P^{\varepsilon 1} + P^{\varepsilon 2} + P^{\varepsilon 3} + P^{\varepsilon 4}$$

where :

$$P^{\varepsilon 1} = -2\nu \overline{u_{i,m} u_{j,m}} \overline{V}_{i,j} \quad (\text{A.28a})$$

$$P^{\varepsilon 2} = -2\nu \overline{u_{m,i} u_{m,j}} \overline{V}_{i,j} \quad (\text{A.28b})$$

$$P^{\varepsilon 3} = -2\nu \overline{u_j u_{i,m}} \overline{V}_{i,jm} \quad (\text{A.28c})$$

$$P^{\varepsilon 4} = -2\nu \overline{u_{i,j} u_{i,m} u_{j,m}} \quad (\text{A.28d})$$

The source due to body forces is expressed as :

$$G^\varepsilon = 2\nu \overline{u_{i,j} f_{i,j}} \quad (\text{A.29})$$

whereas the viscous destruction can be written as :

$$E^\varepsilon = 2\nu^2 \overline{u_{i,jm} u_{i,jm}} \quad (\text{A.30})$$

The pressure correlation term is trace-free, but the diffusion terms reduces to :

$$d^{\varepsilon u} = -\nu (\overline{u_m u_{i,j} u_{i,j}})_{,m} \quad (\text{A.31a})$$

$$d^{\varepsilon p} = 2\nu \frac{1}{\rho} (\overline{p_{,i} u_{m,i}})_{,m} \quad (\text{A.31b})$$

$$d^{\varepsilon \nu} = \nu \varepsilon_{,mm} \quad (\text{A.31c})$$

with the same interpretation as those for the dissipation rate tensor.

Appendix B

Governing equations for the two-point statistical closure

B.1 Two - point description with centered variables

The principal rules associated with the variable transformation used in the derivation of the two-point closures in the main text are summarized in this Appendix.

B.1.1 Rules of the centered transformation

The points M' and M'' are taken respectively at the coordinates (\underline{x}', t) and (\underline{x}'', t) . The transformation then is given by :

$$x'_k = X_k - \frac{1}{2} r_k \quad (B.1a)$$

$$x''_k = X_k + \frac{1}{2} r_k \quad (B.1b)$$

where \underline{X} is the location of the midpoint between M' and M'' and \underline{r} describes the distance between M' and M'' defined as

$$X_k = \frac{1}{2} (x'_k + x''_k) \quad (B.2a)$$

$$r_k = (x''_k - x'_k) \quad (B.2b)$$

The Jacobian matrix

The derivation of the Jacobian matrix associated with the transformation is :

$$\frac{\partial}{\partial x'_k} = \frac{1}{2} \frac{\partial}{\partial X_k} - \frac{\partial}{\partial r_k} \quad (B.3a)$$

$$\frac{\partial}{\partial x''_k} = \frac{1}{2} \frac{\partial}{\partial X_k} + \frac{\partial}{\partial r_k} \quad (B.3b)$$

Notations

In order to simplify the expressions, the following notations are adopted for any dependent variables ϕ :

$$\frac{\partial\phi}{\partial x'_k} = \phi_{,k'} \quad (B.4a)$$

$$\frac{\partial\phi}{\partial x''_k} = \phi_{,k''} \quad (B.4b)$$

and

$$\frac{\partial\phi}{\partial X_k} = \phi_{,k} \quad (B.5a)$$

$$\frac{\partial\phi}{\partial r_k} = \phi_{|k} \quad (B.5b)$$

Property of the derivatives

It should be pointed out that any quantity evaluated at M' does not depend on \underline{x}'' and *vice-versa*, hence

$$\phi'_{,k''} = 0 \quad (B.6a)$$

$$\phi''_{,k'} = 0 \quad (B.6b)$$

For example,

$$\begin{aligned} \phi'_{,k''} &= \frac{\partial\phi(\underline{x}')}{\partial x''_k} \\ &= \frac{\partial x'_p}{\partial x''_k} \left[\frac{\partial\phi}{\partial x'_p} \right] (\underline{x}') \end{aligned}$$

Because, \underline{x}' and \underline{x}'' are considered as independent variables, it follows that

$$\phi'_{,k''} = 0$$

This implies that any correlation between two derivatives of fluctuating quantities at points M' and M'' can be written as higher order multiple derivative of the corresponding two-point quantity, as long as the derivatives are in an Euclidean space. This is the case, for example, with the two-point dissipation rate tensor, where :

$$u'_{i,k'} u''_{j,k''} = [u'_i u''_j]_{,k'k''}$$

Using this property, we can now derive higher order derivatives of the flow quantities in the transformed coordinates . The principal ones are given in the next section.

Expressions for derivatives

Second order derivatives :

$$\frac{\partial^2}{\partial x'_m \partial x'_n} = \frac{1}{4} \cdot, mn - \frac{1}{2} [\cdot, m|n + \cdot |m, n] + \cdot |mn \quad (B.8a)$$

$$\frac{\partial^2}{\partial x''_m \partial x''_n} = \frac{1}{4} \cdot, mn + \frac{1}{2} [\cdot, m|n + \cdot |m, n] + \cdot |mn \quad (B.8b)$$

$$\frac{\partial^2}{\partial x'_m \partial x''_n} = \frac{1}{4} \cdot, mn + \frac{1}{2} [\cdot, m|n - \cdot |m, n] - \cdot |mn \quad (B.8c)$$

and in the contracted form

$$\frac{\partial^2}{\partial x'^2_k} = \frac{1}{4} \cdot, kk - \cdot, k|k + \cdot |kk \quad (B.9a)$$

$$\frac{\partial^2}{\partial x''^2_k} = \frac{1}{4} \cdot, kk + \cdot, k|k + \cdot |kk \quad (B.9b)$$

$$\frac{\partial^2}{\partial x'_k \partial x''_k} = \frac{1}{4} \cdot, kk - \cdot |kk \quad (B.9c)$$

Third order derivatives :

$$\begin{aligned} \frac{\partial^3}{\partial x'_n \partial x'_m \partial x''_k} &= \frac{1}{8} \cdot, nmk + \frac{1}{4} [\cdot, kn|m + \cdot, km|n] + \frac{1}{2} \cdot, k|nm \\ &+ \frac{1}{4} \cdot, nm|k - \frac{1}{2} [\cdot, n|km + \cdot, m|kn] + \cdot |nmk \end{aligned} \quad (B.10a)$$

$$\begin{aligned} \frac{\partial^3}{\partial x''_n \partial x''_m \partial x'_k} &= \frac{1}{8} \cdot, nmk - \frac{1}{4} [\cdot, kn|m + \cdot, km|n] + \frac{1}{2} \cdot, k|nm \\ &- \frac{1}{4} \cdot, nm|k - \frac{1}{2} [\cdot, n|km + \cdot, m|kn] - \cdot |nmk \end{aligned} \quad (B.10b)$$

and, after contraction of m and k in the contracted form

$$\frac{\partial^3}{\partial x'_n \partial x'_k \partial x''_k} = \frac{1}{8} \cdot, nkk + \cdot |nkk - \frac{1}{4} \cdot, kk|n - \frac{1}{2} \cdot, n|kk \quad (B.11a)$$

$$\frac{\partial^3}{\partial x''_n \partial x'_k \partial x''_k} = \frac{1}{8} \cdot, nkk - \cdot |nkk + \frac{1}{4} \cdot, kk|n - \frac{1}{2} \cdot, n|kk \quad (B.11b)$$

Fourth order derivatives

$$\begin{aligned} \frac{\partial^4}{\partial x'_n \partial x'_n \partial x'_k \partial x''_k} &= \frac{1}{16} \cdot ,nmkk + \frac{1}{4} [\cdot ,kk|nn - \cdot |kk,nn] - \cdot |kknn \\ &\quad - \frac{1}{4} \cdot ,kkn|n + \cdot |kkn,n \end{aligned} \quad (B.12a)$$

$$\begin{aligned} \frac{\partial^4}{\partial x''_n \partial x''_n \partial x'_k \partial x''_k} &= \frac{1}{16} \cdot ,nmkk + \frac{1}{4} [\cdot ,kk|nn - \cdot |kk,nn] - \cdot |kknn \\ &\quad + \frac{1}{4} \cdot ,kkn|n - \cdot |kkn,n \end{aligned} \quad (B.12b)$$

Incompressibility constraint

The incompressibility condition allows to write some relations between the derivative with respect to \underline{X} and to \underline{r} .

Continuity and fluctuations

$$\frac{1}{2} u'_{i,i} - u'_{i|i} = 0 \quad (B.13a)$$

$$\frac{1}{2} u''_{i,i} + u''_{i|i} = 0 \quad (B.13b)$$

Continuity and double correlations

The two-point second order tensor is defined as :

$$R_{i,j} = \overline{u'_i u''_j} \quad (B.14)$$

so that the incompressible relations can be written as :

$$\frac{1}{2} R_{i,j,i} - R_{i,j|i} = 0 \quad (B.15a)$$

$$\frac{1}{2} R_{i,j,j} + R_{i,j|j} = 0 \quad (B.15b)$$

Continuity and triple correlations

The two-points triple correlations tensors are defined as :

$$T^u_{ik,j} = -\overline{u'_i u'_k u''_j} \quad (B.16a)$$

$$T^u_{i,jk} = -\overline{u'_i u''_j u''_k} \quad (B.16b)$$

which satisfy the following constraints :

$$\frac{1}{2} T_{ik,j}^u + T_{ik,j|j}^u = 0 \quad (B.17a)$$

$$\frac{1}{2} T_{i,jk,i}^u - T_{i,jk|i}^u = 0 \quad (B.17b)$$

Continuity and pressure-velocity correlations

$$\frac{1}{2} (\overline{p''u'_i})_{,i} - (\overline{p''u'_i})_{|i} = 0 \quad (B.18a)$$

$$\frac{1}{2} (\overline{p'u''_j})_{,j} + (\overline{p'u''_j})_{|j} = 0 \quad (B.18b)$$

These relations are used to evaluate the pressure-velocity transport equation.

Mean quantities

In two-point equations the mean velocity components are evaluated at two distinct points, but they remain the one-point quantities. The expressions for their gradients or their second order derivative can therefore be simplified. Of course, they satisfy all kinematic properties introduced earlier in this Appendix. It should be pointed out that in the transformed coordinates system there is no need to write any derivative of the mean flow with respect to \underline{r} as they can be explicitly written as function of the derivative with respect to \underline{X} .

Gradients

Gradients of the mean velocity can be transformed as :

$$\overline{V}'_{k,n'} = \frac{1}{2} \overline{V}'_{k,n} - \overline{V}'_{k|n} \quad (B.19a)$$

$$\overline{V}''_{k,n''} = \frac{1}{2} \overline{V}''_{k,n} + \overline{V}''_{k|n} \quad (B.19b)$$

Since these are one-point quantities, it follows also that :

$$\overline{V}'_{k,n''} = 0 \quad (B.20a)$$

$$\overline{V}''_{k,n'} = 0 \quad (B.20b)$$

Those relations can be written in the transformed variables system, so that :

$$0 = \frac{1}{2} \overline{V}'_{k,n} + \overline{V}'_{k|n} \quad (B.21a)$$

$$0 = \frac{1}{2} \overline{V}''_{k,n} - \overline{V}''_{k|n} \quad (B.21b)$$

Therefore :

$$\overline{V}'_{k|n} = -\frac{1}{2}\overline{V}'_{k,n} \quad (B.22a)$$

$$\overline{V}''_{k|n} = \frac{1}{2}\overline{V}''_{k,n} \quad (B.22b)$$

and

$$\overline{V}'_{k,n'} = \overline{V}'_{k,n} \quad (B.23a)$$

$$\overline{V}''_{k,n''} = \overline{V}''_{k,n} \quad (B.23b)$$

These relations can be applied to the mean velocity derivatives of any order.

Second derivative

It is interesting to note that the above relations lead to :

$$\overline{V}'_{k|nn} = \frac{1}{4}\overline{V}'_{k,nn} \quad (B.24a)$$

$$\overline{V}''_{k|nn} = \frac{1}{4}\overline{V}''_{k,nn} \quad (B.24b)$$

These expressions are especially useful for showing explicitly the departure of the two-point equations from the corresponding one-point expressions.

B.1.2 Transport equations

Evolution equation for the two-point second order tensor

Convective terms

The transformation of the convective terms in the transformed system is straightforward :

$$C_{i'.j''} = \frac{1}{2}[\overline{V}'_k + \overline{V}''_k] R_{i,j,k} + [\overline{V}''_k - \overline{V}'_k] R_{i,j|k} \quad (B.25)$$

Since $R_{i,j}$ is the two-point analogue of the one-point Reynolds stress tensor, the above expression can be split in two contributions

$$C_{i'.j''} = C_{i,j} + C_{i,j}^{\cdot\cdot} \quad (B.26)$$

where

$$C_{i,j} = \frac{1}{2}[\overline{V}'_k + \overline{V}''_k] R_{i,j,k} \quad (B.27a)$$

$$C_{i,j}^{\cdot\cdot} = [\overline{V}''_k - \overline{V}'_k] R_{i,j|k} \quad (B.27b)$$

The second tensor, $C_{i,j}^{\cdot\cdot}$, becomes zero when the distance between the two points is reduced to zero. This is a two-point contribution which does not affect the one-point convection tensor, but only its partition between the two points. That is why this contribution is denoted by two dots in the exponent. It should be noted here that the first tensor remains a two point tensor as long as \underline{r} does not go to zero. The derivation of the different terms in the one point limit will be explained later.

The two-point material derivative is :

$$\frac{\bar{d}}{dt} = \frac{\partial}{\partial t} + \frac{1}{2} [\bar{V}'_k + \bar{V}''_k] \frac{\partial}{\partial X_k} \quad (\text{B.28})$$

Production terms

The production terms do not need any transformation and they show clearly to two-point contribution. The mean velocity gradients can, however, be transformed so that the terms reduce to :

$$P_{i'.j''} = -R_{i.k} \bar{V}''_{j,k} - R_{k.j} \bar{V}'_{i,k} \quad (\text{B.29})$$

Body force term

As long as the only body force of interest in this study is the Coriolis force, its two-point tensor can be written as :

$$\begin{aligned} G_{i'.j''} &= G_{i,j} \\ &= -2\Omega_k [\varepsilon_{kmi} R_{m,j} + \varepsilon_{kmj} R_{i,m}] \end{aligned} \quad (\text{B.30})$$

The Coriolis force is indeed linear with respect to the velocity.

Triple velocity correlation tensor

The triple velocity correlation contribution may be written as :

$$d_{i'.j''}^u = T_{i.jk,k''}^u + T_{ik.j,k'}^u \quad (\text{B.31})$$

so that its transformed expression is :

$$d_{i'.j''}^u = \frac{1}{2} [T_{i.jk,k}^u + T_{ik.j,k}^u] + [T_{i.jk|k}^u - T_{ik.j|k}^u] \quad (\text{B.32})$$

Just as the convective terms, this expression can be split in order to exhibit the part which vanish in the one-point limit.

$$d_{i'.j''}^u = d_{i,j}^u + d_{i,j}^{u..} \quad (\text{B.33})$$

with

$$d_{i,j}^u = \frac{1}{2} [T_{i.jk,k}^u + T_{ik.j,k}^u] \quad (\text{B.34a})$$

$$d_{i,j}^{u..} = [T_{i.jk|k}^u - T_{ik.j|k}^u] \quad (\text{B.34b})$$

Pressure-velocity tensor

The correlations between the velocity and the gradient of fluctuating pressure lead to the following term :

$$\Pi_{i'.j''} = -\frac{1}{\rho} \overline{(u'_i p''_{,j''} + u''_j p'_{,i'})} \quad (\text{B.35})$$

to which the variables transformation can be also applied. It is interesting to point out that the classical decomposition into a pressure-strain tensor and a pressure diffusion term does not seem to be relevant in the two-point context at least for this tensor. The term can be rewritten as :

$$\Pi_{i'.j''} = -\frac{1}{\rho} [(\overline{u'_i p''})_{,j''} + (\overline{u''_j p'})_{,i'}] \quad (\text{B.36})$$

without any further assumption because of the following property :

$$u'_{i,j''} = 0 \quad (\text{B.37a})$$

$$u''_{i,j'} = 0 \quad (\text{B.37b})$$

It is, therefore, more interesting to transform the above expression for the pressure-velocity tensor, so that :

$$\Pi_{i'.j''} = -\frac{1}{2} \frac{1}{\rho} [(\overline{u'_i p''})_{,j} + (\overline{u''_j p'})_{,i}] - \frac{1}{\rho} [(\overline{u'_i p''})_{|j} - (\overline{u''_j p'})_{|i}] \quad (\text{B.38})$$

This time the second part of the expression goes to zero in the one-point limit. Nevertheless, this expression is of interest to simplifying the overall contribution of the transport terms in the source equation. These terms can be expressed as :

$$\Pi_{i'.j''} = \frac{1}{2} [T_{i.jk,k}^p + T_{ik.j,k}^p] + [T_{i.jk|k}^p - T_{ik.j|k}^p] \quad (\text{B.39})$$

with

$$T_{i.jk}^p = -\frac{1}{\rho} \overline{u'_i p''} \delta_{jk} \quad (\text{B.40a})$$

$$T_{ik.j}^p = -\frac{1}{\rho} \overline{u''_j p'} \delta_{ik} \quad (\text{B.40b})$$

These contributions can be grouped with the terms representing the transport by fluctuating velocity so that

$$d_{i'.j''}^u + \Pi_{i'.j''} = \frac{1}{2} [T_{i.jk}^t + T_{ik.j}^t]_{,k} + [T_{i.jk}^t - T_{ik.j}^t]_{|k} \quad (\text{B.41})$$

with

$$T_{i.jk}^t = -\overline{u'_i u''_j u''_k} - \frac{1}{\rho} \overline{u'_i p''} \delta_{jk} \quad (\text{B.42a})$$

$$T_{ik.j}^t = -\overline{u'_i u''_k u''_j} - \frac{1}{\rho} \overline{u''_j p'} \delta_{ik} \quad (\text{B.42b})$$

Viscosity contribution

The viscous term can be written directly in the two-point form as :

$$d_{i'.j''}^{\nu} = \nu [(\overline{u'_i u''_j})_{,k'k'} + (\overline{u'_i u''_j})_{,k''k''}] \quad (\text{B.43})$$

The evaluation of the second order derivative leads to the following transformed expressions :

$$d_{i'.j''}^{\nu} = \nu \left[\frac{1}{2} R_{i.j,kk} + 2 R_{i.j|kk} \right] \quad (\text{B.44})$$

This term can be written in the form which exhibits the viscous diffusion :

$$d_{i'.j''}^{\nu} = \nu R_{i.j,kk} + \nu \left[-\frac{1}{2} R_{i.j,kk} + 2 R_{i.j|kk} \right] \quad (\text{B.45})$$

The first term obviously corresponds to the two-point viscous diffusion contribution, whereas the last part is the two-point expression for the dissipation tensor.

The two-point dissipation tensor is defined as :

$$\varepsilon_{i.j} = 2 \nu \overline{u'_{i,k'} u''_{j,k''}} \quad (\text{B.46})$$

Applying the rules introduced earlier, the cross-derivatives with respect to M' and M'' can be expressed as :

$$\check{\varepsilon}_{i.j} = \frac{1}{4} R_{i.j,kk} - R_{i.j|kk} \quad (\text{B.47})$$

where

$$\check{\varepsilon}_{i.j} = \frac{1}{2\nu} \varepsilon_{i.j} \quad (\text{B.48})$$

is a normalized form of the two point dissipation rate tensor.

Summary and remarks

The transformed equation for the two-point second-order correlation tensor can now be summarized :

$$\begin{aligned} \frac{\overline{d}R_{i.j}}{dt} &= -R_{i.k} \overline{V''_{j,k}} - R_{k.j} \overline{V'_{i,k}} \\ &- 2 \Omega_k [\varepsilon_{kmi} R_{m.j} + \varepsilon_{kmj} R_{i.m}] \\ &+ \frac{1}{2} [T_{i.jk}^t + T_{ik.j}^t]_{,k} \\ &+ [T_{i.jk}^t - T_{ik.j}^t]_{|k} \\ &+ \nu R_{i.j,kk} \\ &- 2 \nu \left[\frac{1}{4} R_{i.j,kk} - R_{i.j|kk} \right] \\ &- [\overline{V''_k} - \overline{V'_k}] R_{i.j|k} \end{aligned} \quad (\text{B.49})$$

Remarks :

The transport equation for the two-point second order tensor differs from the one point equation by two features.

First, the only unknowns terms in the two-point second-moment equation are the triple correlations tensor as long as the pressure-velocity correlation can be obtained by means of the Poisson equation derived from the Reynolds stress transport.

The second feature is the presence of terms that vanish in the one-point limit. These terms only influence the partition of the two-point correlation in the two-point context. They then can be seen as typical inhomogeneous two-point transfer terms.

Evolution equation for the two-point dissipation tensor

As seen in the previous section the two-point dissipation tensor is defined by :

$$\varepsilon_{i,j} = 2\nu \overline{u'_{i,k'} u''_{j,k''}} \quad (B.50)$$

For convenience, we also eliminated viscosity by introducing the normalized quantity

$$\check{\varepsilon}_{i,j} = \frac{1}{2\nu} \varepsilon_{i,j} \quad (B.51)$$

The transport equation for the two-point dissipation tensor may now be written as :

$$\begin{aligned} \check{\varepsilon}_{i,j,t} + \overline{V}'_n \check{\varepsilon}_{i,j,n'} + \overline{V}''_n \check{\varepsilon}_{i,j,n''} &= -[\overline{u'_{n,k'} u''_{j,k''}} \overline{V}'_{i,n'} + \overline{u''_{n,k''} u'_{i,k'}} \overline{V}''_{j,n''}] \quad (B.52) \\ &- [\overline{u'_{i,n'} u''_{j,k''}} \overline{V}'_{n,k'} + \overline{u''_{j,n''} u'_{i,k'}} \overline{V}''_{n,k''}] \\ &- [\overline{u'_n u''_{j,k''}} \overline{V}'_{i,n'k'} + \overline{u''_n u'_{i,k'}} \overline{V}'_{j,n''k''}] \\ &- [\overline{u'_{i,n'} u'_{n,k'} u''_{j,k''}} + \overline{u''_{j,n''} u''_{n,k''} u'_{i,k'}}] \\ &+ [\overline{f'_{i,n'} u''_{j,k''}} + \overline{f''_{j,n''} u'_{i,k'}}] \\ &- [\overline{u'_n u'_{i,n'k'} u''_{j,k''}} + \overline{u''_n u''_{j,n''k''} u'_{i,k'}}] \\ &- \frac{1}{\rho} [\overline{p'_{i'k'} u''_{j,k''}} + \overline{u'_{i,k'} p''_{j''k''}}] \\ &+ \nu [(\overline{u'_{i,k'} u''_{j,k''}})_{,n'n'} + (\overline{u''_{j,k''} u'_{i,k'}})_{,n''n''}] \end{aligned}$$

Several contributions of the source term can be grouped into a tensor similar to the one adopted in the one-point description.

$$C_{i'.j''}^\varepsilon = \overline{V}'_n \check{\varepsilon}_{i,j,n'} + \overline{V}''_n \check{\varepsilon}_{i,j,n''} \quad (B.53a)$$

$$P_{i'.j''}^{\varepsilon 1} = -\check{\varepsilon}_{i,k} \overline{V}''_{j,k''} - \check{\varepsilon}_{k,j} \overline{V}'_{i,k'} \quad (B.53b)$$

$$P_{i'.j''}^{\varepsilon 2} = -[\overline{u'_{i,n'} u''_{j,k''}} \overline{V}'_{n,k'} + \overline{u''_{j,n''} u'_{i,k'}} \overline{V}''_{n,k''}] \quad (B.53c)$$

$$P_{i'.j''}^{\varepsilon 3} = -[\overline{u'_n u''_{j,k''}} \overline{V}'_{i,n'k'} + \overline{u''_n u'_{i,k'}} \overline{V}'_{j,n''k''}] \quad (B.53d)$$

$$P_{i'.j''}^{\varepsilon 4} = -[\overline{u'_{i,n'} u'_{n,k'} u''_{j,k''}} + \overline{u''_{j,n''} u''_{n,k''} u'_{i,k'}}] \quad (B.53e)$$

$$G_{i'.j''}^\varepsilon = \overline{[f'_{i,n'}u''_{j,k''} + f''_{j,n''}u'_{i,k'}]} \quad (B.53f)$$

$$d_{i'.j''}^{\varepsilon u} = -\overline{[u'_n u'_{i,n'} u''_{j,k''} + u''_n u''_{j,n''} u'_{i,k'}]} \quad (B.53g)$$

$$\Pi_{i'.j''}^\varepsilon = -\frac{1}{\rho} \overline{[p'_{i',k'} u''_{j,k''} + u'_{i,k'} p''_{j'',k''}]} \quad (B.53h)$$

$$d_{i'.j''}^{\varepsilon \nu} = \nu [\check{\varepsilon}_{i,j,n'n''} + \check{\varepsilon}_{i,j,n''n''}] \quad (B.53i)$$

The transport equation can be transformed into the new coordinate system in the same way as the previous Reynolds stress equation. We consider again term by term.

Convective terms

The transformed convective term yields the same definition of the material derivative :

$$C_{i'.j''}^\varepsilon = \frac{1}{2} [\overline{V}'_k + \overline{V}''_k] \check{\varepsilon}_{i,j,k} + [\overline{V}''_k - \overline{V}'_k] \check{\varepsilon}_{i,j|k} \quad (B.54)$$

The same splitting is also applicable :

$$C_{i'.j''}^\varepsilon = C_{i,j}^\varepsilon + C_{i,j}^{\varepsilon..} \quad (B.55)$$

where

$$C_{i,j}^\varepsilon = \frac{1}{2} [\overline{V}'_k + \overline{V}''_k] \check{\varepsilon}_{i,j,k} \quad (B.56a)$$

$$C_{i,j}^{\varepsilon..} = [\overline{V}''_k - \overline{V}'_k] \check{\varepsilon}_{i,j|k} \quad (B.56b)$$

In the one-point limit the second term vanishes.

Production terms

The production terms are usually split into four contributions as in the one-point description and these notations are also chosen here. Each term is treated separately.

The $P_{i'.j''}^{\varepsilon 1}$ term

As in the equation for the two-point Reynolds stress tensor, this contribution clearly shows the way the term reduces to the one point limit. However it is interesting to write its expression in the form which shows the explicit contribution of the Reynolds stress tensor itself :

$$\begin{aligned} P_{i'.j''}^{\varepsilon 1} &= \frac{1}{4} [-R_{n,j,kk} \overline{V}'_{i,n'} - R_{i,n,kk} \overline{V}''_{j,n''}] \\ &\quad - [-R_{n,j|kk} \overline{V}'_{i,n'} - R_{i,n|kk} \overline{V}''_{j,n''}] \end{aligned} \quad (B.57)$$

When the mean velocity gradient is also transformed, this expression is simplified to :

$$\begin{aligned} P_{i'.j''}^{\varepsilon 1} &= \frac{1}{4} [-R_{n,j,kk} \overline{V}'_{i,n} - R_{i,n,kk} \overline{V}''_{j,n}] \\ &\quad - [-R_{n,j|kk} \overline{V}'_{i,n} - R_{i,n|kk} \overline{V}''_{j,n}] \end{aligned} \quad (B.58)$$

The $P_{i'.j''}^{\varepsilon^2}$ term

The evaluation of the cross-derivative with respect to both variables leads to the following terms :

$$\begin{aligned} P_{i'.j''}^{\varepsilon^2} &= \frac{1}{4} [-R_{i.j,nk} \bar{V}'_{n,k'} - R_{i.j,kn} \bar{V}''_{n,k''}] \\ &\quad - [-R_{i.j|nk} \bar{V}'_{n,k'} - R_{i.j|kn} \bar{V}''_{n,k''}] \\ &\quad + \frac{1}{2} [R_{i.j,n|k} - R_{i.j,k|n}] (\bar{V}''_{n,k''} - \bar{V}'_{n,k'}) \end{aligned} \quad (\text{B.59})$$

or again :

$$\begin{aligned} P_{i'.j''}^{\varepsilon^2} &= \frac{1}{4} [-R_{i.j,nk} \bar{V}'_{n,k} - R_{i.j,kn} \bar{V}''_{n,k}] \\ &\quad - [-R_{i.j|nk} \bar{V}'_{n,k} - R_{i.j|kn} \bar{V}''_{n,k}] \\ &\quad + \frac{1}{2} [R_{i.j,n|k} - R_{i.j,k|n}] (\bar{V}''_{n,k} - \bar{V}'_{n,k}) \end{aligned} \quad (\text{B.60})$$

The $P_{i'.j''}^{\varepsilon^3}$ term

This term can be rewritten as :

$$P_{i'.j''}^{\varepsilon^3} = -[(\overline{u'_n u''_j})_{,k''} \bar{V}'_{i,n'k'} + (\overline{u''_n u'_i})_{,k'} \bar{V}'_{j,n''k''}] \quad (\text{B.61})$$

so that the transformation leads directly to :

$$P_{i'.j''}^{\varepsilon^3} = \frac{1}{2} [-R_{n.j,k} \bar{V}'_{i,n'k'} - R_{i.n,k} \bar{V}''_{j,n''k''}] \quad (\text{B.62})$$

$$- [-R_{i.n|k} \bar{V}''_{j,n''k''} + R_{n.j|k} \bar{V}'_{i,n'k'}] \quad (\text{B.63})$$

and, after the mean velocity gradients are also transformed :

$$P_{i'.j''}^{\varepsilon^3} = \frac{1}{2} [-R_{n.j,k} \bar{V}'_{i,nk} - R_{i.n,k} \bar{V}''_{j,nk}] \quad (\text{B.64})$$

$$- [-R_{i.n|k} \bar{V}''_{j,nk} + R_{n.j|k} \bar{V}'_{i,nk}] \quad (\text{B.65})$$

The $P_{i'.j''}^{\varepsilon^4}$ term

This term is expressed in the form which mimics the splitting used in the one-point context even though this form is not very convenient for the two-point framework :

$$P_{i'.j''}^{\varepsilon^4} = -[\overline{u'_{i,n'} u'_{n,k'} u''_{j,k''}} + \overline{u''_{j,n''} u''_{n,k''} u'_{i,k'}}] \quad (\text{B.66})$$

or,

$$\begin{aligned} P_{i'.j''}^{\varepsilon^4} &= -[\overline{(u'_i u'_n)_{,k'n'} u''_{j,k''}} + \overline{(u''_j u''_n)_{,n''k''} u'_{i,k'}}] \\ &\quad + [\overline{(u'_n u'_{i,k'n'}) u''_{j,k''}} + \overline{(u''_n u''_{j,n''k''}) u'_{i,k'}}] \end{aligned} \quad (\text{B.67})$$

The latter expression can be conveniently joint with the turbulent transport term. The sum of the two term reduces to :

$$P_{i'.j''}^{\varepsilon^4} + d_{i'.j''}^{\varepsilon u} = -[\overline{(u'_i u'_n)_{,k'n'} u''_{j,k''}} + \overline{(u''_j u''_n)_{,n''k''} u'_{i,k'}}] \quad (\text{B.68})$$

which can be rewritten as :

$$P_{i'.j''}^{\varepsilon^4} + d_{i'.j''}^{\varepsilon u} = -[\overline{(u'_i u'_n u''_j)_{,k'k''n'}} + \overline{(u'_i u''_j u''_n)_{,k'k''n''}}] \quad (\text{B.69})$$

Body force term

If we confine our attention to only Coriolis force, the expression of this term is straightforward and need not be further elaborated.

Triple velocity correlation tensor

This tensor is grouped with the production by selfstretching of turbulent vortex filament, as described above. The sum of these two contributions can be expressed in terms of the two-point triple correlation tensor as :

$$P_{i'.j''}^{\varepsilon^4} + d_{i'.j''}^{\varepsilon u} = [T_{in,j,k'k''n'}^u + T_{i,jn,k'k''n''}^u] \quad (\text{B.70})$$

Using the transformation of the triple derivative leads to :

$$\begin{aligned} P_{i'.j''}^{\varepsilon^4} + d_{i'.j''}^{\varepsilon u} &= \frac{1}{4} \left[\frac{1}{2} (T_{i,jn}^u + T_{in,j}^u)_{,nkk} + (T_{i,jn}^u - T_{in,j}^u)_{|n,kk} \right] \\ &- \left[\frac{1}{2} (T_{i,jn}^u + T_{in,j}^u)_{,n|kk} + (T_{i,jn}^u - T_{in,j}^u)_{|nkk} \right] \end{aligned} \quad (\text{B.71})$$

Pressure-velocity tensor

The same kind of expression can be applied to the pressure-velocity term :

$$\Pi_{i'.j''}^{\varepsilon} = -\frac{1}{\rho} [\overline{(p' u''_j)_{,i'k'k''}} + \overline{(u'_i p'')_{,j''k'k''}}] \quad (\text{B.72})$$

Applying the transformation leads to :

$$\begin{aligned} \Pi_{i'.j''}^{\varepsilon} &= \frac{1}{\rho} \frac{1}{4} \left[-\frac{1}{2} (\overline{p' u''_j} \delta_{in} + \overline{p'' u'_i} \delta_{jn})_{,nkk} - (\overline{p'' u'_i} \delta_{jn} - \overline{p' u''_j} \delta_{in})_{|n,kk} \right] \\ &- \frac{1}{\rho} \left[-\frac{1}{2} (\overline{p' u''_j} \delta_{in} + \overline{p'' u'_i} \delta_{jn})_{,n|kk} - (\overline{p'' u'_i} \delta_{jn} - \overline{p' u''_j} \delta_{in})_{|nkk} \right] \end{aligned} \quad (\text{B.73})$$

It is clear that the transport terms may be lumped together with the third order velocity correlation tensor in the same way as in the Reynolds stress equation. Denoting the joint terms as

$$T_{in,j}^t = -\overline{u'_i u'_n u''_j} - \frac{1}{\rho} \overline{p' u''_j} \delta_{in} \quad (\text{B.74a})$$

$$T_{i,jn}^t = -\overline{u'_i u''_n u''_j} - \frac{1}{\rho} \overline{p'' u'_i} \delta_{jn} \quad (\text{B.74b})$$

we can now express the sum of the following three terms :

$$\begin{aligned} P_{i'.j''}^{\varepsilon^4} + d_{i'.j''}^{\varepsilon^u} + \Pi_{i'.j''}^{\varepsilon} &= \frac{1}{4} \left[\frac{1}{2} (T_{i.jn}^t + T_{in.j}^t)_{,nkk} + (T_{i.jn}^t - T_{in.j}^t)_{|n,kk} \right] \\ &- \left[\frac{1}{2} (T_{i.jn}^t + T_{in.j}^t)_{,n|kk} + (T_{i.jn}^t - T_{in.j}^t)_{|nkk} \right] \end{aligned} \quad (\text{B.75})$$

Viscous terms

The viscous effect appears in the equation within the following term :

$$d_{i'.j''}^{\varepsilon\nu} = \nu [\check{\varepsilon}_{i,j,n'n'} + \check{\varepsilon}_{i,j,n''n''}] \quad (\text{B.76})$$

that can be directly transformed as :

$$d_{i'.j''}^{\varepsilon\nu} = \nu \left[\frac{1}{2} \check{\varepsilon}_{i,j,nn} + 2 \check{\varepsilon}_{i,j|nn} \right] \quad (\text{B.77})$$

Once again, this term can be split into :

$$d_{i'.j''}^{\varepsilon\nu} = \nu \check{\varepsilon}_{i,j,nn} + \nu \left[-\frac{1}{2} \check{\varepsilon}_{i,j,nn} + 2 \check{\varepsilon}_{i,j|nn} \right] \quad (\text{B.78})$$

which displays clearly the two constituents, the viscous diffusion term, defined by

$$d_{i,j}^{\varepsilon\nu} = \nu \check{\varepsilon}_{i,j,nn} \quad (\text{B.79})$$

and the viscous destruction term

$$E_{i,j}^{\varepsilon} = 2 \nu \left[\frac{1}{4} \check{\varepsilon}_{i,j,nn} - \check{\varepsilon}_{i,j|nn} \right] \quad (\text{B.80})$$

The total viscous term can also be rewritten as a function of the two-point Reynolds stress tensor :

$$d_{i,j}^{\varepsilon\nu} - E_{i,j}^{\varepsilon} = \nu \left[\frac{1}{8} R_{i,j,kknn} - 2 R_{i,j|kknn} \right] \quad (\text{B.81})$$

Summary and remarks

The entire equation can now be written taking into account the respective contribution of each part after the term splitting as :

$$\begin{aligned} &\frac{1}{4} \left[R_{i,j,kk,t} + \frac{1}{2} [\overline{V}'_k + \overline{V}''_k] R_{i,j,nnk} \right] \\ &- \left[\boxed{R_{i,j|kk,t} + \frac{1}{2} [\overline{V}'_k + \overline{V}''_k] R_{i,j|nn,k}} \right] \end{aligned} \quad (\text{B.82})$$

=

$$\begin{aligned}
& \frac{1}{4} \left[-(\bar{V}_k'' - \bar{V}_k') R_{i,j,nn|k} \right] \\
& - \left[\boxed{-(\bar{V}_k'' - \bar{V}_k') R_{i,j|nnk}} \right] \\
& + \frac{1}{4} \left[-R_{n,j,kk} \bar{V}'_{i,n} - R_{i,n,kk} \bar{V}''_{j,n} \right] \\
& - \left[\boxed{-R_{n,j|kk} \bar{V}'_{i,n} - R_{i,n|kk} \bar{V}''_{j,n}} \right] \\
& + \frac{1}{4} \left[-R_{i,j,nk} \bar{V}'_{n,k} - R_{i,j,kn} \bar{V}''_{n,k} \right] \\
& - \left[\boxed{-R_{i,j|nk} \bar{V}'_{n,k} - R_{i,j|kn} \bar{V}''_{n,k}} \right] \\
& + \frac{1}{4} \left[-2(\bar{V}_k'' - \bar{V}_k')_{,n} R_{i,j|k,n} \right] \\
& - \left[\boxed{-\frac{1}{2}(\bar{V}_k'' - \bar{V}_k')_{,n} R_{i,j,k|n}} \right] \\
& + \frac{1}{4} \left[2[-R_{n,j,k} \bar{V}'_{i,nk} - R_{i,n,k} \bar{V}''_{j,nk}] \right] \\
& - \left[\boxed{-R_{i,n|k} \bar{V}''_{j,nk} + R_{n,j|k} \bar{V}'_{i,nk}} \right] \\
& + \frac{1}{4} \left[\frac{1}{2} (T_{i,jn}^t + T_{in,j}^t)_{,nkk} + (T_{i,jn}^t - T_{in,j}^t)_{|n,kk} \right] \\
& - \left[\boxed{\frac{1}{2} (T_{i,jn}^t + T_{in,j}^t)_{,n|kk} + (T_{i,jn}^t - T_{in,j}^t)_{|nkk}} \right] \\
& + \frac{1}{4} \left[-2\nu \check{\xi}_{i,j,nn} + \nu R_{i,j,kknn} \right] \\
& - \left[\boxed{-2\nu \check{\xi}_{i,j|nn} + \nu R_{i,j,kk|nn}} \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4} \left[-\frac{1}{2} (\bar{V}_k' + \bar{V}_k'')_{,nn} R_{i,j,k} \right] \\
& - \left[\boxed{-\frac{1}{8} (\bar{V}_k' + \bar{V}_k'')_{,nn} R_{i,j,k}} \right]
\end{aligned}$$

$$\begin{aligned}
& +\frac{1}{4} \left[-R_{i.k} \overline{V''}_{j,knn} - R_{k.j} \overline{V}'_{i,knn} \right] \\
& - \left[\boxed{-\frac{1}{4} R_{i.k} \overline{V''}_{j,knn} - \frac{1}{4} R_{k.j} \overline{V}'_{i,knn}} \right] \\
& +\frac{1}{4} \left[-(\overline{V''}_k - \overline{V}'_k)_{,nn} R_{i.j|k} \right] \\
& - \left[\boxed{-\frac{1}{4} (\overline{V''}_k - \overline{V}'_k)_{,nn} R_{i.j|k}} \right]
\end{aligned}$$

The boxes indicate the contributions of the transport equation for $R_{i.j|nn}$ to the transport equation for the dissipation tensor. The last free terms (below the line) exist also in the transport equation for $R_{i.j,kk}$: they complete the equation so that their contribution need not be taken into account in the $\check{\xi}_{i,j}$ equation. However the following equation is always satisfied :

$$\frac{d\check{\xi}_{i,j}}{dt} = \frac{1}{4} \frac{dR_{i.j,kk}}{dt} - \frac{dR_{i.j|kk}}{dt} \quad (\text{B.83})$$

where the material derivative is again

$$\overline{d.} = \frac{\partial.}{\partial t} + \frac{1}{2} [\overline{V}'_k + \overline{V}''_k] \frac{\partial.}{\partial X_k} \quad (\text{B.84})$$

as previously defined. The development of each of these contributions can be found in the next section.

Remarks

It should be pointed out that the splitting here adopted ensures a clear separation between the action of the viscous destruction and the viscous diffusion in the equation for $\check{\xi}_{i,j}$. Both physical processes are also clearly displayed in each, the $R_{i.j,kk}$ and $R_{i.j|kk}$ transport equations. The definition of the viscous diffusion of the dissipation tensor are recalled again :

$$d_{i,j}^{\varepsilon\nu} = \nu \check{\xi}_{i,j,nn} \quad (\text{B.85})$$

The transformation allows to rewrite this in terms of the two-point Reynolds stress tensor as :

$$d_{i,j}^{\varepsilon\nu} = \nu \left[\frac{1}{4} R_{i.j,kk} - R_{i.j|kk} \right]_{,nn} \quad (\text{B.86})$$

or,

$$d_{i,j}^{\varepsilon\nu} = \frac{1}{4} [\nu R_{i.j,kknn}] - [\nu R_{i.j|kk,nn}] \quad (\text{B.87})$$

where the first term on the right hand side is the viscous diffusion of $R_{i.j,kk}$, whereas the second term is the viscous diffusion of $R_{i.j|kk}$.

The destruction term can also be decomposed as :

$$E_{i,j}^\varepsilon = 2\nu \left[\frac{1}{4} \check{\varepsilon}_{i,j,nn} - \check{\varepsilon}_{i,j|nn} \right] \quad (\text{B.88})$$

which can further be written as :

$$E_{i,j}^\varepsilon = \frac{1}{4} [2\nu \check{\varepsilon}_{i,j,nn}] - [2\nu \check{\varepsilon}_{i,j|nn}] \quad (\text{B.89})$$

displaying clearly the viscous dissipation of $R_{i,j,kk}$ and $R_{i,j|kk}$ respectively.

It is also interesting to lump the viscous contributions in the $R_{i,j|kk}$ equation :

$$-2\nu \check{\varepsilon}_{i,j|nn} + \nu R_{i,j,kk|nn} \quad (\text{B.90})$$

which again can be rewritten in terms of $R_{i,j}$ only :

$$-2\nu \left[\frac{1}{4} R_{i,j,kk} - R_{i,j|kk} \right]_{nn} + \nu R_{i,j|nn,kk} \quad (\text{B.91})$$

or, after rearrangement,

$$\frac{1}{2} \nu R_{i,j|nn,kk} + 2\nu R_{i,j|kknn} \quad (\text{B.92})$$

The above transformation brings in a substantial simplification, which comes from the fact that one part of the dissipation tensor has the same mathematical form as the viscous diffusion operator. It should be kept in mind, however, that this does not mean that the viscous diffusion coefficient of $R_{i,j|kk}$ is one half of ν , since $2\nu R_{i,j|kknn}$ does not represent the entire viscous dissipation process. This issue will be further discussed in the context of the one-point limit.

B.1.3 Evolution equation for each part of the dissipation tensor using the centered transformation

This section presents the transport equation for $R_{i,j|nn}$ and $R_{i,j,nn}$ derived from the transport equation for the two-point Reynolds stress tensor. Using the centered transformation, this equation can be written as :

$$\begin{aligned} \frac{\overline{d}R_{i,j}}{dt} &= -R_{i,k} \overline{V}''_{j,k} - R_{k,j} \overline{V}'_{i,k} \\ &- 2\Omega_k [\varepsilon_{kmi} R_{m,j} + \varepsilon_{kmj} R_{i,m}] \\ &+ \frac{1}{2} [T_{i,jk}^t + T_{ik,j}^t]_{,k} \\ &+ [T_{i,jk}^t - T_{ik,j}^t]_{|k} \\ &+ \nu R_{i,j,kk} \\ &- 2\nu \left[\frac{1}{4} R_{i,j,kk} - R_{i,j|kk} \right] \\ &- [\overline{V}''_k - \overline{V}'_k] R_{i,j|k} \end{aligned} \quad (\text{B.93})$$

with

$$\frac{\bar{d}}{dt} = \frac{\partial}{\partial t} + \frac{1}{2} [\bar{V}'_k + \bar{V}''_k] \frac{\partial}{\partial X_k}$$

Homogeneous part

The double derivation leads to :

$$\begin{aligned} \frac{\partial R_{i,j|nn}}{\partial t} + \frac{1}{2} [\bar{V}'_k + \bar{V}''_k] R_{i,j|nn,k} & \quad (B.94) \\ + \frac{1}{2} [\bar{V}'_k + \bar{V}''_k]_{|nn} R_{i,j,k} + [\bar{V}'_k + \bar{V}''_k]_{|n} R_{i,j|n,k} & = -R_{i,k|nn} \bar{V}''_{j,k} - R_{k,j|nn} \bar{V}'_{i,k} \\ & - R_{i,k} \bar{V}''_{j,k|nn} - R_{k,j} \bar{V}'_{i,k|nn} \\ & - 2 R_{i,k|n} \bar{V}''_{j,k|n} - 2 R_{k,j|n} \bar{V}'_{i,k|n} \\ & - 2 \Omega_k [\varepsilon_{kmi} R_{m,j|nn} + \varepsilon_{kmj} R_{i,m|nn}] \\ & + \frac{1}{2} [T_{i,jk}^t + T_{ik,j}^t]_{,k|nn} \\ & + [T_{i,jk}^t - T_{ik,j}^t]_{|knn} \\ & + \nu R_{i,j|nn,kk} \\ & - 2 \nu \left[\frac{1}{4} R_{i,j|nn,kk} - R_{i,j|kknn} \right] \\ & - [\bar{V}''_k - \bar{V}'_k] R_{i,j|knn} \\ & - [\bar{V}''_k - \bar{V}'_k]_{|nn} R_{i,j|k} \\ & - 2 [\bar{V}''_k - \bar{V}'_k]_{|n} R_{i,j|kn} \end{aligned}$$

The first line represents the material two-point derivative in the centered transformation. The last two terms on the left hand side are additive terms originating from the second derivation. They need to be modified before any further interpretation. The first three lines of the right hand side come from the production term. The very first one can be seen as the production of $R_{i,j|nn}$ by the mean velocity gradient. The two other are inhomogeneous productions. The Coriolis terms is expressed assuming the constant rotation rate. The terms in the fifth and sixth lines on the right hand side represent the transport of $R_{i,j|nn}$ and need no further elaboration. The viscous terms is also clear. The seventh line represents the diffusion process whereas the eight one is the viscous destruction of $R_{i,j|nn}$. The last three lines come from the two-point contribution of the convective term.

The equation involves derivatives of the mean velocity components with respect to $\underline{\mathbf{r}}$, that can be expressed in terms of derivatives with respect to $\underline{\mathbf{X}}$. With this transformation the equation can be rewritten as :

$$\begin{aligned} \frac{\bar{d}R_{i,j|nn}}{dt} & = -\frac{1}{8} [\bar{V}'_k + \bar{V}''_k]_{,nn} R_{i,j,k} - \frac{1}{2} [\bar{V}''_k - \bar{V}'_k]_{,n} R_{i,j|n,k} \\ & - R_{i,k|nn} \bar{V}''_{j,k} - R_{k,j|nn} \bar{V}'_{i,k} \end{aligned} \quad (B.95)$$

$$\begin{aligned}
 & - \frac{1}{4} R_{i,k} \bar{V}''_{j,knn} - \frac{1}{4} R_{k,j} \bar{V}'_{i,knn} \\
 & - [R_{i,k|n} \bar{V}''_{j,kn} - R_{k,j|n} \bar{V}'_{i,kn}] \\
 & - 2 \Omega_k [\varepsilon_{kmi} R_{m,j|nn} + \varepsilon_{kmj} R_{i,m|nn}] \\
 & + \frac{1}{2} [T_{i,jk}^t + T_{ik,j}^t]_{,k|nn} \\
 & + [T_{i,jk}^t - T_{ik,j}^t]_{|knn} \\
 & + \nu R_{i,j|nn,kk} \\
 & - 2 \nu \check{\xi}_{i,j|nn} \\
 & - [\bar{V}''_k - \bar{V}'_k] R_{i,j|knn} \\
 & - \frac{1}{4} [\bar{V}''_k - \bar{V}'_k]_{,nn} R_{i,j|k} \\
 & - [\bar{V}''_k + \bar{V}'_k]_{,n} R_{i,j|kn}
 \end{aligned}$$

The contribution of the convective terms, previously on the left hand side of the equation, are put here on the right hand side. The first of these terms remains in the one-point limit and can be seen as an inhomogeneous contribution. It does not appear in the $\check{\xi}_{i,j}$ equation because that term also exists in the transport equation for $R_{i,j,kk}$. The second term is a typical two-point term that vanishes in the one-point limit. A similar term exists in the equation for $R_{i,j,kk}$, but comes from the derivation of the two-point convective term instead of the derivation of the material derivative here. The third term is the production of $R_{i,j|nn}$ by the mean velocity gradients. The next term is an inhomogeneous production whose contribution is not seen in the equation of $\check{\xi}_{i,j}$ because it vanishes with the corresponding term of the $R_{i,j,kk}$ equation, also coming from the production contribution. The next term is also an inhomogeneous production that forms a part of the inhomogeneous production of $\check{\xi}_{i,j}$ and the contribution of which is not zero in the one-point limit. The fourth next term remains unchanged as compared to the previous equation. The three contributions from the inhomogeneous convective part can be better understood using this transformed expression. The first one vanishes in the one-point limit represents the inhomogeneous convection term of $R_{i,j|kk}$. The second one also disappears in the one-point limit; its contribution is taken into account in the $\check{\xi}_{i,j}$ equation since the corresponding term also exists in the $R_{i,j,kk}$ equation that conceals its contribution. The very last term is actually a part of the second production term in the $\check{\xi}_{i,j}$. Its contribution does not vanish in the one-point limit.

Inhomogeneous part

The double derivation of the Reynolds stress equation with respect to \underline{X} leads directly to :

$$\begin{aligned}
 \frac{\bar{d}R_{i,j,nn}}{dt} & = -\frac{1}{2} [\bar{V}'_k + \bar{V}''_k]_{,nn} R_{i,j,k} - [\bar{V}'_k + \bar{V}''_k]_{,n} R_{i,j,nk} \\
 & - R_{i,k,nn} \bar{V}''_{j,k} - R_{k,j,nn} \bar{V}'_{i,k}
 \end{aligned} \tag{B.96}$$

$$\begin{aligned}
& - R_{i.k} \bar{V}''_{j,knn} - R_{k.j} \bar{V}'_{i,knn} \\
& - 2 R_{i.k,n} \bar{V}''_{j,kn} - 2 R_{k.j,n} \bar{V}'_{i,kn} \\
& - 2 \Omega_k [\varepsilon_{kmi} R_{m.j,nn} + \varepsilon_{kmj} R_{i.m,nn}] \\
& + \frac{1}{2} [T_{i.jk}^t + T_{ik.j}^t]_{,knn} \\
& + [T_{i.jk}^t - T_{ik.j}^t]_{|k,nn} \\
& + \nu R_{i.j,nnkk} \\
& - 2\nu \left[\frac{1}{4} R_{i.j,nnkk} - R_{i.j|kk,nn} \right] \\
& - [\bar{V}''_k - \bar{V}'_k] R_{i.j|k,nn} \\
& - [\bar{V}''_k - \bar{V}'_k]_{,nn} R_{i.j|k} \\
& - 2 [\bar{V}''_k - \bar{V}'_k]_{,n} R_{i.j|k,n}
\end{aligned}$$

where, as earlier, the first two lines of the right hand side represent the contributions of the second derivative of the convective term. In contrast to the equation for $R_{i.j|nn}$, there is no need to transform this equation. The interpretation of each term is similar. The expressions for the complementary terms to form the $\check{\xi}_{i,j}$ transport equation can easily be derived, since

$$\check{\xi}_{i,j} = \frac{1}{4} R_{i.j|nn} - R_{i.j,nn}$$

B.2 Two - point description with decentered variables

The principal rules associated with the decentered variable transformation are summarized in this Appendix. It is illustrated that the centered transformation chosen in this report leads to much more clear expressions for the two-points tensors as the derivative with respect to the reference location and to the distance between the two points are more easily separated.

B.2.1 Rules of the decentered transformation

The points M' and M'' are taken respectively at the coordinates (\underline{x}', t) and (\underline{x}'', t) . The decentered transformation is defined by :

$$x'_k = X_k \tag{B.97a}$$

$$x''_k = X_k + r_k \tag{B.97b}$$

where \underline{X} is the location of the first point M' and \underline{r} describes the distance between M' and M'' . These position vectors can be written as :

$$X_k = x'_k \tag{B.98a}$$

$$r_k = (x''_k - x'_k) \tag{B.98b}$$

The Jacobian matrix

The Jacobian matrix, associated with the transformation, follows directly from the definition of the position vectors :

$$\frac{\partial}{\partial x'_k} = \frac{\partial}{\partial X_k} - \frac{\partial}{\partial r_k} \quad (B.99a)$$

$$\frac{\partial}{\partial x''_k} = \frac{\partial}{\partial r_k} \quad (B.99b)$$

Notations

The same notations as those used in the centered transformation are adopted here for any dependent variable ϕ :

$$\frac{\partial \phi}{\partial x'_{k'}} = \phi_{,k'} \quad (B.100a)$$

$$\frac{\partial \phi}{\partial x''_k} = \phi_{,k''} \quad (B.100b)$$

Hence,

$$\frac{\partial \phi}{\partial X_k} = \phi_{,k} \quad (B.101a)$$

$$\frac{\partial \phi}{\partial r_k} = \phi_{|k} \quad (B.101b)$$

Property of the derivatives

Any quantities evaluated on M' still does not depend on \underline{x}'' and *vice-versa*.

$$\phi'_{,k''} = 0 \quad (B.102a)$$

$$\phi''_{,k'} = 0 \quad (B.102b)$$

Expressions for the derivatives

Second order derivatives

$$\frac{\partial^2 \cdot}{\partial x'_m \partial x'_n} = \cdot_{,mn} - [\cdot_{,m|n} + \cdot_{|m,n}] + \cdot_{|mn} \quad (B.103a)$$

$$\frac{\partial^2 \cdot}{\partial x''_m \partial x''_n} = \cdot_{|mn} \quad (B.103b)$$

$$\frac{\partial^2 \cdot}{\partial x'_m \partial x''_n} = \cdot_{,n|m} - \cdot_{|mn} \quad (B.103c)$$

APPENDIX B. GOVERNING EQUATIONS FOR THE TWO-POINT STATISTICAL CLOSURE

or, in the contracted form

$$\frac{\partial^2 \cdot}{\partial x'_k{}^2} = \cdot,_{kk} - 2 \cdot,_{k|k} + \cdot|_{kk} \quad (B.104a)$$

$$\frac{\partial^2 \cdot}{\partial x''_k{}^2} = \cdot|_{kk} \quad (B.104b)$$

$$\frac{\partial^2 \cdot}{\partial x'_k \partial x''_k} = \cdot,_{k|k} - \cdot|_{kk} \quad (B.104c)$$

Third order derivatives

$$\frac{\partial^3 \cdot}{\partial x'_n \partial x'_m \partial x''_k} = \cdot,_{nm|k} - [\cdot,_{n|mk} + \cdot,_{m|nk}] + \cdot|_{nmk} \quad (B.105a)$$

$$\frac{\partial^3 \cdot}{\partial x''_n \partial x''_m \partial x'_k} = \cdot,_{k|nm} - \cdot|_{nmk} \quad (B.105b)$$

and

$$\frac{\partial^3 \cdot}{\partial x'_n \partial x'_k \partial x''_k} = \cdot,_{nk|k} - [\cdot,_{n|kk} + \cdot,_{k|kn}] + \cdot|_{nkk} \quad (B.106a)$$

$$\frac{\partial^3 \cdot}{\partial x''_n \partial x'_k \partial x''_k} = \cdot,_{k|kn} - \cdot|_{nkk} \quad (B.106b)$$

Fourth order derivatives

$$\frac{\partial^4 \cdot}{\partial x'_n \partial x'_n \partial x'_k \partial x''_k} = \cdot,_{nnk|k} - 2 \cdot,_{nk|nk} + \cdot,_{k|nnk} \quad (B.107a)$$

$$\begin{aligned} & - \cdot,_{nn|kk} + 2 \cdot,_{n|nkk} \cdot |_{nnkk} \\ \frac{\partial^4 \cdot}{\partial x''_n \partial x''_n \partial x'_k \partial x''_k} & = \cdot,_{k|knn} - \cdot|_{kkn} \end{aligned} \quad (B.107b)$$

Incompressibility constraint

The incompressibility condition enables to derive some relations between the derivative with respect to \underline{X} and to \underline{r} .

Continuity and fluctuations

$$u'_{i,i} - u'_{i|i} = 0 \quad (B.108a)$$

$$u''_{i|i} = 0 \quad (B.108b)$$

Continuity and double correlations

The two-point second order tensor is defined as :

$$R_{i,j} = \overline{u'_i u''_j} \quad (B.109)$$

so that the incompressible condition leads to :

$$R_{i,j,i} - R_{i,j|i} = 0 \quad (B.110a)$$

$$R_{i,j|j} = 0 \quad (B.110b)$$

Continuity and triple correlations

The two-point triple correlations tensors are defined as :

$$T_{ik,j}^u = -\overline{u'_i u'_k u''_j} \quad (B.111a)$$

$$T_{i,jk}^u = -\overline{u'_i u''_j u''_k} \quad (B.111b)$$

leading to the following constraints :

$$T_{ik,j|j}^u = 0 \quad (B.112a)$$

$$T_{i,jk,i}^u - T_{i,jk|i}^u = 0 \quad (B.112b)$$

Continuity and pressure-velocity correlations

$$(\overline{p'' u'_i})_{,i} - (\overline{p'' u'_i})_{|i} = 0 \quad (B.113a)$$

$$(\overline{p' u''_j})_{|j} = 0 \quad (B.113b)$$

These relations are mainly used to evaluate the pressure-velocity transport terms.

Mean quantities

The derivatives of the mean velocity components with respect to \underline{X} and \underline{r} can also be simplified.

Gradients

The gradients of the mean velocity can be transformed as :

$$\overline{V'}_{k,n'} = \overline{V'}_{k,n} \quad (B.114a)$$

$$\overline{V''}_{k,n''} = \overline{V''}_{k,n} \quad (B.114b)$$

and

$$\overline{V'}_{k|n} = 0 \quad (B.115a)$$

$$\overline{V''}_{k|n} = \overline{V''}_{k,n} \quad (B.115b)$$

These relations can be applied to any order of the derivatives of the mean velocity.

Second derivative

$$\bar{V}'_{k,n'm'} = \bar{V}'_{k,nm} \quad (B.116a)$$

$$\bar{V}''_{k,n''m''} = \bar{V}''_{k,nm} \quad (B.116b)$$

These expressions are especially useful to explicitly show the departure of the two-point equations from the corresponding one-point equations.

It should be noted that the expressions for the operators seem to be much simpler with the decentered transformation than with the centred one. The transformed equations are, however, more difficult to interpret as shown in Appendix B.

B.2.2 Transport equations

Evolution equation for the two-point second order tensor

Convective terms

The transformation is straightforward :

$$C_{i'.j''} = [\bar{V}'_k] R_{i,j,k} + [\bar{V}''_k - \bar{V}'_k] R_{i,j|k} \quad (B.117)$$

This transformation leads to an expression that shows in an obvious manner the difference from the one-point expression. The material derivative is :

$$\frac{\bar{d}}{dt} = \frac{\partial}{\partial t} + [\bar{V}'_k] \frac{\partial}{\partial X_k} \quad (B.118)$$

Production terms

There is no difference between the productions term in the centered or decentered variable system :

$$P_{i'.j''} = -R_{i.k} \bar{V}''_{j,k} - R_{k.j} \bar{V}'_{i,k} \quad (B.119)$$

Body force term

The same applies also to the Coriolis term : there is no difference in formulation between the centered and decentered transformation.

Triple velocity correlation tensor

The triple velocity correlation term can be written as :

$$d_{i'.j''}^u = T_{i.jk,k''}^u + T_{ik,j,k'}^u \quad (B.120)$$

and the transformation yields :

$$d_{i'.j''}^u = [T_{ik,j,k}^u] + [T_{i.jk|k}^u - T_{ik.j|k}^u] \quad (B.121)$$

so that the departure from the one-point limit is again straightforward.

Pressure-velocity tensor

The same form of the expression can be derived for the transformed pressure-velocity tensor :

$$d_{i'.j''}^p = [T_{ik.j,k}^p] + [T_{i.jk|k}^p - T_{ik.j|k}^p] \quad (B.122)$$

Hence, it is justified to write the transport term together again with the same notations :

$$T_{i.jk}^t = -\overline{u'_i u''_j u''_k} - \frac{1}{\rho} \overline{u'_i p''} \delta_{jk} \quad (B.123a)$$

$$T_{ik.j}^t = -\overline{u'_i u'_k u''_j} - \frac{1}{\rho} \overline{u''_j p'} \delta_{ik} \quad (B.123b)$$

Compared to the expressions obtained with the centered transformation the equation is here written in a simpler form without any change in the physical interpretation of each contribution to the source term.

Viscosity contribution

The difference between the two transformations appears in the expressions for the viscous action and allows to choose one or the other expression depending on the relative importance attributed to this term. The kinematic transformation rule already shows that second derivative expressions in the original two point description include cross derivative in the decentered variable system. Then the viscous term can be written as :

$$d_{i'.j''}^\nu = \nu [R_{i.j,kk} - 2 R_{i.j,k|k} + 2 R_{i.j|kk}] \quad (B.124)$$

The two-point dissipation tensor is transformed as :

$$\check{\varepsilon}_{i.j} = R_{i.j|k,k} - R_{i.j|kk} \quad (B.125)$$

that allows to write the viscous contribution directly in the form :

$$d_{i'.j''}^\nu = \nu R_{i.j,kk} - 2 \nu [R_{i.j|k,k} - R_{i.j|kk}] \quad (B.126)$$

The viscous diffusion term is then clearly defined by :

$$d_{i.j}^\nu = \nu R_{i.j,kk} \quad (B.127)$$

Summary and remarks

The synthesis of the above derivation yields the transport equation for the second-order velocity correlation in decentered coordinate system :

$$\begin{aligned} \frac{\overline{d}R_{i.j}}{dt} &= -R_{i.k} \overline{V''_{j,k}} - R_{k.j} \overline{V'_{i,k}} \\ &- 2 \Omega_k [\varepsilon_{kmi} R_{m.j} + \varepsilon_{kmj} R_{i.m}] \end{aligned} \quad (B.128)$$

$$\begin{aligned}
 & + T_{ik,j,k}^t \\
 & + [T_{i,jk}^t - T_{ik,j}^t]_{|k} \\
 & + \nu R_{i,j,kk} \\
 & - 2\nu [R_{i,j,k|k} - R_{i,j|kk}] \\
 & - [\bar{V}_k'' - \bar{V}_k'] R_{i,j|k}
 \end{aligned}$$

Remarks :

The two-point transport equation for the Reynolds stress tensor in decentered coordinate system is simpler than the one obtained with the centered transformation and it is closer in its outlook to its one-point limit. Moreover, the decentered transformation allows the splitting of the viscous term into dissipation and viscous diffusion in a quite straightforward manner. However, as seen below, the expression for the dissipation tensor in decentered system does not separate clearly the pure homogeneous contribution from the non-homogeneous one.

Evolution equation for the two-point dissipation tensor

The application of the decentered transformation to the two-point equation for the dissipation tensor leads to similar results compared to its centered transformation as obtained with the Reynolds stress transport.

Convective terms

The transformation of the convective term is straightforward :

$$C_{i'.j''}^\varepsilon = C_{i,j}^\varepsilon + C_{i,j}^{\varepsilon..} \quad (B.129)$$

where

$$C_{i,j}^\varepsilon = [\bar{V}_k'] \check{\varepsilon}_{i,j,k} \quad (B.130a)$$

$$C_{i,j}^{\varepsilon..} = [\bar{V}_k'' - \bar{V}_k'] \check{\varepsilon}_{i,j|k} \quad (B.130b)$$

Production terms

The $P_{i'.j''}^{\varepsilon 1}$ term

Once the splitting of the transformed dissipation tensor is applied, the first production term can be written as :

$$\begin{aligned}
 P_{i'.j''}^{\varepsilon 1} & = [-R_{n,j,k|k} \bar{V}_{i,n}' - R_{i,n,k|k} \bar{V}_{j,n}''] \\
 & - [-R_{n,j|kk} \bar{V}_{i,n}' - R_{i,n|kk} \bar{V}_{j,n}'']
 \end{aligned} \quad (B.131)$$

The $P_{i'.j''}^{\varepsilon 2}$ term

The expression for this term is here apparently simpler, as compared to the centered transformation :

$$\begin{aligned} P_{i'.j''}^{\varepsilon 2} &= [-R_{i.j,n|k} \bar{V}'_{n,k} - R_{i.j,k|n} \bar{V}''_{n,k}] \\ &- [-R_{i.j|nk} \bar{V}'_{n,k} - R_{i.j|kn} \bar{V}''_{n,k}] \end{aligned} \quad (\text{B.132})$$

The $P_{i'.j''}^{\varepsilon 3}$ term

This contribution to the production takes now an unusual form with the decentered transformation, because it has an apparent asymmetric term as a consequence of the kinematic transformation rules :

$$P_{i'.j''}^{\varepsilon 3} = -R_{i.n,k} \bar{V}''_{j,nk} \quad (\text{B.133})$$

$$- [-R_{i.n|k} \bar{V}''_{j,nk} + R_{n.j|k} \bar{V}'_{i,nk}] \quad (\text{B.134})$$

The $P_{i'.j''}^{\varepsilon 4}$ term

This term is transformed together with the turbulent transport, exactly as done in the centered transformation.

$$P_{i'.j''}^{\varepsilon 4} + d_{i'.j''}^{\varepsilon u} = -[(\overline{u'_i u'_n u''_j})_{,k'k''n'} + (\overline{u'_i u''_j u''_n})_{,k'k''n''}] \quad (\text{B.135})$$

Body force term

The transformation of this term is straightforward, if only the Coriolis force is considered :

Triple velocity correlation tensor

The transformation of the triple derivation gives the following expression for the turbulent transport :

$$\begin{aligned} P_{i'.j''}^{\varepsilon 4} + d_{i'.j''}^{\varepsilon u} &= [T_{in.j,n|k,k}^u + T_{in.j|nkk}^u - T_{in.j,n|kk}^u - T_{in.j|nk,k}^u] \\ &+ [T_{i.nj|nk,k}^u - T_{i.nj|nkk}^u] \end{aligned} \quad (\text{B.136})$$

Pressure-velocity tensor

The same transformation applies to the pressure-velocity tensor :

$$\begin{aligned} \Pi_{i'.j''}^{\varepsilon} &= [T_{in.j,n|k,k}^p + T_{in.j|nkk}^p - T_{in.j,n|kk}^p - T_{in.j|nk,k}^p] \\ &+ [T_{i.nj|nk,k}^p - T_{i.nj|nkk}^p] \end{aligned} \quad (\text{B.137})$$

APPENDIX B. GOVERNING EQUATIONS FOR THE TWO-POINT STATISTICAL CLOSURE

The same grouping of terms applied earlier may be applied, leading to a further simplification as follows :

$$\begin{aligned} \Pi_{i'.j''}^\varepsilon + P_{i'.j''}^{\varepsilon 4} + d_{i'.j''}^{\varepsilon u} &= [T_{in.j,n}^t + (T_{i.nj}^t - T_{in.j}^t)|_n]_{,k|k} \\ &- [T_{in.j,n}^t + (T_{i.nj}^t - T_{in.j}^t)|_n]_{|kk} \end{aligned} \quad (\text{B.138})$$

Viscosity contribution

The transformation of the viscous contribution $d_{i'.j''}^{\varepsilon \nu}$ leads again to a mixed expression where cross-derivative are present, but where the splitting into diffusion and dissipation is obvious :

$$d_{i'.j''}^{\varepsilon \nu} = \nu [\check{\varepsilon}_{i.j,nn} - 2\check{\varepsilon}_{i.j|n,n} + 2\check{\varepsilon}_{i.j|nn}] \quad (\text{B.139})$$

with

$$d_{i.j}^{\varepsilon \nu} = \nu \check{\varepsilon}_{i.j,nn} \quad (\text{B.140})$$

and

$$E_{i.j}^\varepsilon = 2\nu [\check{\varepsilon}_{i.j|n,n} - \check{\varepsilon}_{i.j|nn}] \quad (\text{B.141})$$

Summary and remarks

Exactly as in the case of the centered transformation, the transport equation for the dissipation tensor can be seen as the sum of two transport equations :

$$\frac{d\check{\varepsilon}_{i.j}}{dt} = \frac{dR_{i.j,k|k}}{dt} - \frac{dR_{i.j|kk}}{dt} \quad (\text{B.142})$$

It should be pointed out, however, that the interpretation of this decomposition is less easy. The transport equation for $R_{i.j|nn}$ is, of course, not the same as for the centered transformation, and this time the contributions of each part of the splitting to the entire transport equation for the dissipation can be written as :

$$\begin{aligned} & \left[R_{i.j,k|k,t} + \overline{V}'_k R_{i.j,n|n,k} \right] \\ & - \left[\boxed{R_{i.j|kk,t} + \overline{V}'_k R_{i.j|nn,k}} \right] \\ & = \\ & \left[-(\overline{V}''_k - \overline{V}'_k) R_{i.j,n|nk} \right] \\ & - \left[\boxed{-(\overline{V}''_k - \overline{V}'_k) R_{i.j|nnk}} \right] \\ & + \left[-R_{n.j,k|k} \overline{V}'_{i,n} - R_{i.n,k|k} \overline{V}''_{j,n} \right] \end{aligned} \quad (\text{B.143})$$

$$\begin{aligned}
 & - \left[\boxed{-R_{n,j|kk} \bar{V}'_{i,n} - R_{i,n|kk} \bar{V}''_{j,n}} \right] \\
 & + \left[-R_{i,j,n|k} \bar{V}'_{n,k} - R_{i,j,k|n} \bar{V}''_{n,k} \right] \\
 & - \left[\boxed{-2 R_{i,j|nk} \bar{V}''_{n,k}} + (\bar{V}''_{k,n} - \bar{V}'_{k,n}) R_{i,j|kn} \right] \\
 & \quad + \left[-R_{i,k} \bar{V}''_{j,k|n} \right] \\
 & - \left[R_{i,n|k} \bar{V}''_{j,nk} + R_{n,j|k} \bar{V}'_{i,nk} \boxed{-2 R_{i,n|k} \bar{V}''_{j,nk}} \right] \\
 & \quad + \left[T_{in,j,n}^t + (T_{i,jn}^t - T_{in,j}^t)|_n \right]_{,k|k} \\
 & - \left[\boxed{T_{in,j,n}^t + (T_{i,jn}^t - T_{in,j}^t)|_n} \right]_{|kk} \\
 & \quad + \left[-2 \nu \check{\xi}_{i,j|k,k} + \nu R_{i,j|n,n,kk} \right] \\
 & - \left[\boxed{-2 \nu \check{\xi}_{i,j|kk} + \nu R_{i,j|nn,kk}} \right]
 \end{aligned}$$

$$\begin{aligned}
 & - \left[R_{i,k} \bar{V}''_{j,knn} \right] \\
 & - \left[\boxed{-R_{i,k} \bar{V}''_{j,knn}} \right] \\
 & - \left[R_{i,j|k} \bar{V}''_{k,nn} \right] \\
 & - \left[\boxed{-R_{i,j|k} \bar{V}''_{k,nn}} \right]
 \end{aligned}$$

As in the notations adopted for the centered transformation, the terms that constitutes the transport equation for $R_{i,j|nn}$ are put into the boxes whereas the other ones come from the equation for $R_{i,j|n,n}$. These equations contain also the additional terms the contribution of which is not seen in the dissipation equation because their effects are compensated by each other. It should be pointed out here that in the second and third production term, the contribution that actually comes from the transport equation for $R_{i,j|n,n}$ does involve terms with $R_{i,j|pq}$.

The viscous contributions are clearly separated. The splitting of the dissipation and diffusion processes is here again straightforward, as it was in the centered transformation. The viscous diffusion can be written as :

$$d_{i,j}^{\varepsilon\nu} = \nu \check{\varepsilon}_{i,j,nn} \quad (\text{B.144})$$

It can be further rewritten as

$$d_{i,j}^{\varepsilon\nu} = [\nu R_{i,j,k|k,nn}] - [\nu R_{i,j|kk,nn}] \quad (\text{B.145})$$

where the first term is the viscous diffusion of $R_{i,j,k|k}$ and the second the viscous diffusion of $R_{i,j|kk}$.

The treatment of the dissipation operator is very much alike, since the contributions to $E_{i,j}^{\varepsilon}$ can be split into two dissipation processes, that of $R_{i,j,k|k}$ and of $R_{i,j|kk}$:

$$E_{i,j}^{\varepsilon} = [2\nu \check{\varepsilon}_{i,j,k|k}] - [2\nu \check{\varepsilon}_{i,j|kk}] \quad (\text{B.146})$$

The exploitation of this equation remains, however, less simple than the one obtained with the centered transformation, since the interpretation of the $R_{i,j,k|k}$ quantity and, moreover, of its transport equation, is not straightforward within the one-point context.

It should be pointed out that the entire viscous contribution to the $R_{i,j|nn}$ equation can be written here as :

$$-2\nu \check{\varepsilon}_{i,j|kk} + \nu R_{i,j|nn,kk} \quad (\text{B.147})$$

or, again

$$-2\nu R_{i,j|nnk,k} + 2\nu R_{i,j|kknn} + \nu R_{i,j|nn,kk} \quad (\text{B.148})$$

and that here no further simplification is possible, since none of the parts of the dissipation process has the same form as the viscous diffusion operator.

B.2.3 Evolution equation for each part of the dissipation tensor in the decentered transformation

This section presents the transport equations for $R_{i,j|nn}$ and $R_{i,j,n|n}$ obtained by the derivation of the transport equation for the two-point Reynolds stress tensor. With the decentered transformation, this equation can be written as :

$$\begin{aligned} \frac{\bar{d}R_{i,j}}{dt} &= -R_{i,k} \bar{V}_{j,k}'' - R_{k,j} \bar{V}_{i,k}' \\ &- 2\Omega_k [\varepsilon_{kmi} R_{m,j} + \varepsilon_{kmj} R_{i,m}] \\ &+ [T_{ik,j}^t]_{,k} \\ &+ [T_{i,jk}^t - T_{ik,j}^t]_{|k} \\ &+ \nu R_{i,j,kk} \\ &- 2\nu [R_{i,j,k|k} - R_{i,j|kk}] \\ &- [\bar{V}_k'' - \bar{V}_k'] R_{i,j|k} \end{aligned} \quad (\text{B.149})$$

where

$$\frac{\bar{d}}{dt} = \frac{\partial}{\partial t} + \bar{V}'_k \frac{\partial}{\partial X_k}$$

Double derivative part

The double derivation leads to :

$$\begin{aligned} & \frac{\partial R_{i,j|nn}}{\partial t} + \bar{V}'_k R_{i,j|nn,k} \\ + \bar{V}'_{k|nn} R_{i,j,k} + 2 \bar{V}'_{k|n} R_{i,j|n,k} &= -R_{i,k|nn} \bar{V}''_{j,k} - R_{k,j|nn} \bar{V}'_{i,k} \\ & - R_{i,k} \bar{V}''_{j,k|nn} - R_{k,j} \bar{V}'_{i,k|nn} \\ & - 2 R_{i,k|n} \bar{V}''_{j,k|n} - 2 R_{k,j|n} \bar{V}'_{i,k|n} \\ & - 2 \Omega_k [\varepsilon_{kmi} R_{m,j|nn} + \varepsilon_{kmj} R_{i,m|nn}] \\ & + [T_{ik,j}^t]_{,k|nn} \\ & + [T_{i,jk}^t - T_{ik,j}^t]_{,knn} \\ & + \nu R_{i,j|nn,kk} \\ & - 2 \nu [R_{i,j|nnk,k} - R_{i,j|kknn}] \\ & - [\bar{V}''_k - \bar{V}'_k] R_{i,j|knn} \\ & - [\bar{V}''_k - \bar{V}'_k]_{,nn} R_{i,j|k} \\ & - 2 [\bar{V}''_k - \bar{V}'_k]_{,n} R_{i,j|kn} \end{aligned} \tag{B.150}$$

The same interpretation of the terms as in the Appendix B can be given here. The mean velocity gradients can again be modified in order to simplify the interpretation, yielding :

$$\begin{aligned} \frac{\bar{d}R_{i,j|nn}}{dt} &= -R_{i,k|nn} \bar{V}''_{j,k} - R_{k,j|nn} \bar{V}'_{i,k} \\ & - R_{i,k} \bar{V}''_{j,knn} \\ & - 2 R_{i,k|n} \bar{V}''_{j,kn} \\ & - 2 \Omega_k [\varepsilon_{kmi} R_{m,j|nn} + \varepsilon_{kmj} R_{i,m|nn}] \\ & + [T_{ik,j}^t]_{,k|nn} \\ & + [T_{i,jk}^t - T_{ik,j}^t]_{,knn} \\ & + \nu R_{i,j|nn,kk} \\ & - 2 \nu [R_{i,j|nnk,k} - R_{i,j|kknn}] \\ & - [\bar{V}''_k - \bar{V}'_k] R_{i,j|knn} \\ & - \bar{V}''_{k,nn} R_{i,j|k} \\ & - 2 \bar{V}''_{k,n} R_{i,j|kn} \end{aligned} \tag{B.151}$$

Their simplification comes from the fact that any derivative of \bar{V}' with respect to \underline{r} is zero. This means that there are no additional terms originating from the derivation of

the material derivative, and that the contributions from the non-homogeneous convective terms are greatly simplified.

Cross derivative part

As a first step, the first derivative of the Reynolds stress transport equation with respect to \underline{r} can be written, using the transformed expressions of the mean velocity gradients, as :

$$\begin{aligned}
 \frac{\bar{d}R_{i,j|n}}{dt} &= -R_{i,k|n} \bar{V}''_{j,k} - R_{k,j|n} \bar{V}'_{i,k} \\
 &- R_{i,k} \bar{V}''_{j,kn} \\
 &- 2 \Omega_k [\varepsilon_{kmi} R_{m,j|n} + \varepsilon_{kmj} R_{i,m|n}] \\
 &+ [T_{ik,j}^t]_{,k|n} \\
 &+ [T_{i,jk}^t - T_{ik,j}^t]_{|kn} \\
 &+ \nu R_{i,j,kk|n} \\
 &- 2 \nu [R_{i,j,k|kn} - R_{i,j|kkn}] \\
 &- [\bar{V}''_k - \bar{V}'_k] R_{i,j|kn} \\
 &- \bar{V}''_{k,n} R_{i,j|k}
 \end{aligned} \tag{B.152}$$

Applying then the cross derivation leads to :

$$\begin{aligned}
 \frac{\bar{d}R_{i,j|n,n}}{dt} &= -\bar{V}'_{k,n} R_{i,j,k|n} \\
 &- R_{i,k|n,n} \bar{V}''_{j,k} - R_{k,j|n,n} \bar{V}'_{i,k} \\
 &- R_{i,k|n} \bar{V}''_{j,kn} - R_{k,j|n} \bar{V}'_{i,kn} \\
 &- R_{i,k,n} \bar{V}''_{j,kn} \\
 &- R_{i,k} \bar{V}''_{j,knn} \\
 &- 2 \Omega_k [\varepsilon_{kmi} R_{m,j|n,n} + \varepsilon_{kmj} R_{i,m|n,n}] \\
 &+ [T_{ik,j}^t]_{,k|n,n} \\
 &+ [T_{i,jk}^t - T_{ik,j}^t]_{|kn,n} \\
 &+ \nu R_{i,j,kkn|n} \\
 &- 2 \nu [R_{i,j,nk|kn} - R_{i,j|kkn,n}] \\
 &- [\bar{V}''_k - \bar{V}'_k] R_{i,j|kn,n} \\
 &- [\bar{V}''_k - \bar{V}'_k]_{,n} R_{i,j|kn} \\
 &- \bar{V}''_{k,nn} R_{i,j|k} \\
 &- \bar{V}''_{k,n} R_{i,j|k,n}
 \end{aligned} \tag{B.153}$$

Those two equations enable to obtain again the transport equation for the dissipation tensor described earlier in the decentered transformation system.

Appendix C

Application of the Fourier transform

This Appendix presents the notations used for the Fourier transform of the two-point equations and of the Poisson equation obtained by taking the divergence of the second order tensor.

C.1 Basic notations

$$\Phi_{ij}(\underline{X}, \underline{r}) = \frac{1}{(2\pi)^3} \int e^{-I \underline{k} \cdot \underline{r}} R_{i,j}(\underline{x}', \underline{x}'') d\underline{r} \quad (C.1)$$

The complex conjugate is denoted by a *, so that :

$$\Phi_{ij}^*(\underline{X}, \underline{k}) = \Phi_{ji}(\underline{X}, -\underline{k}) \quad (C.2)$$

The operations of gradient and multiplication by the coordinate transform is as follows :

$$\text{TF}\left[\frac{\partial}{\partial r_i}\right] = I k_i \quad (C.3a)$$

$$\text{TF}[r_i] = I \frac{\partial}{\partial k_i} \quad (C.3b)$$

Thus,

$$\text{TF}\left[\frac{\partial}{\partial x'_i}\right] = \frac{1}{2} \frac{\partial}{\partial X_i} - I k_i \quad (C.4a)$$

$$\text{TF}\left[\frac{\partial}{\partial x''_i}\right] = \frac{1}{2} \frac{\partial}{\partial X_i} + I k_i \quad (C.4b)$$

In a condensed notation, these derivations are denoted as follows :

$$\nabla_i^* = \frac{1}{2} \frac{\partial}{\partial X_i} - I k_i \quad (C.5a)$$

$$\nabla_i = \frac{1}{2} \frac{\partial}{\partial X_i} + I k_i \quad (C.5b)$$

C.2 Expansion of the velocity components

In the physical space the Taylor expansion of the velocity components can be written using the following operators :

$$\bar{V}'_i = \exp\left(-\frac{1}{2} r_n \frac{\partial}{\partial X_n}\right) \bar{V}_i \quad (C.6a)$$

$$\bar{V}''_i = \exp\left(\frac{1}{2} r_n \frac{\partial}{\partial X_n}\right) \bar{V}_i \quad (C.6b)$$

Their transformed expression in the spectral space are written by means of the transformed operators defined by :

$$\Sigma(\underline{X}, -\underline{k}) = \exp\left(-\frac{I}{2} \frac{\partial^{\bar{V}}}{\partial X_n} \frac{\partial}{\partial k_n}\right) \quad (C.7a)$$

$$\Sigma(\underline{X}, \underline{k}) = \exp\left(\frac{I}{2} \frac{\partial^{\bar{V}}}{\partial X_n} \frac{\partial}{\partial k_n}\right) \quad (C.7b)$$

The second order expansion of the Σ operator lead to :

$$\Sigma(\underline{X}, \underline{k}) = 1 + \frac{I}{2} \frac{\partial}{\partial k_n} \frac{\partial^{\bar{V}}}{\partial X_n} - \frac{1}{8} \frac{\partial^2}{\partial k_n \partial k_m} \frac{\partial^{2\bar{V}}}{\partial X_n \partial X_m} + O(3) \quad (C.8a)$$

Its conjugate form can then be written as :

$$\Sigma(\underline{X}, \underline{k})^* = 1 - \frac{I}{2} \frac{\partial}{\partial k_n} \frac{\partial^{\bar{V}}}{\partial X_n} - \frac{1}{8} \frac{\partial^2}{\partial k_n \partial k_m} \frac{\partial^{2\bar{V}}}{\partial X_n \partial X_m} + O(3) \quad (C.8b)$$

The superscript \bar{V} is here introduced to remind that the derivation with respect to X_n applies only to the mean velocity components. Hence, the expansion leads to the following expressions :

$$2 \Sigma \bar{V}_n \Phi_{im} = 2 \bar{V}_n \Phi_{im} + I \bar{V}_{n,l} \frac{\partial \Phi_{im}}{\partial k_l} - \frac{1}{4} \bar{V}_{n,pq} \frac{\partial^2 \Phi_{im}}{\partial k_p \partial k_q} + O(3) \quad (C.9a)$$

$$2 \Sigma^* \bar{V}_n \Phi_{mj} = 2 \bar{V}_n \Phi_{mj} - I \bar{V}_{n,l} \frac{\partial \Phi_{mj}}{\partial k_l} - \frac{1}{4} \bar{V}_{n,pq} \frac{\partial^2 \Phi_{mj}}{\partial k_p \partial k_q} + O(3) \quad (C.9b)$$

These expansions are to be used for the derivation of the pressure-velocity terms.

C.3 Expression for the pressure-velocity correlations

The pressure-velocity correlations can be obtained in terms of the spectral tensor and the triple correlations through the derivation of the Poisson equations, obtained by taking the divergence of the second order tensor equation. This equation can then be solved either in the spectral space (Laporta, 1995) or in the physical space (Besnard *et al.*, 1992). The expression for the solutions is recalled in the following paragraphs.

C.3.1 Expression in the physical space

The two-point evolution equation for the Reynolds stresses can be written as :

$$\begin{aligned}
 R_{i,j,t} + \overline{V}'_k R_{i,j,k'} + \overline{V}''_k R_{i,j,k''} &= -R_{i,k} \overline{V}''_{j,k''} - R_{k,j} \overline{V}'_{i,k'} \\
 &+ T_{i,jk,k''}^u + T_{ik,j,k'}^u \\
 &- \frac{1}{\rho} [\overline{u'_i p''_{j''}} + \overline{u''_j p'_{i'}}] \\
 &+ \nu [R_{i,j,k'k'} + R_{i,j,k''k''}]
 \end{aligned} \tag{C.10}$$

When the divergence is taken relative to the first point, the resulting Poisson equation can be simplified to :

$$[2\overline{V}'_i R_{k,j} - T_{ik,j}^u]_{i'k'} = -\frac{1}{\rho} \overline{u''_j p'_{i'j'}} \tag{C.11a}$$

and relative to the second point :

$$[2\overline{V}''_j R_{i,k} - T_{i,jk}^u]_{j''k''} = -\frac{1}{\rho} \overline{u'_i p''_{j''j''}} \tag{C.11b}$$

These Poisson equations can be solved in the physical space using Green functions and then the spectral tensor of the pressure-velocity correlations is obtained by a Fourier transform of the solution. This procedure has been used by Besnard *et al.* (1992). The alternative procedure consists in solving the Poisson equation in the spectral space ; the latter approach is usually followed when the turbulence is homogeneous. This solution is more straightforward than the previous one, however it requires a careful treatment of the transformation in order to simplify the calculations (Laporta, 1995). This procedure is outlined first, and after that we recall the previous approach.

C.3.2 Solution of the Poisson equation in the spectral space

As for the Reynolds stress transport equation, it is more convenient to apply the Fourier transform after the independent variables (\underline{x}' , \underline{x}'') have been transformed into a set of (\underline{X} , \underline{r}) variables. In this way we can better differentiate the local from non-local contributions.

It appears here, however, that the centered transformation

$$X_k = \frac{1}{2}(x'_k + x''_k) \tag{C.12a}$$

$$r_k = (x''_k - x'_k) \tag{C.12b}$$

is not simple. The double derivation involves all kinds of second derivatives with respect to \underline{x} and \underline{r} .

$$\frac{\partial^2}{\partial x'_m \partial x'_n} = \frac{1}{4} \cdot_{,mn} - \frac{1}{2} [\cdot_{,m|n} + \cdot_{|m,n}] + \cdot_{|mn} \tag{C.13a}$$

$$\frac{\partial^2}{\partial x''_m \partial x''_n} = \frac{1}{4} \cdot_{,mn} + \frac{1}{2} [\cdot_{,m|n} + \cdot_{|m,n}] + \cdot_{|mn} \tag{C.13b}$$

In contrast, the decentered transformation

$$X_k = x'_k \quad (C.14a)$$

$$r_k = (x''_k - x'_k) \quad (C.14b)$$

leads to a simple expression, because

$$\frac{\partial^2 \cdot}{\partial x'_m \partial x'_n} = \cdot_{,mn} - [\cdot_{,m|n} + \cdot_{|m,n}] + \cdot_{|mn} \quad (C.15a)$$

$$\frac{\partial^2 \cdot}{\partial x''_m \partial x''_n} = \cdot_{|mn} \quad (C.15b)$$

As the quantities are conjugate in the spectral space, it suffices to focus on one of them. The decentered transformation allows to work with the second Poisson equation :

$$[2\bar{V}''_j R_{i.k} - T_{i.jk}^u]_{j''k''} = -\frac{1}{\rho} \overline{u'_i p''}_{,j''j''} \quad (C.16a)$$

whose transformation with the decentered coordinates can be written as :

$$[2\bar{V}''_j R_{i.k} - T_{i.jk}^u]_{|jk} = -\frac{1}{\rho} \overline{u'_i p''}_{|jj} \quad (C.16b)$$

It is noted that the formulation of boundary conditions associated with the partial differential equations depends on the physical boundaries. In the spectral space the boundary conditions are implicitly written in the equations through a suitable choice of the Fourier transform. It has been assumed here that there is no wall so that the usual definition of the Fourier transform can be applied. The transformation of the Poisson equation can then be written as :

$$k_j k_k [2\tilde{V}''_j \tilde{\Phi}_{ik} - \tilde{T}_{i.jk}^u] = -k^2 \frac{1}{\rho} \widehat{u'_i p''} \quad (C.16c)$$

where the $\tilde{\cdot}$ denotes the decentered Fourier transform of the variables. Assuming that the boundary conditions are zero at infinity, the solution to this equation can be written as

$$\begin{aligned} \tilde{\Pi}_i^* &= \left[-\frac{1}{\rho} \widehat{u'_i p''} \right] \\ &= \frac{k_j k_k}{k^2} [2\tilde{V}''_j \tilde{\Phi}_{ik} - \tilde{T}_{i.jk}^u] \end{aligned} \quad (C.17a)$$

and, using the decentered Taylor expansion operator,

$$\tilde{\Pi}_i^* = \frac{k_j k_k}{k^2} \left[2 \exp\left(I \frac{\partial \cdot}{\partial k_m} \frac{\partial \tilde{V} \cdot}{\partial X_m}\right) \bar{V}_j \tilde{\Phi}_{ik} - \tilde{T}_{i.jk}^u \right] \quad (C.18a)$$

This is the decentered quantity, linked to the centered one by the following transformation :

$$\Pi_i^* = \exp\left(-\frac{I}{2} \frac{\partial \cdot}{\partial k_m} \frac{\partial \cdot}{\partial X_m}\right) \tilde{\Pi}_i^* \quad (C.19)$$

so that the quantity wanted can be written as :

$$\begin{aligned} \Pi_i^* &= \exp\left(-\frac{I}{2} \frac{\partial}{\partial k_m} \frac{\partial}{\partial X_m}\right) \left[\frac{k_m k_n}{k^2} 2 \exp\left(I \frac{\partial}{\partial k_m} \left\{ \frac{\partial \bar{V}}{\partial X_m} + \frac{1}{2} \frac{\partial \bar{\Phi}}{\partial X_m} \right\}\right) \bar{V}_m \Phi_{in} \right] \\ &- \exp\left(-\frac{I}{2} \frac{\partial}{\partial k_m} \frac{\partial}{\partial X_m}\right) \left[\frac{k_m k_n}{k^2} \exp\left(\frac{I}{2} \frac{\partial}{\partial k_m} \frac{\partial}{\partial X_m}\right) T_{i.nm} \right] \end{aligned} \quad (C.20a)$$

Its conjugate form leads to the expression for Π_j :

$$\begin{aligned} \Pi_j &= \exp\left(\frac{I}{2} \frac{\partial}{\partial k_m} \frac{\partial}{\partial X_m}\right) \left[\frac{k_m k_n}{k^2} 2 \exp\left(-I \frac{\partial}{\partial k_m} \left\{ \frac{\partial \bar{V}}{\partial X_m} + \frac{1}{2} \frac{\partial \bar{\Phi}}{\partial X_m} \right\}\right) \bar{V}_m \Phi_{nj} \right] \\ &- \exp\left(\frac{I}{2} \frac{\partial}{\partial k_m} \frac{\partial}{\partial X_m}\right) \left[\frac{k_m k_n}{k^2} \exp\left(-\frac{I}{2} \frac{\partial}{\partial k_m} \frac{\partial}{\partial X_m}\right) T_{nm.j} \right] \end{aligned} \quad (C.20b)$$

Various operators remain to be derived in order to get the expanded expressions of the pressure-velocity terms.

C.3.3 Solution of the Poisson equation in the physical space

The two possible Poisson equations resulting from the divergence of the Reynolds stress transport in the two-point physical description of the turbulence can be written as :

$$[2 \bar{V}'_i R_{k.j} - T_{ik.j}]_{i'k'} = -\frac{1}{\rho} \overline{u'_j p'}_{,i'i'} \quad (C.21a)$$

$$[2 \bar{V}''_j R_{i.k} - T_{i.jk}]_{j''k''} = -\frac{1}{\rho} \overline{u'_i p''}_{,j''j''} \quad (C.21b)$$

Using the Green theorem leads directly to their integration :

$$-\frac{1}{\rho} \overline{u'_j p'}(\underline{x}', \underline{x}'') = -\frac{1}{4\pi} \int G(\underline{y}, \underline{x}'') \frac{\partial}{\partial y_i} \frac{\partial}{\partial y_k} [2 \bar{V}'_i(\underline{y}) R_{k.j}(\underline{y}, \underline{x}'') - T_{ik.j}(\underline{y}, \underline{x}'')] d\underline{y} \quad (C.22a)$$

$$-\frac{1}{\rho} \overline{u'_i p''}(\underline{x}', \underline{x}'') = -\frac{1}{4\pi} \int G(\underline{x}', \underline{y}) \frac{\partial}{\partial y_j} \frac{\partial}{\partial y_k} [2 \bar{V}'_j(\underline{y}) R_{i.k}(\underline{x}', \underline{y}) - T_{i.jk}(\underline{x}', \underline{y})] d\underline{y} \quad (C.22b)$$

where \underline{y} is the current point of evaluation of the integrals and the surface integrals have been neglected assuming the boundary conditions to be zero at the infinity. The treatment of such integrals has already be discussed by Chou (1945). Assuming local homogeneity, the Green function can take the following form :

$$G(\underline{x}', \underline{x}'') = \frac{1}{\|\underline{x}' - \underline{x}''\|} \quad (C.23)$$

Again, it suffices to focus on only one of the pressure-velocity correlation. Following the same approach as earlier, we focus our attention on the second equality. This can be transformed into the Fourier space yielding (Besnard *et al.*, 1992) :

$$\begin{aligned} \Pi_i^*(\underline{X}, \underline{k}) &= -\nabla_j(\underline{X}, \underline{k}) \nabla_k(\underline{X}, \underline{k}) \cdot \\ &\frac{1}{4\pi} \int \hat{G}(\underline{X} - \underline{y}, \underline{k}) [2 \Sigma(\underline{y}, \underline{k}) \bar{V}_j(\underline{y}) \Phi_{ik}(\underline{y}, \underline{k}) - T_{i.jk}(\underline{y}, \underline{k})] \end{aligned} \quad (C.24)$$

The transformed Green function is :

$$-\nabla_n \nabla_n \hat{G}(\underline{X} - \underline{y}, \underline{k}) = \delta(\underline{X} - \underline{y}) \quad (C.25)$$

The operator can then be written as :

$$\Pi_i^*(\underline{X}, \underline{k}) = -\nabla_j \nabla_k G [2 \Sigma \bar{V}_j \Phi_{ik} - T_{i,jk}] \quad (C.26)$$

and the Green function here is defined by the following inverse :

$$G = \left[-\left(\frac{1}{2} \frac{\partial}{\partial X_n} + I k_n \right)^2 \right]^{-1} \quad (C.27)$$

It is then clear that the pressure-velocity correlations can be written as :

$$\Pi_i^*(\underline{X}, \underline{k}) = -\nabla_m \nabla_n G [2 \Sigma \bar{V}_n \Phi_{im} - T_{i,mn}] \quad (C.28a)$$

$$\Pi_j(\underline{X}, \underline{k}) = -\nabla_m^* \nabla_n^* G^* [2 \Sigma^* \bar{V}_n \Phi_{mj} - T_{mn,j}] \quad (C.28b)$$

The Taylor expansion of these expressions remains to be done.

C.4 Expansion of the operators involved in the Poisson equations

The Green function is defined as :

$$G = \left[-\left(\frac{1}{2} \frac{\partial}{\partial X_n} + I k_n \right)^2 \right]^{-1} \quad (C.29)$$

so that its expansion can be expressed as :

$$G = \frac{1}{k^2} \left[1 + I \frac{k_l}{k^2} \frac{\partial}{\partial X_l} + \frac{1}{4k^2} \left(\delta_{ls} - 4 \frac{k_l k_s}{k^2} \right) \frac{\partial^2}{\partial X_l \partial X_s} \right] + O(3) \quad (C.30a)$$

and its conjugate form can be written as :

$$G^* = \frac{1}{k^2} \left[1 - I \frac{k_l}{k^2} \frac{\partial}{\partial X_l} + \frac{1}{4k^2} \left(\delta_{ls} - 4 \frac{k_l k_s}{k^2} \right) \frac{\partial^2}{\partial X_l \partial X_s} \right] + O(3) \quad (C.30b)$$

The grouping of the derivative operators leads to the following expression :

$$\begin{aligned} \nabla_j \nabla_m \nabla_n &= -I k_j k_m k_n & (C.31a) \\ &- \frac{1}{2} \left(k_m k_n \frac{\partial}{\partial X_j} + k_j k_m \frac{\partial}{\partial X_n} + k_j k_n \frac{\partial}{\partial X_m} \right) \\ &+ \frac{1}{4} I \left(k_n \frac{\partial^2}{\partial X_j \partial X_m} + k_m \frac{\partial^2}{\partial X_j \partial X_n} + k_j \frac{\partial^2}{\partial X_n \partial X_m} \right) \\ &+ O(3) \end{aligned}$$

and the conjugate form :

$$\begin{aligned}
 \nabla_i^* \nabla_m^* \nabla_n^* &= I k_i k_m k_n & (C.31b) \\
 &- \frac{1}{2} \left(k_m k_n \frac{\partial}{\partial X_i} + k_i k_m \frac{\partial}{\partial X_n} + k_i k_n \frac{\partial}{\partial X_m} \right) \\
 &- \frac{1}{4} I \left(k_n \frac{\partial^2}{\partial X_i \partial X_m} + k_m \frac{\partial^2}{\partial X_i \partial X_n} + k_i \frac{\partial^2}{\partial X_n \partial X_m} \right) \\
 &+ O(3)
 \end{aligned}$$

The combination of these expansions yields the following operators :

$$\begin{aligned}
 \nabla_j \nabla_m \nabla_n G &= -I k_j \frac{k_m k_n}{k^2} & (C.32a) \\
 &+ \frac{k_j k_m k_n k_l}{k^4} \frac{\partial}{\partial X_l} \\
 &- \frac{1}{2} \left(\frac{k_m k_n}{k^2} \frac{\partial}{\partial X_j} + \frac{k_j k_m}{k^2} \frac{\partial}{\partial X_n} + \frac{k_j k_n}{k^2} \frac{\partial}{\partial X_m} \right) \\
 &+ \frac{I}{4} \frac{1}{k^2} \left(k_n \frac{\partial^2}{\partial X_j \partial X_m} + k_m \frac{\partial^2}{\partial X_j \partial X_n} + k_j \frac{\partial^2}{\partial X_n \partial X_m} \right) \\
 &- \frac{I}{2} \frac{k_l}{k^2} \left(\frac{k_m k_n}{k^2} \frac{\partial^2}{\partial X_j \partial X_l} + \frac{k_j k_m}{k^2} \frac{\partial^2}{\partial X_n \partial X_l} + \frac{k_j k_n}{k^2} \frac{\partial^2}{\partial X_m \partial X_l} \right) \\
 &- \frac{I}{4} \frac{1}{k^4} \left(\delta_{ls} - 4 \frac{k_l k_s}{k^2} \right) k_j k_m k_n \frac{\partial^2}{\partial X_l \partial X_s} \\
 &+ O(3)
 \end{aligned}$$

and their conjugate form :

$$\begin{aligned}
 \nabla_i^* \nabla_m^* \nabla_n^* G^* &= I k_i \frac{k_m k_n}{k^2} & (C.32b) \\
 &+ \frac{k_i k_m k_n k_l}{k^4} \frac{\partial}{\partial X_l} \\
 &- \frac{1}{2} \left(\frac{k_m k_n}{k^2} \frac{\partial}{\partial X_i} + \frac{k_i k_m}{k^2} \frac{\partial}{\partial X_n} + \frac{k_i k_n}{k^2} \frac{\partial}{\partial X_m} \right) \\
 &- \frac{I}{4} \frac{1}{k^2} \left(k_n \frac{\partial^2}{\partial X_i \partial X_m} + k_m \frac{\partial^2}{\partial X_i \partial X_n} + k_i \frac{\partial^2}{\partial X_n \partial X_m} \right) \\
 &+ \frac{I}{2} \frac{k_l}{k^2} \left(\frac{k_m k_n}{k^2} \frac{\partial^2}{\partial X_i \partial X_l} + \frac{k_i k_m}{k^2} \frac{\partial^2}{\partial X_n \partial X_l} + \frac{k_i k_n}{k^2} \frac{\partial^2}{\partial X_m \partial X_l} \right) \\
 &+ \frac{I}{4} \frac{1}{k^4} \left(\delta_{ls} - 4 \frac{k_l k_s}{k^2} \right) k_i k_m k_n \frac{\partial^2}{\partial X_l \partial X_s} \\
 &+ O(3)
 \end{aligned}$$

Appendix D

Two-point equations and one-point equations

This appendix summarizes the physical interpretation of various terms in the two-point second moment equation.

D.1 Reynolds stress equation

D.1.1 Expression in the one-point limit

The reduction of the two-point equations to the one-point limit is straightforward in this case. The expression for the mean velocity has to be expanded with respect to the distance \underline{r} . As these are one point quantities, we get :

$$\overline{V}'_k = \overline{V}_k - \frac{1}{2} r_n \overline{V}_{k,n} + O(r^2) \quad (D.1a)$$

$$\overline{V}''_k = \overline{V}_k + \frac{1}{2} r_n \overline{V}_{k,n} + O(r^2) \quad (D.1b)$$

Convective terms

The expansion of $C_{i,j}$ and $C_{i,j}^{\cdot\cdot}$ can be written as :

$$C_{i,j} = \overline{V}_k R_{ij,k} + O(r) \quad (D.2a)$$

$$C_{i,j}^{\cdot\cdot} = r_n \overline{V}_{k,n} (R_{i,j|k})_0 + O(r^2) \quad (D.2b)$$

Production terms

The expansion of the production terms is also straightforward and leads to :

$$P_{i,j} = -R_{ik} \overline{V}_{j,k} - R_{jk} \overline{V}_{i,k} + O(r) \quad (D.3)$$

Triple velocity correlation tensor

For the same reason the triple velocity correlations in the one-point limit are :

$$d_{i,j}^u = T_{ijk,k}^u + O(r) \quad (D.4a)$$

$$d_{i,j}^{u..} = O(r) \quad (D.4b)$$

since

$$T_{ik,j|k}^u - T_{i,jk|k}^u = \overline{(u'_i u''_j)_{|k} (u''_k - u'_k)} + \overline{u'_i u''_j (u''_k - u'_k)_{|k}} \quad (D.5)$$

Pressure-velocity tensor

By definition

$$T_{i,jk,k}^p = -\frac{1}{\rho} \overline{[p'' u'_i \delta_{jn}]_{,n}} \quad (D.6a)$$

$$T_{ik,j,k}^p = -\frac{1}{\rho} \overline{[p'' u'_j \delta_{in}]_{,n}} \quad (D.6b)$$

This notation comes from the fact that the derivative of a quantity in one of the two points with respect to the other point does not need to be taken into account. This tensor corresponds to the one point correlation between the fluctuating velocity and pressure gradient, as follows from the expansion :

$$\frac{1}{2} [T_{i,jk,k}^p + T_{ik,j,k}^p] = -\frac{1}{2} \frac{1}{\rho} \overline{[p u_i \delta_{jn} + p u_j \delta_{in}]_{,n}} + O(r) \quad (D.7a)$$

$$T_{i,jk|k}^p - T_{ik,j|k}^p = -\frac{1}{2} \frac{1}{\rho} \overline{[p u_i \delta_{jn} + p u_j \delta_{in}]_{,n}} + \frac{1}{\rho} \overline{[p(u_{i,j} + u_{j,i})]} + O(r) \quad (D.7b)$$

and it can be reduced to :

$$\frac{1}{2} [T_{i,jk,k}^p + T_{ik,j,k}^p] = \frac{1}{2} d_{ij}^p + O(r) \quad (D.8a)$$

$$T_{i,jk|k}^p - T_{ik,j|k}^p = \frac{1}{2} d_{ij}^p + \phi_{ij} + O(r) \quad (D.8b)$$

The sum leads to :

$$\frac{1}{2} [T_{i,jk,k}^p + T_{ik,j,k}^p] + T_{i,jk|k}^p - T_{ik,j|k}^p = d_{ij}^p + \phi_{ij} + O(r) \quad (D.9)$$

being

$$\frac{1}{2} [T_{i,jk,k}^p + T_{ik,j,k}^p] + T_{i,jk|k}^p - T_{ik,j|k}^p = \Phi_{ij} + O(r) \quad (D.10)$$

Viscosity contribution

The viscous diffusion can be written as :

$$d_{i,j}^\nu = \nu R_{ij,kk} + O(r) \quad (D.11)$$

whereas the dissipation part may be written as :

$$\check{\varepsilon}_{i,j} = \frac{1}{4} R_{ij,kk} - (R_{i.j|kk})_0 + O(r) \quad (D.12)$$

The one-point dissipation tensor can be interpreted as to be composed of an inhomogeneous term of the same mathematical form as the viscous diffusion process, and of an homogeneous contribution which comes from the two-point description.

D.1.2 Remarks

There is no need to recall here the one-point equation for the Reynolds stress tensor. The notations are described in Appendix A. It is, however, interesting to point out that the difference of the derivatives with respect to \underline{r} of the pressure transport term does not entirely vanish in the one-point limit, but contributes to the pressure redistribution and the pressure diffusion tensors.

The splitting of $\varepsilon_{i,j}$ is the main reason why the centered transformation seems to be the most appropriate description of the two-point processes.

D.2 Dissipation equation

D.2.1 Expression in the one-point limit

As for the two-point Reynolds stress tensor, the one-point equation for the dissipation tensor is here also presented in terms of the one-point limit of its two-point value.

Convective terms

The development of the convective term is obvious and similar to the corresponding development in the $R_{i,j}$ equation.

$$C_{i,j}^\varepsilon = \bar{V}_k \check{\varepsilon}_{ij,k} + O(r) \quad (D.13a)$$

$$C_{i,j}^{\varepsilon..} = O(r) \quad (D.13b)$$

Production terms

The $P_{i'.j''}^{\varepsilon 1}$ term

This production term strictly corresponds to the one point $P_{ij}^{\varepsilon 1}$ tensor.

$$P_{i'.j''}^{\varepsilon 1} = P_{ij}^{\varepsilon 1} + O(r) \quad (D.14)$$

Its expression in terms of the Reynolds stress tensor leads to :

$$\begin{aligned}
 P_{i'.j''}^{\varepsilon 1} &= \frac{1}{4} [-R_{nj,kk} \bar{V}_{i,n} - R_{in,kk} \bar{V}_{j,n}] \\
 &- [-(R_{n.j|kk})_0 \bar{V}_{i,n} - (R_{i.n|kk})_0 \bar{V}_{j,n}] + O(r)
 \end{aligned} \tag{D.15}$$

where $(.)_0$ denotes the limit of the two-point expression when r goes to zero.

The $P_{i'.j''}^{\varepsilon 2}$ term

In this expression, the third two-point contribution entirely vanishes in the one-point limit, so that the remaining terms are :

$$\begin{aligned}
 P_{i'.j''}^{\varepsilon 2} &= \frac{1}{4} [-R_{ij,nk} \bar{V}_{n,k} - R_{ij,kn} \bar{V}_{n,k}] \\
 &- [-(R_{i.j|nk})_0 \bar{V}_{n,k} - (R_{i.j|kn})_0 \bar{V}_{n,k}] + O(r)
 \end{aligned} \tag{D.16}$$

The $P_{i'.j''}^{\varepsilon 3}$ term

As compared with the previous production contribution, this inhomogeneous term does not have a two-point counterpart and its limit is directly obtained :

$$P_{i'.j''}^{\varepsilon 3} = \frac{1}{2} [-R_{nj,k} \bar{V}_{i,nk} - R_{n,k} \bar{V}_{j,nk}] \tag{D.17}$$

$$- [(R_{i.n|k})_0 \bar{V}_{j,nk} - (R_{n.j|k})_0 \bar{V}_{i,nk}] + O(r) \tag{D.18}$$

It should, however, be pointed out that these two terms do not exactly correspond to the expression for the inhomogeneous production of $R_{ij,kk}$ and $(R_{i.j|kk})_0$, respectively, as their inhomogeneous production does include additional terms. Those terms are not typical two-point contributions and they are present in the one-point expression. However, they are identical in both equations and vanish in the equation for the dissipation tensor.

The $P_{i'.j''}^{\varepsilon 4}$ term

This term is no longer treated separately from the turbulent transport term and the fluctuating pressure transport term.

Triple velocity correlation tensor

This term is included in the transport term described below.

Pressure-velocity tensor

This is the last contribution to the transport term which can be written as :

$$d_{i'.j''}^{\varepsilon} = P_{i'.j''}^{\varepsilon 4} + d_{i'.j''}^{\varepsilon u} + \Pi_{i'.j''}^{\varepsilon} \tag{D.19}$$

or again :

$$\begin{aligned} d_{i'.j''}^{\varepsilon} &= \frac{1}{4} \left[\frac{1}{2} (T_{i.jn} + T_{in.j})_{,nkk} + (T_{i.jn} - T_{in.j})_{|n,kk} \right] \\ &- \left[\frac{1}{2} (T_{i.jn} + T_{in.j})_{,n|kk} + (T_{i.jn} - T_{in.j})_{|nkk} \right] \end{aligned} \quad (D.20)$$

with the same notations as for the Reynolds stress equation,

$$T_{i.jk} = -\overline{u'_i u'_j u''_k} - \frac{1}{\rho} \overline{u'_i p''} \delta_{jk} \quad (D.21a)$$

$$T_{ik.j} = -\overline{u'_i u'_k u''_j} - \frac{1}{\rho} \overline{u''_j p'} \delta_{ik} \quad (D.21b)$$

Without further consideration, the simplest way is to keep the previous form, so that in the one-point limit the transport term can be written as :

$$\begin{aligned} d_{i'.j''}^{\varepsilon} &= \frac{1}{4} \left(\left[\frac{1}{2} (T_{i.jn} + T_{in.j})_{,nkk} + (T_{i.jn} - T_{in.j})_{|n,kk} \right]_0 \right. \\ &- \left. \left[\frac{1}{2} (T_{i.jn} + T_{in.j})_{,n|kk} + (T_{i.jn} - T_{in.j})_{|nkk} \right]_0 \right) \end{aligned} \quad (D.22)$$

Viscosity contribution

The viscous contribution consist of the viscous diffusion and dissipation. When these term are expressed in terms of the Reynolds stress tensor, the viscous term in the one-point limit can be written as :

$$d_{i.j}^{\varepsilon\nu} - E_{i.j}^{\varepsilon} = \nu \left[\frac{1}{8} R_{ij,kknn} - 2 (R_{i.j|kknn})_0 \right] + O(r) \quad (D.23)$$

with

$$\begin{aligned} d_{i.j}^{\varepsilon\nu} &= \nu \check{\varepsilon}_{ij,nn} + O(r) \\ &= \nu \left[\frac{1}{4} R_{ij,kknn} - (R_{i.j|kk,nn})_0 \right] + O(r) \end{aligned} \quad (D.24a)$$

$$E_{i.j}^{\varepsilon} = 2\nu \left[\frac{1}{16} R_{ij,kknn} - \frac{1}{2} (R_{i.j|kk,nn})_0 + (R_{i.j|kknn})_0 \right] + O(r) \quad (D.24b)$$

D.2.2 Remarks

The one-point transport equation for the so-called homogeneous part of $\check{\varepsilon}_{ij}$, defined here by :

$$\check{\varepsilon}_{ij}^h = \check{\varepsilon}_{ij} - \frac{1}{4} R_{ij,kk} \quad (D.25)$$

can be derived by taking the one-point limit of the two-point equation of $R_{i.j|kk}$:

$$\begin{aligned}
 \frac{\bar{d}(R_{i.j|nn})_0}{dt} &= -\frac{1}{4}\bar{V}_{k,nn}R_{ij,k} & (D.26) \\
 &- (R_{i.k|nn})_0\bar{V}_{j,k} - (R_{k.j|nn})_0\bar{V}_{i,k} \\
 &- \frac{1}{4}R_{ik}\bar{V}_{j,knn} - \frac{1}{4}R_{kj}\bar{V}_{i,knn} \\
 &- [(R_{i.k|n})_0\bar{V}_{j,kn} - (R_{k.j|n})_0\bar{V}_{i,kn}] \\
 &- 2\Omega_k[\varepsilon_{kmi}(R_{m.j|nn})_0 + \varepsilon_{kmj}(R_{i.m|nn})_0] \\
 &+ \frac{1}{2}[[T_{i.jk} + T_{ik.j}]_{|nn})_0],k \\
 &+ ([T_{i.jk} - T_{ik.j}]_{|knn})_0 \\
 &+ \nu[(R_{i.j|nn})_0],kk \\
 &- 2\nu(\check{\xi}_{i.j|nn})_0 \\
 &- 2\bar{V}_{k,n}(R_{i.j|kn})_0
 \end{aligned}$$

This equation can now be rewritten as :

$$\begin{aligned}
 \frac{\bar{d}\check{\xi}_{ij}^h}{dt} &= \frac{1}{4}\bar{V}_{k,nn}R_{ij,k} & (D.27) \\
 &- \check{\xi}_{ik}^h\bar{V}_{j,k} - \check{\xi}_{kj}^h\bar{V}_{i,k} \\
 &+ \frac{1}{4}R_{ik}\bar{V}_{j,knn} + \frac{1}{4}R_{kj}\bar{V}_{i,knn} \\
 &+ [(R_{i.k|n})_0\bar{V}_{j,kn} - (R_{k.j|n})_0\bar{V}_{i,kn}] \\
 &- 2\Omega_k[\varepsilon_{kmi}\check{\xi}_{mj}^h + \varepsilon_{kmj}\check{\xi}_{im}^h] \\
 &- \frac{1}{2}[[T_{i.jk} + T_{ik.j}]_{|nn})_0],k \\
 &- ([T_{i.jk} - T_{ik.j}]_{|knn})_0 \\
 &+ \nu[\check{\xi}_{ij}^h],kk \\
 &- \frac{1}{2}\nu[\check{\xi}_{ij}^k],kk - 2\nu(R_{i.j|nnkk})_0 \\
 &+ 2\bar{V}_{k,n}(R_{i.j|kn})_0
 \end{aligned}$$

Appendix E

Spectral equations and the one-point equations

In this Appendix we summarize the physical interpretation of various terms in the spectral second moment equation. Once integrated over the wave number space, this equation leads to the Reynolds stress equation in the physical space.

E.1 Reynolds stress equation

The transport equation for the second moment in the physical space can be written as :

$$\begin{aligned}
 R_{ij,t} &+ \overline{V}_k R_{ij,k} = & \nu R_{ij,kk} & - & 2\nu \overline{u_{i,k}u_{j,k}} \\
 \text{unsteadiness} & \text{convection} & \text{viscous diffusion} & & \text{viscous dissipation} \\
 & & & - & \overline{V}_{j,k} R_{ik} - \overline{V}_{i,k} R_{kj} \\
 & & & & \text{production by mean velocity gradients} \\
 & & + \frac{1}{\rho} \overline{p(u_{i,j} + u_{j,i})} & & \\
 & & \text{redistribution (pressure strain term)} & & \\
 & + \left[\begin{array}{cc} T_{ijm}^u & + \\ & T_{ijm}^p \end{array} \right]_{,m} \\
 & & \text{turbulent diffusion} & & \text{pressure diffusion}
 \end{aligned}$$

with

$$T_{ijk}^u = -\overline{u_i u_j u_k} \tag{E.2a}$$

$$T_{ijk}^p = -\frac{1}{\rho} \overline{p(u_i \delta_{jk} + u_j \delta_{ik})} \tag{E.2b}$$

The transformed equation in the spectral space can be written as :

$$\begin{aligned}
 \Phi_{ij,t} + \frac{1}{2} [\Sigma + \Sigma^*] \bar{V}_k \Phi_{ij,k} &= \nu \Phi_{ij,kk} - 2\nu \left[\frac{1}{4} \Phi_{ij,kk} + k^2 \Phi_{ij} \right] \\
 \text{unsteadiness} \quad \text{convection} \quad \text{viscous diffusion} \quad \text{viscous dissipation} & \\
 &- [\Sigma - \Sigma^*] \bar{V}_k (I k_k \Phi_{ij}) \\
 &\quad \text{convection in the spectral space} \\
 &- [\Sigma \bar{V}_j]_{,k} \Phi_{ik} - [\Sigma^* \bar{V}_i]_{,k} \Phi_{kj} \\
 &\quad \text{production by mean velocity gradients} \\
 &+ \nabla_i^* \Pi_j + \nabla_j \Pi_i^* \\
 &\quad \text{pressure-velocity terms} \\
 &+ \nabla_m^* T_{im,j} + \nabla_m T_{i,jm} \\
 &\quad \text{turbulent diffusion}
 \end{aligned}$$

where ∇_m is the gradient operator in the spectral space and ∇_m^* its conjugate defined as :

$$\nabla_m = \frac{1}{2} \{.\}_{,m} + I k_m \tag{E.4a}$$

and Σ is the operator of the Taylor expansion applied on the velocity gradients, defined as :

$$\Sigma = \exp \left(\frac{1}{2} \{.\}_{|n} \{.\}_{,n}^{\bar{V}} \right) \tag{E.4b}$$

E.1.1 Expression in the one-point limit

Integration of the second order spectral correlations over the wave numbers yields the Reynolds stress tensor at a point in the physical space. We consider now term by term. The integration of the viscous term is straightforward. The identification of the viscous diffusion is immediate and the expression for the viscous dissipation requires an integration by parts :

$$\int \nu \Phi_{ij,kk} d\underline{k} = \nu R_{ij,kk} \tag{E.5}$$

and

$$\begin{aligned}
 \int 2\nu \left[\frac{1}{4} \Phi_{ij,kk} + k^2 \Phi_{ij} \right] d\underline{k} &= 2\nu \int R_{i,j|nn} e^{-I \underline{k}r} d\underline{r} d\underline{k} \\
 &= 2\nu \left[R_{i,j|nn} \right]_{r=0}
 \end{aligned} \tag{E.6}$$

The turbulent diffusion can be split into two terms. The first term representing the first order spatial inhomogeneity :

$$\nabla_m^* T_{im,j} + \nabla_m T_{i,jm} = \frac{1}{2} [T_{im,j} + T_{i,jm}]_{,m} + I k_m [T_{im,j} - T_{i,jm}] \tag{E.7}$$

Here the first contribution acts as a diffusion term in the spectral space and corresponds to the turbulent diffusion process in the physical space.

$$\int \frac{1}{2} [T_{im,j} + T_{i,jm}]_m d\mathbf{k} = [T_{imj}]_m \quad (\text{E.8})$$

whereas the second contribution is a typical spectral term and represents the non linear transfer process. Its integration over all the wave numbers brings no contribution to the Reynolds stress equation.

$$\begin{aligned} \int I k_m [T_{im,j} - T_{i,jm}] d\mathbf{k} &= \int I k_m [T_{im,j}^u - T_{i,jm}^u] e^{-I\mathbf{k}\mathbf{r}} d\mathbf{r} d\mathbf{k} \\ &= \int [T_{im,j}^u - T_{i,jm}^u]_m e^{-I\mathbf{k}\mathbf{r}} d\mathbf{r} d\mathbf{k} \\ &= \left[[T_{im,j}^u - T_{i,jm}^u]_m \right]_{\mathbf{r}=0} \\ &= 0 \end{aligned} \quad (\text{E.9})$$

In the physical space the pressure terms are usually split into a redistributive and a diffusive contribution, as described in the first equation. Following the same splitting as the one used for the turbulent transport, the spectral pressure terms can be written as :

$$\nabla_i^* \Pi_j + \nabla_j \Pi_i^* = \frac{1}{2} [\Pi_{j,i} + \Pi_{i,j}^*] - I [k_i \Pi_j + k_j \Pi_i^*] \quad (\text{E.10})$$

The integration of the first part is again straightforward :

$$\begin{aligned} \int \frac{1}{2} [\Pi_{j,i} + \Pi_{i,j}^*] d\mathbf{k} &= \frac{1}{\rho} \int \frac{1}{2} [\overline{p'u_j''} + \overline{p''u_{i,j}'}] e^{-I\mathbf{k}\mathbf{r}} d\mathbf{r} d\mathbf{k} \\ &= \frac{1}{\rho} \frac{1}{2} [-\overline{(pu_j)_{,i}} + \overline{(pu_j)_{,i}}] d\mathbf{k} \end{aligned} \quad (\text{E.11})$$

whereas the remaining term can be integrated as :

$$\begin{aligned} \int I [k_i \Pi_j + k_j \Pi_i^*] d\mathbf{k} &= \frac{1}{\rho} \int I [k_i \overline{p'u_j''} + k_j \overline{p''u_{i,j}'}] e^{-I\mathbf{k}\mathbf{r}} d\mathbf{r} d\mathbf{k} \\ &= \frac{1}{\rho} \left[-\frac{1}{2} \overline{(pu_j)_{,i}} + \overline{pu_{j,i}} - \frac{1}{2} \overline{(pu_i)_{,j}} + \overline{pu_{i,j}} \right] \end{aligned} \quad (\text{E.12})$$

Hence, the transport term by fluctuating pressure can be transformed as :

$$-\frac{1}{\rho} [\overline{(pu_j)_{,i}} + \overline{(pu_i)_{,j}}] = TF^{-1} [\Pi_{j,i} + \Pi_{i,j}^*] \quad (\text{E.13})$$

and the redistribution tensor as :

$$\frac{1}{\rho} \overline{p(u_{i,j} + u_{j,i})} = TF^{-1} \left[-\frac{1}{2} (\Pi_{j,i} + \Pi_{i,j}^*) - I (k_i \Pi_j - k_j \Pi_i^*) \right] \quad (\text{E.14})$$

The remaining terms in the equation involve mean velocity gradients. In the spectral equations these terms depend on the spectral derivative $\{.\}_{|n}$. Therefore, their contribution to the Reynolds stress equation vanishes as long as the order of the spectral derivative is higher than two (according to the Ostrogradsky property, these terms are transformed into surface integrals where all spectral terms go to zero, as noted by Laporta (1995)). Thus the only terms that need to be taken into account have at most spectral derivatives of the first order. They can be written as :

$$\begin{aligned}\hat{S}_{ij}^{\bar{V}} &= -(\nabla_k + \nabla_k^*) \bar{V}_k \Phi_{ij} \\ &+ (\nabla_k + \nabla_k^*) (\bar{V}_i \Phi_{kj} + \bar{V}_j \Phi_{ik}) \\ &+ k_n (\bar{V}_{n,p} \Phi_{ij|p}) \\ &+ \frac{1}{2} k_n (\bar{V}_{i,p} \Phi_{nj|p} - \bar{V}_{j,p} \Phi_{in|p})\end{aligned}$$

The non-zero integrations over the wave numbers result in the convection and production of the Reynolds stress equation.

E.1.2 Remarks

The comparison with the two-point equations in the physical space is straightforward, as it can be seen in the previous Appendix.

Appendix F

Classical one point closure

This appendix recalls the derivation of the classical closure for the homogeneous pressure-strain term using the fourth order tensor X_{ijpq} .

The pressure-strain term is usually expressed as :

$$\phi_{ij}^R = 2(S_{mn} + W_{mn}) \left[X_{imnj} + X_{mjni} \right] \quad (F.1)$$

The classical closure for X_{ijpq} is the following functional :

$$\begin{aligned} \frac{1}{2\varphi} X_{ijpq} &= C_1 \delta_{ij} \delta_{pq} + C_2 (\delta_{ip} \delta_{jq} + \delta_{jp} \delta_{iq}) \\ &+ C_3 \delta_{ij} h_{pq} + C_4 h_{ij} \delta_{pq} \\ &+ C_5 (\delta_{ip} h_{jq} + \delta_{jp} h_{iq} + h_{ip} \delta_{jq} + h_{jp} \delta_{iq}) \\ &+ C_6 \delta_{ij} h_{.pq}^2 + C_7 h_{.ij}^2 \delta_{pq} \\ &+ C_8 (\delta_{ip} h_{.jq}^2 + \delta_{jp} h_{.iq}^2 + h_{.ip}^2 \delta_{jq} + h_{.jp}^2 \delta_{iq}) \\ &+ C_9 h_{ij} h_{.pq} + C_{10} (h_{ip} h_{jq} + h_{jp} h_{iq}) \\ &+ C_{11} h_{ij} h_{.pq}^2 + C_{12} h_{.ij}^2 h_{pq} \\ &+ C_{13} (h_{ip} h_{.jq}^2 + h_{jp} h_{.iq}^2 + h_{.ip}^2 h_{jq} + h_{.jp}^2 h_{iq}) \\ &+ C_{14} h_{.ij}^2 h_{.pq}^2 + C_{15} (h_{.ip}^2 h_{.jq}^2 + h_{.jp}^2 h_{.iq}^2) \end{aligned} \quad (F.2)$$

This expression satisfies all symmetry properties of the X_{ijpq} tensor for an homogeneous turbulent flow. They are, respectively

$$X_{ijpq} = X_{ijqp} \quad (F.3a)$$

because of the second derivative permutation, and

$$X_{ijpq} = X_{jipq} \quad (F.3b)$$

because of the Hermitian character of the spectral tensor Φ_{ij} in the case of an homogeneous turbulence, that follows once the spherical integration is applied to a symmetric tensor. Furthermore the fact that the flow is incompressible allows to impose :

$$X_{njnq} = 0 \quad (F.3c)$$

which gives three relations between the coefficients :

$$C_1 + 4C_2 - 2II_h C_8 + III_h (C_{11} + C_{12} + 2C_{13}) = 0 \quad (F.4a)$$

$$C_3 + C_4 + 5C_5 - II_h (C_{11} + C_{12} + 4C_{13}) + III_h (C_{14} + C_{15}) = 0 \quad (F.4b)$$

$$C_6 + C_7 + 5C_8 + C_9 + C_{10} - II_h (C_{14} + 3C_{15}) = 0 \quad (F.4c)$$

Using the Cayley-Hamilton theorem to reduce the order of all tensor products (Spencer, 1971), the rapid term can be written as :

$$\begin{aligned} \frac{1}{2\varphi} \phi_{ij}^R &= \beta_1 h_{ij} \\ &+ \beta_2 [h_{.ij}^2 + 2/3 II_h \delta_{ij}] \\ &+ \beta_3 S_{ij} \\ &+ \beta_4 [h_{ik} S_{kj} + h_{jk} S_{ik} - 2/3 h_{mn} S_{mn} \delta_{ij}] \\ &+ \beta_5 [h_{.ik}^2 S_{kj} + h_{.jk}^2 S_{ik} - 2/3 h_{.mn}^2 S_{mn} \delta_{ij}] \\ &+ \beta_6 [h_{ik} W_{jk} + h_{jk} W_{ik}] \\ &+ \beta_7 [h_{.ik}^2 W_{jk} + h_{.jk}^2 W_{ik}] \\ &+ \beta_8 [h_{ik} W_{kp} h_{.pj}^2 + h_{.ik}^2 W_{pk} h_{jp}] \end{aligned} \quad (F.5)$$

with

$$\begin{aligned} \beta_1 &= 4 \{bS\} (C_9 + 2C_{10}) + 2 \{b^2S\} (C_{11} + C_{12} + 4C_{13}) \\ \beta_2 &= 2 \{bS\} (C_{11} + C_{12} + 4C_{13}) + 4 \{b^2S\} (C_{14} + 2C_{15}) \\ \beta_3 &= 4(C_1 + C_2) - 4II_h (C_9 + C_{10}) - 2III_h (C_{11} + C_{12} + 2C_{13}) - 4II_h^2 (C_{14} + C_{15}) \\ \beta_4 &= -6C_5 + 2II_h (C_{11} + C_{12} + 4C_{13}) - 6III_h (C_{14} + C_{15}) \\ \beta_5 &= -6(C_8 + C_9 + C_{10}) - 2II_h (C_{14} - C_{15}) \\ \beta_6 &= 2(-2C_4 - 5C_5 + II_h (C_{11} + C_{12} + 4C_{13}) - III_h (C_{14} + C_{15})) \\ \beta_7 &= 2(-2C_7 + C_9 + C_{10} + 5C_8) + II_h (C_{14} + 3C_{15}) \\ \beta_8 &= 2(C_{11} - C_{12}) \end{aligned}$$

where the notation $\{.\}$ stands for the trace of the tensor to which it is applied.

The closure of the pressure-strain term through X_{ijpq} allows to reduce the number of the unknown coefficients because of the properties of X_{ijpq} . This is, for instance interesting to force the model to respect the fact that the spectral tensor is a contraction of the fourth order tensor. This is the so-called normalization condition which can be written as

$$X_{ijnm} = h_{ij} + \frac{1}{3} \delta_{ij} \quad (F.7)$$

This is the only normalization condition to be satisfied as long as the model involves only the h_{ij} anisotropy tensor. This condition brings three additional equations :

$$3C_1 + 2C_2 - 2II_h C_6 + 4III_h C_{13} = \frac{1}{3} \quad (F.8a)$$

$$3C_4 + 4C_5 - 2II_h (C_{11} + 2C_{13}) + 2III_h C_{15} = 1 \quad (F.8b)$$

$$3C_7 + 4C_8 + 2C_{10} - 2II_h (C_{14} + C_{15}) = 0 \quad (F.8c)$$

Using all those constraints allows to reduce the number of unknown coefficients to nine. If they are chosen to be :

$$C_5 \quad C_8 \quad C_9 \quad C_{10} \quad C_{11} \quad C_{12} \quad C_{13} \quad C_{14} \quad C_{15} \quad (F.9)$$

then the solution of the system of constraints leads to the following relations :

$$C_1 = \frac{2}{15} - 2II_h \left(\frac{5}{3} C_8 + \frac{2}{5} C_9 + \frac{2}{15} C_{10} \right) + \frac{1}{5} III_h (C_{11} + C_{12} - 6 C_{13}) + \frac{4}{15} II_h^2 (C_{14} + 7 C_{15}) \quad (F.10a)$$

$$C_2 = -\frac{1}{30} + II_h \left(\frac{4}{3} C_8 + \frac{1}{5} C_9 + \frac{1}{15} C_{10} \right) - \frac{1}{5} III_h \left[\frac{3}{2} (C_{11} + C_{12}) + C_{13} \right] - \frac{1}{15} II_h^2 (C_{14} + 7 C_{15}) \quad (F.10b)$$

$$C_3 = -\frac{1}{3} - \frac{11}{3} C_5 + II_h \left(\frac{1}{3} C_{11} + C_{12} + \frac{8}{3} C_{13} \right) - III_h \left(C_{14} + \frac{1}{3} C_{15} \right) \quad (F.10c)$$

$$C_4 = \frac{1}{3} - \frac{4}{3} C_5 + \frac{2}{3} II_h (C_{11} + 2 C_{13}) - \frac{2}{3} III_h C_{15} \quad (F.10d)$$

$$C_6 = -\frac{11}{3} C_8 - C_9 - \frac{1}{3} C_{10} + \frac{1}{3} II_h (C_{14} + 7 C_{15}) \quad (F.10e)$$

$$C_7 = -\frac{4}{3} C_8 - \frac{2}{3} C_{10} + \frac{2}{3} II_h (C_{14} + C_{15}) \quad (F.10f)$$

Appendix G

Choice of the multiple-scale variables

This appendix details the integration of the spherical variables and the approximations used to evaluate the variables wanted.

G.1 Closures assumptions

Let us recall the closure of the model of the spherically integrated second order tensor (Schiestel, 1993) :

$$\varphi_{ij}(k) = f(k) E(k) \frac{R_{ij}}{K} + (1 - f(k)) \frac{2}{3} E(k) \delta_{ij} \quad (\text{G.1})$$

as well as the adopted approximation of the one-dimensional spectrum : and the damping function associated with the assumed isotropy of the small structures :

G.2 Integration

The quantities to be integrated are defined as

$$I^{(m)} = \int_0^{+\infty} k^m \varphi_{ij} dk \quad (\text{G.2})$$

where $m > -5$ in order to have a convergent integral. This integration can be rewritten as :

$$I^{(m)} = \int_0^{k_L} k^m \varphi_{ij} dk + \int_{k_L}^{k_\eta} k^m \varphi_{ij} dk \quad (\text{G.3})$$

which leads to the definition of the two integrals applied on the low and the high wave numbers regions of the spectrum :

$$I^{(m)<<} = \int_0^{k_L} k^m \varphi_{ij} dk \quad (\text{G.4a})$$

$$I^{(m)>>} = \int_{k_L}^{k_\eta} k^m \varphi_{ij} dk \quad (\text{G.4b})$$

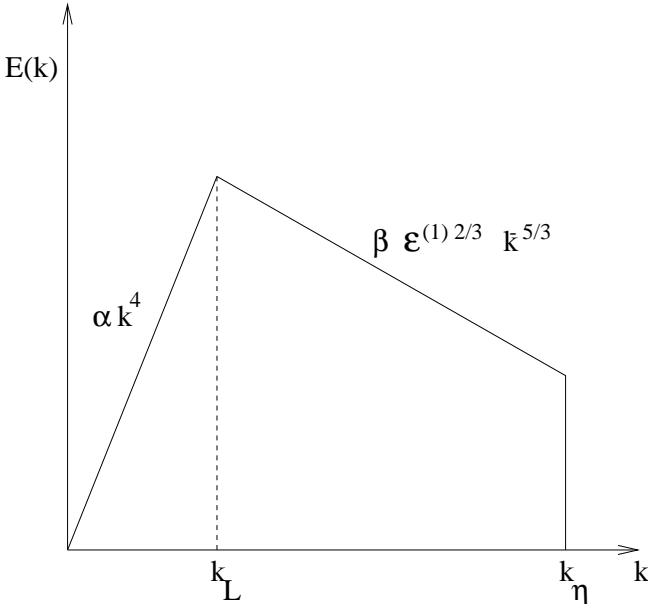


Figure G.1 – Schematical representation of a high-Reynolds kinetic energy spectrum.

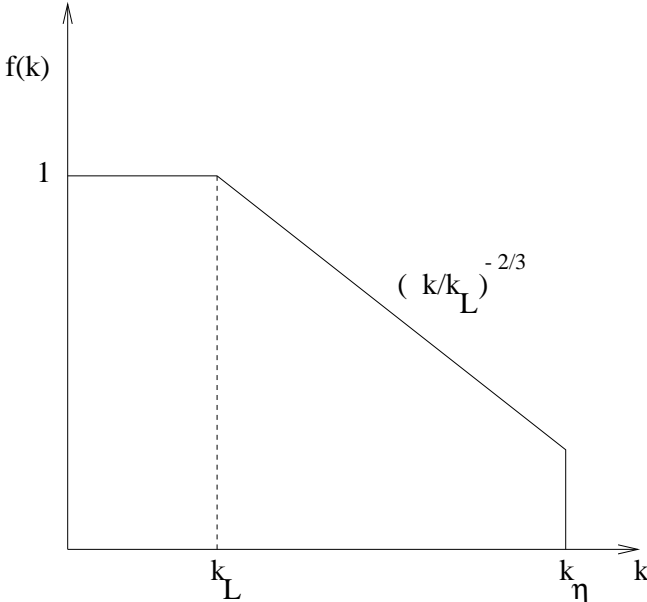


Figure G.2 – Schematics of the damping of the anisotropy of small scales.

G.2.1 Large scales contribution

According to the closures adopted, when the scales of the eddy structures are larger than the integral longitudinal length scale, the spherical spectral tensor is defined by

$$\varphi_{ij}(k) = E(k) \frac{R_{ij}}{K} \quad (\text{G.5})$$

where

$$E(k) = \alpha k^4 \quad (\text{G.6})$$

so that the integral can be written as :

$$I^{(m)<<} = \alpha \frac{R_{ij}}{K} \int_0^{k_L} k^{(m+4)} dk \quad (\text{G.7})$$

Taking into account the continuity of the spectrum allows to express α in terms of β , so that :

$$I^{(m)<<} = \beta \varepsilon^{(1)2/3} k_L^{-17/3} \frac{R_{ij}}{K} \int_0^{k_L} k^{(m+4)} dk \quad (\text{G.8})$$

The integration can, therefore, be performed for $m > -5$, which simplifies to

$$\boxed{I^{(m)<<} = \frac{\beta}{(m+5)} \frac{R_{ij}}{K} \varepsilon^{(1)2/3} k_L^{(m-2/3)} \quad \text{when} \quad m > -5} \quad (\text{G.9})$$

This expression enables us to see which values of m might be of interest. The energy-containing length scale can indeed be rewritten in term of the kinetic energy and the transfer rate through the spectrum, i.e. :

$$I^{(m)<<} = \frac{\beta}{(m+5)} \frac{R_{ij}}{K} \varepsilon^{(1)m} K^{(-3/2m+1)} \quad (\text{G.10})$$

or, again :

$$I^{(m)<<} = \frac{\beta}{(m+5)} R_{ij} \varepsilon^{(1)m} K^{-3/2m} \quad (\text{G.11})$$

Therefore, when $m = 0$, this function yields the Reynolds stress tensor R_{ij} . The derivation of a variable where $\varepsilon^{(1)}$ is entirely separated from the kinetic energy, seems to be difficult. However, taking into account the definition of the length scale, this integral reduces to $R_{ij} L$ when $m = -1$. Finally, in order to treat the small scales, we can consider the dissipation rate, which is recovered from the above general integral by taking $m = 2$.

G.2.2 Small scales contribution

The spherical tensor, introduced earlier

$$\varphi_{ij}(k) = f(k) E(k) \frac{R_{ij}}{K} + (1 - f(k)) \frac{2}{3} E(k) \delta_{ij}$$

reflects the assumption that the information on the anisotropy of the turbulence comes only from the Reynolds tensor, and that this anisotropic character decreases with an increase in the wave number (smaller structures). This statement becomes more transparent if the above spherical tensor is rewritten as :

$$\varphi_{ij}(k) - \frac{2}{3} E(k) \delta_{ij} = f(k) E(k) \left[\frac{R_{ij}}{K} - \frac{2}{3} \delta_{ij} \right]$$

The integration of the small scales contributions can then be split into two parts :

$$I^{(m) \gg i} = \int_{k_L}^{k_\eta} k^m \frac{2}{3} E(k) \delta_{ij} dk \quad (G.12a)$$

$$I^{(m) \gg a} = \int_{k_L}^{k_\eta} k^m f(k) E(k) \left[\frac{R_{ij}}{K} - \frac{2}{3} \delta_{ij} \right] dk \quad (G.12b)$$

Isotropic part

The expression for the isotropic part can be written as :

$$I^{(m) \gg i} = \frac{2}{3} \delta_{ij} \beta \varepsilon^{(1)2/3} \int_{k_L}^{k_\eta} k^{(m-5/3)} dk \quad (G.13)$$

For m different from $\frac{2}{3}$ this expression can be written as :

$$I^{(m) \gg i} = \frac{\beta}{(m-2/3)} \frac{2}{3} \delta_{ij} \varepsilon^{(1)2/3} \left[k^{(m-2/3)} \right]_{k_L}^{k_\eta} \quad (G.14)$$

or again

$$I^{(m) \gg i} = \frac{\beta}{(m-2/3)} \frac{2}{3} \delta_{ij} \varepsilon^{(1)2/3} k_\eta^{(m-2/3)} \left[1 - \left(\frac{k_L}{k_\eta} \right)^{(m-2/3)} \right] \quad (G.15)$$

When $m < 2/3$, the scales of the energy containing eddies are predominant so that this integral can be approximated by

$$\boxed{I^{(m) \gg i} \sim -\frac{\beta}{(m-2/3)} \frac{2}{3} \delta_{ij} \varepsilon^{(1)2/3} k_L^{(m-2/3)} \quad \text{when} \quad m < 2/3} \quad (G.16a)$$

whereas when $m > 2/3$, the small scales processes are dominant and

$$\boxed{I^{(m) \gg i} \sim \frac{\beta}{(m-2/3)} \frac{2}{3} \delta_{ij} \varepsilon^{(1)2/3} k_\eta^{(m-2/3)} \quad \text{when} \quad 2/3 < m} \quad (G.16b)$$

Anisotropic part

The expression for the anisotropic part can be written as :

$$I^{(m) \gg a} = \left[\frac{R_{ij}}{K} - \frac{2}{3} \delta_{ij} \right] \beta \varepsilon^{(1)2/3} k_L^{2/3} \int_{k_L}^{k_\eta} k^{(m-7/3)} dk \quad (G.17)$$

When the exponent m differs from $\frac{4}{3}$, this expression can be written as :

$$I^{(m)>>a} = \frac{\beta}{(m-4/3)} \left[\frac{R_{ij}}{K} - \frac{2}{3} \delta_{ij} \right] \varepsilon^{(1)2/3} k_L^{2/3} \left[k^{(m-4/3)} \right]_{k_L}^{k_\eta} \quad (G.18)$$

or again

$$I^{(m)>>a} = \frac{\beta}{(m-4/3)} \left[\frac{R_{ij}}{K} - \frac{2}{3} \delta_{ij} \right] \varepsilon^{(1)2/3} k_\eta^{(m-2/3)} \left(\frac{k_L}{k_\eta} \right)^{2/3} \left[1 - \left(\frac{k_L}{k_\eta} \right)^{(m-4/3)} \right] \quad (G.19)$$

When $m < 4/3$, the energy containing scales are predominant so that this integral can be approximated by

$$\boxed{I^{(m)>>a} \sim -\frac{\beta}{(m-4/3)} \left[\frac{R_{ij}}{K} - \frac{2}{3} \delta_{ij} \right] \varepsilon^{(1)2/3} k_L^{(m-2/3)} \quad \text{when } m < 4/3} \quad (G.20a)$$

whereas, when $m > 4/3$, the small scales processes are dominant and

$$\boxed{I^{(m)>>a} \sim \frac{\beta}{(m-4/3)} \left[\frac{R_{ij}}{K} - \frac{2}{3} \delta_{ij} \right] \varepsilon^{(1)2/3} k_\eta^{(m-2/3)} \left(\frac{k_L}{k_\eta} \right)^{2/3} \quad \text{when } 4/3 < m} \quad (G.20b)$$

The decomposition into the isotropic and anisotropic part chosen here is based on a physical interpretation, though a mathematical decomposition based on the polynomial extension of the functional terms is also possible. It is interesting to note that the anisotropic terms can be rewritten as :

$$I^{(m)>>a} \sim -\frac{\beta}{(m-4/3)} \left[\frac{R_{ij}}{K} - \frac{2}{3} \delta_{ij} \right] \varepsilon^{(1)2/3} k_L^{2/3} k_L^{(m-4/3)} \quad \text{when } m < 4/3 \quad (G.21a)$$

and

$$I^{(m)>>a} \sim \frac{\beta}{(m-4/3)} \left[\frac{R_{ij}}{K} - \frac{2}{3} \delta_{ij} \right] \varepsilon^{(1)2/3} k_L^{2/3} k_\eta^{(m-4/3)} \quad \text{when } 4/3 < m \quad (G.21b)$$

which make the comparison with the isotropic part more obvious and display more clearly the relevant scale associated to each of the parts.

G.2.3 Synthesis

To summarize, the whole integral $I^{(m)}$ can be written as

$$\begin{aligned} \int_0^{+\infty} k^m \varphi_{ij} dk &= \frac{\beta}{(m+5)} \frac{R_{ij}}{K} \varepsilon^{(1)2/3} k_\eta^{(m-2/3)} \left(\frac{k_L}{k_\eta} \right)^{(m-2/3)} \\ &+ \frac{\beta}{(m-2/3)} \frac{2}{3} \delta_{ij} \varepsilon^{(1)2/3} k_\eta^{(m-2/3)} \left[1 - \left(\frac{k_L}{k_\eta} \right)^{(m-2/3)} \right] \\ &+ \frac{\beta}{(m-4/3)} \left[\frac{R_{ij}}{K} - \frac{2}{3} \delta_{ij} \right] \varepsilon^{(1)2/3} k_\eta^{(m-2/3)} \left(\frac{k_L}{k_\eta} \right)^{2/3} \left[1 - \left(\frac{k_L}{k_\eta} \right)^{(m-4/3)} \right] \end{aligned} \quad (G.22)$$

whenever $m > -5$ and differs from $\frac{2}{3}$ and from $\frac{4}{3}$.

G.2.4 Approximations

As already seen, depending on the value of m , different forms of the integral $I^{(m)}$ are obtained which enables to choose suitable dependent variables. The integration of the large scale contribution points out at interesting values of m , $m = 0$, $m = -1$ and $m = 2$. Therefore, within the domain of validity of m , only two intervals can be considered, the one for $-5 < m < 2/3$ and the other for $4/3 < m$. The simplifications involved in these intervals are discussed in the following paragraphs.

Energy containing eddies and transfer contribution

The evaluation of each part of the integral when $-5 < m < 2/3$ has shown that each of the contributions has the same scaling which is :

$$\varepsilon^{(1)2/3} k_L^{(m-2/3)} \quad (\text{G.23})$$

Therefore, the integral must be evaluated using the entire wave number contribution. For this purpose it is useful to write the integral as a functional expansion in terms of the Reynolds stress :

$$\int_0^{+\infty} k^m \varphi_{ij} dk = \varepsilon^{(1)2/3} k_L^{(m-2/3)} \left(I_0^{(m)} \frac{2}{3} \delta_{ij} + I_1^{(m)} \frac{R_{ij}}{K} \right) \quad (\text{G.24})$$

where

$$I_0^{(m)} = \left(-\frac{\beta}{(m-4/3)} \frac{-2/3}{(m-2/3)} \right) \left[1 + \frac{(m-4/3)}{2/3} \left(\frac{k_\eta}{k_L} \right)^{(m-2/3)} - \frac{(m-2/3)}{2/3} \left(\frac{k_\eta}{k_L} \right)^{(m-4/3)} \right]$$

$$I_1^{(m)} = \left(-\frac{\beta}{(m-4/3)} \frac{19/3}{(m+5)} \right) \left[1 - \frac{(m+5)}{19/3} \left(\frac{k_\eta}{k_L} \right)^{(m-4/3)} \right]$$

As a first approximation only the zero order term can be retained and the second integral has a higher order scalar coefficient for $m = 0$ and $m = -1$, so that for these values, the integral can be approximated by :

$$\int_0^{+\infty} k^m \varphi_{ij} dk \sim \varepsilon^{(1)2/3} k_L^{(m-2/3)} \frac{R_{ij}}{K} \quad (\text{G.26})$$

that is :

$$\boxed{\int_0^{+\infty} \varphi_{ij} dk \sim R_{ij}}$$

and

$$\boxed{\int_0^{+\infty} k^{-1} \varphi_{ij} dk \sim \frac{K^{3/2}}{\varepsilon^{(1)}} R_{ij}}$$

This parameter yields in an indirect manner the spectral energy transfer rate $\varepsilon^{(1)}$, which can be related to the length scale of the energy containing eddies $L = K^{3/2}/\varepsilon^{(1)}$.

The above result corresponds also to the simplified closure on the length scale tensor defined by Donaldson and Sandri (1981),

$$\Lambda_{ij} = L \frac{R_{ij}}{K}$$

which can be extended to account for both the isotropic and anisotropic contributions :

$$\Lambda_{ij} = L \left[\frac{2}{3} \delta_{ij} + \sigma \left(\frac{R_{ij}}{K} - \frac{2}{3} \delta_{ij} \right) \right]$$

Finally, following the above arguments, we can generalize the interpretation of the latter integral by writing

$$\int_0^{+\infty} k^{-1} \varphi_{ij} dk \sim \Lambda_{ij} K$$

The tensorial closure is here written as a function of the physical Reynolds stress. This is due to the spectral model of the spherical tensor.

Small scales contribution

In the case where $4/3 < m$, the evaluation of various terms remains valid, though each of the terms has a different scaling. Adopting the same splitting presented in the preceding section, we can first group the terms in the following manner :

$$\begin{aligned} \int_0^{+\infty} k^m \varphi_{ij} dk &\sim c_K^{(m)>>} \frac{\beta}{(m+5)} \frac{R_{ij}}{K} \varepsilon^{(1)2/3} k_\eta^{(m-2/3)} \left(\frac{k_L}{k_\eta} \right)^{(m-2/3)} \\ &+ c_{\varepsilon i}^{(m)>>} \frac{\beta}{(m-2/3)} \frac{2}{3} \delta_{ij} \varepsilon^{(1)2/3} k_\eta^{(m-2/3)} \\ &+ c_{\varepsilon a}^{(m)>>} \frac{\beta}{(m-4/3)} \left[\frac{R_{ij}}{K} - \frac{2}{3} \delta_{ij} \right] \varepsilon^{(1)2/3} k_\eta^{(m-2/3)} \left(\frac{k_L}{k_\eta} \right)^{2/3} \end{aligned} \quad (\text{G.27})$$

The first contribution can be neglected, so that the integral can be written as :

$$\int_0^{+\infty} k^m \varphi_{ij} dk \sim \varepsilon^{(1)2/3} k_\eta^{(m-2/3)} \left[f_{\varepsilon^{(m)}} \frac{R_{ij}}{K} + (1 - f_{\varepsilon^{(m)}}) \frac{2}{3} \delta_{ij} \right] \quad (\text{G.28})$$

where

$$f_{\varepsilon^{(m)}} = \left(\frac{k_L}{k_\eta} \right)^{2/3} \quad (\text{G.29})$$

It is interesting to note that the ratio of the two length scales delimiting the spectrum can be rewritten in terms of the turbulent Reynolds number

$$\frac{k_L}{k_\eta} = R_t^{-3/4} \left(\frac{\varepsilon^{(1)}}{\varepsilon^{(2)}} \right)^{3/2} \quad (\text{G.30})$$

so that this function might be seen as a damping function expressed in terms of the spectrum imbalance and the turbulent Reynolds number. The scale of this expression can also be transformed using the molecular viscosity as :

$$\varepsilon^{(1)2/3} k_\eta^{(m-2/3)} = \nu^{- (3/4 m-1/2)} \varepsilon^{(2) (3/4 m-1/2)} \varepsilon^{(1) (1-m/2)} \quad (\text{G.31})$$

When $m = 2$, those parameters are reduced to

$$f_{\varepsilon^{(2)}} = R_t^{-1/2} \left(\frac{\varepsilon^{(1)}}{\varepsilon^{(2)}} \right) \quad (\text{G.32})$$

and

$$\varepsilon^{(1)2/3} k_\eta^{2/3} = \nu^{-1} \varepsilon^{(2)} \quad (\text{G.33})$$

Hence, an expression for a suitable variable describing the small scales ($\varepsilon^{(2)}$) can be derived :

$$\boxed{\int_0^{+\infty} \nu k^2 \varphi_{ij} dk \sim \varepsilon^{(2)} \left[f_{\varepsilon^{(2)}} \frac{R_{ij}}{K} + (1 - f_{\varepsilon^{(2)}}) \frac{2}{3} \delta_{ij} \right] \quad \text{where} \quad f_{\varepsilon^{(2)}} = R_t^{-1/2} \left(\frac{\varepsilon^{(1)}}{\varepsilon^{(2)}} \right)}$$

This gives an expression of the dissipation rate tensor very much like the standard expression used in one-point second order models. This is not surprising, since the spectral model chosen also matches the same tensorial derivation.

Intermediate case

The intermediate case, where $2/3 < m < 4/3$, is of no use in the choice of the variables. This exponent appears, however, in the diffusive terms in the transport equations and, for that reason, we give a brief outline of such a case. For this interval of m ($0 < m-2/3 < 2/3$) and for high Reynolds numbers, the integral can be approximated by :

$$\int_0^{+\infty} k^m \varphi_{ij} dk \sim \frac{2}{3} \delta_{ij} \varepsilon^{(1)2/3} k_\eta^{(m-2/3)} \quad (\text{G.34})$$

Bibliography

- ANDRÉ, J.C., AND LESIEUR, M. (1977) Influence of helicity on the evolution of isotropic turbulence at high Reynolds number. *J. Fluid Mech.*, **81**, pp. 187–207.
- BERTOGLIO, J.C., AND JEANDEL, D. (1993) A simplified spectral closure for inhomogeneous turbulence : application to the boundary layer. In *Fifth Symposium on Turbulent Shear Flows, Cornell*, Vol. 10-1.
- BESNARD, D.C., HARLOW, E.H., RAUENZAHN, R.M., AND ZEMACH, C. (1992) Spectral transport model for turbulence. Technical Report 92-1666, Los Alamos national laboratory report.
- CAMBON, C., JEANDEL, D., AND MATHIEU, J. (1981) Spectral modelling of homogeneous non-isotropic turbulence. *J. Fluid Mech.*, **104**.
- CAMBON, C., JACQUIN, L., AND LUBRANO, J.L. (1991) Toward a new Reynolds stress model for rotating turbulent flows. *Phys. Fluids, Series A*, **4**(4), pp. 812–824.
- CAMBON, C. (1979) *Modélisation spectrale en turbulence homogène anisotrope*. PhD thesis, Lyon I.
- CAMBON, C. (1982) *Etude Spectrale d'un Champ Turbulent Incompressible, soumis a des Effets couplés de Déformation et de Rotation, imposés exterieurement*. PhD thesis, Lyon I.
- CHOU, P.Y. (1945) On velocity correlations and the solutions of the equations of turbulent fluctuations. *Quart. of Appl. Math.*, **3**, pp. 38–54.
- CLARK, T.T., AND ZEMACH, C. (1995) A spectral model applied to homogeneous turbulence. *Phys. Fluids*, **7**(7), pp. 1674–1694.
- CROW, S.C. (1968) Viscoelastic properties of fine-grained incompressible turbulence. *J. Fluid Mech.*, **33**(1), pp. 1–15.
- DAVIDOV, B.I. (1961) Statistical dynamics of an incompressible turbulent fluid. *Dokl. Akad. Nauk. SSSR*, **36**.
- DONALDSON, CDUP., AND SANDRI, G. (1981) On the inclusion of information on eddy structure in second order closure models of turbulent flows. In *AGARD*, Vol. 308, pp. 25.1–25.14.
- DURBIN, P.A. (1993) A Reynolds stress model for near-wall turbulence. *J. Fluid Mech.*, **249**, pp. 465–498.

- HANJALIĆ, K., AND LAUNDER, B.E. (1972) A Reynolds stress model of turbulence and its application to thin shear flow. *J. Fluid Mech.*, **54**(4), pp. 609–638.
- HANJALIĆ, K., AND LAUNDER, B.E. (1976) Contribution towards a Reynolds stress closure for low Reynolds number turbulence. *J. Fluid Mech.*, **74**, pp. 593–610.
- HANJALIĆ, K., LAUNDER, B.E., AND SCHIESTEL, R. (1979) Multiple scale concepts in turbulent transport modelling. In *Second Symposium on Turbulent Shear Flows, London*, pp. 36–49.
- HINZE, J.O. (1975) *Turbulence*. Mc Graw-Hill, second edition.
- JEANDEL, D., BRISON, J.F., AND MATHIEU, J. (1978) Modeling methods in physical and spectral space. *Phys. Fluids*, **21**, pp. 169–182.
- JOVANOVIĆ, J., YE, Q.Y., AND DURST, F. (1992) Refinement of the equation for the determination of turbulent micro-scale. Technical Report 349, LSTM.
- JOVANOVIĆ, J., YE, Q.Y., AND DURST, F. (1995) Statistical interpretation of the turbulent dissipation rate in wall-bounded flows. *J. Fluid Mech.*, **293**, pp. 321–347.
- KÁRMÁN, T. VON, AND HOWARTH, L. (1938) On the statistical theory of isotropic turbulence. *Proc. Roy. Soc.*, **164**, pp. 192–215.
- KIM, J., MOIN, P., AND MOSER, R. (1987) Turbulence statistics in fully developed channel flow at low Reynolds number. *J. Fluid Mech.*, **177**, pp. 133–166.
- KOLMOGOROV, A.N. (1941) Local structure of turbulence in incompressible viscous fluid for very large Reynolds number. *Doklady AN. SSSR*, **30**, pp. 299–303.
- KOVASZNAY, L.S.G. (1948) Spectrum of locally isotropic turbulence. *J. Aero. Sci.*, **15**, pp. 745–753.
- KRAICHNAN, R.H. (1958) Irreversible statistical mechanics of incompressible hydromagnetic turbulence. *Phys. Rev.*, **109**, pp. 1407–1422.
- KRAICHNAN, R.H. (1959) The structure of isotropic turbulence at very high Reynolds numbers. *J. Fluid Mech.*, **5**, pp. 497–543.
- KRAICHNAN, R.H. (1965) Lagrangian history closure approximation for turbulence. *Phys. Fluids*, **8**, pp. 575–598.
- KRAICHNAN, R.H. (1972) Model for inhomogeneous turbulence. *J. Fluid Mech.*, **56**, pp. 287–304.
- LAPORTA, A. (1995) *Etude Spectrale et modélisation de la turbulence inhomogène*. PhD thesis, Lyon I.
- LAUNDER, B.E., AND SCHIESTEL, R. (1978) Sur l'utilisation d'échelles temporelles multiples en modélisations des écoulements turbulents. *C. R. Acad. Sc. Paris*, **t286**(A), pp. 709–712.
- LAUNDER, B.E., REECE, G.J., AND RODI, W. (1975) Progress in the development of a Reynolds stress turbulent closure. *J. Fluid Mech.*, **183**, pp. 63–63.
- LAUNDER, B.E. (1991) Current capabilities for modelling turbulence in industrial flows. *Applied Scientific Research*, **48**, pp. 247–269.

- LEITH, C.E. (1967) Diffusion approximation to inertial energy transfer in isotropic turbulence. *Phys. Fluids*, **10**(7), pp. 1409–1416.
- LEITH, C.E. (1971) Atmospheric predictability and two-dimensional turbulence. *J. Atmosph.*, **28**, pp. 145–161.
- LIN, A., AND WOLFSHTEIN, M. (1980) Tensorial volume of turbulence. *Phys. Fluids*, **23**(3), pp. 644–646.
- MILLIONSCHTCHIKOV, M.D. (1941) On the theory of homogeneous isotropic turbulence. *C. R. Acad. Sc. URSS*, **32**(615).
- NAOT, D., SHAVIT, A., AND WOLFSTHEIN, M. (1973) Two-point correlation model and the redistribution of Reynolds stresses. *Phys. Fluids*, **16**, pp. 738–743.
- OBERLACK, M., AND PETERS, N. (1993) Closure of the dissipation tensor equation and the pressure velocity correlation based on the two-point correlation equation. In *Ninth Symposium on Turbulent Shear Flows, Tokyo*, Vol. 23-3, pp. 1–6.
- OBERLACK, M. (1994) *Herleitung und Lösung einer Langenmass und Dissipationstensorgleichung für turbulente strömungen*. PhD thesis, Köln.
- OBERLACK, M. (1997) Non-isotropic dissipation in non-homogeneous turbulence. *J. Fluid Mech.*, **350**, pp. 351–374.
- O'BRIEN, E.E., AND FRANCIS, G.C. (1962) A consequence of zero fourth cumulant approximation. *J. Fluid Mech.*, **13**, pp. 369–382.
- OGURA, Y. (1963) A consequence of zero fourth cumulant approximation in the decay of isotropic turbulence. *J. Fluid Mech.*, **16**, pp. 33–40.
- ORSZAG, S.A. (1970) Analytical theory of turbulence. *J. Fluid Mech.*, **41**, pp. 363–386.
- POUQUET, A., LESIEUR, M., ANDRÉ, J.C., AND BASDEVANT, C. (1975) Evolution of high Reynolds number two-dimensionnal turbulence. *J. Fluid Mech.*, **72**, pp. 305–319.
- REYNOLDS, O. (1895) On the dynamical theory of incompressible viscous fluids and the determination of the criterion. *Philosophical Transactions of the Royal Society of London, Serie A*, **186**, pp. 123–164.
- ROTTA, J.C. (1951) Statistische theorie nichthomogener turbulenz. *ZAMM*, **5**, pp. 136–139.
- ROTTA, J.C. (1972) *Turbulente Strömungen*. B.G. Teubner, Stuttgart.
- SCHIESTEL, R. (1983) Sur le concept d'échelles multiples en modélisation des écoulements turbulents. 1. schéma multi-échelles pour l'énergie cinétique de la turbulence et la variance d'un scalaire passif. *Journal de Mécanique théorique et appliquée*, **2**(3), pp. 417–449.
- SCHIESTEL, R. (1987) Multiple time scale modeling of turbulent flows in one point closures. *Phys. Fluids*, **30**(3), pp. 722–731.
- SCHIESTEL, R. (1993) *Modélisation et simulation des écoulements turbulents*. Hermès.

- SHIMA, N. (1995) Modeling of asymptotic near-wall behavior of the Reynolds stress dissipation. In *Tenth Symposium on Turbulent Shear Flows, Pennsylvania*, Vol. 2, pp. 13–18.
- SPENCER, A.J.M. (1971) Theory of invariants. In *Continuum physics*, Vol. 1. Eringen.
- SPEZIALE, C.G., AND GATSKI, T.B. (1997) Analysis and modeling of anisotropies in the dissipation rate of turbulence. *J. Fluid Mech.*, **344**, pp. 155–180.
- STEINKAMP, M.J., CLARK, T.T., AND HARLOW, F.H. (1995) Stochastic interpenetration of fluids. Technical Report LA-13016-MS, Los Alamos, New Mexico.
- TAGAWA, M., NAGANO, Y., AND TSUJI, T. (1991) Turbulence model for the dissipation components of Reynolds stresses. In *Eighth Symposium on Turbulent Shear Flows, Munchen*, Vol. 29-3, pp. 1–6.
- WILCOX, D.C. (1988) Reassessment of the scale-determining equation for advanced turbulence models. *AIAA Journal*, **26**, pp. 1299–1310.
- YOSHIKAWA, A. (1987) Statistical modeling of a transport equation for the kinetic energy dissipation rate. *Phys. Fluids*, **30**(3), pp. 628–631.