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A canonical structure on the tangent bundle of a pseudo- or para-Kähler manifold

Henri Anciaux*, Pascal Romon†

Abstract

It is a classical fact that the cotangent bundle $T^*\mathcal{M}$ of a differentiable manifold \mathcal{M} enjoys a canonical symplectic form Ω^* . If $(\mathcal{M}, J, g, \omega)$ is a pseudo-Kähler or para-Kähler $2n$ -dimensional manifold, we prove that the tangent bundle $T\mathcal{M}$ also enjoys a natural pseudo-Kähler or para-Kähler structure $(\tilde{J}, \tilde{g}, \Omega)$, where Ω is the pull-back by g of Ω^* and \tilde{g} is a pseudo-Riemannian metric with neutral signature $(2n, 2n)$. We investigate the curvature properties of the pair (\tilde{J}, \tilde{g}) and prove that: \tilde{g} is scalar-flat, is not Einstein unless g is flat, has nonpositive (resp. nonnegative) Ricci curvature if and only if g has nonpositive (resp. nonnegative) Ricci curvature as well, and is locally conformally flat if and only if $n = 1$ and g has constant curvature, or $n > 2$ and g is flat. We also check that (i) the holomorphic sectional curvature of (\tilde{J}, \tilde{g}) is not constant unless g is flat, and (ii) in $n = 1$ case, that \tilde{g} is never anti-self-dual, unless conformally flat.

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Keywords: tangent bundle, pseudo-Kähler geometry, para-Kähler geometry, self-duality and anti-self-duality.

Introduction

It is a classical fact that given any differentiable manifold \mathcal{M} , its cotangent bundle $T^*\mathcal{M}$ enjoys a canonical symplectic structure Ω^* .

Moreover, given a linear connection ∇ on a manifold \mathcal{M} , (e.g. the Levi-Civita connection of a Riemannian metric), the bundle $T\mathcal{M}$ splits into a direct sum of two subbundles $H\mathcal{M}$ and $V\mathcal{M}$, both isomorphic to $T\mathcal{M}$. This allows to define an almost complex structure J by setting $J(X_h, X_v) := (-X_v, X_h)$, where, for $X \in T\mathcal{M} = H\mathcal{M} \oplus V\mathcal{M}$, we write $X \simeq (X_h, X_v) \in T\mathcal{M} \times T\mathcal{M}$. Analogously, one may introduce a natural almost para-complex (or bi-Lagrangian) structure, setting $J'(X_h, X_v) := (X_v, X_h)$.

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It is also well known that the tangent bundle of a Riemannian manifold (\mathcal{M}, g) can be given a natural Riemannian structure, called *Sasaki metric*. A simple way to understand this construction, which extends *verbatim* to the case of a pseudo-Riemannian metric g with signature $(p, m - p)$, is as follows: using the splitting $T\mathcal{M} = H\mathcal{M} \oplus V\mathcal{M}$, we set:

$$G((X_h, X_v), (Y_h, Y_v)) := g(X_h, Y_h) + g(X_v, Y_v).$$

This metric has signature $(2p, 2(m - p))$ and is well behaved with respect to J in two ways: (i) G is compatible with J , i.e. $G(., .) = G(J., J.)$, and (ii) the symplectic form $\Omega := G(J., .)$ is nothing but the pull-back of Ω^* by the musical isomorphism between $T\mathcal{M} \simeq_g T^*\mathcal{M}$. In other words, the triple (J, G, Ω) defines an “almost pseudo-Kähler” structure¹ on $T\mathcal{M}$.

Unfortunately, this construction suffers two flaws: J is not integrable unless ∇ is flat and the metric G is somewhat “rigid”: for example, if G has constant scalar curvature, then g is flat (see [15]). We refer to [5, 16] and the survey [9] for more detail on the Sasaki metric.

Another construction can be made in the case where \mathcal{M} is complex (resp. para-complex): in this case both $T\mathcal{M}$ and $T^*\mathcal{M}$ enjoy a canonical complex (resp. para-complex) structure which are defined as follows: given a family of holomorphic (resp. para-holomorphic²) local charts $\varphi : \mathcal{M} \rightarrow \mathcal{U} \subset \mathbb{R}^{2n}$ on \mathcal{M} , we define holomorphic (resp. para-holomorphic) local charts $\bar{\varphi} : T\mathcal{M} \rightarrow \mathcal{U} \times \mathbb{R}^{2n}$ by $\bar{\varphi}(x, V) = (\varphi(x), d\varphi_x(V))$, $\forall (x, V) \in T\mathcal{M}$ for the tangent bundle, and $\bar{\varphi} : T^*\mathcal{M} \rightarrow \mathcal{U} \times \mathbb{R}^{2n}$ by $\bar{\varphi}(x, \xi) = (\varphi(x), ((d\varphi_x)^t)^{-1}(\xi))$, $\forall (x, \xi) \in T^*\mathcal{M}$ for the cotangent bundle. In the first section, we shall see that if \mathcal{M} is merely almost complex (resp. almost para-complex), then a more subtle argument allows to define again a canonical almost complex structure (resp. almost para-complex structure) on $T\mathcal{M}$. On the other hand, we shall prove in the second section that if \mathcal{M} is pseudo- or para-Kähler, the corresponding structure on $T\mathcal{M}$ can also be constructed using the splitting $H\mathcal{M} \oplus V\mathcal{M}$ induced by the Levi-Civita connection of the Kählerian metric.

Combining the canonical symplectic structure Ω^* of $T^*\mathcal{M}$ with the canonical complex (resp. para-complex) structure \tilde{J}^* just defined, it is natural to introduce a 2-tensor \tilde{g}^* by the formula

$$\tilde{g}^* := \Omega^*(., \tilde{J}^*.).$$

However, it turns out that Ω^* is not compatible with \tilde{J}^* , since it turns out that $\Omega^*(\tilde{J}^*., \tilde{J}^*.) = -\varepsilon\Omega^*$ instead of the required formula $\Omega^*(\tilde{J}^*., \tilde{J}^*.) = \varepsilon\Omega^*$ (here and in the following, in order to deal simultaneously with the complex

¹We might also define an “almost para-Kähler” structure on $T\mathcal{M}$ by introducing the *para-Sasaki metric*

$$G'((X_h, X_v), (Y_h, Y_v)) := g(X_h, Y_h) - g(X_v, Y_v).$$

This metric has neutral signature (m, m) (m being the dimension of \mathcal{M}), is compatible with J' and verifies $\Omega := -G'(J'., .)$.

²The terminology *split-holomorphic* is sometimes used.

and para-complex cases, we define ε to be such that $(\tilde{J}^*)^2 = -\varepsilon \text{Id}$, i.e. $\varepsilon = 1$ in the complex case and $\varepsilon = -1$ in the para-complex case). It follows that the tensor \tilde{g}^* is not symmetric and therefore we failed in constructing a canonical pseudo-Riemannian structure on $T^*\mathcal{M}$.

On the other hand, the same idea works well if one considers, instead of the cotangent bundle, the tangent bundle of a pseudo- or para-Kähler manifold (\mathcal{M}, J, g) , thus obtaining a canonical pseudo- or para-Kähler structure. The purpose of this note is to investigate in detail this construction and to study its curvature properties. The results are summarized in the following:

Main Theorem *Let $(\mathcal{M}, J, g, \omega)$ be a pseudo- or para-Kähler manifold. Then $T\mathcal{M}$ enjoys a natural pseudo- or para-Kähler structure $(\tilde{J}, \tilde{g}, \Omega)$ with the following properties:*

- \tilde{J} is the canonical complex or para-complex structure of $T\mathcal{M}$ induced from that of \mathcal{M} ;
- Ω is the pull-back of Ω^* by the metric isomorphism $T\mathcal{M} \simeq_g T^*\mathcal{M}$;
- The pseudo-Riemannian metric \tilde{g} can be recovered from \tilde{J} and Ω by the equation $\tilde{g}(\cdot, \cdot) := \Omega(\cdot, \tilde{J}\cdot)$;
- According to the splitting $TT\mathcal{M} = H\mathcal{M} \oplus V\mathcal{M}$ induced by the Levi-Civita connection of g , the triple $(\tilde{J}, \tilde{g}, \Omega)$ takes the following expression:

$$\begin{aligned} \tilde{J}(X_h, X_v) &:= (JX_h, JX_v) \\ \tilde{g}((X_h, X_v), (Y_h, Y_v)) &:= g(X_v, JY_h) - g(X_h, JY_v) \\ \Omega((X_h, X_v), (Y_h, Y_v)) &:= g(X_v, Y_h) - g(X_h, Y_v); \end{aligned}$$

- The pseudo-Riemannian metric \tilde{g} has the following properties:
 - (i) \tilde{g} has neutral signature neutral $(2n, 2n)$ and is scalar flat;
 - (ii) $(T\mathcal{M}, \tilde{g})$ is Einstein if and only if (\mathcal{M}, g) is flat, and therefore $(T\mathcal{M}, \tilde{g})$ is flat as well;
 - (iii) the Ricci curvature $\widetilde{\text{Ric}}$ of \tilde{g} has the same sign as the Ricci curvature Ric of g ;
 - (iv) $(T\mathcal{M}, \tilde{g})$ is locally conformally flat if and only if $n = 1$ and g has constant curvature, or $n > 2$ and g is flat; if $n = 1$, \tilde{g} is always self-dual, so anti-self-duality is equivalent to conformal flatness;
 - (v) the pair (\tilde{J}, \tilde{g}) has constant holomorphic curvature if and only if g is flat.

Remark 1. *We use in (iv) the general property that four-dimensional neutral pseudo-Kähler or para-Kähler manifolds are self-dual if and only if their scalar curvature vanishes. This is analogous to the case of Kähler four-dimensional manifolds, except that self-duality is exchanged with anti-self-duality. A proof of this statement is given in Theorem A.2 in the appendix.*

This result is a generalization of previous work on the tangent bundle of a Riemannian surface (see [10], [11], [3]). The authors wish to thank Brendan Guilfoyle for his valuable suggestions and comments.

1 Almost complex and para-complex structures on the tangent bundle

Given a manifold \mathcal{M} endowed with an almost complex or almost para-complex structure J , it is only natural to ask whether its tangent or cotangent bundle inherit such a structure. The answer is positive:

Proposition 1. *Let (\mathcal{M}, J) be an almost complex (resp. para-complex) manifold. Then its tangent bundle admits a canonical almost complex (resp. para-complex) structure \tilde{J} . Furthermore, if J is complex (resp. para-complex), so is \tilde{J} .*

Remark 2. *Such a result has been proven already by Lempert & Szöke [14] for the tangent bundle in the almost complex case. Their construction uses the jets over \mathcal{M} and is quite a bit more technical than our proof. However it gives an interesting interpretation of the meaning of \tilde{J} . We shall see below in Proposition 2 a different and simpler way of defining and understanding \tilde{J} , provided \mathcal{M} is a pseudo- or para-Kähler manifold.*

Proof. We prove the result using coordinate charts, which amounts to showing that \tilde{J} can be defined independently of any change of variable. Let $y = \varphi(x)$ be a local change of coordinates on \mathbb{R}^n and write ξ and η respectively for the tangent coordinates induced by the charts (i.e. $\sum_i \xi^i \partial/\partial x^i = \sum_i \eta^i \partial/\partial y^i$). The change of tangent coordinates at x is $\xi \mapsto \eta = d\varphi(x)\xi$, in other words φ induces a chart Φ on \mathbb{R}^{2n} , $\Phi : (x, \xi) \mapsto (\varphi(x), d\varphi(x)\xi)$. The tangent coordinates at (x, ξ) (resp. (y, η)) are denoted by (X, Ξ) (resp. (Y, H)) and the change of (doubly) tangent coordinates is

$$d\Phi(x, \xi) : (X, \Xi) \mapsto (Y, H) = (d\varphi(x)X, d^2\varphi(x)(X, \xi) + d\varphi(x)\Xi).$$

Assume moreover that we have a $(1, 1)$ tensor, which reads in the x coordinate as the matrix $J(x)$ and in the y coordinate as the matrix $J'(y) = J'(\varphi(x)) = d\varphi(x) \circ J(x) \circ (d\varphi(x))^{-1}$. Equivalently for any X and $Y = d\varphi(x)X$, we have $J'(y)Y = J'(\varphi(x))d\varphi(x)X = d\varphi(x)J(x)X$. Differentiating this equality along ξ yields

$$\begin{aligned} (D_{d\varphi(x)\xi}J')(\varphi(x))d\varphi(x)X + J'(\varphi(x))d^2\varphi(x)(X, \xi) \\ = d\varphi(x)(D_\xi J)(x)X + d^2\varphi(x)(J(x)X, \xi), \end{aligned} \quad (1)$$

where $(D_\xi J)(x)$ denotes in this proof the directional derivative of the matrix J at x in the direction ξ (not a covariant derivative).

We now define the (1, 1) tensor \tilde{J} in the (x, ξ) coordinate by

$$\tilde{J}(x, \xi) : (X, \Xi) \mapsto (J(x)X, J(x)\Xi + D_\xi J(x)X).$$

Let us prove that this definition is coordinate-independent (for greater readability we will often write J, J' for $J(x), J'(y)$). Using (1) and the symmetry of the second order differential $d^2\varphi(x)$,

$$\begin{aligned} d\Phi(x, \xi)(J(X, \Xi)) &= d\Phi(x, \xi)(JX, J\Xi + D_\xi J(x)X) \\ &= (d\varphi(x)JX, d^2\varphi(x)(JX, \xi) + d\varphi(x)(J\Xi + D_\xi J(x)X)) \\ &= (J'Y, J'd\varphi(x)\Xi \\ &\quad + (D_{d\varphi(x)\xi}J')(\varphi(x))d\varphi(x)X + J'd^2\varphi(x)(X, \xi)) \\ &= (J'Y, J'(d\varphi(x)\Xi + d^2\varphi(x)(X, \xi)) \\ &\quad + (D_{d\varphi(x)\xi}J')(\varphi(x))d\varphi(x)X) \\ &= (J'Y, J'H + D_\eta J'(y)Y) = \tilde{J}'(y, \eta)(Y, H), \end{aligned}$$

where \tilde{J}' denotes the map corresponding to \tilde{J} in the (y, η) coordinates. Consequently the tensor on \mathcal{M} extends naturally to $T\mathcal{M}$.

We have so far defined a (1, 1) tensor on $T\mathcal{M}$ without extra assumptions. Suppose now that J is an almost complex (resp. para-complex) structure, so that $J^2 = -\varepsilon \text{Id}$. Differentiating this property yields $J D_\xi J + (D_\xi J)J = 0$. Then

$$\begin{aligned} \tilde{J}^2(X, \Xi) &= (J^2X, J(J\Xi + D_\xi JX) + D_\xi J(JX)) \\ &= (-\varepsilon X, -\varepsilon\Xi + J(dJ\xi)X + (dJ\xi)(JX)) = -\varepsilon(X, \Xi) \end{aligned}$$

so that \tilde{J} is also an almost complex (resp. para-complex) structure.

Finally if J is a complex (resp. para-complex) structure then we can use complex (resp. para-complex) coordinate charts, which amounts to saying that J is a constant matrix. Then \tilde{J} defined in the associated charts on $T\mathcal{M}$ takes a simpler expression, and is also constant:

$$\tilde{J}(x, \xi) : (X, \Xi) \mapsto (JX, J\Xi)$$

and that characterizes a complex (resp. para-complex) structure. \square

Remark 3. *Finding a similar almost-complex structure on $T^*\mathcal{M}$ is much more difficult, and may not be true in all generality. The Reader will note that, whenever \mathcal{M} is endowed with a pseudo-Riemannian metric, we have a musical correspondence between $T\mathcal{M}$ and $T^*\mathcal{M}$, and \tilde{J} induces a corresponding structure \tilde{J}^* on $T^*\mathcal{M}$. However different metrics will yield different structures on $T^*\mathcal{M}$. There is one unambiguous case, which will be the setting in the remainder of this article, namely when J is integrable.*

2 The Kähler structure

Let \mathcal{M} be a differentiable manifold. We denote by π and π^* the canonical projections $T\mathcal{M} \rightarrow \mathcal{M}$ and $T\mathcal{M}^* \rightarrow \mathcal{M}$. The subbundle $\ker(d\pi) := V\mathcal{M}$ of $TT\mathcal{M}$ (it is thus a bundle over $T\mathcal{M}$) will be called *the vertical bundle*.

Assume now that \mathcal{M} is equipped with a linear connection ∇ . The corresponding horizontal bundle is defined as follows: let \bar{X} be a tangent vector to $T\mathcal{M}$ at some point (x_0, V_0) . This implies that there exists a curve $\gamma(s) = (x(s), V(s))$ such that $(x(0), V(0)) = (x_0, V_0)$ and $\gamma'(0) = \bar{X}$. If $\bar{X} \notin V\mathcal{M}$ (which implies $x'(0) \neq 0$), we define the connection map (see [7], [3]) $K : TT\mathcal{M} \rightarrow T\mathcal{M}$ by $K\bar{X} = \nabla_{x'(0)}V(0)$, where ∇ denotes the Levi-Civita connection of the metric g . If \bar{X} is vertical, we may assume that the curve γ stays in a fiber so that $V(s)$ is a curve in a vector space. We then define $K\bar{X}$ to be simply $V'(0)$. The horizontal bundle is then $\text{Ker}(K)$ and we have a direct sum

$$\begin{aligned} TT\mathcal{M} &= H\mathcal{M} \oplus V\mathcal{M} \simeq T\mathcal{M} \oplus T\mathcal{M} \\ \bar{X} &\simeq (\Pi\bar{X}, K\bar{X}). \end{aligned} \quad (2)$$

Here and in the following, Π is a shorthand notation for $d\pi$.

Lemma 1. [7] *Given a vector field X on (\mathcal{M}, ∇) there exists exactly one vector field X^h and one vector field X^v on $T\mathcal{M}$ such that $(\Pi X^h, KX^h) = (X, 0)$ and $(\Pi X^v, KX^v) = (0, X)$. Moreover, given two vector fields X and Y on (\mathcal{M}, ∇) , we have, at the point (x, V) :*

$$\begin{aligned} [X^v, Y^v] &= 0, \\ [X^h, Y^v] &= (\nabla_X Y)^v \simeq (0, \nabla_X Y), \\ [X^h, Y^h] &\simeq ([X, Y], -R(X, Y)V), \end{aligned}$$

where R denotes the curvature of ∇ and we use the direct sum notation (2).

The Reader should not confuse the horizontal *lift* X^h , which is a vector field on $T\mathcal{M}$ constructed from a vector field $X \in \mathfrak{X}(\mathcal{M})$, with the notation $\bar{X}_h = \Pi\bar{X}$ denoting the horizontal *part* of $\bar{X} \in \mathfrak{X}(T\mathcal{M})$. Similarly, the vertical lift X^v is *not* the vertical projection $\bar{X}_v = K\bar{X}$.

We say that a vector field \bar{X} on $T\mathcal{M}$ is *projectable* if it is constant on the fibres, i.e. $(\Pi\bar{X}, K\bar{X})(x, V) = (\Pi\bar{X}, K\bar{X})(x, V')$. According to the lemma above, it is equivalent to the fact that there exists two vector fields X_1 and X_2 on \mathcal{M} such that $\bar{X} = (X_1)^h + (X_2)^v$.

Assume now that \mathcal{M} is equipped with a pseudo-Riemannian metric g , i.e. a non-degenerate bilinear form. By the non-degeneracy assumption, we can identify $T^*\mathcal{M}$ with $T\mathcal{M}$ by the following (musical) isomorphism:

$$\begin{aligned} \iota : T\mathcal{M} &\rightarrow T\mathcal{M}^* \\ (x, V) &\mapsto (x, \xi), \end{aligned}$$

where ξ is defined by

$$\xi(W) = g(V, W), \quad \forall W \in T_x\mathcal{M}.$$

The *Liouville form* $\alpha \in \Omega^1(T^*\mathcal{M})$ is the 1-form defined by $\alpha_{(x,\xi)}(\bar{X}) = \xi_x(d\pi^*(\bar{X}))$, where \bar{X} is a tangent vector at the point (x, ξ) of $T^*\mathcal{M}$. The canonical symplectic form on $T\mathcal{M}^*$ is defined to be $\Omega^* := -d\alpha$. There is an elegant, explicit formula for the symplectic form $\Omega := \iota^*(\Omega^*)$ in terms of the metric g and the splitting induced by the Levi-Civita connection ∇ (see [2], [13]):

Lemma 2. *Let \bar{X} and \bar{Y} be two tangent vectors to $T\mathcal{M}$; we have*

$$\Omega(\bar{X}, \bar{Y}) = g(\mathbf{K}\bar{X}, \Pi\bar{Y}) - g(\Pi\bar{X}, \mathbf{K}\bar{Y}).$$

Proposition 2. *Let $(\mathcal{M}, \mathbf{J}, g)$ be a pseudo- or para-Kähler manifold. The canonical structure $\tilde{\mathbf{J}}$ satisfies*

$$\tilde{\mathbf{J}}\bar{X} \simeq \tilde{\mathbf{J}}(\Pi\bar{X}, \mathbf{K}\bar{X}) = (\mathbf{J}\Pi\bar{X}, \mathbf{J}\mathbf{K}\bar{X}).$$

Corollary 1. *Let $(\mathcal{M}, \mathbf{J}, g)$ be a pseudo- or para-Kähler manifold. The 2-tensor $\tilde{g}(\cdot, \cdot) := \Omega(\cdot, \tilde{\mathbf{J}}\cdot)$ satisfies*

$$\tilde{g}(\bar{X}, \bar{Y}) = g(\mathbf{K}\bar{X}, \mathbf{J}\Pi\bar{Y}) - g(\Pi\bar{X}, \mathbf{J}\mathbf{K}\bar{Y}).$$

Moreover, \tilde{g} is symmetric and therefore defines a pseudo-Riemannian metric on $T\mathcal{M}$.

Proof of Proposition 2. Let us write the splitting of $TT\mathcal{M}$ in a local coordinate x as in the proof of Proposition 1 ⁽³⁾. The Levi-Civita connection is expressed through its connection form μ : $\nabla_X Y = dY(X) + \mu(X)Y$. Consequently, if $(X, \Xi) \in T_{(x,\xi)}T\mathcal{M}$, $\Pi(X, \Xi) = X$ and $\mathbf{K}(X, \Xi) = \Xi + \mu(X)\xi$. Thus

$$\Pi(\tilde{\mathbf{J}}(X, \Xi)) = \mathbf{J}X \text{ and } \mathbf{K}(\tilde{\mathbf{J}}(X, \Xi)) = \mathbf{J}(x)\Xi + (d\mathbf{J}(x)\xi)X + \mu(\mathbf{J}(x)X)\xi.$$

Because \mathbf{J} is integrable, we may choose x to be a complex coordinate, so that \mathbf{J} is a constant endomorphism, and $d\mathbf{J}(x)\xi$ vanishes. Because \mathcal{M} is Kähler, we know that $\mu(X)$ commutes with \mathbf{J} . However, ∇ being without torsion, $\mu(X)Y = \mu(Y)X$, so

$$\mathbf{K}(\tilde{\mathbf{J}}(X, \Xi)) = \mathbf{J}\Xi + \mathbf{J}\mu(X)\xi = \mathbf{J}\mathbf{K}(X, \Xi). \quad \square$$

Corollary 2. *The symplectic form Ω is compatible with the complex or para-complex structure $\tilde{\mathbf{J}}$.*

Proof. Using Lemma 2, we compute

$$\begin{aligned} \Omega(\tilde{\mathbf{J}}\bar{X}, \tilde{\mathbf{J}}\bar{Y}) &= g(\mathbf{K}\tilde{\mathbf{J}}\bar{X}, \Pi\tilde{\mathbf{J}}\bar{Y}) - g(\Pi\tilde{\mathbf{J}}\bar{X}, \mathbf{K}\tilde{\mathbf{J}}\bar{Y}) \\ &= g(\mathbf{J}\mathbf{K}\bar{X}, \mathbf{J}\Pi\bar{Y}) - g(\mathbf{J}\Pi\bar{X}, \mathbf{J}\mathbf{K}\bar{Y}) \\ &= \varepsilon g(\mathbf{K}\bar{X}, \Pi\bar{Y}) - \varepsilon g(\Pi\bar{X}, \mathbf{K}\bar{Y}) \\ &= \varepsilon \Omega(\bar{X}, \bar{Y}). \end{aligned}$$

□

³ The Reader should be aware of the conflicting notation: the splitting of $TT\mathcal{M} \simeq \mathbb{R}^{4n}$ as $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$ induced by the coordinate charts (e.g. $\bar{X} \simeq ((x, \xi), (X, \Xi))$) differs a priori from the connection-induced splitting $\bar{X} \simeq (\Pi\bar{X}, \mathbf{K}\bar{X})$.

3 The Levi-Civita connection of \tilde{g}

The following lemma describes the Levi-Civita connection $\tilde{\nabla}$ of \tilde{g} in terms of the direct decomposition of $T\mathcal{M}$, the Levi-Civita connection ∇ of g and its curvature tensor R .

Lemma 3. *Let \bar{X} and \bar{Y} be two vector fields on $T\mathcal{M}$ and assume that \bar{Y} is projectable, then at the point (x, V) we have*

$$(\tilde{\nabla}_{\bar{X}}\bar{Y})|_V = (\nabla_{\Pi\bar{X}}\Pi\bar{Y}, \nabla_{\Pi\bar{X}}K\bar{Y} - T_1(\Pi\bar{X}, \Pi\bar{Y}, V)),$$

where

$$T_1(X, Y, V) = \frac{1}{2} \left(R(X, Y)V - \varepsilon R(V, JX)JY - \varepsilon R(V, JY)JX \right)$$

Moreover, if \mathcal{M} is a pseudo-Riemannian surface with Gaussian curvature c , we have

$$T_1(X, Y, V) = \begin{cases} -2cg(V, X)Y & \text{in the Kähler case,} \\ +2cg(V, Y)X & \text{in the para-Kähler case.} \end{cases}$$

Proof. We use Lemma 1 together with the Koszul formula:

$$\begin{aligned} 2\tilde{g}(\tilde{\nabla}_{\bar{X}}\bar{Y}, \bar{Z}) &= \bar{X}\tilde{g}(\bar{Y}, \bar{Z}) + \bar{Y}\tilde{g}(\bar{X}, \bar{Z}) - \bar{Z}\tilde{g}(\bar{X}, \bar{Y}) + \tilde{g}([\bar{X}, \bar{Y}], \bar{Z}) \\ &\quad - \tilde{g}([\bar{X}, \bar{Z}], \bar{Y}) - \tilde{g}([\bar{Y}, \bar{Z}], \bar{X}), \end{aligned}$$

where X, Y and Z are three vector fields on $T\mathcal{M}$. From the fact that $[X^v, Y^v]$ and $\tilde{g}(X^v, Y^v)$ vanish we have:

$$\begin{aligned} 2\tilde{g}(\tilde{\nabla}_{X^v}Y^v, Z^v) &= X^v\tilde{g}(Y^v, Z^v) + Y^v\tilde{g}(X^v, Z^v) - Z^v\tilde{g}(X^v, Y^v) \\ &\quad + \tilde{g}([X^v, Y^v], Z^v) - \tilde{g}([X^v, Z^v], Y^v) - \tilde{g}([Y^v, Z^v], X^v) \\ &= 0. \end{aligned}$$

Moreover, taking into account that $\tilde{g}(Y^v, Z^h)$ and similar quantities are constant on the fibres, we obtain

$$\begin{aligned} 2\tilde{g}(\tilde{\nabla}_{X^v}Y^v, Z^h) &= X^v\tilde{g}(Y^v, Z^h) + Y^v\tilde{g}(X^v, Z^h) - Z^h\tilde{g}(X^v, Y^v) \\ &\quad + \tilde{g}([X^v, Y^v], Z^h) - \tilde{g}([X^v, Z^h], Y^v) - \tilde{g}([Y^v, Z^h], X^v) \\ &= -\tilde{g}(-(\nabla_Z X)^v, Y^v) - \tilde{g}(-(\nabla_Z Y)^v, X^v) \\ &= 0. \end{aligned}$$

From these last two equations we deduce that $\tilde{\nabla}_{X^v}Y^v$ vanishes. Analogous computations show that $\tilde{\nabla}_{X^v}Y^h$ vanishes as well. From Lemma 1 and the formula $[\bar{X}, \bar{Y}] = \tilde{\nabla}_{\bar{X}}\bar{Y} - \tilde{\nabla}_{\bar{Y}}\bar{X}$, we deduce that

$$\tilde{\nabla}_{X^h}Y^v \simeq (0, \nabla_X Y). \quad (3)$$

Finally, introducing

$$T_1(X, Y, V) := \frac{1}{2} \left(R(X, Y)V - \varepsilon R(V, JY)JX - \varepsilon R(V, JX)JY \right),$$

we compute that

$$\begin{aligned}
2\tilde{g}(\tilde{\nabla}_{X^h}Y^h, Z^h) &= -g(\mathbf{R}(X, Y)V, \mathbf{J}Z) + g(\mathbf{R}(X, Z)V, \mathbf{J}Y) + g(\mathbf{R}(Y, Z)V, \mathbf{J}X) \\
&= -g(\mathbf{R}(X, Y)V, \mathbf{J}Z) + g(\mathbf{R}(V, \mathbf{J}Y)X, Z) + g(\mathbf{R}(V, \mathbf{J}X)Y, Z) \\
&= -g(\mathbf{R}(X, Y)V, \mathbf{J}Z) + \varepsilon g(\mathbf{R}(V, \mathbf{J}Y)\mathbf{J}X, \mathbf{J}Z) + \varepsilon g(\mathbf{R}(V, \mathbf{J}X)\mathbf{J}Y, \mathbf{J}Z) \\
&= -g(2\mathbf{T}_1(X, Y, V), \mathbf{J}Z)
\end{aligned}$$

and

$$\tilde{g}(\tilde{\nabla}_{X^h}Y^h, Z^v) = -g(\nabla_X Y, \mathbf{J}Z),$$

from which we deduce that

$$\tilde{\nabla}_{X^h}Y^h(V) = (\nabla_X Y, -\mathbf{T}_1(X, Y, V)). \quad (4)$$

From (3) and (4) we deduce the required formula for $\tilde{\nabla}_{\tilde{X}}\tilde{Y}$.

If $n = 1$, we have $\mathbf{R}(X, Y)Z = c(g(Y, Z)X - g(X, Z)Y)$, hence the tensor \mathbf{T}_1 becomes:

$$\begin{aligned}
2\mathbf{T}_1(X, Y, V) &= \mathbf{R}(X, Y)V + \varepsilon\mathbf{J}\mathbf{R}(V, \mathbf{J}X)Y + \varepsilon\mathbf{J}\mathbf{R}(V, \mathbf{J}Y)X \\
&= c\left(g(Y, V)X - g(X, V)Y \right. \\
&\quad \left. - \varepsilon\mathbf{J}(g(\mathbf{J}X, Y)V - g(V, Y)\mathbf{J}X + g(\mathbf{J}Y, X)V - g(V, X)\mathbf{J}Y)\right) \\
&= c\left(g(Y, V)X - g(X, V)Y \right. \\
&\quad \left. - \varepsilon(g(\mathbf{J}X, Y)\mathbf{J}V + g(V, Y)X + g(\mathbf{J}Y, X)\mathbf{J}V + g(V, X)Y)\right) \\
&= c((1 - \varepsilon)g(V, Y)X - (1 + \varepsilon)g(V, X)Y).
\end{aligned}$$

□

Remark 4. *It should be noted that covariant derivatives with respect to a projectable vertical field X^v always vanish.*

Proposition 3. *The structure $\tilde{\mathbf{J}}$ is parallel with respect to $\tilde{\nabla}$.*

Proof. It can be seen as a trivial consequence of the fact that $\tilde{\mathbf{J}}$ is complex (resp. para-complex) and Ω is closed, but can also be checked directly, using the equivariance properties of \mathbf{J} w.r.t. the connection ∇ and the curvature tensor \mathbf{R} . Using the definition of $\tilde{\mathbf{J}}$ and Lemma 3, $\tilde{\nabla}_{\tilde{X}}\tilde{\mathbf{J}}\tilde{Y}$ is obvious provided $\mathbf{T}_1(X, \mathbf{J}Y, V) = \mathbf{J}\mathbf{T}_1(X, Y, V)$. That is indeed the case since

$$\begin{aligned}
2(\mathbf{T}_1(X, \mathbf{J}Y, V) - \mathbf{J}\mathbf{T}_1(X, Y, V)) &= \mathbf{R}(X, \mathbf{J}Y)V + \mathbf{R}(V, \mathbf{J}X)Y + \mathbf{R}(V, Y)\mathbf{J}X \\
&\quad - \mathbf{R}(X, Y)\mathbf{J}V - \mathbf{R}(V, \mathbf{J}X)Y - \mathbf{R}(V, \mathbf{J}Y)X \\
&= \mathbf{R}(X, \mathbf{J}Y)V + \mathbf{R}(\mathbf{J}Y, V)X + \mathbf{J}(\mathbf{R}(V, Y)X + \mathbf{R}(Y, X)V) \\
&= \mathbf{R}(V, X)\mathbf{J}Y + \mathbf{J}\mathbf{R}(X, V)Y = 0,
\end{aligned}$$

where we have used Bianchi's identity. □

4 Curvature properties of $(\tilde{\mathcal{J}}, \tilde{g})$

4.1 The Riemannian curvature tensor of \tilde{g}

Proposition 4. *The curvature tensor $\widetilde{\text{Rm}} := -\tilde{g}(\tilde{R}.,.)$ of \tilde{g} at (x, V) is given by the formula*

$$\begin{aligned} \widetilde{\text{Rm}}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}) &= g(T_2(\Pi\tilde{X}, \Pi\tilde{Y}, \Pi\tilde{Z}, V), \text{J}\Pi\tilde{W}) \\ &\quad - \text{Rm}(\Pi\tilde{X}, \Pi\tilde{Y}, \Pi\tilde{Z}, \text{JK}\tilde{W}) - \text{Rm}(\Pi\tilde{X}, \Pi\tilde{Y}, \text{JK}\tilde{Z}, \Pi\tilde{W}) \\ &\quad - \text{Rm}(\Pi\tilde{X}, \text{JK}\tilde{Y}, \Pi\tilde{Z}, \Pi\tilde{W}) + \text{Rm}(\text{JK}\tilde{X}, \Pi\tilde{Y}, \Pi\tilde{Z}, \Pi\tilde{W}), \end{aligned}$$

where

$$T_2(X, Y, Z, V) := (\nabla_X T_1)(Y, Z, V) - (\nabla_Y T_1)(X, Z, V).$$

Moreover, $(T\mathcal{M}, \tilde{g})$ is scalar flat and the Ricci tensor of \tilde{g} is

$$\widetilde{\text{Ric}}(\tilde{X}, \tilde{Y}) = 2\text{Ric}(\Pi\tilde{X}, \Pi\tilde{Y}).$$

Corollary 3. *$(T\mathcal{M}, \tilde{g})$ is Einstein if and only if (\mathcal{M}, g) is flat. Moreover $(T\mathcal{M}, \tilde{g})$ has nonnegative (resp. nonpositive) Ricci curvature if and only if (\mathcal{M}, g) has nonnegative (resp. nonpositive) Ricci curvature as well.*

Proof of Proposition 4. We will compute the curvature tensor for projectable vector fields, and need only do so for the following six cases, due to the symmetries of $\widetilde{\text{Rm}}$. Remark 4 simplifies computations greatly, since most vertical derivatives vanish, except when the derived vector field is not projectable. In particular $\tilde{R}(X^v, Y^v)$ vanishes as endomorphism, hence:

$$\begin{aligned} \widetilde{\text{Rm}}(X^v, Y^v, Z^v, W^v) &= 0 \\ \widetilde{\text{Rm}}(X^v, Y^v, Z^v, W^h) &= 0 \\ \widetilde{\text{Rm}}(X^v, Y^v, Z^h, W^v) &= 0 \end{aligned}$$

To obtain the last three combinations, let us first derive $\tilde{R}(X^h, Y^h)Z^h$. This is more delicate since we have to covariantly differentiate non-projectable quantities. Indeed

$$\begin{aligned} \tilde{R}(X^h, Y^h)Z^h &= \tilde{\nabla}_{X^h} \tilde{\nabla}_{Y^h} Z^h - \tilde{\nabla}_{Y^h} \tilde{\nabla}_{X^h} Z^h - \tilde{\nabla}_{[X^h, Y^h]} Z^h \\ &= \tilde{\nabla}_{X^h} (\nabla_Y Z, -T_1(Y, Z, V)) - \tilde{\nabla}_{Y^h} (\nabla_X Z, -T_1(X, Z, V)) \\ &\quad - \tilde{\nabla}_{([X, Y], -R(X, Y)V)} Z^h \\ &= (\nabla_X \nabla_Y Z, -T_1(X, \nabla_Y Z, V)) - D_{X^h}(0, T_1(Y, Z, V)) \\ &\quad - (\nabla_Y \nabla_X Z, -T_1(Y, \nabla_X Z, V)) + D_{Y^h}(0, T_1(X, Z, V)) \\ &\quad - (\nabla_{[X, Y]} Z, -T_1([X, Y], Z, V)) \\ &= (R(X, Y)Z, 0) \\ &\quad - (0, T_1(X, \nabla_Y Z, V)) - T_1(Y, \nabla_X Z, V) - T_1([X, Y], Z, V) \\ &\quad - \tilde{\nabla}_{X^h}(0, T_1(Y, Z, V)) + \tilde{\nabla}_{Y^h}(0, T_1(X, Z, V)) \end{aligned}$$

Recalling the lemma⁴ in [12], there exists a vector field U on M such that $U(x) = V$ and $(\nabla_X U)(x) = 0$. Then the vertical lift of $T_1(Y, Z, U)$ is seen to agree to first order with

$$(x, V) \mapsto (0, T_1(X(x), Z(x), V))$$

thus allowing us to use the formula in Lemma 3:

$$\begin{aligned} \tilde{\nabla}_{X^h}(0, T_1(Y, Z, \cdot)) &= \tilde{\nabla}_{X^h}(T_1(Y, Z, U)^v) \\ &= (0, \nabla_X(T_1(Y, Z, U))) \\ &= (0, (\nabla_X T_1)(Y, Z, U) + T_1(\nabla_X Y, Z, U) \\ &\quad + T_1(Y, \nabla_X Z, U) + T_1(Y, Z, \nabla_X U)) \end{aligned}$$

which, evaluated at (x, V) , yields

$$\tilde{\nabla}_{X^h}(0, T_1(Y, Z, \cdot))|_{(x, V)} = (0, (\nabla_X T_1)(Y, Z, V) + T_1(\nabla_X Y, Z, V) + T_1(Y, \nabla_X Z, V)).$$

Summing up,

$$\begin{aligned} \tilde{R}(X^h, Y^h)Z^h|_{(x, V)} &= \left(R(X, Y)Z, \right. \\ &\quad -T_1(X, \nabla_Y Z, V) + T_1(Y, \nabla_X Z, V) \\ &\quad + T_1([X, Y], Z, V) - (\nabla_X T_1)(Y, Z, V) \\ &\quad - T_1(\nabla_X Y, Z, V) - T_1(Y, \nabla_X Z, V) \\ &\quad + (\nabla_Y T_1)(X, Z, V) + T_1(\nabla_Y X, Z, V) \\ &\quad \left. + T_1(X, \nabla_Y Z, V) \right) \\ &= \left(R(X, Y)Z, -(\nabla_X T_1)(Y, Z, V) + (\nabla_Y T_1)(X, Z, V) \right) \\ &= \left(R(X, Y)Z, -T_2(X, Y, Z, V) \right). \end{aligned}$$

From that we deduce directly

$$\begin{aligned} \widetilde{Rm}(X^h, Y^h, Z^h, W^v) &= -Rm(X, Y, Z, JW) \\ \widetilde{Rm}(X^h, Y^h, Z^h, W^h)|_{(x, V)} &= g(T_2(X, Y, Z, V), JW). \end{aligned}$$

On the other hand, using repeatedly Remark 4,

$$\begin{aligned} \widetilde{Rm}(X^h, Y^v, Z^h, W^v) &= \tilde{g}(\tilde{\nabla}_{X^h} \tilde{\nabla}_{Y^v} W^v - \tilde{\nabla}_{Y^v} \tilde{\nabla}_{X^h} W^v - \tilde{\nabla}_{[X^h, Y^v]} W^v, Z^h) \\ &= \tilde{g}(-\tilde{\nabla}_{Y^v}(0, \nabla_X W), Z^h) = \tilde{g}(0, Z^h) = 0. \end{aligned}$$

The claimed formula is easily deduced using the symmetries of the curvature tensor.

⁴Note that computations in [12] are done for the Sasaki metric, hence direct results do not apply.

In order to calculate the Ricci curvature of \tilde{g} , we consider a Hermitian pseudo-orthonormal basis (e_1, \dots, e_{2n}) of $T_x\mathcal{M}$, i.e. $g(e_a, e_b) = \varepsilon_a \delta_{ab}$, where $\varepsilon_a = \pm 1$, and $e_{n+a} = Je_a$. In particular, $\varepsilon_{n+a} = \varepsilon \varepsilon_a$. This gives a (non-orthonormal) basis of $T_{(x,V)}T\mathcal{M}$:

$$\bar{e}_a := (e_a)_h \quad \bar{e}_{2n+a} := (e_a)^v.$$

A calculation using Corollary 1 shows that the expression of \tilde{g} in this basis is:

$$[\tilde{g}_{\mu\nu}]_{1 \leq \mu, \nu \leq 4n} := \begin{pmatrix} 0 & 0 & 0 & \Delta \\ 0 & 0 & -\Delta & 0 \\ 0 & -\Delta & 0 & 0 \\ \Delta & 0 & 0 & 0 \end{pmatrix},$$

where $\Delta = \varepsilon \text{diag}(\varepsilon_1, \dots, \varepsilon_n) = \text{diag}(\varepsilon_{n+1}, \dots, \varepsilon_{2n})$. It follows that $\widetilde{\text{Ric}}(X^v, Y^v)$ and $\widetilde{\text{Ric}}(X^h, Y^v)$ vanish.

Moreover, noting that $\tilde{g}^{\mu\nu} = \tilde{g}_{\mu\nu}$,

$$\begin{aligned} \widetilde{\text{Ric}}(X^h, Y^h) &= \sum_{\mu, \nu=1}^{4n} \tilde{g}^{\mu\nu} \widetilde{\text{Rm}}(X^h, \bar{e}_\mu, Y^h, \bar{e}_\nu) \\ &= \sum_{a=1}^n \varepsilon \varepsilon_a \left(\widetilde{\text{Rm}}(X^h, (e_a)^h, Y^h, (Je_a)^v) - \widetilde{\text{Rm}}(X^h, (Je_a)^h, Y^h, (e_a)^v) \right. \\ &\quad \left. - \widetilde{\text{Rm}}(X^h, (e_a)^v, Y^h, (Je_a)^h) + \widetilde{\text{Rm}}(X^h, (Je_a)^v, Y^h, (e_a)^h) \right) \\ &= \sum_{a=1}^n \varepsilon \varepsilon_a \left(-\text{Rm}(X, e_a, Y, J^2 e_a) + \text{Rm}(X, Je_a, Y, Je_a) \right. \\ &\quad \left. + \text{Rm}(Y, Je_a, X, Je_a) - \text{Rm}(Y, e_a, X, J^2 e_a) \right) \\ &= 2 \sum_{a=1}^n \left(\varepsilon_a \text{Rm}(X, e_a, Y, e_a) + \varepsilon_{a+n} \text{Rm}(X, e_{a+n}, Y, e_{a+n}) \right) \\ &= 2 \sum_{k=1}^{2n} \varepsilon_k \text{Rm}(X, e_k, Y, e_k) = 2 \text{Ric}(X, Y). \end{aligned}$$

We see easily that $\widetilde{\text{Ric}}$ vanishes whenever one of the vectors is along the vertical fiber, thus the expected formula.

Finally the scalar curvature

$$\widetilde{\text{Scal}} = \sum_{\mu, \nu=1}^4 \tilde{g}^{\mu\nu} \widetilde{\text{Ric}}(\bar{e}_\mu, \bar{e}_\nu) = 0,$$

since $\tilde{g}^{\mu\nu}$ vanishes as soon as both \bar{e}_μ, \bar{e}_ν are both horizontal. \square

4.2 The Weyl curvature tensor of \tilde{g}

Proposition 5. *The Weyl tensor \widetilde{W} at (x, V) is given by*

$$\begin{aligned} \widetilde{W}(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) &= \widetilde{\text{Rm}}(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) \\ &\quad - \frac{1}{2n-1} \left(\text{Ric}(\Pi\bar{X}, \Pi\bar{Z})\tilde{g}(\bar{Y}, \bar{W}) + \text{Ric}(\Pi\bar{Y}, \Pi\bar{W})\tilde{g}(\bar{Y}, \bar{W}) \right. \\ &\quad \left. - \text{Ric}(\Pi\bar{X}, \Pi\bar{W})\tilde{g}(\bar{Y}, \bar{Z}) - \text{Ric}(\Pi\bar{Y}, \Pi\bar{Z})\tilde{g}(\bar{X}, \bar{W}) \right). \end{aligned}$$

In particular, if $n = 1$,

$$\widetilde{W}(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) = g(\text{T}_2(\Pi\bar{X}, \Pi\bar{Y}, \Pi\bar{Z}, V), \text{J}\Pi\bar{W}).$$

Corollary 4. *$(\mathcal{TM}, \tilde{g})$ is locally conformally flat if and only if $n = 1$ and g has constant curvature, or $n \geq 2$ and g is flat.*

Remark 5. *This result has been proved in the case $n = 1$ and $\varepsilon = 1$ in [10].*

Proof of Proposition 5. Since the scalar curvature vanishes, we have

$$\widetilde{W} = \widetilde{\text{Rm}} - \frac{1}{4n-2} \widetilde{\text{Ric}} \circledast \tilde{g},$$

where \circledast denotes the Kulkarni–Nomizu product. Recall that $\widetilde{\text{Ric}}(\bar{X}, \bar{Y}) = 0$ if one of the two vectors \bar{X} and \bar{Y} is vertical. Consequently

$$\begin{aligned} \widetilde{\text{Ric}} \circledast \tilde{g}(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) &= 2 \left(\text{Ric}(\Pi\bar{X}, \Pi\bar{Z})\tilde{g}(\bar{Y}, \bar{W}) + \text{Ric}(\Pi\bar{Y}, \Pi\bar{W})\tilde{g}(\bar{Y}, \bar{W}) \right. \\ &\quad \left. - \text{Ric}(\Pi\bar{X}, \Pi\bar{W})\tilde{g}(\bar{Y}, \bar{Z}) - \text{Ric}(\Pi\bar{Y}, \Pi\bar{Z})\tilde{g}(\bar{X}, \bar{W}) \right). \end{aligned}$$

The expression of the Weyl tensor follows easily.

In the case $n = 1$ of a surface with Gaussian curvature c , we have $\text{Ric}(X, Y) = cg(X, Y)$ and $\text{Rm}(X, Y, Z, W) = c(g(X, Z)g(Y, W) - g(X, W)g(Y, Z))$. Hence using Proposition 4, the expression of Weyl tensor simplifies and we get the claimed formula. \square

Proof of Corollary 4. We first deal with the case $n = 1$. Lemma 3 implies that $\text{T}_1(X, Y, Z) = -2cg(Z, X)Y$ when $\varepsilon = 1$ (resp. $2cg(Z, Y)X$ when $\varepsilon = -1$). Therefore, if $\varepsilon = 1$,

$$\begin{aligned} \text{T}_2(X, Y, Z, W) &= \nabla_X \text{T}_1(Y, Z, W) - \nabla_Y \text{T}_1(X, Z, W) \\ &= -2(X.c)g(W, Y)Z + 2(Y.c)g(W, X)Z \\ &= 2g\left((Y.c)X - (X.c)Y, W\right)Z, \end{aligned}$$

which vanishes if and only if $(X.c)Y = (Y.c)X$ for all vectors X, Y , i.e. the curvature c is constant. Analogously, if $\varepsilon = -1$,

$$\begin{aligned} \text{T}_2(X, Y, Z, W) &= \nabla_X \text{T}_1(Y, Z, W) - \nabla_Y \text{T}_1(X, Z, W) \\ &= 2(X.c)g(W, Z)Y - 2(Y.c)g(W, Z)X \\ &= 2\left((X.c)Y - (Y.c)X\right)g(W, Z), \end{aligned}$$

which again vanishes if and only if the curvature c is constant.

Assume now that $(T\mathcal{M}, \tilde{g})$ is conformally flat with $n \geq 2$. Thus in particular

$$\begin{aligned} \widetilde{W}(X^h, Y^h, Z^h, W^v) &= -\text{Rm}(X, Y, Z, JW) \\ &\quad - \frac{1}{2n-1} \left(-\text{Ric}(X, Z)g(Y, JW) + \text{Ric}(Y, Z)g(X, JW) \right) \end{aligned}$$

vanishes, so

$$\text{Rm}(X, Y, Z, JW) = \frac{1}{2n-1} \left(\text{Ric}(X, Z)g(Y, JW) - \text{Ric}(Y, Z)g(X, JW) \right).$$

(Observe that this equation always holds if \mathcal{M} is a surface.) Let us apply the symmetry property of the curvature tensor to this equation with $Z = X$ and $JW = Y$, assuming furthermore that X and Y are two non-null vectors:

$$\begin{aligned} 0 &= (2n-1)(\text{Rm}(X, Y, X, Y) - \text{Rm}(Y, X, Y, X)) \\ &= \text{Ric}(X, X)g(Y, Y) - \text{Ric}(Y, X)g(X, Y) \\ &\quad - \text{Ric}(Y, Y)g(X, X) + \text{Ric}(X, Y)g(Y, X) \\ &= \text{Ric}(X, X)g(Y, Y) - \text{Ric}(Y, Y)g(X, X). \end{aligned}$$

Hence

$$\frac{\text{Ric}(X, X)}{g(X, X)} = \frac{\text{Ric}(Y, Y)}{g(Y, Y)}.$$

The set of non null vectors being dense in $T\mathcal{M}$, it follows by continuity that g is Einstein. We deduce that

$$\begin{aligned} \text{Rm}(X, Y, X, Y) &= \frac{1}{2n-1} \left(\text{Ric}(X, X)g(Y, Y) - \text{Ric}(Y, X)g(X, Y) \right) \\ &= c \left(g(X, X)g(Y, Y) - g(X, Y)g(X, Y) \right), \end{aligned}$$

so g has constant curvature. But since \mathcal{M} is Kähler and has dimension $2n \geq 4$, it must be flat. \square

Finally, we recall the general result linking the Weyl tensor to the scalar curvature in dimension four: for a neutral pseudo-Kähler or para-Kähler metric, self-duality is equivalent to scalar flatness (see Theorem A.2 in annex). We can therefore conclude

Corollary 5. *In dimension four ($n = 1$), the metric \tilde{g} is anti-self-dual if and only the curvature c of g is constant.*

Proof. Thanks to proposition 4, we know that \tilde{g} is scalar flat, hence self-dual (W^- vanishes identically). In order for \tilde{g} to be also anti-self-dual, the Weyl tensor has to vanish completely, which amounts, following corollary 4, to having constant (sectional) curvature c on \mathcal{M} . \square

4.3 The holomorphic sectional curvature of (\tilde{J}, \tilde{g})

Proposition 6. (\tilde{J}, \tilde{g}) has constant holomorphic sectional curvature if and only if g is flat.

Proof. Define the holomorphic sectional curvature tensor of \tilde{g} by $\widetilde{\text{Hol}}(\bar{X}) := \widetilde{\text{Rm}}(\bar{X}, \tilde{J}\bar{X}, \bar{X}, \tilde{J}\bar{X})$. Writing any doubly tangent vector \bar{X} as the sum of a horizontal and a vertical factor, we will compute $\widetilde{\text{Hol}}(X^h + Y^v)$. We deduce from Proposition 4 that $\widetilde{\text{Rm}}$ vanishes whenever two or more entries are vertical. Hence, using the antisymmetric properties of the Riemann tensor w.r.t. the complex or para-complex structure,

$$\begin{aligned} \widetilde{\text{Hol}}(X^h + Y^v) &= \widetilde{\text{Rm}}(X^h, JX^h, X^h, JX^h) \\ &\quad + \widetilde{\text{Rm}}(X^h, JX^h, X^h, JY^v) + \widetilde{\text{Rm}}(X^h, JX^h, Y^v, JX^h) \\ &\quad + \widetilde{\text{Rm}}(X^h, JY^v, X^h, JX^h) + \widetilde{\text{Rm}}(Y^v, JX^h, X^h, JX^h) \\ &= \widetilde{\text{Rm}}(X^h, JX^h, X^h, JX^h) + 4\widetilde{\text{Rm}}(X^h, JX^h, X^h, JY^v) \\ &= g(\text{T}_2(X, JX, X, V) - 4\varepsilon\text{R}(X, Y)X, JX). \end{aligned}$$

In particular,

$$\begin{aligned} \widetilde{\text{Hol}}(X^v) &= 0 \\ \widetilde{\text{Hol}}(X^h + X^v) &= g(\text{T}_2(X, JX, X, V), JX) \\ \widetilde{\text{Hol}}(X^h + (JX)^v) &= g(\text{T}_2(X, JX, X, V), JX) + 4\varepsilon\text{Hol}(X). \end{aligned}$$

It follows from the first equation that if $\widetilde{\text{Hol}}$ is constant, it must be zero. Hence, from the second and third equation we deduce that Hol must vanish, i.e. g is flat. \square

5 Examples

The simplest examples where we may apply the construction above is where $(\mathcal{M}, J, g, \omega)$ is the plane \mathbb{R}^2 equipped with the flat metric $g := dq_1^2 + \varepsilon dq_2^2$ and the complex or para-complex structure J defined by $J(\partial_{q_1}, \partial_{q_2}) = (-\varepsilon\partial_{q_2}, \partial_{q_1})$. In other words, \mathbb{R}^2 is identified with the complex plane \mathbb{C} or the para-complex plane \mathbb{D} . We recall that \mathbb{D} , called the algebra of double numbers, is the two-dimensional real vector space \mathbb{R}^2 endowed with the commutative algebra structure whose product rule is given by

$$(u, v).(u', v') = (uu' + vv', uv' + u'v).$$

The number $(0, 1)$, whose square is $(1, 0)$ and not $(-1, 0)$, will be denoted by τ .

We claim that in the complex case $\varepsilon = 1$, the structure $(\tilde{J}, \tilde{g}, \Omega)$ just constructed on $T\mathbb{C}$ is equivalent to that of the standard complex pseudo-Euclidean

plane $(\mathbb{C}^2, \bar{J}, \langle \cdot, \cdot \rangle_2, \omega_1)$, where \bar{J} is the canonical complex structure, $(z_1 = x_1 + iy_1, z_2 = x_2 + iy_2)$ are the canonical coordinates and

$$\begin{aligned}\langle \cdot, \cdot \rangle_2 &:= -dx_1^2 - dx_2^2 + dx_2^2 + dy_2^2 \\ \omega_1 &:= -dx_1 \wedge dy_1 + dx_2 \wedge dy_2.\end{aligned}$$

To see this, it is sufficient to consider the following complex change of coordinates

$$\begin{cases} z_1 &:= \frac{\sqrt{2}}{2}((p_1 + ip_2) + i(q_1 + iq_2)) \\ z_2 &:= \frac{\sqrt{2}}{2}(p_1 + ip_2 - i(q_1 + iq_2)), \end{cases}$$

which preserves the symplectic form, since we have

$$\omega_1 := -dx_1 \wedge dy_1 + dx_2 \wedge dy_2 = dq_1 \wedge dp_1 + dq_2 \wedge dp_2 = \Omega,$$

where Ω is the canonical symplectic form of $T^*\mathbb{C} \simeq_g T\mathbb{C}$. The metric of a pseudo-Kähler structure being determined by the complex structure and the symplectic form through the formula $\tilde{g} = \Omega(\cdot, \bar{J}\cdot)$, we have the required identification.

Analogously, in the para-complex case $\varepsilon = -1$, the structure $(\tilde{J}, \tilde{g}, \Omega)$ constructed on $T\mathbb{D}$ is equivalent to that of the standard para-complex plane $(\mathbb{D}^2, \bar{J}, \langle \cdot, \cdot \rangle_*, \omega_*)$, where \bar{J} is the canonical para-complex structure, $(w_1 = u_1 + \tau u_1, w_2 = u_2 + \tau y_2)$ are the canonical coordinates and

$$\begin{aligned}\langle \cdot, \cdot \rangle_* &:= du_1^2 - dv_1^2 + du_2^2 - dv_2^2 \\ \omega_* &:= du_1 \wedge dv_1 + du_2 \wedge dv_2.\end{aligned}$$

Here we have to be careful with the identification of $T^*\mathbb{D}$ with $T\mathbb{D}$: since the metric g is $dq_1^2 - dq_2^2$, we have $q_1 := dp_1 \simeq_g \partial_{p_1}$ and $q_2 := dp_2 \simeq_g -\partial_{q_2}$. Hence $\Omega^* = dq_1 \wedge dp_1 + dq_2 \wedge dp_2$ and $\Omega = dq_1 \wedge dp_1 - dq_2 \wedge dp_2$. Introducing the change of para-complex coordinates

$$\begin{cases} w_1 &:= \frac{\sqrt{2}}{2}((p_1 + \tau p_2) - \tau(q_1 + \tau q_2)) \\ w_2 &:= \frac{\sqrt{2}}{2}(\tau(p_1 + \tau p_2) + (q_1 + \tau q_2)), \end{cases}$$

we check that

$$\omega_* = du_1 \wedge dv_1 + du_2 \wedge dv_2 = dq_1 \wedge dp_1 - dq_2 \wedge dp_2 = \Omega,$$

hence we obtain the identification between $(T\mathbb{D}, \tilde{J}, \tilde{g}, \Omega)$ and $(\mathbb{D}^2, \bar{J}, \langle \cdot, \cdot \rangle_*, \omega_*)$. Of course the metrics considered in these two examples are flat.

The next simplest examples of pseudo-Riemannian surfaces are the two-dimensional space forms, namely the sphere \mathbb{S}^2 , the hyperbolic plane $\mathbb{H}^2 := \{x_1^2 + x_2^2 - x_3^2 = -1\}$ and the de Sitter surface $d\mathbb{S}^2 := \{x_1^2 + x_2^2 - x_3^2 = 1\}$. Their tangent bundles enjoy a interesting geometric interpretation (see [10]):

the tangent bundle TS^2 is canonically identified with the set of oriented lines of Euclidean three-space:

$$L(\mathbb{R}^3) \ni \{V + tx | t \in \mathbb{R}\} \simeq (x, V - \langle V, x \rangle_0 x) \in TS^2.$$

Analogously, the tangent bundle $T\mathbb{H}^2$ is canonically identified with the set of oriented negative (timelike) lines of three-space endowed with the metric $\langle \cdot, \cdot \rangle_1 := dx_1^2 + dx_2^2 - dx_3^2$:

$$\mathbb{L}_{1,-}^3 \ni \{V + tx | t \in \mathbb{R}\} \simeq (x, V - \langle V, x \rangle_1 x) \in T\mathbb{H}^2,$$

Finally, the tangent bundle $Td\mathbb{S}^2$ is canonically identified with the set of oriented positive (spacelike) lines of three-space endowed with the metric $\langle \cdot, \cdot \rangle_1$:

$$\mathbb{L}_{1,+}^3 \ni \{V + tx | t \in \mathbb{R}\} \simeq (x, V - \langle V, x \rangle_1 x) \in Td\mathbb{S}^2.$$

Observe that the metric constructed on TS^2 (resp. $T\mathbb{H}^2$) has non-negative (resp. non-positive) Ricci curvature.

A The Weyl tensor in the pseudo-Kähler or para-Kähler cases

The Riemann curvature tensor Rm of a pseudo-Riemannian manifold \mathcal{N} may be seen as a symmetric form R on bivectors of $\Lambda^2 T\mathcal{N}$ (see [4] for references). Splitting R along the eigenspaces $\Lambda^+ \oplus \Lambda^-$ of the Hodge operator $*$ on $\Lambda^2 T\mathcal{N}$, yields the following block decomposition

$$R = \begin{pmatrix} W^+ + \frac{Scal}{12}I & Z \\ Z^* & W^- + \frac{Scal}{12}I \end{pmatrix}$$

where Z^* denotes the adjoint w.r.t. the induced metric on $\Lambda^2 T\mathcal{N}$, so that $W = W^+ \oplus W^-$, the Weyl tensor seen as a 2-form on $\Lambda^2 T\mathcal{N}$, is the traceless, Hodge-commuting part of the Riemann curvature operator R . Hence the following formula

$$W = Rm - \frac{1}{2}Ric \otimes g + \frac{Scal}{12}g \otimes g.$$

If, additionally, \mathcal{N} is a four dimensional Kähler manifold, then

Theorem A.1. W^+ can be written as a multiple of the scalar curvature by a parallel non-trivial 2-form on $\Lambda^2 T\mathcal{N}$.

See Prop. 2 in [6] for a proof and the explicit formula for the tensor involved. We do not need it explicitly since we are only interested in the following

Corollary 6. (\mathcal{N}, g, J) is anti-self-dual ($W^+ = 0$) if and only if the scalar curvature vanishes.

The result extends to the two cases considered in this article: (1) neutral pseudo-Kähler manifolds and (2) para-Kähler manifolds, with a slight twist: W^+ is replaced by W^- . Precisely:

Theorem A.2. *Let (\mathcal{N}, g, J) be a four dimensional manifold endowed with a pseudo-Kähler neutral metric (respectively a para-Kähler metric, necessarily neutral). Then the Weyl tensor W commutes with the Hodge operator and \mathcal{N} is self-dual ($W^- = 0$) if and only if the scalar curvature vanishes.*

The result for neutral pseudo-Kähler manifolds is probably known and relates to representation theory (see [4] for introduction and references), but since we could not find an explicit proof in the literature⁵, we will give a simple one below. To our knowledge, the proof for the para-Kähler case is new (albeit similar).

A.1 The pseudo-Kähler case

We will write explicitly the Weyl tensor in a given positively oriented orthonormal frame, denoted by $(e_1, e_{1'}, e_2, e_{2'})$, where $e_{1'} = Je_1$, $e_{2'} = Je_2$, $g(e_1) = g(e_{1'}) = -1$ and $g(e_2) = g(e_{2'}) = +1$. (For brevity, $g(X)$ denotes the norm $g(X, X)$.) The pseudo-metric g extends to bivectors, has signature $(2, 4)$, and will be again denoted by g : $g(e_a \wedge e_b) = g(e_a)g(e_b) - g(e_a, e_b)^2 = g(e_a)g(e_b)$, so that $\mathcal{B} = (e_1 \wedge e_{1'}, e_1 \wedge e_2, e_1 \wedge e_{2'}, e_{1'} \wedge e_2, e_{1'} \wedge e_{2'}, e_2 \wedge e_{2'})$ is an orthonormal frame of Λ^2 , with $g(e_a \wedge e_b) = -1$, except for $g(e_1 \wedge e_{1'}) = g(e_2 \wedge e_{2'}) = +1$. (Note that the other convention, taking $-g$ does not change the induced metric on Λ^2 .)

Since the volume $e_1 \wedge e_{1'} \wedge e_2 \wedge e_{2'}$ is positively oriented, we construct an orthonormal eigenbasis for the Hodge star on $\Lambda^2 T\mathcal{N}$:

$$\begin{cases} E_1^\pm = \frac{\sqrt{2}}{2}(e_1 \wedge e_{1'} \pm e_2 \wedge e_{2'}) \\ E_2^\pm = \frac{\sqrt{2}}{2}(e_1 \wedge e_2 \pm e_{1'} \wedge e_{2'}) \\ E_3^\pm = \frac{\sqrt{2}}{2}(e_1 \wedge e_{2'} \mp e_{1'} \wedge e_2) \end{cases}$$

so that Λ^\pm is generated by $E_1^\pm, E_2^\pm, E_3^\pm$.

The Kähler condition implies

$$\text{Rm}(JX, JY, Z, T) = \text{Rm}(X, Y, Z, T) = \text{Rm}(X, Y, JZ, JT),$$

because J is isometric and parallel. The matrix of the symmetric 2-form R in

⁵On the contrary, some authors seem to imply that scalar flatness is equivalent to anti-self-duality, see [8]). However this contradiction could possibly come from a different choice of orientation, which would exchange self-dual with anti-self-dual.

the orthonormal frame \mathcal{B} is

	$e_{11'}$	e_{12}	$e_{12'}$	$e_{1'2}$	$e_{1'2'}$	$e_{22'}$
$e_{11'}$	$R_{11'11'}$	$R_{11'12}$	$R_{11'12'}$	$R_{11'1'2} = -R_{11'12'}$	$R_{11'1'2'} = R_{11'12}$	$R_{11'22'}$
e_{12}		R_{1212}	$R_{1212'}$	$R_{121'2} = -R_{1212'}$	$R_{131'2'} = R_{1212}$	$R_{1222'}$
$e_{12'}$			$R_{12'12'}$	$R_{12'1'2} = -R_{12'12'}$	$R_{12'1'2'} = R_{1212'}$	$R_{12'22'}$
$e_{1'2}$				$R_{1'21'2} = R_{12'12'}$	$R_{1'21'2'} = -R_{1212'}$	$R_{1'222'} = -R_{12'22'}$
$e_{1'2'}$					$R_{1'2'1'2'} = R_{1212}$	$R_{1'2'22'} = R_{1222'}$
$e_{22'}$						$R_{22'22'}$

where e_{ab} stands for $e_a \wedge e_b$, for greater legibility. We have written the matrix as a table for clarity and to make symmetries more obvious, and because R is symmetric we need only write half the matrix. We have used the internal symmetries of R , to choose among equivalent coefficients the ones lowest in the lexicographic order of the indices.

The Weyl tensor satisfies *some* of the J-symmetries of R : indeed

$$\begin{aligned} \text{Ric}(JX, JY) &= \sum_{i=1}^4 g(e_i) \text{Rm}(JX, e_i, JY, e_i) = \sum_{i=1}^4 g(e_i) \text{Rm}(X, Je_i, Y, Je_i) \\ &= \sum_{i=1}^4 g(Je_i) \text{Rm}(X, Je_i, Y, Je_i) = \text{Ric}(X, Y) \end{aligned}$$

because (Je_i) is again an orthonormal basis. In particular, this invariance implies $r_{11'} = \text{Ric}(e_1, e_{1'}) = r_{1'1} = -r_{11'}$, so $r_{11'}$ vanish (and so does $r_{22'}$). For the Kulkarni–Nomizu product,

$$\begin{aligned} \text{Ric} \circledast g(JX, Y, Z, T) &= \text{Ric}(JX, Z)g(Y, T) + \text{Ric}(Y, T)g(JX, Z) \\ &\quad - \text{Ric}(JX, T)g(Y, Z) - \text{Ric}(Y, Z)g(JX, T) \\ &= -\text{Ric}(X, JZ)g(JY, JT) - \text{Ric}(JY, JT)g(X, JZ) \\ &\quad + \text{Ric}(X, JT)g(JY, JZ) + \text{Ric}(JY, JZ)g(X, JT) \\ &= -\text{Ric} \circledast g(X, JY, JZ, JT) \end{aligned}$$

so

$$\text{Ric} \circledast g(JX, JY, Z, T) = -\text{Ric} \circledast g(X, J^2Y, JZ, JT) = \text{Ric} \circledast g(X, Y, JZ, JT).$$

Hence the following symmetries (fewer than for Rm) in the coefficients of $\text{Ric} \circledast g$,

$g \otimes g$ and Rm , and therefore W :

	$e_{11'}$	$e_1 \wedge e_2$	$e_{12'}$	$e_{1'} \wedge e_2$	$e_{1'2'}$	$e_{22'}$
$e_{11'}$	$W_{11'11'}$	$W_{11'12}$	$W_{11'12'}$	$W_{11'1'2} = -W_{11'12'}$	$W_{11'1'2'} = W_{11'12}$	$W_{11'22'}$
e_{12}		W_{1212}	$W_{1212'}$	$W_{121'2}$	$W_{121'2'}$	$W_{1222'}$
$e_{12'}$			$W_{12'12'}$	$W_{12'1'2}$	$W_{12'1'2'} = -W_{121'2}$	$W_{12'22'}$
$e_{1'2}$				$W_{1'21'2} = W_{12'12'}$	$W_{1'21'2'} = -W_{1212'}$	$W_{1'222'} = -W_{12'22'}$
$e_{1'2'}$					$W_{1'2'1'2'} = W_{1212}$	$W_{1'2'22'} = W_{1222'}$
$e_{22'}$						$W_{22'22'}$

Expanding on the above eigenbasis of $\Lambda^+ \oplus \Lambda^-$ (which differs from the one in the positive definite case) yields the following Weyl tensor coefficients, which we have simplified using the symmetries above (up to a factor 1/2 due to normalization):

	E_1^+	E_2^+	E_3^+
E_1^+	$W_{11'11'} + W_{22'22'} + 2W_{11'22'}$	$2(W_{11'12} + W_{1222'})$	$2(W_{11'12'} + W_{12'22'})$
E_2^+		$2(W_{1212} + W_{121'2'})$	$2(W_{1212'} - W_{121'2})$
E_3^+			$2(W_{12'12'} - W_{12'1'2})$
E_1^-			
E_2^-			
E_3^-			

	E_1^-	E_2^-	E_3^-
E_1^-	$W_{11'11'} - W_{22'22'}$	0	0
E_2^-	$2(W_{11'12} - W_{1222'})$	0	0
E_3^-	$2(W_{11'12'} - W_{12'22'})$	0	0
E_1^-	$W_{11'11'} + W_{22'22'} - 2W_{11'22'}$	0	0
E_2^-		$2(W_{1212} - W_{121'2'})$	$2(W_{1212'} + W_{121'2})$
E_3^-			$2(W_{12'12'} + W_{12'1'2})$

(Again only half the coefficients are written down.) Further simplifications come from computing W , and using

$$\begin{aligned}
\text{Scal} &= -r_{11} - r_{1'1'} + r_{22} + r_{2'2'} = 2(r_{22} - r_{11}) \\
&= 2(-(-R_{11'11'} + R_{1212} + R_{12'12'}) + (-R_{1212} - R_{1'21'2} + R_{22'22'})) \\
&= 2(R_{11'11'} - 2(R_{1212} + R_{12'12'}) + R_{22'22'}).
\end{aligned}$$

First prove that the Hodge star commutes with W by considering $W(\Lambda^+, \Lambda^-)$:

$$\begin{aligned}
W_{11'11'} &= R_{11'11'} + \frac{1}{2}(r_{11} + r_{1'1'}) + \frac{\text{Scal}}{6} = R_{11'11'} + r_{11} + \frac{\text{Scal}}{6} \\
&= R_{1212} + R_{12'12'} + \frac{\text{Scal}}{6}
\end{aligned}$$

$$\begin{aligned}
W_{22'22'} &= R_{22'22'} - \frac{1}{2}(r_{22} + r_{2'2'}) + \frac{\text{Scal}}{6} = R_{22'22'} - r_{22} + \frac{\text{Scal}}{6} \\
&= R_{1212} + R_{12'12'} + \frac{\text{Scal}}{6}
\end{aligned}$$

so that $W_{11'11'} - W_{22'22'} = 0$. Similarly

$$W_{11'12} = R_{11'12} + \frac{r_{1'2}}{2}, \quad W_{1222'} = R_{1222'} + \frac{r_{12'}}{2} = R_{1222'} - \frac{r_{1'2}}{2}$$

so

$$\begin{aligned}
W_{11'12} - W_{1222'} &= R_{11'12} - R_{1222'} + r_{1'2} = 0 \\
W_{11'12'} &= R_{11'12'} + \frac{r_{1'2'}}{2} = R_{11'12'} + \frac{r_{12}}{2}, \quad W_{12'22'} = R_{12'22'} - \frac{r_{12}}{2}, \\
W_{11'12'} - W_{12'22'} &= R_{11'12'} - R_{12'22'} + r_{12} = 0.
\end{aligned}$$

That proves that W is block-diagonal.

The W^- term satisfies

$$\begin{aligned}
W_{11'11'} + W_{22'22'} - 2W_{11'22'} &= R_{11'11'} + r_{11} + R_{22'22'} - r_{22} + \frac{\text{Scal}}{3} \\
&\quad - 2R_{11'22'} \\
&= R_{11'11'} + R_{22'22'} - 2R_{11'22'} - \frac{\text{Scal}}{6} \\
&= R_{11'11'} + R_{22'22'} - 2(R_{1212} + R_{12'12'}) \\
&\quad - \frac{\text{Scal}}{6} \\
&= \frac{\text{Scal}}{2} - \frac{\text{Scal}}{6} = \frac{\text{Scal}}{3}
\end{aligned}$$

using the first Bianchi identity (and the invariance of Rm):

$$R_{11'22'} = -R_{1'212'} - R_{211'2'} = R_{12'12'} + R_{1212}.$$

$$\begin{aligned}
W_{1212} - W_{121'2'} &= R_{1212} + \frac{r_{22} - r_{11}}{2} - \frac{\text{Scal}}{6} - R_{121'2'} = \frac{\text{Scal}}{4} - \frac{\text{Scal}}{6} \\
&= \frac{\text{Scal}}{12}
\end{aligned}$$

$$W_{12'12'} + W_{12'1'2} = R_{12'12'} + \frac{\text{Scal}}{4} - \frac{\text{Scal}}{6} + R_{12'1'2} = \frac{\text{Scal}}{12}$$

$$W_{1212'} + W_{121'2} = R_{1212'} + \frac{r_{22'}}{2} + R_{121'2} - \frac{r_{11'}}{2} = \frac{1}{2}(r_{22'} - r_{11'}) = 0.$$

Finally,

$$W^- = \text{Scal} \begin{pmatrix} 1/3 & & \\ & 1/6 & \\ & & 1/6 \end{pmatrix} = \frac{\text{Scal}}{6} \text{Id} + \frac{\text{Scal}}{6} E_1^- \otimes E_1^-$$

(and indeed this matrix is traceless w.r.t. the pseudo-metric g). One should note that the above expression differs from the Riemannian case, where

$$W^+ = \text{Scal} \begin{pmatrix} 1/3 & & \\ & -1/6 & \\ & & -1/6 \end{pmatrix} = -\frac{\text{Scal}}{6} \text{Id} + \frac{\text{Scal}}{3} E_1^+ \otimes E_1^+.$$

We let the Reader check that in the neutral case, the W^+ part is not a multiple of the scalar curvature, which completes the proof of Theorem A.2.

A.2 The para-Kähler case

The computations are almost identical, but the results differ from the pseudo-Kähler setup, because the para-complex structure J is now an anti-isometry: $R(JX, JY)Z = -R(X, Y)Z$. We pick an orthonormal basis $(e_1, e_{1'}, e_2, e_{2'})$ with $e_{1'} = Je_1$, $e_{2'} = Je_2$, and $g(e_1) = g(e_2) = +1$, $g(e_{1'}) = g(e_{2'}) = -1$. The frame $\mathcal{B} = (e_1 \wedge e_{1'}, e_1 \wedge e_2, e_1 \wedge e_{2'}, e_{1'} \wedge e_2, e_{1'} \wedge e_{2'}, e_2 \wedge e_{2'})$ of $\Lambda^2 T\mathcal{N}$ is also orthonormal w.r.t. the induced metric on Λ^2 , again denoted by g , which has signature $(2, 4)$: $g(e_a \wedge e_b) = g(e_a)g(e_b) = -1$, except for $g(e_1 \wedge e_2) = g(e_{1'} \wedge e_{2'}) = +1$.

An orthonormal eigenbasis for the Hodge operator is the following:

$$\begin{cases} E_1^\pm = \frac{\sqrt{2}}{2}(e_1 \wedge e_{1'} \mp e_2 \wedge e_{2'}) \\ E_2^\pm = \frac{\sqrt{2}}{2}(e_1 \wedge e_2 \mp e_{1'} \wedge e_{2'}) \\ E_3^\pm = \frac{\sqrt{2}}{2}(e_1 \wedge e_{2'} \mp e_{1'} \wedge e_2) \end{cases}$$

where the E_a^+ (resp. E_a^-) span Λ^+ (resp. Λ^-). (Note the sign differences w.r.t. the pseudo-Kähler case.)

Since J is anti-isometric and parallel,

$$\text{Rm}(JX, JY, Z, T) = -\text{Rm}(X, Y, Z, T) = \text{Rm}(X, Y, JZ, JT).$$

Hence the following symmetries of the riemannian curvature operator R , expressed in the frame \mathcal{B} (for symmetry reasons and greater legibility, lower left coefficients are not written in this and the subsequent matrices):

	$e_{11'}$	e_{12}	$e_{12'}$	$e_{1'} \wedge e_2$	$e_{1'2'}$	$e_{22'}$
$e_{11'}$	$R_{11'11'}$	$R_{11'12}$	$R_{11'12'}$	$R_{11'1'2}$ $= -R_{11'12'}$	$R_{11'1'2'}$ $= -R_{11'12}$	$R_{11'22'}$
e_{12}		R_{1212}	$R_{1212'}$	$R_{121'2}$ $= -R_{1212'}$	$R_{121'2'}$ $= -R_{1212}$	$R_{1222'}$
$e_{12'}$			$R_{12'12'}$	$R_{12'1'2}$ $= -R_{12'12'}$	$R_{12'1'2'}$ $= -R_{1212'}$	$R_{12'22'}$
$e_{1'2}$				$R_{1'21'2}$ $= R_{12'12'}$	$R_{1'21'2'}$ $= R_{1212'}$	$R_{1'2'22'}$ $= -R_{12'22'}$
$e_{1'2'}$					$R_{1'2'1'2'}$ $= R_{1212}$	$R_{1'2'22'}$ $= -R_{1222'}$
$e_{22'}$						$R_{22'22'}$

(Note again the similarity with the pseudo-Kähler case: only a few signs change.)

The Weyl tensor satisfies *some* of the J-symmetries of Rm since

$$\begin{aligned}\text{Ric}(JX, JY) &= \sum_{i=1}^4 g(e_i) \text{Rm}(JX, e_i, JY, e_i) = \sum_{i=1}^4 g(e_i) \text{Rm}(X, Je_i, Y, Je_i) \\ &= - \sum_{i=1}^4 g(Je_i) \text{Rm}(X, Je_i, Y, Je_i) = -\text{Ric}(X, Y)\end{aligned}$$

since (Je_i) is also an orthonormal basis. In particular this invariance implies $r_{1'1} = r_{11'} = -r_{1'1}$, so $r_{11'}$ vanishes (and so does $r_{22'}$). Finally,

$$\frac{\text{Scal}}{2} = r_{11} + r_{22} = -R_{11'11'} + 2(R_{1212} - R_{12'12'}) - R_{22'22'}.$$

The Kulkarni–Nomizu product $\text{Ric} \otimes g$ satisfies

$$\begin{aligned}\text{Ric} \otimes g(JX, Y, Z, T) &= \text{Ric}(JX, Z)g(Y, T) + \text{Ric}(Y, T)g(JX, Z) \\ &\quad - \text{Ric}(JX, T)g(Y, Z) - \text{Ric}(Y, Z)g(JX, T) \\ &= \text{Ric}(X, JZ)g(JY, JT) + \text{Ric}(JY, JT)g(X, JZ) \\ &\quad - \text{Ric}(X, JT)g(JY, JZ) - \text{Ric}(JY, JZ)g(X, JT) \\ &= \text{Ric} \otimes g(X, JY, JZ, JT)\end{aligned}$$

so

$$\text{Ric} \otimes g(JX, JY, Z, T) = \text{Ric} \otimes g(X, J^2Y, JZ, JT) = \text{Ric} \otimes g(X, Y, JZ, JT)$$

and the same property holds for $g \otimes g$. Hence the following symmetries (fewer than for Rm) in the coefficients of $\text{Ric} \otimes g$, $g \otimes g$ and Rm, and therefore W:

	$e_{11'}$	e_{12}	$e_{12'}$	$e_{1'2}$	$e_{1'2'}$	$e_{22'}$
$e_{11'}$	$W_{11'11'}$	$W_{11'12}$	$W_{11'12'}$	$W_{11'1'2} = -W_{11'12'}$	$W_{11'1'2'} = -W_{11'12}$	$W_{11'22'}$
e_{12}		W_{1212}	$W_{1212'}$	$W_{121'2}$	$W_{121'2'}$	$W_{1222'}$
$e_{12'}$			$W_{12'12'}$	$W_{12'1'2}$	$W_{12'1'2'} = W_{121'2}$	$W_{12'22'}$
$e_{1'2}$				$W_{1'21'2} = W_{12'12'}$	$W_{1'21'2'} = W_{1212'}$	$W_{1'222'} = -W_{12'22'}$
$e_{1'2'}$					$W_{1'2'1'2'} = W_{1212}$	$W_{1'2'22'} = -W_{1222'}$
$e_{22'}$						$W_{22'22'}$

Let us now express W in the Hodge basis defined earlier, using the above sym-

metries (up to a factor 1/2 due to normalization).

	E_1^+	E_2^+	E_3^+
E_1^+	$W_{11'11'} + W_{22'22'} - 2W_{11'22'}$	$2(W_{11'12} - W_{1222'})$	$2(W_{11'12'} - W_{12'22'})$
E_2^+		$2(W_{1212} - W_{121'2'})$	$2(W_{1212'} - W_{121'2})$
E_3^+			$2(W_{12'12'} - W_{12'1'2})$
E_1^-			
E_2^-			
E_3^-			

	E_1^-	E_2^-	E_3^-
E_1^+	$W_{11'11'} - W_{22'22'}$	0	0
E_2^+	$2(W_{11'12} + W_{1222'})$	0	0
E_3^+	$2(W_{11'12'} + W_{12'22'})$	0	0
E_1^-	$W_{11'11'} + W_{22'22'} + 2W_{11'22'}$	0	0
E_2^-		$2(W_{1212} + W_{121'2'})$	$2(W_{1212'} + W_{121'2})$
E_3^-			$2(W_{12'12'} + W_{12'1'2})$

Only three terms in the off-block-diagonal part are not obviously zero.

$$W_{11'11'} = R_{11'11'} - \frac{1}{2}(-r_{11} + r_{1'1'}) - \frac{\text{Scal}}{6} = R_{11'11'} + r_{11} - \frac{\text{Scal}}{6}$$

$$W_{22'22'} = R_{22'22'} - \frac{1}{2}(-r_{22} + r_{2'2'}) - \frac{\text{Scal}}{6} = R_{22'22'} + r_{22} - \frac{\text{Scal}}{6}$$

but $r_{11} = -R_{11'11'} + R_{1212} - R_{12'12'}$ and $r_{22} = R_{2121} - R_{21'21'} - R_{22'22'} = R_{1212} - R_{12'12'} - R_{22'22'}$ so that

$$W_{11'11'} - W_{22'22'} = R_{11'11'} - R_{22'22'} + r_{11} - r_{22} = 0.$$

Similarly

$$W_{11'12} + W_{1222'} = R_{11'12} - \frac{r_{1'2}}{2} + R_{1222'} + \frac{r_{12'}}{2} = R_{11'12} + R_{1222'} - r_{1'2} = 0$$

$$W_{11'12'} + W_{12'22'} = R_{11'12'} - \frac{r_{1'2'}}{2} + R_{12'22'} + \frac{r_{12}}{2} = R_{11'12'} + R_{12'22'} + r_{12} = 0$$

which proves that W is block-diagonal, i.e. commutes with the Hodge operator.

Let us now look more closely at the W^- term

$$\begin{pmatrix} W_{11'11'} + W_{22'22'} + 2W_{11'22'} & 0 & 0 \\ & 2(W_{1212} + W_{121'2'}) & 2(W_{1212'} + W_{121'2}) \\ & & 2(W_{12'12'} + W_{12'1'2}) \end{pmatrix}$$

$$\begin{aligned}
& W_{11'11'} + W_{22'22'} + 2W_{11'22'} \\
&= R_{11'11'} + r_{11} - \frac{\text{Scal}}{6} + R_{22'22'} + r_{22} - \frac{\text{Scal}}{6} + 2R_{11'22'} \\
&= R_{11'11'} + R_{22'22'} + 2R_{11'22'} + \frac{\text{Scal}}{2} - \frac{\text{Scal}}{3} \\
&= R_{11'11'} + R_{22'22'} + 2(-R_{1212} + R_{12'12'}) + \frac{\text{Scal}}{6} = -\frac{\text{Scal}}{3}
\end{aligned}$$

where we have used the first Bianchi identity (and the invariance of Rm)

$$R_{11'22'} = -R_{1'212'} - R_{211'2'} = R_{12'12'} - R_{1212}.$$

$$\begin{aligned}
W_{1212} + W_{121'2'} &= R_{1212} - \frac{r_{22} + r_{11}}{2} + \frac{\text{Scal}}{6} + R_{121'2'} \\
&= R_{1212} - \frac{\text{Scal}}{4} + \frac{\text{Scal}}{6} + R_{121'2'} = -\frac{\text{Scal}}{12}
\end{aligned}$$

$$W_{12'12'} + W_{12'1'2} = R_{12'12'} + \frac{\text{Scal}}{4} - \frac{\text{Scal}}{6} + R_{12'1'2} = \frac{\text{Scal}}{12}$$

$$W_{1212'} + W_{121'2} = R_{1212'} - \frac{r_{22'}}{2} + R_{121'2} - \frac{r_{11'}}{2} = 0.$$

Finally,

$$W^- = \text{Scal} \begin{pmatrix} -1/3 & & \\ & -1/6 & \\ & & 1/6 \end{pmatrix}$$

vanishes if and only if $\text{Scal} = 0$. (The Reader will check that this matrix is indeed traceless w.r.t. the pseudo-metric g .)

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