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► **To cite this version:**

Henri Anciaux, Pascal Romon. A canonical structure on the tangent bundle of a pseudo- or para-Kähler manifold. 2013. hal-00778411v1

**HAL Id: hal-00778411**

**<https://hal.science/hal-00778411v1>**

Preprint submitted on 20 Jan 2013 (v1), last revised 2 Sep 2013 (v3)

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# A canonical structure on the tangent bundle of a pseudo- or para-Kähler manifold

Henri Anciaux\*, Pascal Romon†

## Abstract

Let  $(\mathcal{M}, J, g, \omega)$  be a pseudo-Kähler or para-Kähler  $2n$ -dimensional manifold. We prove that the tangent bundle  $T\mathcal{M}$  enjoys a natural pseudo-Kähler or para-Kähler structure  $(\tilde{J}, \tilde{g}, \Omega)$ , where  $\Omega$  is the pull-back by  $g$  of the canonical symplectic structure on  $T^*\mathcal{M}$ , and  $\tilde{g}$  is a pseudo-Riemannian metric with neutral signature  $(2n, 2n)$ . We investigate the curvature properties of the pair  $(\tilde{J}, \tilde{g})$ : we prove that  $\tilde{g}$  is scalar-flat, is not Einstein unless  $g$  is flat, has nonpositive (resp. nonnegative) Ricci curvature if and only if  $g$  has nonpositive (resp. nonnegative) Ricci curvature as well, and is locally conformally flat if and only if  $n = 1$  and  $g$  has constant curvature, or  $n > 2$  and  $g$  is flat. We also check that (i) the holomorphic sectional curvature of  $(\tilde{J}, \tilde{g})$  is not constant unless  $g$  is flat, and (ii) in  $n = 1$  case, that  $\tilde{g}$  is never anti-self-dual, unless conformally flat.

2000 MSC: 32Q15, 53D05

## Introduction

It is a classical fact that given any differentiable manifold  $\mathcal{M}$ , its cotangent bundle  $T^*\mathcal{M}$  enjoys a canonical symplectic structure  $\Omega^*$ .

Moreover, given a linear connection  $\nabla$  on a manifold  $\mathcal{M}$ , (e.g. the Levi-Civita connection of a Riemannian metric), the bundle  $T\mathcal{M}$  splits into a direct sum of two subbundles  $H\mathcal{M}$  and  $V\mathcal{M}$  isomorphic to  $T\mathcal{M}$ . This allows to define an almost complex structure  $J$  by setting  $J(X_h, X_v) := (-X_v, X_h)$ , where, for  $X \in T\mathcal{M} = H\mathcal{M} \oplus V\mathcal{M}$ , we write  $X \simeq (X_h, X_v) \in T\mathcal{M} \times T\mathcal{M}$ . Analogously, one may introduce a natural almost para-complex (or bi-Lagrangian) structure, setting  $J'(X_h, X_v) := (X_v, X_h)$ .

It is also well known that the tangent bundle of a Riemannian manifold  $(\mathcal{M}, g)$  can be given a natural Riemannian structure, called *Sasaki metric*. A simple way to understand this construction, which extends *verbatim* to the case of a pseudo-Riemannian metric  $g$  with signature  $(p, m - p)$ , is as follows: using the splitting  $T\mathcal{M} = H\mathcal{M} \oplus V\mathcal{M}$ , we set:

$$G((X_h, X_v), (Y_h, Y_v)) := g(X_h, Y_h) + g(X_v, Y_v).$$

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This metric has signature  $(2p, 2(m-p))$  and is well behaved with respect to  $J$  in two ways: (i)  $G$  is compatible with  $J$ , i.e.  $G(.,.) = G(J., J.)$ , and (ii) the symplectic form  $\Omega := G(J., .)$  is nothing but the pull-back of  $\Omega^*$  by the musical isomorphism between  $T\mathcal{M} \simeq_g T^*\mathcal{M}$ . In other words, the triple  $(J, G, \Omega)$  defines an “almost pseudo-Kähler” structure<sup>1</sup> on  $T\mathcal{M}$ .

Unfortunately, this construction suffers two flaws:  $J$  is not integrable unless  $\nabla$  is flat and the metric  $G$  is somewhat “rigid”: for example, if  $G$  has constant scalar curvature, then  $g$  is flat (see [MT]). We refer to [BV, YI] and references therein for more detail on the Sasaki metric.

Another construction can be made in the case where  $\mathcal{M}$  is complex (resp. para-complex): in this case both  $T\mathcal{M}$  and  $T^*\mathcal{M}$  enjoy a canonical complex (resp. para-complex) structure which are defined as follows: given a family of holomorphic (resp. para-holomorphic<sup>2</sup>) local charts  $\varphi : \mathcal{M} \rightarrow \mathcal{U} \subset \mathbb{R}^{2n}$  on  $\mathcal{M}$ , we define holomorphic (resp. para-holomorphic) local charts  $\bar{\varphi} : T\mathcal{M} \rightarrow \mathcal{U} \times \mathbb{R}^{2n}$  by  $\bar{\varphi}(x, V) = (\varphi(x), d\varphi_x(V))$ ,  $\forall (x, V) \in T\mathcal{M}$  for the tangent bundle, and  $\bar{\varphi} : T^*\mathcal{M} \rightarrow \mathcal{U} \times \mathbb{R}^{2n}$  by  $\bar{\varphi}(x, \xi) = (\varphi(x), ((d\varphi_x)^t)^{-1}(\xi))$ ,  $\forall (x, \xi) \in T^*\mathcal{M}$  for the cotangent bundle. In the first section we shall prove that these canonical structures can also be constructed using the splitting  $H\mathcal{M} \oplus V\mathcal{M}$ , when  $\mathcal{M}$  is endowed with a linear connection.

Combining the canonical symplectic structure  $\Omega^*$  of  $T^*\mathcal{M}$  with the canonical complex (resp. para-complex) structure  $\tilde{J}^*$  just defined, it is natural to introduce a 2-tensor  $\tilde{g}^*$  by the formula

$$\tilde{g}^* := \Omega^*(., \tilde{J}^*.).$$

However, it turns out that  $\Omega^*$  is not compatible with  $\tilde{J}^*$ , since it turns out that  $\Omega^*(\tilde{J}^*., \tilde{J}^*.) = -\varepsilon\Omega^*$  instead of the required formula  $\Omega^*(\tilde{J}^*., \tilde{J}^*.) = \varepsilon\Omega^*$  (here and in the following, in order to deal simultaneously with the complex and para-complex cases, we define  $\varepsilon$  to be such that  $(\tilde{J}^*)^2 = -\varepsilon\text{Id}$ , i.e.  $\varepsilon = 1$  in the complex case and  $\varepsilon = -1$  in the para-complex case). It follows that the tensor  $\tilde{g}^*$  is not symmetric and therefore we failed in constructing a canonical pseudo-Riemannian structure on  $T^*\mathcal{M}$ .

On the other hand, the same idea works well if one considers, instead of the cotangent bundle, the tangent bundle of a pseudo- or para-Kähler manifold  $(\mathcal{M}, J, g)$ , thus obtaining a canonical pseudo- or para-Kähler structure. The purpose of this note is to investigate in detail this construction and to study its curvature properties. The results are summarized in the following:

<sup>1</sup>We might also define an “almost para-Kähler” structure on  $T\mathcal{M}$  by introducing the *para-Sasaki metric*

$$G'((X_h, X_v), (Y_h, Y_v)) := g(X_h, Y_h) - g(X_v, Y_v).$$

This metric has neutral signature  $(m, m)$  ( $m$  being the dimension of  $\mathcal{M}$ ), is compatible with  $J'$  and verifies  $\Omega := -G'(J'., .)$ .

<sup>2</sup> The terminology *split-holomorphic* is sometimes used.

**Main Theorem** *Let  $(\mathcal{M}, J, g, \omega)$  be a pseudo- or para-Kähler manifold. Then  $T\mathcal{M}$  enjoys a natural pseudo- or para-Kähler structure  $(\tilde{J}, \tilde{g}, \Omega)$  with the following properties:*

- $\tilde{J}$  is the canonical complex or para-complex structure of  $T\mathcal{M}$  induced from that of  $\mathcal{M}$ ;
- $\Omega$  is the pull-back of  $\Omega^*$  by the metric isomorphism  $T\mathcal{M} \simeq_g T^*\mathcal{M}$ ;
- The pseudo-Riemannian metric  $\tilde{g}$  can be recovered from  $\tilde{J}$  and  $\Omega$  by the equation  $\tilde{g}(\cdot, \cdot) := \Omega(\cdot, \tilde{J}\cdot)$ ;
- According to the splitting  $TT\mathcal{M} = H\mathcal{M} \oplus V\mathcal{M}$  induced by the Levi-Civita connection of  $g$ , the triple  $(\tilde{J}, \tilde{g}, \Omega)$  takes the following expression:

$$\begin{aligned} \tilde{J}(X_h, X_v) &:= (JX_h, JX_v) \\ \tilde{g}((X_h, X_v), (Y_h, Y_v)) &:= g(X_v, JY_h) - g(X_h, JY_v) \\ \Omega((X_h, X_v), (Y_h, Y_v)) &:= g(X_v, Y_h) - g(X_h, Y_v); \end{aligned}$$

- The pseudo-Riemannian metric  $\tilde{g}$  has the following properties:
  - (i)  $\tilde{g}$  has neutral signature neutral  $(2n, 2n)$  and is scalar flat;
  - (ii)  $(\mathcal{M}, \tilde{g})$  is Einstein if and only if  $(\mathcal{M}, g)$  is flat;
  - (iii) the Ricci curvature  $\widetilde{\text{Ric}}$  of  $\tilde{g}$  has the same sign as the Ricci curvature  $\text{Ric}$  of  $g$ ;
  - (iv)  $(\mathcal{M}, \tilde{g})$  is locally conformally flat if and only if  $n = 1$  and  $g$  has constant curvature, or  $n > 2$  and  $g$  is flat; if  $n = 1$ ,  $\tilde{g}$  is always self-dual, so anti-self-duality is equivalent to conformal flatness;
  - (v) the pair  $(\tilde{J}, \tilde{g})$  has constant holomorphic curvature if and only if  $g$  is flat.

**Remark 1.** *We use in (iv) the general property that four-dimensional neutral pseudo-Kähler or para-Kähler manifolds are self-dual if and only if their scalar curvature vanishes. This is analogous to the case of Kähler four-dimensional manifolds, except that self-duality is exchanged with anti-self-duality. A proof of this statement is given in theorem A.2 in the appendix.*

This result is a generalization of previous work on the tangent bundle of a Riemannian surface (see [GK1], [GK2], [AGR]). The authors wish to thank Brendan Guilfoyle for his valuable suggestions and comments.

# 1 Almost complex and para-complex structures on the tangent and cotangent bundles

Given a manifold  $\mathcal{M}$  endowed with an almost complex or almost para-complex structure  $J$ , it is only natural to ask whether its tangent or cotangent bundle inherit such a structure. The answer is positive:

**Proposition 1.** *Let  $(\mathcal{M}, J)$  be an almost complex (resp. para-complex) manifold. Then the tangent and cotangent bundles admit a canonical almost complex (resp. para-complex) structure  $\tilde{J}$ . Furthermore, if  $J$  is complex (resp. para-complex), then so is  $\tilde{J}$ .*

*Proof.* We prove the result using coordinate charts, which amounts to showing that  $\tilde{J}$  can be defined independently of any change of variable. Let  $y = \varphi(x)$  be a local change of coordinates on  $\mathbb{R}^n$  and write  $\xi$  and  $\eta$  respectively for the tangent coordinates induced by the charts (i.e.  $\sum_i \xi^i \partial / \partial x^i = \sum_i \eta^i \partial / \partial y^i$ ). The change of tangent coordinates at  $x$  is  $\xi \mapsto \eta = d\varphi(x)\xi$ , in other words  $\varphi$  induces a chart  $\Phi$  on  $\mathbb{R}^{2n}$ ,  $\Phi : (x, \xi) \mapsto (\varphi(x), d\varphi(x)\xi)$ . The tangent coordinates at  $(x, \xi)$  (resp.  $(y, \eta)$ ) are denoted by  $(X, \Xi)$  (resp.  $(Y, H)$ ) and the change of (doubly) tangent coordinates is

$$d\Phi(x, \xi) : (X, \Xi) \mapsto (Y, H) = (d\varphi(x)X, d^2\varphi(x)(X, \xi) + d\varphi(x)\Xi).$$

Assume moreover that we have a  $(1, 1)$  tensor, which reads in the  $x$  coordinate as the matrix  $J(x)$  and in the  $y$  coordinate as the matrix  $J'(y) = J'(\varphi(x)) = d\varphi(x) \circ J(x) \circ (d\varphi(x))^{-1}$ . Equivalently for any  $X$  and  $Y = d\varphi(x)X$ , we have  $J'(y)Y = J'(\varphi(x))d\varphi(x)X = d\varphi(x)J(x)X$ . Differentiating this equality along  $\xi$  yields

$$\begin{aligned} (D_{d\varphi(x)\xi}J')(\varphi(x))d\varphi(x)X + J'(\varphi(x))d^2\varphi(x)(X, \xi) \\ = d\varphi(x)(D_\xi J)(x)X + d^2\varphi(x)(J(x)X, \xi), \end{aligned} \quad (1)$$

where  $(D_\xi J)(x)$  denotes in this proof the directional derivative of the matrix  $J$  at  $x$  in the direction  $\xi$  (not a covariant derivative).

We now define the  $(1, 1)$  tensor  $\tilde{J}$  in the  $(x, \xi)$  coordinate by

$$\tilde{J}(x, \xi) : (X, \Xi) \mapsto (J(x)X, J(x)\Xi + D_\xi J(x)X).$$

Let us prove that this definition is coordinate-independent (for greater readability we will often write  $\tilde{J}, \tilde{J}'$  for  $J(x), J'(y)$ ). Using (1) and the symmetry of the

second order differential  $d^2\varphi(x)$ ,

$$\begin{aligned}
d\Phi(x, \xi)(J(X, \Xi)) &= d\Phi(x, \xi)(JX, J\Xi + D_\xi J(x)X) \\
&= (d\varphi(x)JX, d^2\varphi(x)(JX, \xi) + d\varphi(x)(J\Xi + D_\xi J(x)X)) \\
&= (J'Y, J'd\varphi(x)\Xi \\
&\quad + (D_{d\varphi(x)\xi}J')(\varphi(x))d\varphi(x)X + J'd^2\varphi(x)(X, \xi)) \\
&= (J'Y, J'(d\varphi(x)\Xi + d^2\varphi(x)(X, \xi)) \\
&\quad + (D_{d\varphi(x)\xi}J')(\varphi(x))d\varphi(x)X) \\
&= (J'Y, J'H + D_\eta J'(y)Y) = \tilde{J}'(y, \eta)(Y, H),
\end{aligned}$$

where  $\tilde{J}'$  denotes the map corresponding to  $\tilde{J}$  in the  $(y, \eta)$  coordinates. Consequently the tensor on  $\mathcal{M}$  extends naturally to  $T\mathcal{M}$ .

The proof for the cotangent bundle is more involved. Let us denote by  $(x, a)$  a local coordinate on  $T^*\mathcal{M}$ , where  $a = (a_1, \dots, a_n)$  corresponds to  $\sum a_i dx^i$  at  $x$ . We write again  $\Phi$  for the change of variable induced by  $y = \varphi(x)$ , so that  $\Phi(x, a) = (y, b)$  with  $b = {}^t(d\varphi(x))^{-1}a$ . The change of variable induced on  $TT^*\mathcal{M}$  is therefore

$$d\Phi(x, a) : (X, A) \mapsto (Y, B) = (d\varphi(x)X, {}^t(d\varphi(x))^{-1}A + \Psi(X)a)$$

where  $\Psi(X) = D_X {}^t(d\varphi(x))^{-1}$ . The matrix  $\Psi(X)$  cannot be written explicitly but its value is obtained through duality: for all  $\eta, a$ ,  ${}^t\eta\Psi(X)a = D_X({}^t\eta({}^t(d\varphi(x))^{-1}a))$ , so, dropping the  $a$  and transposing,

$${}^t\Psi(X)\eta = D_X(d\varphi(x))^{-1}\eta = D_X(d\varphi^{-1}(\varphi(x))\eta) = d^2\varphi^{-1}(y)(d\varphi(x)X, \eta).$$

Let  $J(x), J'(y)$  be defined as above. A consequence of  $d\varphi$  exchanging  $J$  with  $J'$  is

$$\begin{aligned}
{}^t\Psi(jX)\eta &= d^2\varphi^{-1}(y)(d\varphi(x)jX, \eta) = d^2\varphi^{-1}(y)(J'd\varphi(x)X, \eta) \\
&= Jd^2\varphi^{-1}(y)(Y, \eta) + D_\xi jX - d\varphi^{-1}(y)D_\eta j'Y
\end{aligned}$$

where we have applied (1) to  $\varphi^{-1}$ :

$$(D_\xi j)X + Jd^2\varphi^{-1}(y)(Y, \eta) = d\varphi^{-1}(y)(D_\eta j')Y + d^2\varphi^{-1}(y)(J'Y, \eta).$$

Now set

$$\tilde{J}^*(x, a) : (X, A) \mapsto (JX, {}^tJA + \sigma a)$$

where  $\sigma = \sigma(x, a, X)$  is the matrix such that, for all  $\xi$

$${}^t\sigma\xi = (D_X J)(x)\xi - (D_\xi J)(x)X$$

(which exists since  $\xi \mapsto (D_\xi J)(x)X$  is linear in  $\xi$ ). We write  $\sigma'$  for the corresponding term in the  $(y, b)$  coordinate. We need only check that this definition is independent of the chart. Namely that

$$\begin{aligned}
d\Phi(x, a)(\tilde{J}^*(X, A)) &= (d\varphi(x)JX, {}^t(d\varphi(x))^{-1}({}^tJA + \sigma a) + \Psi(JX)a) \\
&= (J'Y, {}^tJ'({}^t(d\varphi(x))^{-1}A + {}^t(d\varphi(x))^{-1}\sigma a) + \Psi(JX)a)
\end{aligned}$$

is equal to

$$\begin{aligned}\tilde{J}'^*(Y, B) &= (JX, {}^tJB + \sigma'b) \\ &= (J'Y, {}^tJ'({}^t(d\varphi(x))^{-1}A + \Psi(X)a) + \sigma'{}^t(d\varphi(x))^{-1}a).\end{aligned}$$

Indeed, given any vector  $\xi$  and  $\eta = d\varphi(x)\xi$ , and transposing for greater legibility,

$$\begin{aligned}{}^t\sigma(d\varphi(x))^{-1}\eta + {}^t\Psi(JX)\eta &= (D_XJ)\xi - (D_\xi J)X \\ &\quad + Jd^2\varphi^{-1}(y)(Y, \eta) + (D_\xi J)X - d\varphi^{-1}(y)(D_\eta J')Y \\ &= (D_XJ)\xi + Jd^2\varphi^{-1}(y)(Y, \eta) - d\varphi^{-1}(y)(D_\eta J')Y,\end{aligned}$$

while

$$\begin{aligned}(d\varphi(x))^{-1}{}^t\sigma'\eta + {}^t\Psi(X)J'\eta &= (d\varphi(x))^{-1}((D_Y J')\eta - (D_\eta J')Y) + d^2\varphi^{-1}(y)(Y, J'\eta) \\ &= (d\varphi(x))^{-1}((D_Y J')\eta - (D_\eta J')Y) + d^2\varphi^{-1}(y)(J'\eta, Y) \\ &= (d\varphi(x))^{-1}((D_Y J')\eta - (D_\eta J')Y) + (D_XJ)\xi \\ &\quad + Jd^2\varphi^{-1}(y)(\eta, Y) - d\varphi^{-1}(y)(D_Y J')\eta \\ &= (D_XJ)\xi + Jd^2\varphi^{-1}(y)(\eta, Y) - (d\varphi(x))^{-1}((D_\eta J')Y) \\ &= {}^t\sigma(d\varphi(x))^{-1}\eta + {}^t\Psi(JX)\eta.\end{aligned}$$

where again we have used twice the symmetry of  $d^2\varphi^{-1}(y)$  and (1), with  $Y$  and  $\eta$  inverted.

We have so far defined a (1,1) tensor on  $T\mathcal{M}$  and  $T^*\mathcal{M}$  without extra assumptions. Suppose now that  $J$  is an almost complex (resp. para-complex) structure, so that  $J^2 = -\varepsilon\text{Id}$ . Taking the derivative of this property yields  $J D_\xi J + (D_\xi J)J = 0$ . Then

$$\begin{aligned}\tilde{J}^2(X, \Xi) &= (J^2X, J(J\Xi + D_\xi JX) + D_\xi J(JX)) \\ &= (-\varepsilon X, -\varepsilon\Xi + J(dJ\xi)X + (dJ\xi)(JX)) = -\varepsilon(X, \Xi)\end{aligned}$$

so that  $\tilde{J}$  is also an almost complex (resp. para-complex) structure.

Finally if  $J$  is a complex (resp. para-complex) structure then we can use complex (resp. para-complex) coordinate charts, which amounts to saying that  $J$  is a constant matrix. Then  $\tilde{J}$  defined in the associated charts on  $T\mathcal{M}$  takes a simpler expression, and is also constant:

$$\tilde{J}(x, \xi) : (X, \Xi) \mapsto (JX, J\Xi)$$

and that characterizes a complex (resp. para-complex) structure.  $\square$

**Remark 2.** *Such a result has been proven already by Lempert & Szöke [LS] for the tangent bundle in the almost complex case (but not for the cotangent bundle). Their construction uses the jets over  $\mathcal{M}$  and is quite a bit more technical than our proof. However it gives an interesting interpretation of the meaning of  $\tilde{\mathcal{J}}$ . We shall see below in Proposition 2 a different and simpler way of defining and understanding  $\tilde{\mathcal{J}}$ , provided  $\mathcal{M}$  is a pseudo- or para-Kähler manifold.*

## 2 The Kähler structure

Let  $\mathcal{M}$  be a differentiable manifold. We denote by  $\pi$  and  $\pi^*$  the canonical projections  $T\mathcal{M} \rightarrow \mathcal{M}$  and  $T\mathcal{M}^* \rightarrow \mathcal{M}$ . The subbundle  $\ker(d\pi) := V\mathcal{M}$  of  $TT\mathcal{M}$  (it is thus a bundle over  $T\mathcal{M}$ ) will be called *the vertical bundle*.

Assume now that  $\mathcal{M}$  is equipped with a linear connection  $\nabla$ . The corresponding horizontal bundle is defined as follows: let  $\bar{X}$  be a tangent vector to  $T\mathcal{M}$  at some point  $(x_0, V_0)$ . This implies that there exists a curve  $\gamma(s) = (x(s), V(s))$  such that  $(x(0), V(0)) = (x_0, V_0)$  and  $\gamma'(0) = \bar{X}$ . If  $X \notin V\mathcal{M}$  (which implies  $x'(0) \neq 0$ ), we define the connection map (see [Do], [AGR])  $K : TT\mathcal{M} \rightarrow T\mathcal{M}$  by  $K\bar{X} = \nabla_{x'(0)}V(0)$ , where  $\nabla$  denotes the Levi-Civita connection of the metric  $g$ . If  $X$  is vertical, we may assume that the curve  $\gamma$  stays in a fiber so that  $V(s)$  is a curve in a vector space. We then define  $K\bar{X}$  to be simply  $V'(0)$ . The horizontal bundle is then  $\text{Ker}(K)$  and we have a direct sum

$$\begin{aligned} TT\mathcal{M} &= H\mathcal{M} \oplus V\mathcal{M} \simeq T\mathcal{M} \oplus T\mathcal{M} \\ \bar{X} &\simeq (\Pi\bar{X}, K\bar{X}). \end{aligned} \quad (2)$$

Here and in the following,  $\Pi$  is a shorthand notation for  $d\pi$ .

**Lemma 1.** [Do] *Given a vector field  $X$  on  $(\mathcal{M}, \nabla)$  there exists exactly one vector field  $X^h$  and one vector field  $X^v$  on  $T\mathcal{M}$  such that  $(\Pi X^h, KX^h) = (X, 0)$  and  $(\Pi X^v, KX^v) = (0, X)$ . Moreover, given two vector fields  $X$  and  $Y$  on  $(\mathcal{M}, \nabla)$ , we have, at the point  $(x, V)$ :*

$$\begin{aligned} [X^v, Y^v] &= 0, \\ [X^h, Y^v] &= (\nabla_X Y)^v \simeq (0, \nabla_X Y), \\ [X^h, Y^h] &\simeq ([X, Y], -R(X, Y)V), \end{aligned}$$

where  $R$  denotes the curvature of  $\nabla$  and we use the direct sum notation (2).

The Reader should not confuse the horizontal lift  $X^h$ , which is a vector field on  $T\mathcal{M}$  constructed from a vector field  $X \in \mathfrak{X}(\mathcal{M})$ , with the notation  $\bar{X}_h = \Pi\bar{X}$  denoting the horizontal part of  $\bar{X} \in \mathfrak{X}(T\mathcal{M})$ . Similarly, the vertical lift  $X^v$  is not the vertical projection  $\bar{X}_v = K\bar{X}$ .

We say that a vector field  $\bar{X}$  on  $T\mathcal{M}$  is *projectable* if it is constant on the fibres, i.e.  $(\Pi\bar{X}, K\bar{X})(x, V) = (\Pi\bar{X}, K\bar{X})(x, V')$ . According to the lemma above, it is equivalent to the fact that there exists two vector fields  $X_1$  and  $X_2$  on  $\mathcal{M}$  such that  $\bar{X} = (X_1)^h + (X_2)^v$ .



Assume now that  $\mathcal{M}$  is equipped with a pseudo-Riemannian metric  $g$ , i.e. a non-degenerate bilinear form. By the non-degeneracy assumption, we can identify  $T^*\mathcal{M}$  with  $T\mathcal{M}$  by the following (musical) isomorphism:

$$\begin{aligned} \iota : T\mathcal{M} &\rightarrow T\mathcal{M}^* \\ (x, V) &\mapsto (x, \xi), \end{aligned}$$

where  $\xi$  is defined by

$$\xi(W) = g(V, W), \quad \forall W \in T_x\mathcal{M}.$$

The *Liouville form*  $\alpha \in \Omega^1(T^*\mathcal{M})$  is the 1-form defined by  $\alpha_{(x,\xi)}(\bar{X}) = \xi_x(d\pi^*(\bar{X}))$ , where  $\bar{X}$  is a tangent vector at the point  $(x, \xi)$  of  $T^*\mathcal{M}$ . The canonical symplectic form on  $T\mathcal{M}^*$  is defined to be  $\Omega^* := -d\alpha$ . There is an elegant, explicit formula for the symplectic form  $\Omega := \iota^*(\Omega^*)$  in terms of the metric  $g$  and the splitting induced by the Levi-Civita connection  $\nabla$  (see [An], [La]):

**Lemma 2.** *Let  $\bar{X}$  and  $\bar{Y}$  be two tangent vectors to  $T\mathcal{M}$ ; we have*

$$\Omega(\bar{X}, \bar{Y}) = g(K\bar{X}, \Pi\bar{Y}) - g(\Pi\bar{X}, K\bar{Y}).$$

**Proposition 2.** *Let  $(\mathcal{M}, J, g)$  be a pseudo- or para-Kähler manifold. The canonical structure  $\tilde{J}$  satisfies*

$$\tilde{J}\bar{X} \simeq \tilde{J}(\Pi\bar{X}, K\bar{X}) = (J\Pi\bar{X}, JK\bar{X}).$$

**Corollary 1.** *Let  $(\mathcal{M}, J, g)$  be a pseudo- or para-Kähler manifold. The 2-tensor  $\tilde{g}(\cdot, \cdot) := \Omega(\cdot, \tilde{J}\cdot)$  satisfies*

$$\tilde{g}(\bar{X}, \bar{Y}) = g(K\bar{X}, J\Pi\bar{Y}) - g(\Pi\bar{X}, JK\bar{Y}).$$

Moreover,  $\tilde{g}$  is symmetric and therefore defines a pseudo-Riemannian metric on  $T\mathcal{M}$ .

*Proof of Proposition 2.* Let us write the splitting of  $TT\mathcal{M}$  in a local coordinate  $x$  as in the proof of Proposition 1<sup>(3)</sup>. The Levi-Civita connection is expressed through its connection form  $\mu$ :  $\nabla_X Y = dY(X) + \mu(X)Y$ . Consequently, if  $(X, \Xi) \in T_{(x,\xi)}T\mathcal{M}$ ,  $\Pi(X, \Xi) = X$  and  $K(X, \Xi) = \Xi + \mu(X)\xi$ . Thus

$$\Pi(\tilde{J}(X, \Xi)) = JX \quad \text{and} \quad K(\tilde{J}(X, \Xi)) = J(x)\Xi + (dJ(x)\xi)X + \mu(J(x)X)\xi.$$

Because  $J$  is integrable, we may choose  $x$  to be a complex coordinate, so that  $J$  is a constant endomorphism, and  $dJ(x)\xi$  vanishes. Because  $\mathcal{M}$  is Kähler, we know that  $\mu(X)$  commutes with  $J$ . However,  $\nabla$  being without torsion,  $\mu(X)Y = \mu(Y)X$ , so

$$K(\tilde{J}(X, \Xi)) = J\Xi + J\mu(X)\xi = JK(X, \Xi). \quad \square$$

<sup>3</sup> The Reader should be aware of the conflicting notation: the splitting of  $TT\mathcal{M} \simeq \mathbb{R}^{4n}$  as  $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$  induced by the coordinate charts (e.g.  $\bar{X} \simeq ((x, \xi), (X, \Xi))$ ) differs a priori from the connection-induced splitting  $\bar{X} \simeq (\Pi\bar{X}, K\bar{X})$ .

**Corollary 2.** *The symplectic form  $\Omega$  is compatible with the complex or para-complex structure  $\tilde{J}$ .*

*Proof.* Using Lemma 2, we compute

$$\begin{aligned}\Omega(\tilde{J}\bar{X}, \tilde{J}\bar{Y}) &= g(K\tilde{J}\bar{X}, \Pi\tilde{J}\bar{Y}) - g(\Pi\tilde{J}\bar{X}, K\tilde{J}\bar{Y}) \\ &= g(JK\bar{X}, J\Pi\bar{Y}) - g(J\Pi\bar{X}, JK\bar{Y}) \\ &= \varepsilon g(K\bar{X}, \Pi\bar{Y}) - \varepsilon g(\Pi\bar{X}, K\bar{Y}) \\ &= \varepsilon\Omega(\bar{X}, \bar{Y}).\end{aligned}$$

□

### 3 The Levi-Civita connection of $\tilde{g}$

The following lemma describes the Levi-Civita connection  $\tilde{\nabla}$  of  $\tilde{g}$  in terms of the direct decomposition of  $T\mathcal{M}$ , the Levi-Civita connection  $\nabla$  of  $g$  and its curvature tensor  $R$ .

**Lemma 3.** *Let  $\bar{X}$  and  $\bar{Y}$  be two vector fields on  $T\mathcal{M}$  and assume that  $\bar{Y}$  is projectable, then at the point  $(x, V)$  we have*

$$(\tilde{\nabla}_{\bar{X}}\bar{Y})|_V = (\nabla_{\Pi\bar{X}}\Pi\bar{Y}, \nabla_{\Pi\bar{X}}K\bar{Y} - T_1(\Pi\bar{X}, \Pi\bar{Y}, V)),$$

where

$$T_1(X, Y, V) = \frac{1}{2} \left( R(X, Y)V - \varepsilon R(V, JX)JY - \varepsilon R(V, JY)JX \right)$$

Moreover, if  $\mathcal{M}$  is a pseudo-Riemannian surface with Gaussian curvature  $c$ , we have

$$T_1(X, Y, V) = \begin{cases} -2cg(V, X)Y & \text{in the Kähler case} \\ +2cg(V, Y)X & \text{in the para-Kähler case.} \end{cases}$$

*Proof.* We use Lemma 1 together with the Koszul formula:

$$\begin{aligned}2\tilde{g}(\tilde{\nabla}_{\bar{X}}\bar{Y}, \bar{Z}) &= \bar{X}\tilde{g}(\bar{Y}, \bar{Z}) + \bar{Y}\tilde{g}(\bar{X}, \bar{Z}) - \bar{Z}\tilde{g}(\bar{X}, \bar{Y}) + \tilde{g}([\bar{X}, \bar{Y}], \bar{Z}) \\ &\quad - \tilde{g}([\bar{X}, \bar{Z}], \bar{Y}) - \tilde{g}([\bar{Y}, \bar{Z}], \bar{X}),\end{aligned}$$

where  $X, Y$  and  $Z$  are three vector fields on  $T\mathcal{M}$ . From the fact that  $[X^v, Y^v]$  and  $\tilde{g}(X^v, Y^v)$  vanish we have:

$$\begin{aligned}2\tilde{g}(\tilde{\nabla}_{X^v}Y^v, Z^v) &= X^v\tilde{g}(Y^v, Z^v) + Y^v\tilde{g}(X^v, Z^v) - Z^v\tilde{g}(X^v, Y^v) \\ &\quad + \tilde{g}([X^v, Y^v], Z^v) - \tilde{g}([X^v, Z^v], Y^v) - \tilde{g}([Y^v, Z^v], X^v) \\ &= 0.\end{aligned}$$

Moreover, taking into account that  $\tilde{g}(Y^v, Z^h)$  and similar quantities are constant on the fibres, we obtain

$$\begin{aligned}2\tilde{g}(\tilde{\nabla}_{X^v}Y^v, Z^h) &= X^v\tilde{g}(Y^v, Z^h) + Y^v\tilde{g}(X^v, Z^h) - Z^h\tilde{g}(X^v, Y^v) \\ &\quad + \tilde{g}([X^v, Y^v], Z^h) - \tilde{g}([X^v, Z^h], Y^v) - \tilde{g}([Y^v, Z^h], X^v) \\ &= -\tilde{g}(-(\nabla_Z X)^v, Y^v) - \tilde{g}(-(\nabla_Z Y)^v, X^v) \\ &= 0.\end{aligned}$$

From these last two equations we deduce that  $\tilde{\nabla}_{X^v} Y^v$  vanishes. Analogous computations show that  $\tilde{\nabla}_{X^v} Y^h$  vanishes as well. From Lemma 1 and the formula  $[\tilde{X}, \tilde{Y}] = \tilde{\nabla}_{\tilde{X}} \tilde{Y} - \tilde{\nabla}_{\tilde{Y}} \tilde{X}$ , we deduce that

$$\tilde{\nabla}_{X^h} Y^v \simeq (0, \nabla_X Y). \quad (3)$$

Finally, introducing

$$T_1(X, Y, V) := \frac{1}{2} \left( R(X, Y)V - \varepsilon R(V, JY)JX - \varepsilon R(V, JX)JY \right),$$

we compute that

$$\begin{aligned} 2\tilde{g}(\tilde{\nabla}_{X^h} Y^h, Z^h) &= -g(R(X, Y)V, JZ) + g(R(X, Z)V, JY) + g(R(Y, Z)V, JX) \\ &= -g(R(X, Y)V, JZ) + g(R(V, JY)X, Z) + g(R(V, JX)Y, Z) \\ &= -g(R(X, Y)V, JZ) + \varepsilon g(R(V, JY)JX, JZ) + \varepsilon g(R(V, JX)JY, JZ) \\ &= -g(2T_1(X, Y, V), JZ) \end{aligned}$$

and

$$\tilde{g}(\tilde{\nabla}_{X^h} Y^h, Z^v) = -g(\nabla_X Y, jZ),$$

from which we deduce that

$$\tilde{\nabla}_{X^h} Y^h(V) = (\nabla_X Y, -T_1(X, Y, V)). \quad (4)$$

From (3) and (4) we deduce the required formula for  $\tilde{\nabla}_{\tilde{X}} \tilde{Y}$ .

If  $n = 1$ , we have  $R(X, Y)Z = c(g(Y, Z)X - g(X, Z)Y)$ , hence the tensor  $T_1$  becomes:

$$\begin{aligned} 2T_1(X, Y, V) &= R(X, Y)V + \varepsilon JR(V, JX)Y + \varepsilon JR(V, JY)X \\ &= c \left( g(Y, V)X - g(X, V)Y \right. \\ &\quad \left. - \varepsilon J(g(JX, Y)V - g(V, Y)JX + g(JY, X)V - g(V, X)JY) \right) \\ &= c \left( g(Y, V)X - g(X, V)Y \right. \\ &\quad \left. - \varepsilon (g(JX, Y)JV + g(V, Y)X + g(JY, X)JV + g(V, X)Y) \right) \\ &= c((1 - \varepsilon)g(V, Y)X - (1 + \varepsilon)g(V, X)Y). \end{aligned}$$

□

**Remark 3.** *It should be noted that covariant derivatives with respect to a projectable vertical field  $X^v$  always vanish.*

**Proposition 3.** *The structure  $\tilde{J}$  is parallel with respect to  $\tilde{\nabla}$ .*

*Proof.* It can be seen as a trivial consequence of the fact that  $\tilde{J}$  is complex (resp. para-complex) and  $\Omega$  is closed, but can also be checked directly, using

the equivariance properties of  $J$  w.r.t. the connection  $\nabla$  and the curvature tensor  $R$ . Using the definition of  $\tilde{J}$  and Lemma 3,  $\tilde{\nabla}_{\tilde{X}}\tilde{J}\tilde{Y}$  is obvious provided  $T_1(X, JY, V) = JT_1(X, Y, V)$ . That is indeed the case since

$$\begin{aligned}
& 2(T_1(X, JY, V) - JT_1(X, Y, V)) \\
&= R(X, JY)V + R(V, JX)Y + R(V, Y)JX \\
&\quad - R(X, Y)JV - R(V, JX)Y - R(V, JY)X \\
&= R(X, JY)V + R(JY, V)X + J(R(V, Y)X + R(Y, X)V) \\
&= R(V, X)JY + JR(X, V)Y = 0,
\end{aligned}$$

where we have used Bianchi's identity.  $\square$

## 4 Curvature properties of $(\tilde{J}, \tilde{g})$

### 4.1 The Riemannian curvature tensor of $\tilde{g}$

**Proposition 4.** *The curvature tensor  $\widetilde{Rm} := -\tilde{g}(\tilde{R}, \cdot)$  of  $\tilde{g}$  at  $(x, V)$  is given by the formula*

$$\begin{aligned}
\widetilde{Rm}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}) &= g(T_2(\Pi\tilde{X}, \Pi\tilde{Y}, \Pi\tilde{Z}, V), J\Pi\tilde{W}) \\
&\quad - Rm(\Pi\tilde{X}, \Pi\tilde{Y}, \Pi\tilde{Z}, JK\tilde{W}) - Rm(\Pi\tilde{X}, \Pi\tilde{Y}, JK\tilde{Z}, \Pi\tilde{W}) \\
&\quad - Rm(\Pi\tilde{X}, JK\tilde{Y}, \Pi\tilde{Z}, \Pi\tilde{W}) + Rm(JK\tilde{X}, \Pi\tilde{Y}, \Pi\tilde{Z}, \Pi\tilde{W}),
\end{aligned}$$

where

$$T_2(X, Y, Z, V) := (\nabla_X T_1)(Y, Z, V) - (\nabla_Y T_1)(X, Z, V).$$

Moreover,  $(T\mathcal{M}, \tilde{g})$  is scalar flat and the Ricci tensor of  $\tilde{g}$  is

$$\widetilde{Ric}(\tilde{X}, \tilde{Y}) = 2Ric(\Pi\tilde{X}, \Pi\tilde{Y}).$$

**Corollary 3.**  *$(T\mathcal{M}, \tilde{g})$  is Einstein if and only if  $(\mathcal{M}, g)$  is flat. Moreover  $(T\mathcal{M}, \tilde{g})$  has nonnegative (resp. nonpositive) Ricci curvature if and only if  $(\mathcal{M}, g)$  has nonnegative (resp. nonpositive) Ricci curvature as well.*

*Proof of Proposition 4.* We will compute the curvature tensor for projectable vector fields, and need only do so for the following six cases, due to the symmetries of  $Rm$ . Remark 3 simplifies computations greatly, since most vertical derivatives vanish, except when the derived vector field is not projectable. In particular  $\tilde{R}(X^v, Y^v)$  vanishes as endomorphism, hence:

$$\begin{aligned}
\widetilde{Rm}(X^v, Y^v, Z^v, W^v) &= 0 \\
\widetilde{Rm}(X^v, Y^v, Z^v, W^h) &= 0 \\
\widetilde{Rm}(X^v, Y^v, Z^h, W^v) &= 0
\end{aligned}$$

To obtain the last three combinations, let us first derive  $\tilde{\mathbb{R}}(X^h, Y^h)Z^h$ . This is more delicate since we have to covariantly differentiate non-projectable quantities. Indeed

$$\begin{aligned}
\tilde{\mathbb{R}}(X^h, Y^h)Z^h &= \tilde{\nabla}_{X^h}\tilde{\nabla}_{Y^h}Z^h - \tilde{\nabla}_{Y^h}\tilde{\nabla}_{X^h}Z^h - \tilde{\nabla}_{[X^h, Y^h]}Z^h \\
&= \tilde{\nabla}_{X^h}(\nabla_Y Z, -\mathbb{T}_1(Y, Z, V)) - \tilde{\nabla}_{Y^h}(\nabla_X Z, -\mathbb{T}_1(X, Z, V)) \\
&\quad - \tilde{\nabla}_{([X, Y], -\mathbb{R}(X, Y)V)}Z^h \\
&= (\nabla_X \nabla_Y Z, -\mathbb{T}_1(X, \nabla_Y Z, V)) - D_{X^h}(0, \mathbb{T}_1(Y, Z, V)) \\
&\quad - (\nabla_Y \nabla_X Z, -\mathbb{T}_1(Y, \nabla_X Z, V)) + D_{Y^h}(0, \mathbb{T}_1(X, Z, V)) \\
&\quad - (\nabla_{[X, Y]}Z, -\mathbb{T}_1([X, Y], Z, V)) \\
&= (\mathbb{R}(X, Y)Z, 0) \\
&\quad - (0, \mathbb{T}_1(X, \nabla_Y Z, V) - \mathbb{T}_1(Y, \nabla_X Z, V) - \mathbb{T}_1([X, Y], Z, V)) \\
&\quad - \tilde{\nabla}_{X^h}(0, \mathbb{T}_1(Y, Z, V)) + \tilde{\nabla}_{Y^h}(0, \mathbb{T}_1(X, Z, V))
\end{aligned}$$

Recalling the lemma<sup>4</sup> in [Ko], there exists a vector field  $U$  on  $M$  such that  $U(x) = V$  and  $(\nabla_X U)(x) = 0$ . Then the vertical lift of  $\mathbb{T}_1(Y, Z, U)$  is seen to agree to first order with

$$(x, V) \mapsto (0, \mathbb{T}_1(X(x), Z(x), V))$$

thus allowing us to use the formula in Lemma 3:

$$\begin{aligned}
\tilde{\nabla}_{X^h}(0, \mathbb{T}_1(Y, Z, \cdot)) &= \tilde{\nabla}_{X^h}(\mathbb{T}_1(Y, Z, U)^v) \\
&= (0, \nabla_X(\mathbb{T}_1(Y, Z, U))) \\
&= (0, (\nabla_X \mathbb{T}_1)(Y, Z, U) + \mathbb{T}_1(\nabla_X Y, Z, U) \\
&\quad + \mathbb{T}_1(Y, \nabla_X Z, U) + \mathbb{T}_1(Y, Z, \nabla_X U))
\end{aligned}$$

which, evaluated at  $(x, V)$ , yields

$$\tilde{\nabla}_{X^h}(0, \mathbb{T}_1(Y, Z, \cdot))|_{(x, V)} = (0, (\nabla_X \mathbb{T}_1)(Y, Z, V) + \mathbb{T}_1(\nabla_X Y, Z, V) + \mathbb{T}_1(Y, \nabla_X Z, V)).$$

Summing up,

$$\begin{aligned}
\tilde{\mathbb{R}}(X^h, Y^h)Z^h|_{(x, V)} &= \left( \mathbb{R}(X, Y)Z, \right. \\
&\quad - \mathbb{T}_1(X, \nabla_Y Z, V) + \mathbb{T}_1(Y, \nabla_X Z, V) \\
&\quad + \mathbb{T}_1([X, Y], Z, V) - (\nabla_X \mathbb{T}_1)(Y, Z, V) \\
&\quad - \mathbb{T}_1(\nabla_X Y, Z, V) - \mathbb{T}_1(Y, \nabla_X Z, V) \\
&\quad + (\nabla_Y \mathbb{T}_1)(X, Z, V) + \mathbb{T}_1(\nabla_Y X, Z, V) \\
&\quad \left. + \mathbb{T}_1(X, \nabla_Y Z, V) \right) \\
&= \left( \mathbb{R}(X, Y)Z, -(\nabla_X \mathbb{T}_1)(Y, Z, V) + (\nabla_Y \mathbb{T}_1)(X, Z, V) \right) \\
&= \left( \mathbb{R}(X, Y)Z, -\mathbb{T}_2(X, Y, Z, V) \right).
\end{aligned}$$

---

<sup>4</sup>Note that computations in [Ko] are done for the Sasaki metric, hence direct results do not apply.

From that we deduce directly

$$\begin{aligned}\widetilde{\text{Rm}}(X^h, Y^h, Z^h, W^v) &= -\text{Rm}(X, Y, Z, JW) \\ \widetilde{\text{Rm}}(X^h, Y^h, Z^h, W^h)|_{(x, V)} &= g(\text{T}_2(X, Y, Z, V), JW).\end{aligned}$$

On the other hand, using repeatedly Remark 3,

$$\begin{aligned}\widetilde{\text{Rm}}(X^h, Y^v, Z^h, W^v) &= \tilde{g}(\tilde{\nabla}_{X^h}\tilde{\nabla}_{Y^v}W^v - \tilde{\nabla}_{Y^v}\tilde{\nabla}_{X^h}W^v - \tilde{\nabla}_{[X^h, Y^v]}W^v, Z^h) \\ &= \tilde{g}(-\tilde{\nabla}_{Y^v}(0, \nabla_X W), Z^h) = \tilde{g}(0, Z^h) = 0.\end{aligned}$$

The claimed formula is easily deduced using the symmetries of the curvature tensor.

In order to calculate the Ricci curvature of  $\tilde{g}$ , we consider a Hermitian pseudo-orthonormal basis  $(e_1, \dots, e_{2n})$  of  $T_x\mathcal{M}$ , i.e.  $g(e_a, e_b) = \varepsilon_a\delta_{ab}$ , where  $\varepsilon_a = \pm 1$ , and  $e_{n+a} = Je_a$ . In particular,  $\varepsilon_{n+a} = \varepsilon\varepsilon_a$ . This gives a (non-orthonormal) basis of  $T_{(x, V)}T\mathcal{M}$ :

$$\bar{e}_a := (e_a)^h \quad \bar{e}_{2n+a} := (e_a)^v.$$

A calculation using Corollary 1 shows that the expression of  $\tilde{g}$  in this basis is:

$$[\tilde{g}_{\mu\nu}]_{1 \leq \mu, \nu \leq 4n} := \begin{pmatrix} 0 & 0 & 0 & \Delta \\ 0 & 0 & -\Delta & 0 \\ 0 & -\Delta & 0 & 0 \\ \Delta & 0 & 0 & 0 \end{pmatrix},$$

where  $\Delta = \varepsilon \text{diag}(\varepsilon_1, \dots, \varepsilon_n) = \text{diag}(\varepsilon_{n+1}, \dots, \varepsilon_{2n})$ . It follows that  $\widetilde{\text{Ric}}(X^v, Y^v)$  and  $\widetilde{\text{Ric}}(X^h, Y^v)$  vanish.

Moreover, noting that  $\tilde{g}^{\mu\nu} = \tilde{g}_{\mu\nu}$ ,

$$\begin{aligned}\widetilde{\text{Ric}}(X^h, Y^h) &= \sum_{\mu, \nu=1}^{4n} \tilde{g}^{\mu\nu} \widetilde{\text{Rm}}(X^h, \bar{e}_\mu, Y^h, \bar{e}_\nu) \\ &= \sum_{a=1}^n \varepsilon\varepsilon_a \left( \widetilde{\text{Rm}}(X^h, (e_a)^h, Y^h, (Je_a)^v) - \widetilde{\text{Rm}}(X^h, (Je_a)^h, Y^h, (e_a)^v) \right. \\ &\quad \left. - \widetilde{\text{Rm}}(X^h, (e_a)^v, Y^h, (Je_a)^h) + \widetilde{\text{Rm}}(X^h, (Je_a)^v, Y^h, (e_a)^h) \right) \\ &= \sum_{a=1}^n \varepsilon\varepsilon_a \left( -\text{Rm}(X, e_a, Y, J^2e_a) + \text{Rm}(X, Je_a, Y, Je_a) \right. \\ &\quad \left. + \text{Rm}(Y, Je_a, X, Je_a) - \text{Rm}(Y, e_a, X, J^2e_a) \right) \\ &= 2 \sum_{a=1}^n \left( \varepsilon_a \text{Rm}(X, e_a, Y, e_a) + \varepsilon_{a+n} \text{Rm}(X, e_{a+n}, Y, e_{a+n}) \right) \\ &= 2 \sum_{k=1}^{2n} \varepsilon_k \text{Rm}(X, e_k, Y, e_k) = 2\text{Ric}(X, Y).\end{aligned}$$

We see easily that  $\widetilde{\text{Ric}}$  vanishes whenever one of the vectors is along the vertical fiber, thus the expected formula.

Finally the scalar curvature

$$\widetilde{\text{Scal}} = \sum_{\mu, \nu=1}^4 \tilde{g}^{\mu\nu} \widetilde{\text{Ric}}(\bar{e}_\mu, \bar{e}_\nu) = 0,$$

since  $\tilde{g}^{\mu\nu}$  vanishes as soon as both  $\bar{e}_\mu, \bar{e}_\nu$  are both horizontal.  $\square$

## 4.2 The Weyl curvature tensor of $\tilde{g}$

**Proposition 5.** *The Weyl tensor  $\widetilde{W}$  at  $(x, V)$  is given by*

$$\begin{aligned} \widetilde{W}(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) &= \widetilde{\text{Rm}}(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) \\ &\quad - \frac{1}{2n-1} \left( \text{Ric}(\Pi\bar{X}, \Pi\bar{Z})\tilde{g}(\bar{Y}, \bar{W}) + \text{Ric}(\Pi\bar{Y}, \Pi\bar{W})\tilde{g}(\bar{Y}, \bar{W}) \right. \\ &\quad \left. - \text{Ric}(\Pi\bar{X}, \Pi\bar{W})\tilde{g}(\bar{Y}, \bar{Z}) - \text{Ric}(\Pi\bar{Y}, \Pi\bar{Z})\tilde{g}(\bar{X}, \bar{W}) \right). \end{aligned}$$

In particular, if  $n = 1$ ,

$$\widetilde{W}(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) = g(\text{T}_2(\Pi\bar{X}, \Pi\bar{Y}, \Pi\bar{Z}, V), \text{J}\Pi\bar{W}).$$

**Corollary 4.**  *$(T\mathcal{M}, \tilde{g})$  is locally conformally flat if and only if  $n = 1$  and  $g$  has constant curvature, or  $n \geq 2$  and  $g$  is flat.*

**Remark 4.** *This result has been proved in the case  $n = 1$  and  $\varepsilon = 1$  in [GK1].*

*Proof of Proposition 5.* Since the scalar curvature vanishes, we have

$$\widetilde{W} = \widetilde{\text{Rm}} - \frac{1}{4n-2} \widetilde{\text{Ric}} \circledast \tilde{g},$$

where  $\circledast$  denotes the Kulkarni–Nomizu product. Recall that  $\widetilde{\text{Ric}}(\bar{X}, \bar{Y}) = 0$  if one of the two vectors  $\bar{X}$  and  $\bar{Y}$  is vertical. Consequently

$$\begin{aligned} \widetilde{\text{Ric}} \circledast \tilde{g}(\bar{X}, \bar{Y}, \bar{Z}, \bar{W}) &= 2 \left( \text{Ric}(\Pi\bar{X}, \Pi\bar{Z})\tilde{g}(\bar{Y}, \bar{W}) + \text{Ric}(\Pi\bar{Y}, \Pi\bar{W})\tilde{g}(\bar{Y}, \bar{W}) \right. \\ &\quad \left. - \text{Ric}(\Pi\bar{X}, \Pi\bar{W})\tilde{g}(\bar{Y}, \bar{Z}) - \text{Ric}(\Pi\bar{Y}, \Pi\bar{Z})\tilde{g}(\bar{X}, \bar{W}) \right). \end{aligned}$$

The expression of the Weyl tensor follows easily.

In the case  $n = 1$  of a surface with Gaussian curvature  $c$ , we have  $\text{Ric}(X, Y) = cg(X, Y)$  and  $\text{Rm}(X, Y, Z, W) = c(g(X, Z)g(Y, W) - g(X, W)g(Y, Z))$ . Hence using Proposition 4, the expression of Weyl tensor simplifies and we get the claimed formula.  $\square$

*Proof of Corollary 4.* We first deal with the case  $n = 1$ . Lemma 3 implies that  $\text{T}_1(X, Y, Z) = -2cg(Z, X)Y$  when  $\varepsilon = 1$  (resp.  $2cg(Z, Y)X$  when  $\varepsilon = -1$ ). Therefore, if  $\varepsilon = 1$ ,

$$\begin{aligned} \text{T}_2(X, Y, Z, W) &= \nabla_X \text{T}_1(Y, Z, W) - \nabla_Y \text{T}_1(X, Z, W) \\ &= -2(X.c)g(W, Y)Z + 2(Y.c)g(W, X)Z \\ &= 2g\left((Y.c)X - (X.c)Y, W\right)Z, \end{aligned}$$

which vanishes if and only if  $(X.c)Y = (Y.c)X$  for all vectors  $X, Y$ , i.e. the curvature  $c$  is constant. Analogously, if  $\varepsilon = -1$ ,

$$\begin{aligned} T_2(X, Y, Z, W) &= \nabla_X T_1(Y, Z, W) - \nabla_Y T_1(X, Z, W) \\ &= 2(X.c)g(W, Z)Y - 2(Y.c)g(W, Z)X \\ &= 2\left((X.c)Y - (Y.c)X\right)g(W, Z), \end{aligned}$$

which again vanishes if and only if the curvature  $c$  is constant.

Assume now that  $(T\mathcal{M}, \tilde{g})$  is conformally flat with  $n \geq 2$ . Thus in particular

$$\begin{aligned} \widetilde{W}(X^h, Y^h, Z^h, W^v) &= -\text{Rm}(X, Y, Z, JW) \\ &\quad - \frac{1}{2n-1} \left( -\text{Ric}(X, Z)g(Y, JW) + \text{Ric}(Y, Z)g(X, JW) \right) \end{aligned}$$

vanishes, so

$$\text{Rm}(X, Y, Z, JW) = \frac{1}{2n-1} \left( \text{Ric}(X, Z)g(Y, JW) - \text{Ric}(Y, Z)g(X, JW) \right).$$

(Observe that this equation always holds if  $\mathcal{M}$  is a surface.) Let us apply the symmetry property of the curvature tensor to this equation with  $Z = X$  and  $JW = Y$ , assuming furthermore that  $X$  and  $Y$  are two non-null vectors:

$$\begin{aligned} 0 &= (2n-1)(\text{Rm}(X, Y, X, Y) - \text{Rm}(Y, X, Y, X)) \\ &= \text{Ric}(X, X)g(Y, Y) - \text{Ric}(Y, X)g(X, Y) \\ &\quad - \text{Ric}(Y, Y)g(X, X) + \text{Ric}(X, Y)g(Y, X) \\ &= \text{Ric}(X, X)g(Y, Y) - \text{Ric}(Y, Y)g(X, X). \end{aligned}$$

Hence

$$\frac{\text{Ric}(X, X)}{g(X, X)} = \frac{\text{Ric}(Y, Y)}{g(Y, Y)}.$$

The set of non null vectors being dense in  $T\mathcal{M}$ , it follows by continuity that  $g$  is Einstein. We deduce that

$$\begin{aligned} \text{Rm}(X, Y, X, Y) &= \frac{1}{2n-1} \left( \text{Ric}(X, X)g(Y, Y) - \text{Ric}(Y, X)g(X, Y) \right) \\ &= c \left( g(X, X)g(Y, Y) - g(X, Y)g(X, Y) \right), \end{aligned}$$

so  $g$  has constant curvature. But since  $\mathcal{M}$  is Kähler and has dimension  $2n \geq 4$ , it must be flat.  $\square$

Finally, we recall the general result linking the Weyl tensor to the scalar curvature in dimension four: for a neutral pseudo-Kähler or para-Kähler metric, self-duality is equivalent to scalar flatness (see theorem A.2 in annex). We can therefore conclude



**Corollary 5.** *In dimension four ( $n = 1$ ), the metric  $\tilde{g}$  is anti-self-dual if and only the curvature  $c$  of  $g$  is constant.*

*Proof.* Thanks to proposition 4, we know that  $\tilde{g}$  is scalar flat, hence self-dual ( $W^-$  vanishes identically). In order for  $\tilde{g}$  to be also anti-self-dual, the Weyl tensor has to vanish completely, which amounts, following corollary 4, to having constant (sectional) curvature  $c$  on  $M$ .  $\square$

### 4.3 The holomorphic sectional curvature of $(\tilde{J}, \tilde{g})$

**Proposition 6.**  *$(\tilde{J}, \tilde{g})$  has constant holomorphic sectional curvature if and only if  $g$  is flat.*

*Proof.* Define the holomorphic sectional curvature tensor of  $\tilde{g}$  by  $\widetilde{\text{Hol}}(\bar{X}) := \widetilde{\text{Rm}}(\bar{X}, \tilde{J}\bar{X}, \bar{X}, \tilde{J}\bar{X})$ . Writing any doubly tangent vector  $\bar{X}$  as the sum of a horizontal and a vertical factor, we will compute  $\widetilde{\text{Hol}}(X^h + Y^v)$ . We deduce from Proposition 4 that  $\widetilde{\text{Rm}}$  vanishes whenever two or more entries are vertical. Hence, using the antisymmetric properties of the Riemann tensor w.r.t. the complex or para-complex structure,

$$\begin{aligned} \widetilde{\text{Hol}}(X^h + Y^v) &= \widetilde{\text{Rm}}(X^h, JX^h, X^h, JX^h) \\ &\quad + \widetilde{\text{Rm}}(X^h, JX^h, X^h, JY^v) + \widetilde{\text{Rm}}(X^h, JX^h, Y^v, JX^h) \\ &\quad + \widetilde{\text{Rm}}(X^h, JY^v, X^h, JX^h) + \widetilde{\text{Rm}}(Y^v, JX^h, X^h, JX^h) \\ &= \widetilde{\text{Rm}}(X^h, JX^h, X^h, JX^h) + 4\widetilde{\text{Rm}}(X^h, JX^h, X^h, JY^v) \\ &= g(\text{T}_2(X, JX, X, V) - 4\varepsilon\text{R}(X, Y)X, JX). \end{aligned}$$

In particular,

$$\begin{aligned} \widetilde{\text{Hol}}(X^v) &= 0 \\ \widetilde{\text{Hol}}(X^h + X^v) &= g(\text{T}_2(X, JX, X, V), JX) \\ \widetilde{\text{Hol}}(X^h + (JX)^v) &= g(\text{T}_2(X, JX, X, V), JX) + 4\varepsilon\text{Hol}(X). \end{aligned}$$

It follows from the first equation that if  $\widetilde{\text{Hol}}$  is constant, it must be zero. Hence, from the second and third equation we deduce the  $\text{Hol}$  must vanish, i.e.  $g$  is flat.  $\square$

## 5 Examples

The simplest examples where we may apply the construction above is where  $(\mathcal{M}, J, g, \omega)$  is the plane  $\mathbb{R}^2$  equipped with the flat metric  $g := dq_1^2 + \varepsilon dq_2^2$  and the complex or para-complex structure  $J$  defined by  $J(\partial_{q_1}, \partial_{q_2}) = (-\varepsilon\partial_{q_2}, \partial_{q_1})$ . In other words,  $\mathbb{R}^2$  is identified with the complex plane  $\mathbb{C}$  or the para-complex

plane  $\mathbb{D}$ . We recall that  $\mathbb{D}$ , called the algebra of double numbers, is the two-dimensional real vector space  $\mathbb{R}^2$  endowed with the commutative algebra structure whose product rule is given by

$$(u, v).(u', v') = (uu' + vv', uv' + u'v).$$

The number  $(0, 1)$ , whose square is  $(1, 0)$  and not  $(-1, 0)$ , will be denoted by  $\tau$ .

We claim that in the complex case  $\varepsilon = 1$ , the structure  $(\tilde{\mathbb{J}}, \tilde{g}, \Omega)$  just constructed on  $T\mathbb{C}$  is equivalent to that of the standard complex pseudo-Euclidean plane  $(\mathbb{C}^2, \bar{\mathbb{J}}, \langle \cdot, \cdot \rangle_2, \omega_1)$ , where  $\bar{\mathbb{J}}$  is the canonical complex structure,  $(z_1 = x_1 + iy_1, z_2 = x_2 + iy_2)$  are the canonical coordinates and

$$\begin{aligned} \langle \cdot, \cdot \rangle_2 &:= -dx_1^2 - dx_2^2 + dx_2^2 + dy_2^2 \\ \omega_1 &:= -dx_1 \wedge dy_1 + dx_2 \wedge dy_2. \end{aligned}$$

To see this, it is sufficient to consider the following complex change of coordinates

$$\begin{aligned} z_1 &:= \frac{\sqrt{2}}{2}((p_1 + ip_2) + i(q_1 + iq_2)), \\ z_2 &:= \frac{\sqrt{2}}{2}(p_1 + ip_2 - i(q_1 + iq_2)). \end{aligned}$$

which preserves the symplectic form, since we have

$$\omega_1 := -dx_1 \wedge dy_1 + dx_2 \wedge dy_2 = dq_1 \wedge dp_1 + dq_2 \wedge dp_2 = \Omega,$$

where  $\Omega$  is the canonical symplectic form of  $T^*\mathbb{C} \simeq_g T\mathbb{C}$ . The metric of a pseudo-Kähler structure being determined by the complex structure and the symplectic form through the formula  $\tilde{g} = \Omega(\cdot, \tilde{\mathbb{J}}\cdot)$ , we have the required identification.

Analogously, in the para-complex case  $\varepsilon = -1$ , the structure  $(\tilde{\mathbb{J}}, \tilde{g}, \Omega)$  constructed on  $T\mathbb{D}$  is equivalent to that of the standard para-complex plane  $(\mathbb{D}^2, \bar{\mathbb{J}}, \langle \cdot, \cdot \rangle_*, \omega_*)$ , where  $\bar{\mathbb{J}}$  is the canonical para-complex structure,  $(w_1 = u_1 + \tau u_1, w_2 = u_2 + \tau y_2)$  are the canonical coordinates and

$$\begin{aligned} \langle \cdot, \cdot \rangle_* &:= du_1^2 - dv_1^2 + du_2^2 - dv_2^2 \\ \omega_* &:= du_1 \wedge dv_1 + du_2 \wedge dv_2. \end{aligned}$$

Here we have to be careful with the identification of  $T^*\mathbb{D}$  with  $T\mathbb{D}$ : since the metric  $g$  is  $dq_1^2 - dq_2^2$ , we have  $q_1 := dp_1 \simeq_g \partial_{p_1}$  and  $q_2 := dp_2 \simeq_g -\partial_{q_2}$ . Hence  $\Omega^* = dq_1 \wedge dp_1 + dq_2 \wedge dp_2$  and  $\Omega = dq_1 \wedge dp_1 - dq_2 \wedge dp_2$ . Introducing the change of para-complex coordinates

$$\begin{aligned} w_1 &:= \frac{\sqrt{2}}{2}((p_1 + \tau p_2) - \tau(q_1 + \tau q_2)), \\ w_2 &:= \frac{\sqrt{2}}{2}(\tau(p_1 + \tau p_2) + (q_1 + \tau q_2)), \end{aligned}$$

we check that

$$\omega_* = du_1 \wedge dv_1 + du_2 \wedge dv_2 = dq_1 \wedge dp_1 - dq_2 \wedge dp_2 = \Omega,$$

hence we obtain the identification between  $(T\mathbb{D}, \tilde{\mathcal{J}}, \tilde{g}, \Omega)$  and  $(\mathbb{D}^2, \bar{\mathcal{J}}, \langle \cdot, \cdot \rangle_*, \omega_*)$ . Of course the metrics considered in these two examples are flat.

The next simplest examples of pseudo-Riemannian surfaces are the two-dimensional space forms, namely the sphere  $\mathbb{S}^2$ , the hyperbolic plane  $\mathbb{H}^2 := \{x_1^2 + x_2^2 - x_3^2 = -1\}$  and the de Sitter surface  $d\mathbb{S}^2 := \{x_1^2 + x_2^2 - x_3^2 = 1\}$ . Their tangent bundles enjoy a interesting geometric interpretation (see [GK1]): the tangent bundle  $T\mathbb{S}^2$  is canonically identified with the set of oriented lines of Euclidean three-space:

$$L(\mathbb{R}^3) \ni \{V + tx | t \in \mathbb{R}\} \simeq (x, V - \langle V, x \rangle_0 x) \in T\mathbb{S}^2.$$

Analogously, the tangent bundle  $T\mathbb{H}^2$  is canonically identified with the set of oriented negative (timelike) lines of three-space endowed with the metric  $\langle \cdot, \cdot \rangle_1 := dx_1^2 + dx_2^2 - dx_3^2$ :

$$\mathbb{L}_{1,-}^3 \ni \{V + tx | t \in \mathbb{R}\} \simeq (x, V - \langle V, x \rangle_1 x) \in T\mathbb{H}^2,$$

Finally, the tangent bundle  $Td\mathbb{S}^2$  is canonically identified with the set of oriented positive (spacelike) lines of three-space endowed with the metric  $\langle \cdot, \cdot \rangle_1$ :

$$\mathbb{L}_{1,+}^3 \ni \{V + tx | t \in \mathbb{R}\} \simeq (x, V - \langle V, x \rangle_1 x) \in Td\mathbb{S}^2.$$

Observe that the metric constructed on  $T\mathbb{S}^2$  (resp.  $T\mathbb{H}^2$ ) has non-negative (resp. non-positive) Ricci curvature.

## A The Weyl tensor in the pseudo-Kähler or para-Kähler cases

The Riemann curvature tensor  $\text{Rm}$  of a pseudo-riemannian manifold  $\mathcal{N}$  may be seen as a symmetric form  $\text{R}$  on bivectors of  $\Lambda^2 T\mathcal{N}$  (see [Be] for references). Splitting  $\text{R}$  along the eigenspaces  $\Lambda^+ \oplus \Lambda^-$  of the Hodge operator  $*$  on  $\Lambda^2 T\mathcal{N}$ , yields the following block decomposition

$$\text{R} = \begin{pmatrix} \text{W}^+ + \frac{\text{Scal}}{12} I & Z \\ Z^* & \text{W}^- + \frac{\text{Scal}}{12} I \end{pmatrix}$$

where  $Z^*$  denotes the adjoint w.r.t. the induced metric on  $\Lambda^2 T\mathcal{N}$ , so that  $\text{W} = \text{W}^+ \oplus \text{W}^-$ , the Weyl tensor seen as a 2-form on  $\Lambda^2$ , is the traceless, Hodge-commuting part of the Riemann curvature operator  $\text{R}$ . Hence the following formula

$$\text{W} = \text{Rm} - \frac{1}{2} \text{Ric} \otimes g + \frac{\text{Scal}}{12} g \otimes g.$$

If, additionally,  $\mathcal{N}$  is a four dimensional Kähler manifold, then

**Theorem A.1** (Prop. 2 in [De]).  $W^+$  is a multiple of the scalar curvature.

As a consequence,

**Corollary 6.**  $(\mathcal{N}, g, J)$  is anti-self-dual ( $W^+ = 0$ ) if and only if the scalar curvature vanishes.

The result extends to the two cases considered in this article: (1) neutral pseudo-Kähler manifolds and (2) para-Kähler manifolds, with a slight twist:  $W^+$  is replaced by  $W^-$ . Precisely:

**Theorem A.2.** Let  $(\mathcal{N}, g, J)$  be a four dimensional manifold endowed with a pseudo-Kähler neutral metric (respectively a para-Kähler metric, necessarily neutral). Then the Weyl tensor  $W$  commutes with the Hodge operator and  $\mathcal{N}$  is self-dual ( $W^- = 0$ ) if and only if the scalar curvature vanishes.

The result for neutral pseudo-Kähler manifolds is probably known and relates to representation theory (see [Be] for introduction and references), but since we could not find an explicit proof in the literature<sup>5</sup>, we will give a simple one below. To our knowledge, the proof for the para-Kähler case is new (albeit similar).

## A.1 The pseudo-Kähler case

We will write explicitly the Weyl tensor in a given positively oriented orthonormal frame, denoted by  $(e_1, e_{1'}, e_2, e_{2'})$ , where  $e_{1'} = Je_1$ ,  $e_{2'} = Je_2$ ,  $g(e_1) = g(e_{1'}) = -1$  and  $g(e_2) = g(e_{2'}) = +1$ . (For brevity,  $g(X)$  denotes the norm  $g(X, X)$ .) The pseudo-metric  $g$  extends to bivectors, has signature  $(2, 4)$ , and will be again denoted by  $g$ :  $g(e_a \wedge e_b) = g(e_a)g(e_b) - g(e_a, e_b)^2 = g(e_a)g(e_b)$ , so that  $\mathcal{B} = (e_1 \wedge e_{1'}, e_1 \wedge e_2, e_1 \wedge e_{2'}, e_{1'} \wedge e_2, e_{1'} \wedge e_{2'}, e_2 \wedge e_{2'})$  is an orthonormal frame of  $\Lambda^2$ , with  $g(e_a \wedge e_b) = -1$ , except for  $g(e_1 \wedge e_{1'}) = g(e_2 \wedge e_{2'}) = +1$ . (Note that the other convention, taking  $-g$  does not change the induced metric on  $\Lambda^2$ .)

Since the volume  $e_1 \wedge e_{1'} \wedge e_2 \wedge e_{2'}$  is positively oriented, we construct an orthonormal eigenbasis for the Hodge star on  $\Lambda^2 T\mathcal{N}$ :

$$\begin{cases} E_1^\pm = \frac{\sqrt{2}}{2}(e_1 \wedge e_{1'} \pm e_2 \wedge e_{2'}) \\ E_2^\pm = \frac{\sqrt{2}}{2}(e_1 \wedge e_2 \pm e_{1'} \wedge e_{2'}) \\ E_3^\pm = \frac{\sqrt{2}}{2}(e_1 \wedge e_{2'} \mp e_{1'} \wedge e_2) \end{cases}$$

so that  $\Lambda^\pm$  is generated by  $E_1^\pm, E_2^\pm, E_3^\pm$ .

The Kähler condition implies

$$\text{Rm}(JX, JY, Z, T) = \text{Rm}(X, Y, Z, T) = \text{Rm}(X, Y, JZ, JT),$$

---

<sup>5</sup>On the contrary, some authors seem to imply that scalar flatness is equivalent to anti-self-duality, see [DW]). However this contradiction could possibly come from a different choice of orientation, which would exchange self-dual with anti-self-dual.

because  $J$  is isometric and parallel. The matrix of the symmetric 2-form  $R$  in the orthonormal frame  $\mathcal{B}$  is

	$e_{11'}$	$e_{12}$	$e_{12'}$	$e_{1'2}$	$e_{1'2'}$	$e_{22'}$
$e_{11'}$	$R_{11'11'}$	$R_{11'12}$	$R_{11'12'}$	$R_{11'1'2} = -R_{11'12'}$	$R_{11'1'2'} = R_{11'12}$	$R_{11'22'}$
$e_{12}$		$R_{1212}$	$R_{1212'}$	$R_{121'2} = -R_{1212'}$	$R_{131'2'} = R_{1212}$	$R_{1222'}$
$e_{12'}$			$R_{12'12'}$	$R_{12'1'2} = -R_{12'12'}$	$R_{12'1'2'} = R_{1212'}$	$R_{12'22'}$
$e_{1'2}$				$R_{1'21'2} = R_{12'12'}$	$R_{1'21'2'} = -R_{1212'}$	$R_{1'222'} = -R_{12'22'}$
$e_{1'2'}$					$R_{1'2'1'2'} = R_{1212}$	$R_{1'2'22'} = R_{1222'}$
$e_{22'}$						$R_{22'22'}$

where  $e_{ab}$  stands for  $e_a \wedge e_b$ , for greater legibility. We have written the matrix as a table for clarity and to make symmetries more obvious, and because  $R$  is symmetric we need only write half the matrix. We have used the internal symmetries of  $R$ , to choose among equivalent coefficients the ones lowest in the lexicographic order of the indices.

The Weyl tensor satisfies *some* of the  $J$ -symmetries of  $R$ : indeed

$$\begin{aligned} \text{Ric}(JX, JY) &= \sum_i g(e_i) \text{Rm}(JX, e_i, JY, e_i) = \sum_i g(e_i) \text{Rm}(X, Je_i, Y, Je_i) \\ &= \sum_i g(Je_i) \text{Rm}(X, Je_i, Y, Je_i) = \text{Ric}(X, Y) \end{aligned}$$

because  $(Je_i)$  is again an orthonormal basis. In particular, this invariance implies  $r_{11'} = \text{Ric}(e_1, e_{1'}) = r_{1'1} = -r_{11'}$ , so  $r_{11'}$  vanish (and so does  $r_{22'}$ ). For the Kulkarny–Nomizu product,

$$\begin{aligned} \text{Ric} \otimes g(JX, Y, Z, T) &= \text{Ric}(JX, Z)g(Y, T) + \text{Ric}(Y, T)g(JX, Z) \\ &\quad - \text{Ric}(JX, T)g(Y, Z) - \text{Ric}(Y, Z)g(JX, T) \\ &= -\text{Ric}(X, JZ)g(JY, JT) - \text{Ric}(JY, JT)g(X, JZ) \\ &\quad + \text{Ric}(X, JT)g(JY, JZ) + \text{Ric}(JY, JZ)g(X, JT) \\ &= -\text{Ric} \otimes g(X, JY, JZ, JT) \end{aligned}$$

so

$$\text{Ric} \otimes g(JX, JY, Z, T) = -\text{Ric} \otimes g(X, J^2Y, JZ, JT) = \text{Ric} \otimes g(X, Y, JZ, JT).$$

Hence the following symmetries (fewer than for  $\text{Rm}$ ) in the coefficients of  $\text{Ric} \otimes g$ ,

$g \otimes g$  and Rm, and therefore W:

	$e_{11'}$	$e_1 \wedge e_2$	$e_{12'}$	$e_{1'} \wedge e_2$	$e_{1'2'}$	$e_{22'}$
$e_{11'}$	$W_{11'11'}$	$W_{11'12}$	$W_{11'12'}$	$W_{11'1'2} = -W_{11'12'}$	$W_{11'1'2'} = W_{11'12}$	$W_{11'22'}$
$e_{12}$		$W_{1212}$	$W_{1212'}$	$W_{121'2}$	$W_{121'2'}$	$W_{1222'}$
$e_{12'}$			$W_{12'12'}$	$W_{12'1'2}$	$W_{12'1'2'} = -W_{121'2}$	$W_{12'22'}$
$e_{1'2}$				$W_{1'21'2} = W_{12'12'}$	$W_{1'21'2'} = -W_{1212'}$	$W_{1'222'} = -W_{12'22'}$
$e_{1'2'}$					$W_{1'2'1'2'} = W_{1212}$	$W_{1'2'22'} = W_{1222'}$
$e_{22'}$						$W_{22'22'}$

Expanding on the above eigenbasis of  $\Lambda^+ \oplus \Lambda^-$  (which differs from the one in the positive definite case) yields the following Weyl tensor coefficients, which we have simplified using the symmetries above (up to a factor 1/2 due to normalization):

	$E_1^+$	$E_2^+$	$E_3^+$
$E_1^+$	$W_{11'11'} + W_{22'22'} + 2W_{11'22'}$	$2(W_{11'12} + W_{1222'})$	$2(W_{11'12'} + W_{12'22'})$
$E_2^+$		$2(W_{1212} + W_{121'2'})$	$2(W_{1212'} - W_{121'2})$
$E_3^+$			$2(W_{12'12'} - W_{12'1'2})$
$E_1^-$			
$E_2^-$			
$E_3^-$			

	$E_1^-$	$E_2^-$	$E_3^-$
$E_1^-$	$W_{11'11'} - W_{22'22'}$	0	0
$E_2^-$	$2(W_{11'12} - W_{1222'})$	0	0
$E_3^-$	$2(W_{11'12'} - W_{12'22'})$	0	0
$E_1^-$	$W_{11'11'} + W_{22'22'} - 2W_{11'22'}$	0	0
$E_2^-$		$2(W_{1212} - W_{121'2'})$	$2(W_{1212'} + W_{121'2})$
$E_3^-$			$2(W_{12'12'} + W_{12'1'2})$

(Again only half the coefficients are written down.) Further simplifications come from computing W, and using

$$\begin{aligned}
\text{Scal} &= -r_{11} - r_{1'1'} + r_{22} + r_{2'2'} = 2(r_{22} - r_{11}) \\
&= 2(-(-R_{11'11'} + R_{1212} + R_{12'12'}) + (-R_{1212} - R_{1'21'2} + R_{22'22'})) \\
&= 2(R_{11'11'} - 2(R_{1212} + R_{12'12'}) + R_{22'22'}).
\end{aligned}$$

First prove that the Hodge star commutes with W by considering  $W(\Lambda^+, \Lambda^-)$ :

$$\begin{aligned}
W_{11'11'} &= R_{11'11'} + \frac{1}{2}(r_{11} + r_{1'1'}) + \frac{\text{Scal}}{6} = R_{11'11'} + r_{11} + \frac{\text{Scal}}{6} \\
&= R_{1212} + R_{12'12'} + \frac{\text{Scal}}{6}
\end{aligned}$$

$$\begin{aligned}
W_{22'22'} &= R_{22'22'} - \frac{1}{2}(r_{22} + r_{2'2'}) + \frac{\text{Scal}}{6} = R_{22'22'} - r_{22} + \frac{\text{Scal}}{6} \\
&= R_{1212} + R_{12'12'} + \frac{\text{Scal}}{6}
\end{aligned}$$

so that  $W_{11'11'} - W_{22'22'} = 0$ . Similarly

$$W_{11'12} = R_{11'12} + \frac{r_{1'2}}{2}, \quad W_{1222'} = R_{1222'} + \frac{r_{12'}}{2} = R_{1222'} - \frac{r_{1'2}}{2}$$

so

$$\begin{aligned}
W_{11'12} - W_{1222'} &= R_{11'12} - R_{1222'} + r_{1'2} = 0 \\
W_{11'12'} &= R_{11'12'} + \frac{r_{1'2'}}{2} = R_{11'12'} + \frac{r_{12}}{2}, \quad W_{12'22'} = R_{12'22'} - \frac{r_{12}}{2}, \\
W_{11'12'} - W_{12'22'} &= R_{11'12'} - R_{12'22'} + r_{12} = 0.
\end{aligned}$$

That proves that  $W$  is block-diagonal.

The  $W^-$  term satisfies

$$\begin{aligned}
W_{11'11'} + W_{22'22'} - 2W_{11'22'} &= R_{11'11'} + r_{11} + R_{22'22'} - r_{22} + \frac{\text{Scal}}{3} - 2R_{11'22'} \\
&= R_{11'11'} + R_{22'22'} - 2R_{11'22'} - \frac{\text{Scal}}{6} \\
&= R_{11'11'} + R_{22'22'} - 2(R_{1212} + R_{12'12'}) - \frac{\text{Scal}}{6} \\
&= \frac{\text{Scal}}{2} - \frac{\text{Scal}}{6} = \frac{\text{Scal}}{3}
\end{aligned}$$

using the first Bianchi identity (and the invariance of  $\text{Rm}$ ):

$$R_{11'22'} = -R_{1'212'} - R_{211'2'} = R_{12'12'} + R_{1212}.$$

$$W_{1212} - W_{121'2'} = R_{1212} + \frac{r_{22} - r_{11}}{2} - \frac{\text{Scal}}{6} - R_{121'2'} = \frac{\text{Scal}}{4} - \frac{\text{Scal}}{6} = \frac{\text{Scal}}{12}$$

$$W_{12'12'} + W_{12'1'2} = R_{12'12'} + \frac{\text{Scal}}{4} - \frac{\text{Scal}}{6} + R_{12'1'2} = \frac{\text{Scal}}{12}$$

$$W_{1212'} + W_{121'2} = R_{1212'} + \frac{r_{22'}}{2} + R_{121'2} - \frac{r_{11'}}{2} = \frac{1}{2}(r_{22'} - r_{11'}) = 0.$$

Finally,

$$W^- = \text{Scal} \begin{pmatrix} 1/3 & & \\ & 1/6 & \\ & & 1/6 \end{pmatrix} = \frac{\text{Scal}}{6} \text{Id} + \frac{\text{Scal}}{6} E_1^- \otimes E_1^-$$

(and indeed this matrix is traceless w.r.t. the pseudo-metric  $g$ ). One should note that the above expression differs from the riemannian case, where

$$W^+ = \text{Scal} \begin{pmatrix} 1/3 & & \\ & -1/6 & \\ & & -1/6 \end{pmatrix} = -\frac{\text{Scal}}{6} \text{Id} + \frac{\text{Scal}}{3} E_1^+ \otimes E_1^+.$$

We let the Reader check that in the neutral case, the  $W^+$  part is not a multiple of the scalar curvature, which completes the proof of theorem [A.2](#).

## A.2 The para-Kähler case

The computations are almost identical, but the results differ from the pseudo-Kähler setup, because the para-complex structure  $J$  is now an anti-isometry:  $R(JX, JY)Z = -R(X, Y)Z$ . We pick an orthonormal basis  $(e_1, e_{1'}, e_2, e_{2'})$  with  $e_{1'} = Je_1$ ,  $e_{2'} = Je_2$ , and  $g(e_1) = g(e_2) = +1$ ,  $g(e_{1'}) = g(e_{2'}) = -1$ . The frame  $\mathcal{B} = (e_1 \wedge e_{1'}, e_1 \wedge e_2, e_1 \wedge e_{2'}, e_{1'} \wedge e_2, e_{1'} \wedge e_{2'}, e_2 \wedge e_{2'})$  of  $\Lambda^2 T\mathcal{N}$  is also orthonormal w.r.t. the induced metric on  $\Lambda^2$ , again denoted by  $g$ , which has signature  $(2, 4)$ :  $g(e_a \wedge e_b) = g(e_a)g(e_b) = -1$ , except for  $g(e_1 \wedge e_2) = g(e_{1'} \wedge e_{2'}) = +1$ .

An orthonormal eigenbasis for the Hodge operator is the following:

$$\begin{cases} E_1^\pm = \frac{\sqrt{2}}{2}(e_1 \wedge e_{1'} \mp e_2 \wedge e_{2'}) \\ E_2^\pm = \frac{\sqrt{2}}{2}(e_1 \wedge e_2 \mp e_{1'} \wedge e_{2'}) \\ E_3^\pm = \frac{\sqrt{2}}{2}(e_1 \wedge e_{2'} \mp e_{1'} \wedge e_2) \end{cases}$$

where the  $E_a^+$  (resp.  $E_a^-$ ) span  $\Lambda^+$  (resp.  $\Lambda^-$ ). (Note the sign differences w.r.t. the pseudo-Kähler case.)

Since  $J$  is anti-isometric and parallel,

$$\text{Rm}(JX, JY, Z, T) = -\text{Rm}(X, Y, Z, T) = \text{Rm}(X, Y, JZ, JT).$$

Hence the following symmetries of the riemannian curvature operator  $R$ , expressed in the frame  $\mathcal{B}$  (for symmetry reasons and greater legibility, lower left coefficients are not written in this and the subsequent matrices):

	$e_{11'}$	$e_{12}$	$e_{12'}$	$e_{1'} \wedge e_2$	$e_{1'2'}$	$e_{22'}$
$e_{11'}$	$R_{11'11'}$	$R_{11'12}$	$R_{11'12'}$	$R_{11'1'2}$ $= -R_{11'12'}$	$R_{11'1'2'}$ $= -R_{11'12}$	$R_{11'22'}$
$e_{12}$		$R_{1212}$	$R_{1212'}$	$R_{121'2}$ $= -R_{1212'}$	$R_{121'2'}$ $= -R_{1212}$	$R_{1222'}$
$e_{12'}$			$R_{12'12'}$	$R_{12'1'2}$ $= -R_{12'12'}$	$R_{12'1'2'}$ $= -R_{1212'}$	$R_{12'22'}$
$e_{1'2}$				$R_{1'21'2}$ $= R_{12'12'}$	$R_{1'21'2'}$ $= R_{1212'}$	$R_{1'222'}$ $= -R_{12'22'}$
$e_{1'2'}$					$R_{1'2'1'2'}$ $= R_{1212}$	$R_{1'2'22'}$ $= -R_{1222'}$
$e_{22'}$						$R_{22'22'}$

(Note again the similarity with the pseudo-Kähler case: only a few signs change.)

The Weyl tensor satisfies *some* of the  $J$ -symmetries of  $\text{Rm}$  since

$$\begin{aligned} \text{Ric}(JX, JY) &= \sum_i \langle e_i, e_i \rangle \text{Rm}(JX, e_i, JY, e_i) = \sum_i g(e_i) \text{Rm}(X, Je_i, Y, Je_i) \\ &= -\sum_i g(Je_i) \text{Rm}(X, Je_i, Y, Je_i) = -\text{Ric}(X, Y) \end{aligned}$$



since  $(Je_i)$  is also an orthonormal basis. In particular this invariance implies  $r_{1'1} = r_{11'} = -r_{1'1}$ , so  $r_{11'}$  vanishes (and so does  $r_{22'}$ ). Finally,

$$\frac{\text{Scal}}{2} = r_{11} + r_{22} = -R_{11'11'} + 2(R_{1212} - R_{12'12'}) - R_{22'22'}.$$

The Kulkarny–Nomizu product  $\text{Ric} \otimes g$  satisfies

$$\begin{aligned} \text{Ric} \otimes g(JX, Y, Z, T) &= \text{Ric}(JX, Z)g(Y, T) + \text{Ric}(Y, T)g(JX, Z) \\ &\quad - \text{Ric}(JX, T)g(Y, Z) - \text{Ric}(Y, Z)g(JX, T) \\ &= \text{Ric}(X, JZ)g(JY, JT) + \text{Ric}(JY, JT)g(X, JZ) \\ &\quad - \text{Ric}(X, JT)g(JY, JZ) - \text{Ric}(JY, JZ)g(X, JT) \\ &= \text{Ric} \otimes g(X, JY, JZ, JT) \end{aligned}$$

so

$$\text{Ric} \otimes g(JX, JY, Z, T) = \text{Ric} \otimes g(X, J^2Y, JZ, JT) = \text{Ric} \otimes g(X, Y, JZ, JT)$$

and the same property holds for  $g \otimes g$ . Hence the following symmetries (fewer than for  $\text{Rm}$ ) in the coefficients of  $\text{Ric} \otimes g$ ,  $g \otimes g$  and  $\text{Rm}$ , and therefore  $W$ :

	$e_{11'}$	$e_{12}$	$e_{12'}$	$e_{1'2}$	$e_{1'2'}$	$e_{22'}$
$e_{11'}$	$W_{11'11'}$	$W_{11'12}$	$W_{11'12'}$	$W_{11'1'2} = -W_{11'12'}$	$W_{11'1'2'} = -W_{11'12}$	$W_{11'22'}$
$e_{12}$		$W_{1212}$	$W_{1212'}$	$W_{121'2}$	$W_{121'2'}$	$W_{1222'}$
$e_{12'}$			$W_{12'12'}$	$W_{12'1'2}$	$W_{12'1'2'} = W_{121'2}$	$W_{12'22'}$
$e_{1'2}$				$W_{1'21'2} = W_{12'12'}$	$W_{1'21'2'} = W_{1212'}$	$W_{1'222'} = -W_{12'22'}$
$e_{1'2'}$					$W_{1'2'1'2'} = W_{1212}$	$W_{1'2'22'} = -W_{1222'}$
$e_{22'}$						$W_{22'22'}$

Let us now express  $W$  in the Hodge basis defined earlier, using the above symmetries (up to a factor  $1/2$  due to normalization).

	$E_1^+$	$E_2^+$	$E_3^+$
$E_1^+$	$W_{11'11'} + W_{22'22'} - 2W_{11'22'}$	$2(W_{11'12} - W_{1222'})$	$2(W_{11'12'} - W_{12'22'})$
$E_2^+$		$2(W_{1212} - W_{121'2'})$	$2(W_{1212'} - W_{121'2})$
$E_3^+$			$2(W_{12'12'} - W_{12'1'2})$
$E_1^-$			
$E_2^-$			
$E_3^-$			

	$E_1^-$	$E_2^-$	$E_3^-$
$E_1^+$	$W_{11'11'} - W_{22'22'}$	0	0
$E_2^+$	$2(W_{11'12} + W_{1222'})$	0	0
$E_3^+$	$2(W_{11'12'} + W_{12'22'})$	0	0
$E_1^-$	$W_{11'11'} + W_{22'22'} + 2W_{11'22'}$	0	0
$E_2^-$		$2(W_{1212} + W_{121'2'})$	$2(W_{1212'} + W_{121'2})$
$E_3^-$			$2(W_{12'12'} + W_{12'1'2})$

Only three terms in the off-block-diagonal part are not obviously zero.

$$W_{11'11'} = R_{11'11'} - \frac{1}{2}(-r_{11} + r_{1'1'}) - \frac{\text{Scal}}{6} = R_{11'11'} + r_{11} - \frac{\text{Scal}}{6}$$

$$W_{22'22'} = R_{22'22'} - \frac{1}{2}(-r_{22} + r_{2'2'}) - \frac{\text{Scal}}{6} = R_{22'22'} + r_{22} - \frac{\text{Scal}}{6}$$

but  $r_{11} = -R_{11'11'} + R_{1212} - R_{12'12'}$  and  $r_{22} = R_{2121} - R_{21'21'} - R_{22'22'} = R_{1212} - R_{12'12'} - R_{22'22'}$  so that

$$W_{11'11'} - W_{22'22'} = R_{11'11'} - R_{22'22'} + r_{11} - r_{22} = 0.$$

Similarly

$$W_{11'12} + W_{1222'} = R_{11'12} - \frac{r_{1'2}}{2} + R_{1222'} + \frac{r_{12'}}{2} = R_{11'12} + R_{1222'} - r_{1'2} = 0$$

$$W_{11'12'} + W_{12'22'} = R_{11'12'} - \frac{r_{1'2'}}{2} + R_{12'22'} + \frac{r_{12}}{2} = R_{11'12'} + R_{12'22'} + r_{12} = 0$$

which proves that  $W$  is block-diagonal, i.e. commutes with the Hodge operator.

Let us now look more closely at the  $W^-$  term

$$\begin{pmatrix} W_{11'11'} + W_{22'22'} + 2W_{11'22'} & 0 & 0 \\ & 2(W_{1212} + W_{121'2'}) & 2(W_{1212'} + W_{121'2}) \\ & & 2(W_{12'12'} + W_{12'1'2}) \end{pmatrix}$$

$$\begin{aligned} & W_{11'11'} + W_{22'22'} + 2W_{11'22'} \\ &= R_{11'11'} + r_{11} - \frac{\text{Scal}}{6} + R_{22'22'} + r_{22} - \frac{\text{Scal}}{6} + 2R_{11'22'} \\ &= R_{11'11'} + R_{22'22'} + 2R_{11'22'} + \frac{\text{Scal}}{2} - \frac{\text{Scal}}{3} \\ &= R_{11'11'} + R_{22'22'} + 2(-R_{1212} + R_{12'12'}) + \frac{\text{Scal}}{6} = -\frac{\text{Scal}}{3} \end{aligned}$$

where we have used the first Bianchi identity (and the invariance of  $\text{Rm}$ )

$$R_{11'22'} = -R_{1'212'} - R_{211'2'} = R_{12'12'} - R_{1212}.$$

$$\begin{aligned} W_{1212} + W_{121'2'} &= R_{1212} - \frac{r_{22} + r_{11}}{2} + \frac{\text{Scal}}{6} + R_{121'2'} \\ &= R_{1212} - \frac{\text{Scal}}{4} + \frac{\text{Scal}}{6} + R_{121'2'} = -\frac{\text{Scal}}{12} \end{aligned}$$

$$W_{12'12'} + W_{12'1'2} = R_{12'12'} + \frac{\text{Scal}}{4} - \frac{\text{Scal}}{6} + R_{12'1'2} = \frac{\text{Scal}}{12}$$

$$W_{1212'} + W_{121'2} = R_{1212'} - \frac{r_{22'}}{2} + R_{121'2} - \frac{r_{11'}}{2} = 0.$$

Finally,

$$W^- = \text{Scal} \begin{pmatrix} -1/3 & & \\ & -1/6 & \\ & & 1/6 \end{pmatrix}$$

vanishes if and only if  $\text{Scal} = 0$ . (The Reader will check that this matrix is indeed traceless w.r.t. the pseudo-metric  $g$ .)

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