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MULTITRIANGULATIONS, PSEUDOTRIANGULATIONS AND PRIMITIVE SORTING NETWORKS

VINCENT PILAUD AND MICHEL POCCHIOLA

Abstract. We study the set of all pseudoline arrangements with contact points which cover a given support. We define a natural notion of flip between these arrangements and study the graph of these flips. In particular, we provide an enumeration algorithm for arrangements with a given support, based on the properties of certain greedy pseudoline arrangements and on their connection with sorting networks. Both the running time per arrangement and the working space of our algorithm are polynomial.

As the motivation for this work, we provide in this paper a new interpretation of both pseudotriangulations and multitriangulations in terms of pseudoline arrangements on specific supports. This interpretation explains their common properties and leads to a natural definition of multipseudotriangulations, which generalizes both. We study elementary properties of multipseudotriangulations and compare them to iterations of pseudotriangulations.

Keywords. pseudoline arrangement · pseudotriangulation · multitriangulation · flip · sorting network · enumeration algorithm

1. Introduction

The original motivation for this paper is the interpretation of certain families of planar geometric graphs in terms of pseudoline arrangements. As an introductory illustration, we present this interpretation on the family of triangulations of a convex polygon.

Let P be a finite point set in convex position in the Euclidean plane $\mathbb{R}^2$. A *triangulation* T of P is a maximal crossing-free set of edges on P, or equivalently, a decomposition of the convex hull of P into triangles with vertices in P. See Figure 1(a). A *(strict) bisector* of a triangle Δ of T is a line which passes through a vertex of Δ and (strictly) separates its other two vertices. Observe that any triangle has a unique bisector parallel to any direction. Moreover any two triangles of T have a unique common strict bisector, and possibly an additional non-strict bisector if they share an edge.

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An extended abstract of this paper was presented in the *25th European Workshop on Computational Geometry* (Brussels, March 2009) [PP09]. The main results of this paper also appeared in Chapter 3 of the first author's PhD dissertation [Pil10].

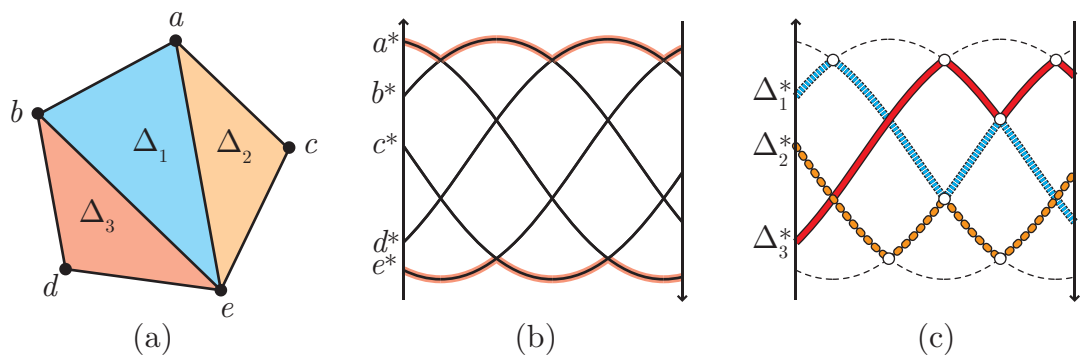

FIGURE 1. (a) A triangulation T of a convex point set P. (b) The dual pseudoline arrangement P^* of P (whose first level is highlighted). (c) The dual pseudoline arrangement T^* of T.

We now switch to the line space $\mathcal{M}$ of the Euclidean plane $\mathbb{R}^2$. Remember that $\mathcal{M}$ is a Möbius strip homeomorphic to the quotient space $\mathbb{R}^2/(\theta, d) \sim (\theta + \pi, -d)$ via the parametrization of an oriented line by its angle θ with the horizontal axis and its algebraic distance d to the origin (see Section 3 for details). For pictures, we represent $\mathcal{M}$ as a vertical band whose boundaries are identified in opposite directions. The dual of a point $p \in P$ is the set of all lines of $\mathbb{R}^2$ passing through p. It is a *pseudoline* of $\mathcal{M}$, *i.e.* a non-separating simple closed curve. The dual of P is the set $P^* := \{p^* \,|\, p \in P\}$. It is a *pseudoline arrangement* of $\mathcal{M}$, since any two pseudolines p^*, q^* of P^* cross precisely once at the line (pq). Call *first level* of P^* the boundary of the external face of the complement of P^*. It corresponds to the supporting lines of the convex hull of P. See Figure 1(b).

As illustrated on Figure 1(c), we observe after [PV94, PV96a] that:

(i) the set Δ^* of all bisectors of a triangle Δ of T is a pseudoline of $\mathcal{M}$;
(ii) the dual pseudolines Δ_1^*, Δ_2^* of any two triangles Δ_1, Δ_2 of T have a unique crossing point (the unique common strict bisector of Δ_1 and Δ_2) and possibly a contact point (when Δ_1 and Δ_2 share a common edge);
(iii) the set $T^* := \{\Delta^* \,|\, \Delta \text{ triangle of } T\}$ is a pseudoline arrangement with contact points; and
(iv) T^* covers P^* minus its first level.

We furthermore prove in this paper that this interpretation is bijective: any pseudoline arrangement with contact points supported by P^* minus its first level is the dual pseudoline arrangement T^* of a triangulation T of P.

Motivated by this interpretation, we study the set of all *pseudoline arrangements with contact points* which cover a given support in the Möbius strip. We define a natural notion of *flip* between them, and study the graph of these flips. In particular, we provide an enumeration algorithm for arrangements with a given support (similar to the enumeration algorithm of [BKPS06] for pseudotriangulations), based on certain *greedy pseudoline arrangements* and their connection with *primitive sorting networks* [Knu73, Section 5.3.4][Knu92, dB74]. The running time per arrangement and the working space of our algorithm are both polynomial.

We are especially interested in the following particular situation. Let L be a pseudoline arrangement and k be a positive integer. We call *k-pseudotriangulation* of L any pseudoline arrangement with contact points that covers L minus its first k levels. These objects provide dual interpretations for two, until now unrelated, classical generalizations of triangulations of a convex polygon (see Section 3 for the definitions and basic properties of these geometric graphs):

Pseudotriangulations ($k = 1$). Introduced for the study of the visibility complex of a set of disjoint convex obstacles in the plane [PV96b, PV96c], pseudotriangulations were used in different contexts such as motion planning and rigidity theory [Str05, HOR+05]. Their combinatorial and geometric structure has been extensively studied in the last decade (number of pseudotriangulations [AAKS04, AOSS08], polytope of pseudotriangulations [RSS03], algorithmic issues [Ber05, BKPS06, HP07], etc.). See [RSS08] for a detailed survey on the subject. As far as pseudotriangulations are concerned, this paper has two main applications: it proves the dual characterization of pseudotriangulations in terms of pseudoline arrangements and provides an interpretation of greedy pseudotriangulations in terms of sorting networks, leading to a new proof of the greedy flip property for points [PV96b, AP03, BKPS06]. The objects studied in this paper have a further (algorithmic) motivation: as a first step to compute the dual arrangement of a set of disjoint convex bodies described only by its chirotope, Habert and Pocchiola raise in [HP07] the question to compute efficiently a pseudotriangulation of a pseudotriangulation of the set, *i.e.* a 2-pseudotriangulation.

Multitriangulations (convex position). Introduced in the context of extremal theory for geometric graphs [CP92], multitriangulations were then studied for their combinatorial structure [Nak00, DKKM01, DKM02, Jon05]. The study of stars in multitriangulations [PS09], generalizing triangles for triangulations, naturally

leads to interpret multitriangulations as multipseudotriangulations of points in convex position. As far as we know, this paper provides the first interpretation of multitriangulations in terms of pseudoline arrangements on the Möbius strip.

The paper is organized as follows. Section 2 is devoted to the study of all pseudoline arrangements with contact points covering a given support. We define the flip and the greedy pseudoline arrangements (2.2) whose properties yield the enumeration algorithm for pseudoline arrangements with a given support (2.3).

In Section 3, we prove that the pseudotriangulations of a finite planar point set P in general position correspond to the pseudoline arrangements with contact points supported by the dual pseudoline arrangement of P minus its first level (3.2). Similarly, we observe that multitriangulations of a planar point set P in convex position correspond to pseudoline arrangements with contact points supported by the dual pseudoline arrangement of P minus its first k levels (3.3).

This naturally yields to the definition of multipseudotriangulations in Section 4. We study the primal of a multipseudotriangulation. We discuss some of its structural properties (4.2) which generalize the cases of pseudotriangulations and multitriangulations: number of edges, pointedness, crossing-freeness. We study in particular the faces of multipseudotriangulations (4.3) which naturally extend triangles in triangulations, pseudotriangles in pseudotriangulations, and stars in multitriangulations.

In Section 5, we compare multipseudotriangulations to iterated pseudotriangulations. We give an example of a 2-triangulation which is not a pseudotriangulation of a triangulation (5.1). We prove however that greedy multipseudotriangulations are iterated greedy pseudotriangulations (5.2), and we study flips in iterated pseudotriangulations (5.3).

Section 6 presents two further topics. The first one (6.1) is a pattern avoiding characterization of greedy multipseudotriangulations related to horizon trees. The second one (6.2) is a discussion on multipseudotriangulations of double pseudoline arrangements, which extend pseudotriangulations of convex bodies in the plane.

Finally, we discuss in Section 7 some related problems and open questions concerning in particular the primal of a multipseudotriangulation, the diameter and the polytopality of the graph of flips, and the number of multipseudotriangulations. Since the submission of this paper, some of these questions were partially answered in [Stu11, SS12, PS12] based on a framework similar to the material presented in this paper.

2. Pseudoline arrangements with the same support

2.1. **Pseudoline arrangements in the Möbius strip.** Let $\mathcal{M}$ denote the *Möbius strip* (without boundary), defined as the quotient set of the plane $\mathbb{R}^2$ under the map $\tau : \mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \mapsto (x+\pi, -y)$. The induced canonical projection will be denoted by $\pi : \mathbb{R}^2 \to \mathcal{M}$.

A *pseudoline* is the image λ under the canonical projection π of the graph $\{(x, f(x)) \,|\, x \in \mathbb{R}\}$ of a continuous and *π-antiperiodic* function $f : \mathbb{R} \to \mathbb{R}$ (that is, which satisfies $f(x + \pi) = -f(x)$ for all $x \in \mathbb{R}$). We say that f represents the pseudoline λ.

When we consider two pseudolines, we always assume that they have a finite number of intersection points. Thus, these intersection points can only be either *crossing points* or *contact points*. See Figure 3(a). Any two pseudolines always have an odd number of crossing points (in particular, at least one). When λ and μ have exactly one crossing point, we denote it by $\lambda \wedge \mu$.

A *pseudoline arrangement with contact points* is a finite set Λ of pseudolines such that any two of them have exactly one crossing point and possibly some contact points. See Figure 2. In this paper, we are only interested in *simple* arrangements, that is, where no three pseudolines meet in a common point. The *support* of Λ is the union of its pseudolines. Observe that Λ is completely determined by its support together with its set of contact points. The *first level* of Λ is the external hull of the support of Λ, *i.e.* the boundary of the external face of the complement of the support of Λ. We define inductively the *kth level* of Λ as the external hull of the support of Λ minus its first $k-1$ levels.

Remark 1. The usual definition of pseudoline arrangements does not allow contact points. In this paper, they play a crucial role since we

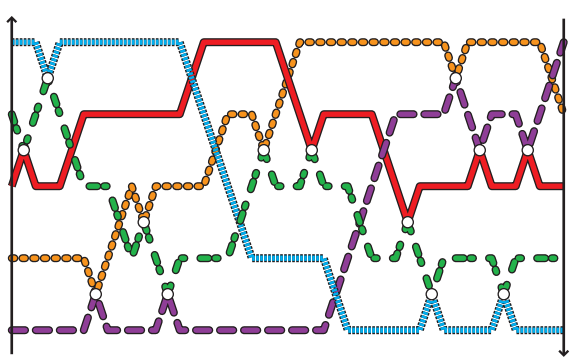

Figure 2. A pseudoline arrangement in the Möbius strip. Its contact points are represented by white circles.

are interested in all pseudoline arrangements which share a common support, and which only differ by their sets of contact points. To simplify the exposition, we omit throughout the paper to specify that we work with pseudoline arrangements *with contact points*.

Pseudoline arrangements are also classically defined on the projective plane rather than the Möbius strip. The projective plane is obtained from the Möbius strip by adding a point at infinity.

For more details on pseudoline arrangements, we refer to the broad literature on the topic [Grü72, BVS+99, Knu92, Goo97].

2.2. **Flip graph and greedy pseudoline arrangements.**

2.2.1. *Flips.* In the following lemma, we use the symbol $\triangle$ for the symmetric difference: $X \triangle Y := (X \smallsetminus Y) \cup (Y \smallsetminus X)$. We refer to Figure 3 for an illustration of this lemma.

Lemma 2. *Let Λ be a pseudoline arrangement, $\mathcal{S}$ be its support, and V be the set of its contact points. Let $v \in V$ be a contact point of two pseudolines of Λ, and w denote their unique crossing point. Then $V \triangle \{v, w\}$ is also the set of contact points of a pseudoline arrangement Λ' supported by $\mathcal{S}$.*

Proof. Let f and g represent the two pseudolines λ and μ of Λ in contact at v. Let x and y be such that $v = \pi(x, f(x))$, $w = \pi(y, f(y))$ and $x < y < x + \pi$. We define two functions f' and g' by

$$f' := \begin{cases} f & \text{on } [x, y] + \mathbb{Z}\pi, \\ g & \text{otherwise,} \end{cases} \quad \text{and} \quad g' := \begin{cases} g & \text{on } [x, y] + \mathbb{Z}\pi, \\ f & \text{otherwise.} \end{cases}$$

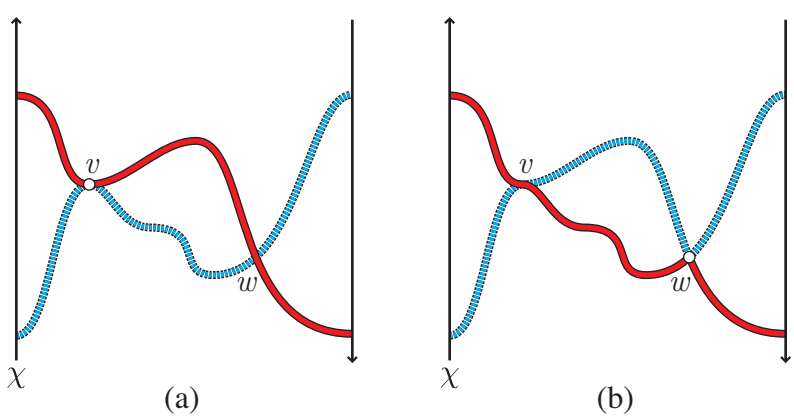

FIGURE 3. (a) A pseudoline arrangement with one contact point v and one crossing point w. (b) Flipping v in the pseudoline arrangement of (a).

These two functions are continuous and π-antiperiodic, and thus define two pseudolines λ' and μ'. These two pseudolines have a contact point at w and a unique crossing point at v, and they cross any pseudoline ν of $\Lambda \smallsetminus \{\lambda, \mu\}$ exactly once (since ν either crosses both λ and μ between v and w, or crosses both λ and μ between w and v). Consequently, $\Lambda' := \Lambda \triangle \{\lambda, \mu, \lambda', \mu'\}$ is a pseudoline arrangement, with support $\mathcal{S}$, and contact points $V \triangle \{v, w\}$. $\square$

Definition 3. *Let Λ be a pseudoline arrangement with support $\mathcal{S}$ and contact points V, let $v \in V$ be a contact point between two pseudolines of Λ which cross at w, and let Λ' be the pseudoline arrangement with support $\mathcal{S}$ and contact points $V \triangle \{v, w\}$. We say that we obtain Λ' by* flipping *v in Λ.*

Note that the starting point of a flip is always a contact point. To this contact point corresponds precisely one crossing point. In contrast, it would be incorrect to try to flip a crossing point: the two pseudolines which cross at this point might have zero or many contact points.

Observe also that the pseudoline arrangements Λ and Λ' are the only two pseudoline arrangements supported by $\mathcal{S}$ whose sets of contact points contain $V \smallsetminus \{v\}$.

Definition 4. *Let $\mathcal{S}$ be the support of a pseudoline arrangement. The* flip graph *of $\mathcal{S}$, denoted by $G(\mathcal{S})$, is the graph whose vertices are all the pseudoline arrangements supported by $\mathcal{S}$, and whose edges are flips between them.*

In other words, there is an edge in the graph $G(\mathcal{S})$ between two pseudoline arrangements if and only if their sets of contact points differ by exactly two points.

Observe that the graph $G(\mathcal{S})$ is regular: there is one edge adjacent to a pseudoline arrangement Λ supported by $\mathcal{S}$ for each contact point of Λ, and two pseudoline arrangements with the same support have the same number of contact points.

Example 5. The flip graph of the support of an arrangement of two pseudolines with p contact points is the complete graph on $p+1$ vertices.

2.2.2. *Acyclic orientations.* Let $\mathcal{S}$ be the support of a pseudoline arrangement and $\bar{\mathcal{S}}$ denote its preimage under the projection π. We orient the graph $\bar{\mathcal{S}}$ along the abscissa increasing direction, and the graph $\mathcal{S}$ by projecting the orientations of the edges of $\bar{\mathcal{S}}$. We denote by $\preccurlyeq$ the induced partial order on the vertex set of $\bar{\mathcal{S}}$ (defined by $z \preccurlyeq z'$ if there exists an oriented path on $\bar{\mathcal{S}}$ from z to z').

A *filter* of $\bar{\mathcal{S}}$ is a proper set F of vertices of $\bar{\mathcal{S}}$ such that $z \in F$ and $z \preccurlyeq z'$ implies $z' \in F$. The corresponding *antichain* is the set

of all edges and faces of $\bar{\mathcal{S}}$ with one vertex in F and one vertex not in F. This antichain has a linear structure, and thus, can be seen as the set of edges and faces that cross a vertical curve $\bar{\chi}$ of $\mathbb{R}^2$. The projection $\chi \coloneqq \pi(\bar{\chi})$ of such a curve is called a *cut* of $\mathcal{S}$. We see the fundamental domain located between the two curves $\bar{\chi}$ and $\tau(\bar{\chi})$ as the result of cutting the Möbius strip along the cut χ. For example, we use such a cut to represent pseudoline arrangements in all figures of this paper. See for example Figure 3.

The cut χ defines a partial order $\preccurlyeq_\chi$ on the vertex set of $\mathcal{S}$: for all vertices v and w of $\mathcal{S}$, we write $v \preccurlyeq_\chi w$ if there is an oriented path in $\mathcal{S}$ which does not cross χ. In other words, $v \preccurlyeq_\chi w$ if $\bar{v} \preccurlyeq \bar{w}$, where $\bar{v}$ (resp. $\bar{w}$) denotes the unique preimage of v (resp. w) between $\bar{\chi}$ and $\tau(\bar{\chi})$. For example, in the arrangements of Figure 3, we have $v \prec_\chi w$.

Let Λ be a pseudoline arrangement supported by $\mathcal{S}$, v be a contact point between two pseudolines of Λ and w denote their crossing point. Since v and w lie on a same pseudoline on $\mathcal{S}$, they are comparable for $\preccurlyeq_\chi$. We say that the flip of v is *χ-increasing* if $v \prec_\chi w$ and *χ-decreasing* otherwise. For example, the flip from (a) to (b) in Figure 3 is χ-increasing. We denote by $G_\chi(\mathcal{S})$ the directed graph of χ-increasing flips on pseudoline arrangements supported by $\mathcal{S}$.

Lemma 6. *The directed graph $G_\chi(\mathcal{S})$ of χ-increasing flips is acyclic.*

Proof. If Λ and Λ' are two pseudoline arrangements supported by $\mathcal{S}$, we write $\Lambda \trianglelefteq_\chi \Lambda'$ if there exists a bijection ϕ between their sets of contact points such that $v \preccurlyeq_\chi \phi(v)$ for any contact point v of Λ. It is easy to see that this relation is a partial order on the vertices of $G_\chi(\mathcal{S})$. Since the edges of $G_\chi(\mathcal{S})$ are oriented according to $\trianglelefteq_\chi$, the graph $G_\chi(\mathcal{S})$ is acyclic. $\square$

Theorem 7 below states that this acyclic graph $G_\chi(\mathcal{S})$ has in fact a unique source, and thus is connected.

2.2.3. *Sorting networks.* Let n denote the number of pseudolines of the arrangements supported by $\mathcal{S}$ and $m \geq \binom{n}{2}$ their number of intersection points (crossing points plus contact points). We consider a chain $F = F_m \supset F_{m-1} \supset \cdots \supset F_1 \supset F_0 = \tau(F)$ of filters of $\bar{\mathcal{S}}$ such that two consecutive of them F_i and F_{i+1} only differ by a single element: $\{\bar{v}_i\} \coloneqq F_{i+1} \smallsetminus F_i$. This corresponds to a (backward) *sweep* $\chi = \chi_0, \chi_1, \dots, \chi_m = \chi$ of the Möbius strip, where each cut χ_{i+1} is obtained from the cut χ_i by sweeping a maximal vertex $v_i \coloneqq \pi(\bar{v}_i)$ of $\mathcal{S}$ (for the partial order $\preccurlyeq_\chi$). For all i, let $e_i^1, e_i^2, \dots, e_i^n$ denote the sequence of edges of $\bar{\mathcal{S}}$ with exactly one vertex in F_i, ordered from top

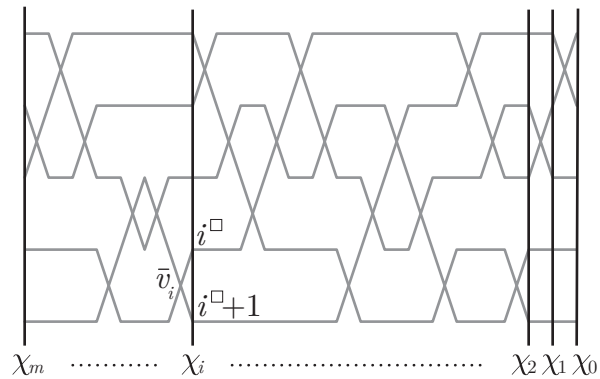

FIGURE 4. A (backward) sweep of the support of the pseudoline arrangement of Figure 2.

to bottom, and let $i^\square$ be the index such that $\bar{v}_i$ is the common point of edges $e_i^{i^\square}$ and $e_i^{i^\square+1}$ (see Figure 4).

Let $\Lambda := \{\lambda_1, \dots, \lambda_n\}$ be a pseudoline arrangement supported by $\mathcal{S}$. For all i, we denote by σ_i^Λ the permutation of $\{1, \dots, n\}$ whose jth entry $\sigma_i^\Lambda(j)$ is the index of the pseudoline supporting e_i^j, *i.e.* such that $\pi(e_i^j) \subset \lambda_{\sigma_i^\Lambda(j)}$. Up to reindexing the pseudolines of Λ, we can assume that σ_0^Λ is the inverted permutation $\sigma_0^\Lambda := [n, n-1, \dots, 2, 1]$, and consequently that σ_m^Λ is the identity permutation. Observe that for all i:

(i) if v_i is a contact point of Λ, then $\sigma_i^\Lambda = \sigma_{i+1}^\Lambda$;
(ii) otherwise, σ_{i+1}^Λ is obtained from σ_i^Λ by inverting its $i^\square$th and $(i^\square+1)$th entries.

The following theorem is illustrated on Figure 5.

Theorem 7. *The directed graph $G_\chi(\mathcal{S})$ has a unique source Γ, characterized by the property that the permutation σ_{i+1}^Γ is obtained from σ_i^Γ by sorting its $i^\square$th and $(i^\square+1)$th entries, for all i.*

Proof. If Γ satisfies the above property, then it is obviously a source of the directed graph $G_\chi(\mathcal{S})$: any flip of Γ is χ-increasing since two of its pseudolines cannot touch before they cross.

Assume conversely that Γ is a source of $G_\chi(\mathcal{S})$. Let $a := \sigma_i^\Gamma(i^\square)$ and $b := \sigma_i^\Gamma(i^\square+1)$. We have two possible situations:

(i) If $a < b$, then the two pseudolines λ_a and λ_b of Γ already cross before v_i. Consequently, v_i is necessarily a contact point of Γ, which implies that $\sigma_{i+1}^\Gamma(i^\square) = a$ and $\sigma_{i+1}^\Gamma(i^\square+1) = b$.

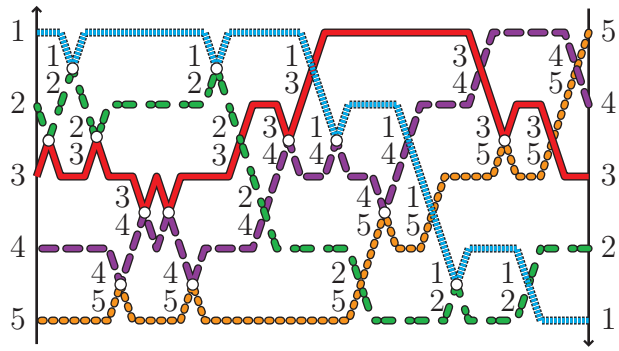

FIGURE 5. The greedy pseudoline arrangement on the support of Figure 2, obtained by sorting the permutation $[5, 4, 3, 2, 1]$. The result of each comparator is written after it (*i.e.* to its left since we sweep backwards).

(ii) If $a > b$, then the two pseudolines λ_a and λ_b of Γ do not cross before v_i. Since Γ is a source of $G_\chi(\mathcal{S})$, v_i is necessarily a crossing point of Γ. Thus $\sigma^\Gamma_{i+1}(i^\square) = b$ and $\sigma^\Gamma_{i+1}(i^\square + 1) = a$.

In both cases, σ^Γ_{i+1} is obtained from σ^Γ_i by sorting its $i^\square$th and $(i^\square+1)$th entries. $\square$

Corollary 8. *The graphs of flips $G(\mathcal{S})$ and $G_\chi(\mathcal{S})$ are connected.*

Definition 9. *The unique source of the directed graph $G_\chi(\mathcal{S})$ is denoted by $\Gamma_\chi(\mathcal{S})$ and called the* χ-greedy pseudoline arrangement *on $\mathcal{S}$.*

Let us reformulate Theorem 7 in terms of sorting networks (see [Knu73, Section 5.3.4] for a detailed presentation; see also [dB74]). Let $i < j$ be two integers. A *comparator* $[i : j]$ transforms a sequence of numbers $(a_1, \dots, a_n)$ by sorting (a_i, a_j), *i.e.* replacing a_i by $\min(a_i, a_j)$ and a_j by $\max(a_i, a_j)$. A comparator $[i : j]$ is *primitive* if $j = i + 1$. A *sorting network* is a sequence of comparators that sorts any sequence $(a_1, \dots, a_n)$.

The support $\mathcal{S}$ of an arrangement of n pseudolines together with a sweep $F_m \supset \cdots \supset F_0$ corresponds to the primitive sorting network $[1^\square : 1^\square + 1], \dots, [m^\square : m^\square + 1]$ (see [Knu92, Section 8]). Theorem 7 affirms that sorting the permutation $[n, n-1, \dots, 2, 1]$ according to this sorting network provides a pseudoline arrangement supported by $\mathcal{S}$, which depends only upon the support $\mathcal{S}$ and the filter F_0 (not on the total order given by the sweep).

2.2.4. *Greedy set of contact points.* The following proposition provides an alternative construction of the greedy pseudoline arrangement $\Gamma_\chi(\mathcal{S})$ of the support $\mathcal{S}$.

Proposition 10. *Let $v_1, \dots, v_q$ be a sequence of vertices of $\mathcal{S}$ constructed recursively by choosing, as long as possible, a remaining vertex v_i of $\mathcal{S}$ minimal (for the partial order $\preccurlyeq_\chi$) such that $\{v_1, \dots, v_i\}$ is a subset of the set of contact points of a pseudoline arrangement supported by $\mathcal{S}$. Then the resulting set $\{v_1, \dots, v_q\}$ is exactly the set of contact points of $\Gamma_\chi(\mathcal{S})$.*

Proof. First of all, $\{v_1, \dots, v_q\}$ is by construction the set of contact points of a pseudoline arrangement Λ supported by $\mathcal{S}$. If Λ is not the (unique) source $\Gamma_\chi(\mathcal{S})$ of the oriented graph $G_\chi(\mathcal{S})$, then there is a contact point v_i of Λ whose flip is χ-decreasing. Let w denote the corresponding crossing point, and Λ' the pseudoline arrangement obtained from Λ by flipping v_i. This implies that $\{v_1, \dots, v_{i-1}, w\}$ is a subset of the set of contact points of Λ' and that $w \prec_\chi v_i$, which contradicts the minimality of v_i. □

Essentially, this proposition affirms that we obtain the same pseudoline arrangement when:

(i) sweeping $\mathcal{S}$ decreasingly and place crossing points as long as possible; or
(ii) sweeping $\mathcal{S}$ increasingly and place contact points as long as possible.

2.2.5. *Constrained flip graph.* We now need to extend the previous results to constrained pseudoline arrangements on $\mathcal{S}$, in which we force a set V of vertices of $\mathcal{S}$ to be contact points.

Theorem 11. *Let V be a subset of vertices of the support $\mathcal{S}$, and let $G_\chi(\mathcal{S} \,|\, V)$ be the subgraph of $G_\chi(\mathcal{S})$ induced by the pseudoline arrangements with support $\mathcal{S}$, whose set of contact points contains V. Then this directed graph $G_\chi(\mathcal{S} \,|\, V)$ is either empty or an acyclic connected graph with a unique source Γ characterized by the property that for all i:*

(i) if $v_i \in V$, then $\sigma^\Gamma_{i+1} = \sigma^\Gamma_i$;
(ii) if $v_i \notin V$, then σ^Γ_{i+1} is obtained from σ^Γ_i by sorting its $i^\square$th and $(i^\square + 1)$th entries.

Proof. We transform our support $\mathcal{S}$ into another one $\mathcal{S}'$ by opening all intersection points of V (the local picture of this transformation is $\rtimes \to \asymp$). If $\mathcal{S}'$ supports at least one pseudoline arrangement, we apply the result of Theorem 7: a pseudoline arrangement supported by $\mathcal{S}'$ corresponds to a pseudoline arrangement with support $\mathcal{S}$ whose set of contact points contains V. □

We denote by $\Gamma_\chi(\mathcal{S} \,|\, V)$ the unique source of the constrained flip graph $G_\chi(\mathcal{S} \,|\, V)$.

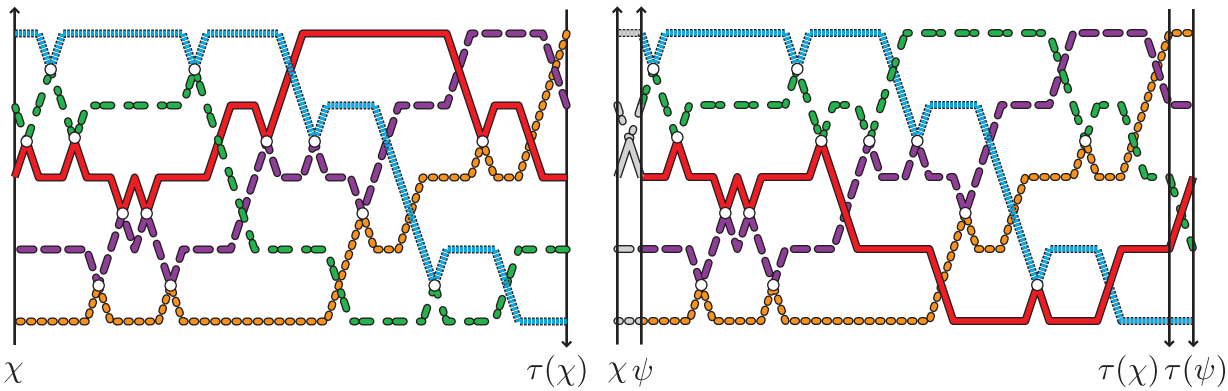

FIGURE 6. The greedy flip property.

In terms of sorting networks, $\Gamma_\chi(\mathcal{S}\,|\,V)$ is the result of the sorting of the inverted permutation $[n, n-1, \dots, 2, 1]$ by the restricted primitive network $([i^\square : i^\square + 1])_{i \in I}$, where $I := \{i \,|\, v_i \notin V\}$.

Observe also that we can obtain, like in the previous subsection, the contact points of $\Gamma_\chi(\mathcal{S}\,|\,V)$ by an iterative procedure: we start from the set V and add recursively a minimal (for the partial order $\preccurlyeq_\chi$) remaining vertex v_i of $\mathcal{S}$ such that $V \cup \{v_1, \dots, v_i\}$ is a subset of the set of contact points of a pseudoline arrangement supported by $\mathcal{S}$. The vertex set produced by this procedure is the set of contact points of the χ-greedy constrained pseudoline arrangement $\Gamma_\chi(\mathcal{S}\,|\,V)$.

2.3. **Greedy flip property and enumeration.**

2.3.1. *Greedy flip property.* We are now ready to state the *greedy flip property* (see Figure 6) that says how to update the greedy pseudoline arrangement $\Gamma_\chi(\mathcal{S}\,|\,V)$ when either χ or V are slightly perturbed.

Theorem 12 (Greedy flip property)**.** *Let $\mathcal{S}$ be the support of a pseudoline arrangement. Let χ be a cut of $\mathcal{S}$, v be a minimal (for the order $\preccurlyeq_\chi$) vertex of $\mathcal{S}$, and ψ denote the cut obtained from χ by sweeping v. Let V be a set of vertices of $\mathcal{S}$ (such that $G(\mathcal{S}\,|\,V)$ is not empty), and $W := V \cup \{v\}$. Then:*

(1) If v is a contact point of $\Gamma_\chi(\mathcal{S}\,|\,V)$ which is not in V, then $\Gamma_\psi(\mathcal{S}\,|\,V)$ is obtained from $\Gamma_\chi(\mathcal{S}\,|\,V)$ by flipping v. Otherwise, $\Gamma_\psi(\mathcal{S}\,|\,V) = \Gamma_\chi(\mathcal{S}\,|\,V)$.

(2) If v is a contact point of $\Gamma_\chi(\mathcal{S}\,|\,V)$, then $\Gamma_\psi(\mathcal{S}\,|\,W) = \Gamma_\chi(\mathcal{S}\,|\,V)$. Otherwise, $G(\mathcal{S}\,|\,W)$ is empty.

Proof. We consider a sweep

$$F_{m+1} = F \supset F_m = F' \supset F_{m-1} \supset \cdots \supset F_2 \supset F_1 = \tau(F) \supset F_0 = \tau(F')$$

such that F (resp. F') is a filter corresponding to the cut χ (resp. ψ). Define the points $\bar{v}_i := F_{i+1} \smallsetminus F_i$ and $v_i := \pi(\bar{v}_i)$, and the index $i^\square$ as previously. Let $\sigma_1, \dots, \sigma_{m+1}$ denote the sequence of permutations corresponding to $\Gamma_\chi(\mathcal{S} \,|\, V)$ on the sweep $F_1 \subset \cdots \subset F_{m+1}$. In other words:

(i) σ_1 is the inverted permutation $[n, n-1, \dots, 2, 1]$;
(ii) if $v_i \in V$, then $\sigma_{i+1} = \sigma_i$;
(iii) otherwise, σ_{i+1} is obtained from σ_i by sorting its $i^\square$th and $(i^\square + 1)$th entries.

Similarly, let $\rho_0, \dots, \rho_m$ and $\omega_0, \dots, \omega_m$ denote the sequences of permutations corresponding to $\Gamma_\psi(\mathcal{S} \,|\, V)$ and $\Gamma_\psi(\mathcal{S} \,|\, W)$ respectively on the sweep $F_0 \subset \cdots \subset F_m$.

Assume first that v is a contact point of $\Gamma_\chi(\mathcal{S} \,|\, V)$, but is not in V. Let j denote the integer such that v_j is the crossing point of the two pseudolines of $\Gamma_\chi(\mathcal{S} \,|\, V)$ that are in contact at v. We claim that in this case $\Gamma_\psi(\mathcal{S} \,|\, V)$ is obtained from $\Gamma_\chi(\mathcal{S} \,|\, V)$ by flipping v, *i.e.* that:

(i) for all $1 \leq i \leq j$, ρ_i is obtained from σ_i by exchanging $m^\square$ and $m^\square + 1$;
(ii) for all $j < i \leq m$, $\rho_i = \sigma_i$.

Indeed, ρ_1 is obtained by exchanging $m^\square$ and $m^\square + 1$ in the sequence $\rho_0 = [n, n-1, \dots, 2, 1] = \sigma_1$ (since $m^\square$ and $m^\square + 1$ are respectively the $(0^\square + 1)$th and $0^\square$th entries of ρ_0). Then, any comparison between two consecutive entries give the same result in ρ_i and in σ_i, until $m^\square$ and $m^\square + 1$ are compared again, *i.e.* until $i = j$. At this stage, $m^\square$ and $m^\square + 1$ are already sorted in ρ_j but not in σ_j. Consequently, we have to exchange $m^\square$ and $m^\square + 1$ in σ_j and not in ρ_j, and we obtain $\sigma_{j+1} = \rho_{j+1}$. After this, all the comparisons give the same result in ρ_i and σ_i, and $\rho_i = \sigma_i$ for all $j < i \leq m$.

We prove similarly that:

- When v is not a crossing point of $\Gamma_\chi(\mathcal{S} \,|\, V)$, or is in V, $\rho_i = \sigma_i$ for all $i \in [m]$, and $\Gamma_\psi(\mathcal{S} \,|\, V) = \Gamma_\chi(\mathcal{S} \,|\, V)$.
- When v is a contact point of $\Gamma_\chi(\mathcal{S} \,|\, V)$, $\omega_i = \sigma_i$ for all $i \in [m]$, and $\Gamma_\psi(\mathcal{S} \,|\, W) = \Gamma_\chi(\mathcal{S} \,|\, V)$.

Finally, we prove that $G(\mathcal{S} \,|\, W)$ is empty when v is not a contact point of $\Gamma_\chi(\mathcal{S} \,|\, V)$. For this, assume that $G(\mathcal{S} \,|\, W)$ is not empty, and consider the greedy arrangement $\Gamma = \Gamma_\chi(\mathcal{S} \,|\, W)$. The flip of any contact point of Γ not in W is χ-increasing. Furthermore, since v is a minimal element for $\preccurlyeq_\chi$, the flip of v is also χ-increasing. Consequently, Γ is a source in the graph $G_\chi(\mathcal{S} \,|\, V)$, which implies that $\Gamma_\chi(\mathcal{S} \,|\, V) = \Gamma$, and thus, v is a contact point of $\Gamma_\chi(\mathcal{S} \,|\, V)$. $\square$

2.3.2. *Enumeration.* From the greedy flip property, we derive a binary tree structure on colored pseudoline arrangements supported by $\mathcal{S}$, whose left-pending leaves are precisely the pseudoline arrangements supported by $\mathcal{S}$. A pseudoline arrangement is *colored* if its contact points are colored in blue, green or red. Green and red contact points are considered to be fixed, while blue ones can be flipped.

Theorem 13. *Let $\mathcal{T}$ be the binary tree on colored pseudoline arrangements supported by $\mathcal{S}$ defined as follows:*

(i) The root of the tree is the χ-greedy pseudoline arrangement on $\mathcal{S}$, entirely colored in blue.
(ii) Any node Λ of $\mathcal{T}$ is a leaf of $\mathcal{T}$ if either it contains a green contact point or it only contains red contact points.
(iii) Otherwise, choose a minimal blue point v of Λ. The right child of Λ is obtained by flipping v and coloring it in blue if the flip is χ-increasing and in green if the flip is χ-decreasing. The left child of Λ is obtained by changing the color of v into red.

Then the set of pseudoline arrangements supported by $\mathcal{S}$ is exactly the set of red-colored leafs of $\mathcal{T}$.

Proof. The proof is similar to that of Theorem 9 in [BKPS06].

We define inductively a cut $\chi(\Lambda)$ for each node Λ of $\mathcal{T}$: the cut of the root is χ, and for each node Λ the cut of its children is obtained from χ by sweeping the contact point v. We also denote $V(\Lambda)$ the set of red contact points of Λ. With these notations, the greedy flip property (Theorem 12) ensures that $\Lambda = \Gamma_{\chi(\Lambda)}(\mathcal{S} \,|\, V(\Lambda))$, for each node Λ of $\mathcal{T}$.

The fact that any red-colored leaf of $\mathcal{T}$ is a pseudoline arrangement supported by $\mathcal{S}$ is obvious. Reciprocally, let us prove that any pseudoline arrangement supported by $\mathcal{S}$ is a red leaf of $\mathcal{T}$. Let Λ be a pseudoline arrangement supported by $\mathcal{S}$. We define inductively a path $\Lambda_0, \dots, \Lambda_p$ in the tree $\mathcal{T}$ as follows: Λ_0 is the root of $\mathcal{T}$ and for all $i \geq 0$, Λ_{i+1} is the left child of Λ_i if the minimal blue contact point of Λ_i is a contact point of Λ, and its right child otherwise (we stop when we reach a leaf). We claim that for all $0 \leq i \leq p$:

- the set $V(\Lambda_i)$ is a subset of contact points of Λ;
- the contact points of Λ not in $V(\Lambda_i)$ are not located between $\chi(\Lambda)$ and χ;

from which we derive that $\Lambda = \Lambda_p$ is a red-colored leaf. □

Visiting the tree $\mathcal{T}$ provides an algorithm to enumerate all pseudoline arrangements with a given support. In the next section, we will see the connection between this algorithm and the enumeration algorithm of [BKPS06] for pseudotriangulations of a point set.

Let us briefly discuss the complexity of this algorithm. We assume that the input of the algorithm is a pseudoline arrangement and we consider a flip as an elementary operation. Then, this algorithm requires a polynomial running time per pseudoline arrangement supported by $\mathcal{S}$. As for many enumeration algorithms, the crucial point of this algorithm is that its working space is also polynomial (while the number of pseudoline arrangements supported by $\mathcal{S}$ is exponential).

3. Dual pseudoline arrangements

In this section, we prove that both the graph of flips on "(pointed) pseudotriangulations of a point set" and the graph of flips on "multitriangulations of a convex polygon" can be interpreted as graphs of flips on "pseudoline arrangements with a given support". This interpretation is based on the classical duality that we briefly recall in the first subsection, and leads to a natural definition of "multipseudotriangulations of a pseudoline arrangement" that we present in Section 4.

3.1. **Dual pseudoline arrangement of a point set.** To a given oriented line in the Euclidean plane, we associate its angle $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ with the horizontal axis and its algebraic distance $d \in \mathbb{R}$ to the origin (*i.e.* the value of $\langle(-v,u)\,|\,.\rangle$ on the line, where (u,v) is its unitary direction vector). Since the same line oriented in the other direction gives an angle $\theta+\pi$ and a distance $-d$, this parametrization naturally associates a point of the Möbius strip $\mathcal{M} := \mathbb{R}^2/(\theta,d) \sim (\theta+\pi,-d)$ to each line of the Euclidean plane. In other words, the line space of the Euclidean plane is (isomorphic to) the Möbius strip.

Via this parametrization, the set of lines passing through a point p forms a pseudoline p^*. The pseudolines p^* and q^* dual to two distinct points p and q have a unique crossing point, namely the line (pq). Thus, for a finite point set P in the Euclidean plane, the set $P^* := \{p^* \,|\, p \in P\}$ is a pseudoline arrangement without contact points (see Figure 7). Again, we always assume that the point set P is in general position (no three points lie in a same line), so that the arrangement P^* is simple (no three pseudolines pass through the same point).

This elementary duality also holds for any topological plane (or $\mathbb{R}^2$-plane, see [SBG+95] for a definition), not only for the Euclidean plane $\mathbb{R}^2$. That is to say, the line space of a topological plane is (isomorphic to) the Möbius strip and the dual of a finite set of points in a topological plane is a pseudoline arrangement without contact points. Let us also recall that any pseudoline arrangement of the Möbius strip without contact points is the dual arrangement of a finite set of points in a certain topological plane [HP09]. Thus, in the rest of this paper,

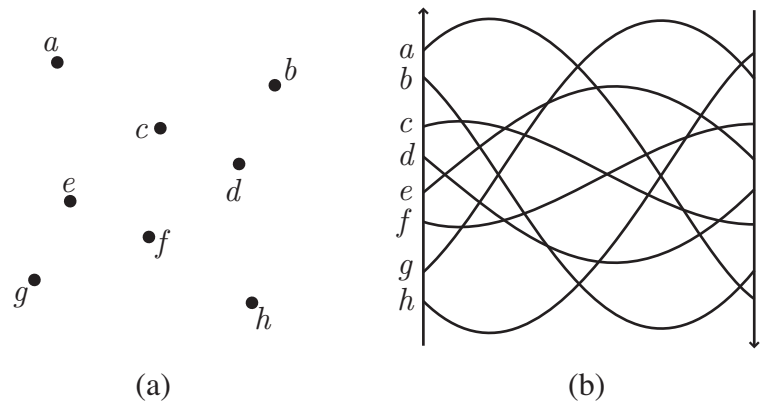

FIGURE 7. A point set P in general position (a) and (a representation of) its dual arrangement P^* (b).

we deal with sets of points and their duals without restriction to the Euclidean plane.

3.2. **Dual pseudoline arrangement of a pseudotriangulation.** We refer to [RSS08] for a detailed survey on pseudotriangulations, and just recall here some basic definitions.

Definition 14. *A* pseudotriangle *is a polygon Δ with only three convex vertices (the* corners *of Δ), joined by three concave polygonal chains (Figure 8). A line is said to be* tangent *to Δ if:*

(i) either it passes through a corner of Δ and separates the two edges incident to it;
(ii) or it passes through a concave vertex of Δ and does not separate the two edges incident to it.

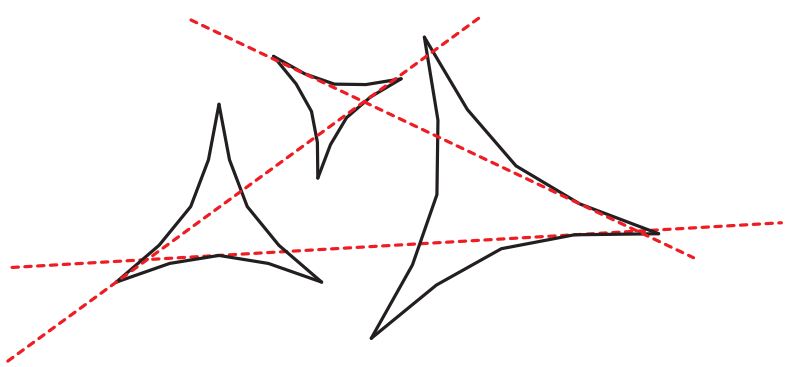

FIGURE 8. Three pseudotriangles and their common tangents.

A pseudotriangulation *of a point set P in general position is a set of edges of P which decomposes the convex hull of P into pseudotriangles. We moreover always assume that pseudotriangulations are* pointed*, meaning that there exists a line passing through any point $p \in P$ and defining a half-plane containing all the edges incident to p.*

The results of this paper only concern pointed pseudotriangulations. Therefore we omit to always specify that pseudotriangulations are pointed. Historically, pseudotriangulations were introduced for families of smooth convex bodies [PV96c] and were therefore automatically pointed. Pseudotriangulations of points, pointed or not, can be regarded as limits of pseudotriangulations of infinitesimally small convex bodies. Note that pointed pseudotriangulations are edge-minimal pseudotriangulations.

Under the pointedness assumption, any two pseudotriangles of a pseudotriangulation have a unique common tangent. This leads to the following observation (see Figure 9):

Observation 15 ([PV94, PV96a])**.** Let T be a pseudotriangulation of a point set P in general position. Then:

(i) the set Δ^* of all tangents to a pseudotriangle Δ of T is a pseudoline;
(ii) the dual pseudolines Δ_1^*, Δ_2^* of any two pseudotriangles Δ_1, Δ_2 of T have a unique crossing point (the unique common tangent to Δ_1 and Δ_2) and possibly a contact point (when Δ_1 and Δ_2 share a common edge);
(iii) the set $T^* := \{\Delta^* \mid \Delta \text{ pseudotriangle of } T\}$ is a pseudoline arrangement (with contact points); and
(iv) T^* is supported by P^* minus its first level (see Figure 9(b)).

In fact, this covering property characterizes pseudotriangulations:

Theorem 16. *Let P be a finite point set in general position in the plane, and P^{*1} denote the support of its dual pseudoline arrangement minus its first level. Then:*

(i) The dual arrangement $T^* := \{\Delta^* \mid \Delta \text{ pseudotriangle of } T\}$ of a pseudotriangulation T of P is supported by P^{*1}.
(ii) The primal set of edges

$$E := \{[p,q] \mid p,q \in P,\ p^* \wedge q^* \text{ contact point of } \Lambda\}$$

of a pseudoline arrangement Λ supported by P^{*1} is a pseudotriangulation of P.

In this section, we provide three proofs of Part (ii) of this result. The first proof is based on flips. First, remember that there is also a simple

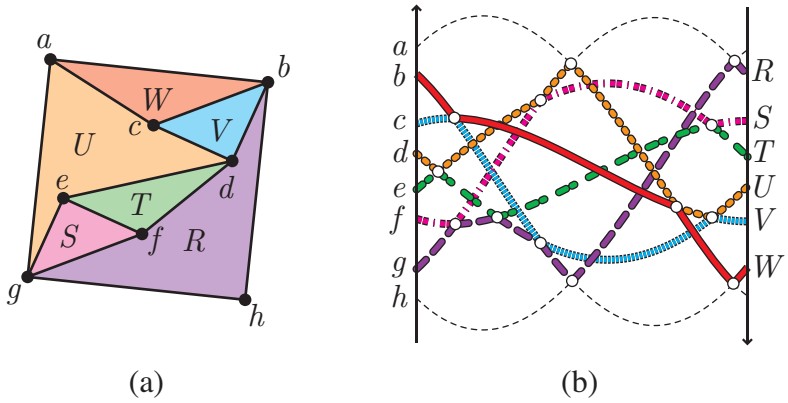

FIGURE 9. (a) A pseudotriangulation T of the point set P of Figure 7(a). (b) The dual arrangement T^* of T, drawn on the dual arrangement P^* of P of Figure 7(b). Each pseudoline of T^* corresponds to a pseudotriangle of T; each contact point in T^* corresponds to an edge in T; each crossing point in T^* corresponds to a common tangent in T.

flip operation on pseudotriangulations of P: replacing any internal edge e in a pseudotriangulation of P by the common tangent of the two pseudotriangles containing e produces a new pseudotriangulation of P. For example, Figure 10 shows two pseudotriangulations of the point set of Figure 7(a), related by a flip, together with their dual pseudoline arrangements. We denote by $G(P)$ the graph of flips on pseudotriangulations of P, whose vertices are pseudotriangulations of P and whose edges are flips between them. In other words, there is an edge in $G(P)$ between two pseudotriangulations of P if and only if their symmetric difference is reduced to a pair.

Proof 1 *of Theorem 16(ii).* The two notions of flips (the primal notion on pseudotriangulations of P and the dual notion on pseudoline arrangements supported by P^{*1}) coincide via duality: an internal edge e of a pseudotriangulation T of P corresponds to a contact point e^* of the dual pseudoline arrangement T^*; the two pseudotriangles Δ_1 and Δ_2 of T containing e correspond to the two pseudolines Δ_1^* and Δ_2^* of T^* in contact at e^*; and the common tangent f of Δ_1 and Δ_2 corresponds to the crossing point f^* of Δ_1^* and Δ_2^*.

Thus, the graph $G(P)$ is a subgraph of $G(\mathcal{S})$. Since both are connected and regular of degree $|P| - 3$, they coincide. In particular, any

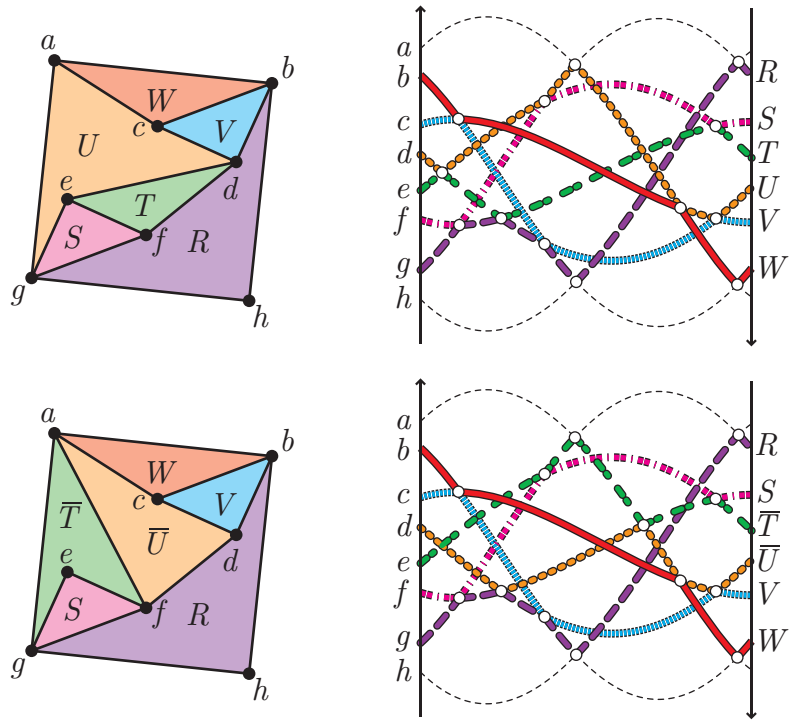

FIGURE 10. A flip in the pseudotriangulation of Figure 9(a) and the corresponding flip in its dual pseudoline arrangement of Figure 9(b).

pseudoline arrangement supported by P^{*1} is the dual of a pseudotriangulation of P. □

Remark 17. Observe that this duality matches our greedy pseudoline arrangement supported by P^{*1} with the greedy pseudotriangulation of [BKPS06]. In particular, the greedy flip property and the enumeration algorithm of Subsection 2.3 are generalizations of results in [BKPS06].

Our second proof of Theorem 16 is slightly longer but more direct, and it introduces a "witness method" that we will repeatedly use throughout this paper. It is based on the following characterization of pseudotriangulations:

Lemma 18 ([Str05])**.** *A graph T on P is a pointed pseudotriangulation of P if and only if it is crossing-free, pointed and has $2|P|-3$ edges.*

Proof 2 *of Theorem 16(ii).* We check that E is crossing-free, pointed and has $2|P|-3$ edges:

Cardinality. First, the number of edges of E equals the difference between the number of crossing points of P^* and of Λ:

$$|E| = \binom{|P^*|}{2} - \binom{|\Lambda|}{2} = \binom{|P|}{2} - \binom{|P|-2}{2} = 2|P| - 3.$$

Crossing-free. Let p, q, r, s be four points of P in convex position. Let t be the intersection of $[p, r]$ and $[q, s]$ (see Figure 11(a)). We use the pseudoline t^* as a *witness* to prove that $[p, r]$ and $[q, s]$ cannot both be in E. For this, we count crossings of t^* with P^* and Λ respectively:

(i) Since P is in general position, the point t is not in P. Therefore $P^* \cup \{t^*\} = (P \cup \{t\})^*$ is a (non-simple) pseudoline arrangement, and t^* crosses P^* exactly $|P|$ times.
(ii) Since t^* is a pseudoline, it crosses each pseudoline of Λ at least once. Thus, it crosses Λ at least $|\Lambda| = |P| - 2$ times.
(iii) For each of the points $p^* \wedge r^*$ and $q^* \wedge s^*$, replacing the crossing point by a contact point removes two crossings with t^*.

Thus, $[p, r]$ and $[q, s]$ cannot both be in E, and E is crossing-free.

Pointed. Let p, q, r, s be four points of P such that s lies inside the convex hull of $\{p, q, r\}$. We first construct a witness pseudoline (see Figure 11(b)) that we use to prove that $[p, s]$, $[q, s]$ and $[r, s]$ cannot all be in E. Let f_p, f_q, f_r and f_s represent p^*, q^*, r^* and s^* respectively. Let $x, y, z \in \mathbb{R}$ be such that $f_p(x) = f_s(x)$, $f_q(y) = f_s(y)$ and $f_r(z) = f_s(z)$. Let g be a continuous and π-antiperiodic function vanishing exactly on $\{x, y, z\} + \mathbb{Z}\pi$ and changing sign each time it vanishes; say for example

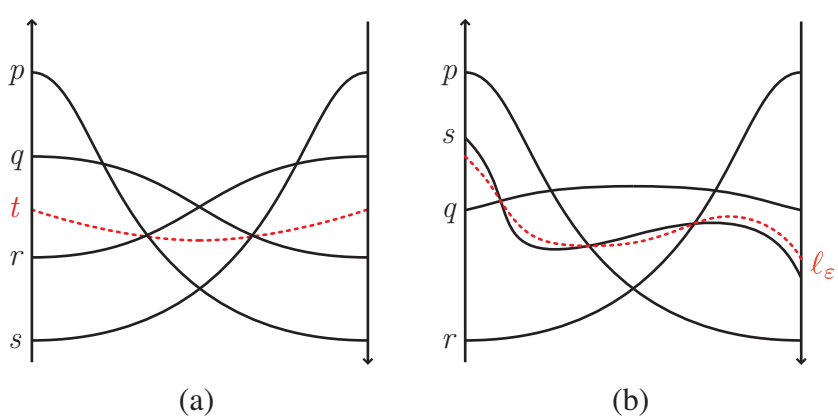

FIGURE 11. (a) Four points p, q, r, s in convex position with the intersection t of $[p, r]$ and $[q, s]$. (b) A point s inside a triangle pqr with the witness pseudoline ℓ_ε.

$g(t) := \sin(t-x)\sin(t-y)\sin(t-z)$. For all $\varepsilon > 0$, we define the function $h_\varepsilon : \mathbb{R} \to \mathbb{R}$ by $h_\varepsilon(t) = f_s(t) + \varepsilon g(t)$. The function h_ε is continuous and π-antiperiodic. The corresponding pseudoline ℓ_ε crosses s^* three times. It is also easy to see that if ε is sufficiently small, then ℓ_ε crosses the pseudolines of $(P \smallsetminus \{s\})^*$ exactly as s^* does (see Figure 11(b)). For such a small ε, we count the crossings of ℓ_ε with P^* and Λ respectively:

(i) ℓ_ε crosses P^* exactly $|P| + 2$ times (it crosses s^* three times and any other pseudoline of P^* exactly once).
(ii) Since ℓ_ε is a pseudoline, it crosses Λ at least $|\Lambda| = |P| - 2$ times.
(iii) For each of the points $p^* \wedge s^*$, $q^* \wedge s^*$ and $r^* \wedge s^*$, replacing the crossing point by a contact point removes two crossings with ℓ_ε.

Thus, $[p, r]$, $[q, s]$ and $[r, s]$ cannot all be in E, and E is pointed. □

Observe that once we know that E is crossing-free, we could also argue its pointedness observing that the pseudolines of Λ would cover the dual pseudoline of a non-pointed vertex twice.

Our third proof of Theorem 16 focusses on pseudotriangles. For every pseudoline λ of the pseudoline arrangement Λ, we denote by

$$S(\lambda) := \{[p, q] \,|\, p, q \in P,\ p^* \wedge q^* \text{ contact point of } \lambda\}$$

the polygonal cycle formed by the edges primal to the contact points of λ. For any point q in the plane, we denote by $\sigma_\lambda(q)$ the *winding number* of $S(\lambda)$ around q, *i.e.* the number of rounds made by $S(\lambda)$ around the point q.

Proof 3 *of Theorem 16(ii).* Consider a point q inside the convex hull of our point set P, and such that $P \cup \{q\}$ be in general position. Hence, its dual pseudoline q^* has exactly $|P|$ crossings with P^*, none of which are on the first level of P^*. For any pseudoline $\lambda \in \Lambda$, let $\tau_\lambda(q)$ denote the number of intersection points between q^* and λ (that is, the number of tangents to $S(\lambda)$ passing through q). Then $\sigma_\lambda(q) = (\tau_\lambda(q) - 1)/2$ and

$$|P| = |q^* \cap P^{*1}| = \sum_{\lambda \in \Lambda} \tau_\lambda(q) = |\Lambda| + 2 \sum_{\lambda \in \Lambda} \sigma_\lambda(q) = |P| - 2 + 2 \sum_{\lambda \in \Lambda} \sigma_\lambda(q).$$

Consequently, $\sum_{\lambda \in \Lambda} \sigma_\lambda(q) = 1$. Hence $\sigma_\lambda(q) = 0$ for all $\lambda \in \Lambda$, except for precisely one pseudoline $\mu \in \Lambda$ which satisfies $\sigma_\mu(q) = 1$.

As a consequence, all polygons $S(\lambda)$ for $\lambda \in \Lambda$ are pseudotriangles (otherwise, we would have points such that $\sigma_\lambda(q) > 1$ for some λ) and they cover the convex hull of P. Consequently, these $|P| - 2$ pseudotriangles form a pointed pseudotriangulation of P. □

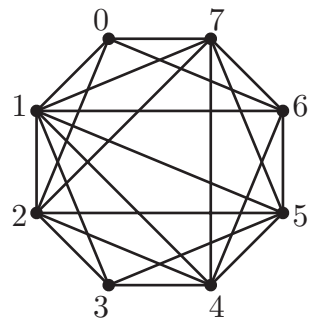

FIGURE 12. A 2-triangulation of the octagon.

3.3. **Dual pseudoline arrangement of a multitriangulation.** Let C_n denote the set of vertices of the convex regular n-gon. We are interested in the following generalization of triangulations, introduced by Capoyleas and Pach [CP92] in the context of extremal theory for geometric graphs (see Figure 12).

Definition 19. *For $\ell \in \mathbb{N}$, an ℓ-crossing is a set of ℓ mutually crossing edges of C_n. A k-triangulation of the n-gon is a maximal set of edges of C_n with no $(k+1)$-crossing.*

Observe that an edge of C_n can be involved in a $(k+1)$-crossing only if there remain at least k vertices on each side. Such an edge is called *k-relevant*. An edge with exactly (resp. strictly less than) $k-1$ vertices on one side is a *k-boundary* edge (resp. a *k-irrelevant* edge). By maximality, every k-triangulation consists of all the nk k-irrelevant plus k-boundary edges and some k-relevant edges.

In [PS09], the triangles and their bisectors are generalized for k-triangulations as follows (see Figure 13):

Definition 20 ([PS09])**.** *A k-star is a star polygon of type $\{2k+1/k\}$, that is, a set of edges of the form $\{s_j s_{j+k} \mid j \in \mathbb{Z}_{2k+1}\}$, where $s_0, s_1, \dots, s_{2k}$*

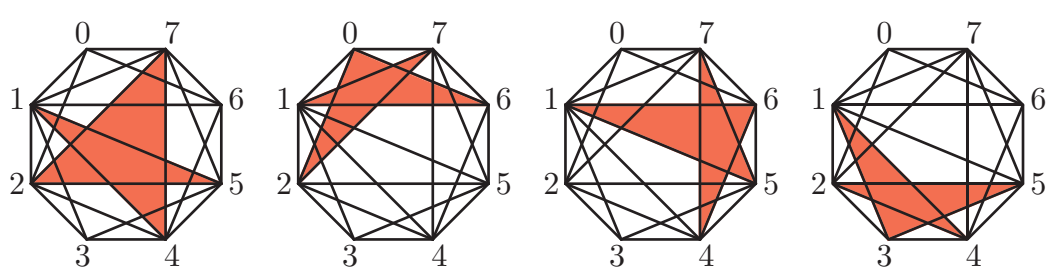

FIGURE 13. The four 2-stars of the 2-triangulation of Figure 12.

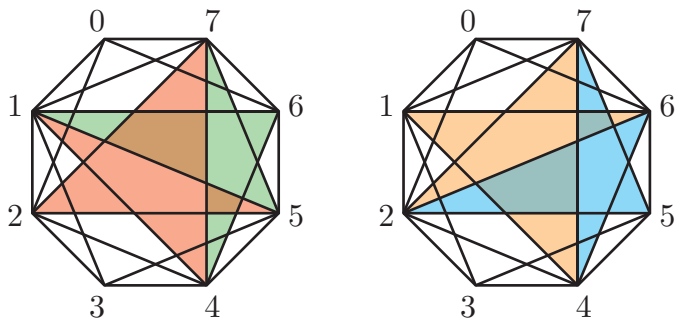

FIGURE 14. A flip in the 2-triangulation of Figure 12.

are cyclically ordered around the unit circle. A (strict) bisector *of a k-star is a (strict) bisector of one of its angles $s_{j-k}s_js_{j+k}$.*

As for $k = 1$, where triangles provide a powerful tool to study triangulations, k-stars are useful to understand k-triangulations. In the following theorem, we point out five properties of stars proved in [PS09]. Figure 13 and Figure 14 illustrate these results on the 2-triangulation of Figure 12.

Theorem 21 ([PS09])**.** *Let T be a k-triangulation of the n-gon. Then*

(i) T contains exactly $n - 2k$ k-stars and $k(n - 2k - 1)$ k-relevant edges.
(ii) Each edge of T belongs to zero, one, or two k-stars, depending on whether it is k-irrelevant, k-boundary, or k-relevant.
(iii) Every pair of k-stars of T has a unique common strict bisector.
(iv) Flipping any k-relevant edge e of T into the common strict bisector f of the two k-stars containing e produces a new k-triangulation $T\triangle\{e, f\}$ of the n-gon. T and $T\triangle\{e, f\}$ are the only two k-triangulations of the n-gon containing $T \setminus \{e\}$.
(v) The flip graph $G_{n,k}$ on k-triangulations of the n-gon is connected and regular of degree $k(n - 2k - 1)$.

Similarly to Observation 15, we can interpret these properties of the stars of the multitriangulations in the dual space (see Figure 15):

Observation 22. Let T be a k-triangulation of a convex n-gon. Then:

(i) the set S^* of all bisectors of a k-star S of T is a pseudoline of the Möbius strip;
(ii) the dual pseudolines S_1^*, S_2^* of any two k-stars S_1, Δ_2 of T have a unique crossing point (the unique common strict bisector of S_1 and S_2) and possibly some contact points (when S_1 and S_2 share common edges);

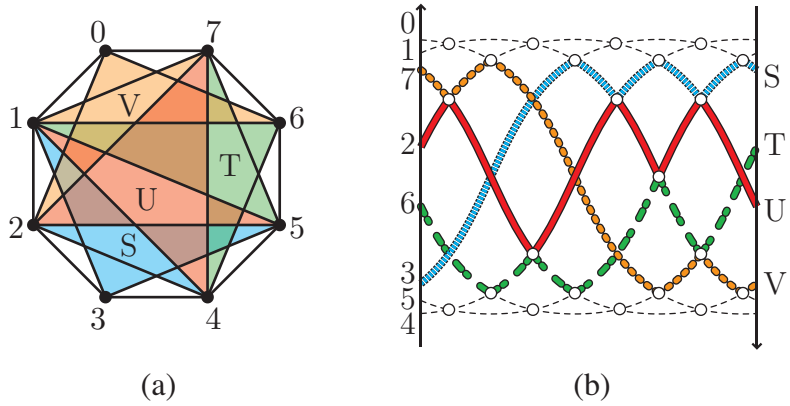

FIGURE 15. A 2-triangulation T of the octagon (a) and its dual pseudoline arrangement T^* (b). Each thick pseudoline of T^* corresponds to a 2-star of T; each contact point in T^* corresponds to an edge in T; each crossing point in T^* corresponds to a common bisector in T.

(iii) the set $T^* := \{S^* \mid S \text{ } k\text{-star of } T\}$ of dual pseudolines of k-stars of T is a pseudoline arrangement (with contact points); and
(iv) T^* is supported by the dual pseudoline arrangement C_n^* of C_n minus its first k levels (see Figure 15(b)).

Again, it turns out that this observation provides a characterization of multitriangulations of a convex polygon:

Theorem 23. *Let C_n denote the set of vertices of a convex n-gon, and C_n^{*k} denote the support of its dual pseudoline arrangement minus its first k levels. Then:*

(i) The dual pseudoline arrangement $T^* := \{S^* \mid S \text{ } k\text{-star of } T\}$ of a k-triangulation T of the n-gon is supported by C_n^{*k}.
(ii) The primal set of edges

$$E := \{[p,q] \mid p,q \in C_n,\ p^* \wedge q^* \text{ contact point of } \Lambda\}$$

of a pseudoline arrangement Λ supported by C_n^{*k} is a k-triangulation of the n-gon.

We provide two proofs of this theorem.

Proof 1 *of Theorem 23(ii).* The two notions of flips (the primal notion on k-triangulations of the n-gon and the dual notion on pseudoline arrangements supported by C_n^{*k}) coincide. Thus, the flip graph $G_{n,k}$ on k-triangulations of the n-gon is a subgraph of $G(C_n^{*k})$. Since they are both connected and regular of degree $k(n-2k-1)$, these two graphs

coincide. In particular, any pseudoline arrangement supported by C_n^{*k} is the dual of a k-triangulation of the n-gon. □

Proof 2 *of Theorem 23(ii).* We follow the method of our second proof of Theorem 16(ii). Since E has the right number of edges (namely $k(2n-2k-1)$), we only have to prove that it is $(k+1)$-crossing-free. We consider $2k+2$ points $p_0, \dots, p_k, q_0, \dots, q_k$ cyclically ordered around the unit circle. Since the definition of crossing (and thus, of ℓ-crossing) is purely combinatorial, *i.e.* depends only on the cyclic order of the points and not on their exact positions, we can move all the vertices of our n-gon on the unit circle while preserving their cyclic order. In particular, we can assume that the lines $(p_i q_i)_{i \in \{0,\dots,k\}}$ all contain a common point t. Its dual pseudoline t^* crosses C_n^* exactly n times and Λ at least $|\Lambda| = n - 2k$ times. Furthermore, for any point $p_i^* \wedge q_i^*$, replacing the crossing point by a contact point removes two crossings with t^*. Thus, the pseudoline t^* provides a witness which proves that the edges $[p_i, q_i]$, $i \in \{0, \dots, k\}$, cannot be all in E, and thus ensures that E is $(k+1)$-crossing-free. □

4. Multipseudotriangulations

Motivated by Theorems 16 and 23, we define in terms of pseudoline arrangements a natural generalization of both pseudotriangulations and multitriangulations. We then study elementary properties of the corresponding set of edges in the primal space.

4.1. **Definition.** We consider the following generalizations of both pseudotriangulations and multitriangulations:

Definition 24. *Let L be a pseudoline arrangement supported by $\mathcal{S}$. Define its k-kernel $\mathcal{S}^k$ to be its support minus its first k levels (which are the iterated external hulls of $\mathcal{S}$). Denote by V^k the set of contact points of L in $\mathcal{S}^k$. A k-pseudotriangulation of L is a pseudoline arrangement whose support is $\mathcal{S}^k$ and whose set of contact points contains V^k.*

Pseudotriangulations of a point set P correspond via duality to 1-pseudotriangulations of the dual pseudoline arrangement P^*. Similarly, k-triangulations of the n-gon correspond to k-pseudotriangulations of the pseudoline arrangement C_n^* in convex position. If L is a pseudoline arrangement with no contact point, then any pseudoline arrangement supported by $\mathcal{S}^k$ is a k-pseudotriangulation of L. In general, the condition that the contact points of L in its k-kernel should be contact points of any k-pseudotriangulation of L is a natural assumption for iterating multipseudotriangulations (see Section 5).

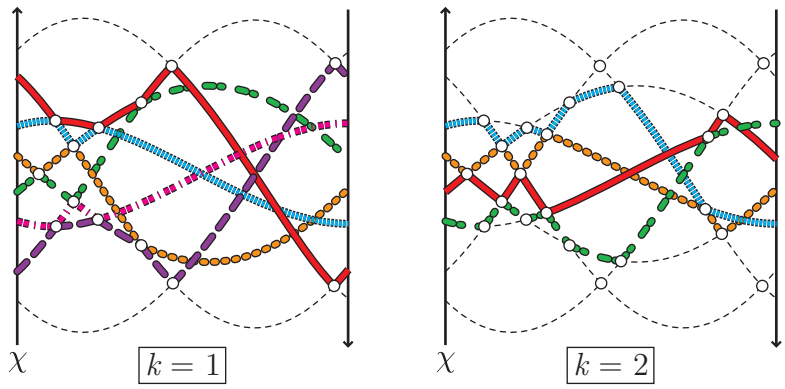

FIGURE 16. The χ-greedy 1-pseudotriangulation and the χ-greedy 2-pseudotriangulation of the pseudoline arrangement of Figure 7(b).

Let Λ be a k-pseudotriangulation of L. We denote by $V(\Lambda)$ the union of the set of contact points of Λ with the set of intersection points of the first k levels of L. In other words, $V(\Lambda)$ is the set of intersection points of L which are not crossing points of Λ. As for pseudoline arrangements, the set $V(\Lambda)$ completely determines Λ.

Flips for multipseudotriangulations are defined as in Lemma 2, with the restriction that the contact points in V^k cannot be flipped. In other words, the flip graph on k-pseudotriangulations of L is exactly the graph $G(\mathcal{S}^k \,|\, V^k)$. Section 2 asserts that the graph of flips is regular and connected, and provides an enumeration algorithm for multipseudotriangulations of L.

Let χ be a cut of (the support of) L. It is also a cut of the k-kernel $\mathcal{S}^k$ of L. A particularly interesting example of k-pseudotriangulation of L is the source of the graph of χ-increasing flips on k-pseudotriangulations of L (see Figure 16 for an illustration):

Definition 25. *The χ-greedy k-pseudotriangulation of L, denoted $\Gamma^k_\chi(L)$, is the greedy pseudoline arrangement $\Gamma_\chi(\mathcal{S}^k \,|\, V^k)$.*

4.2. **Pointedness and crossings.** Let P be a point set in general position. Let Λ be a k-pseudotriangulation of P^* and $V(\Lambda)$ be the set of crossing points of P^* which are not crossing points of Λ. We call *primal of Λ* the set

$$E := \{[p,q] \,|\, p,q \in P,\ p^* \wedge q^* \in V(\Lambda)\}$$

of edges of P primal to $V(\Lambda)$ (see Figure 17). Here, we discuss general properties of primals of multipseudotriangulations. We start with

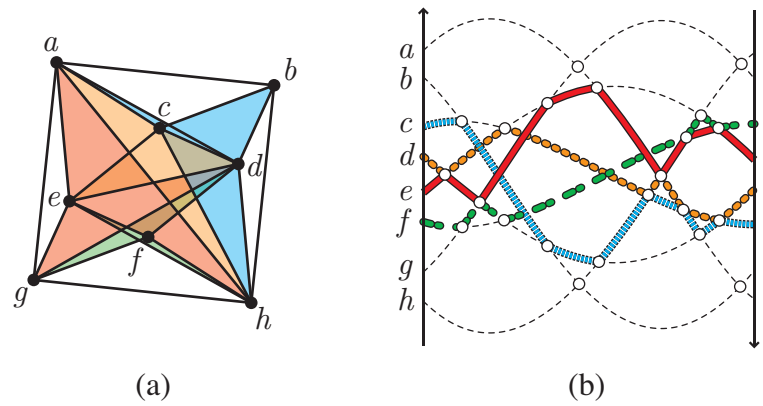

FIGURE 17. The primal set of edges (a) of a 2-pseudotriangulation (b) of the dual pseudoline of the point set of Figure 7(a).

elementary properties that we already observed for the special cases of pseudotriangulations and multitriangulations in the proofs of Theorems 16 and 23:

Lemma 26. *The primal E of Λ has $k(2|P|-2k-1)$ edges.*

Proof. The number of edges of E is the difference between the number of crossing points in the pseudoline arrangements P^* and Λ:

$$|E| = \binom{|P^*|}{2} - \binom{|\Lambda|}{2} = \binom{|P|}{2} - \binom{|P|-2k}{2} = k(2|P|-2k-1). \quad \square$$

We now discuss pointedness of E. We call *k-alternation* any set $\{f_i \mid i \in \mathbb{Z}_{2k+1}\}$ of $2k+1$ edges all incident to a common vertex and whose cyclic order around it is given by

$$f_0 \prec \bar{f}_{1+k} \prec f_1 \prec \bar{f}_{2+k} \prec \ldots \prec f_{2k} \prec \bar{f}_k \prec f_0,$$

where $\bar{f}_i$ denotes the opposite direction of the edge f_i.

Lemma 27. *The primal E of Λ cannot contain a k-alternation.*

Proof. We simply mimic the proof of pointedness in Theorem 16. Let $p_0, \ldots, p_{2k}$ and q be $2k+2$ points of P such that $F := \{[p_i, q] \mid i \in \mathbb{Z}_{2k+1}\}$ is a k-alternation. We prove that F cannot be a subset of E by constructing a witness pseudoline ℓ that separates all the crossing points $p_i^* \wedge q^*$ corresponding to F, while crossing q^* exactly $2k+1$ times and the other pseudolines of P^* exactly as q^* does. (We skip the precise construction, since it is exactly the same as in the proof of Theorem 16.) Counting the crossings of ℓ with P^* and Λ, we obtain:

(i) ℓ crosses P^* exactly $|P|+2k$ times;

(ii) ℓ crosses Λ at least $|\Lambda| = |P| - 2k$ times;
(iii) for each of the points $p_i^* \wedge q^*$, replacing the crossing point by a contact point removes two crossings with ℓ.

Thus the edges $[p_i, q]$ cannot all be contained in E. □

Remark 28. Observe that a set of edges is pointed if and only if it is 1-alternation-free. In contrast, we want to observe the difference between k-alternation-freeness and the following natural notion of k-pointedness: we say that a set F of edges with vertices in P is *k-pointed* if for all p in P, there exists a line which passes through p and defines a half-plane that contains at most $k-1$ segments of F adjacent to p. Observe that a k-pointed set is automatically k-alternation-free but that the converse statement does not hold (see Figure 18(a)).

Finally, contrarily to pseudotriangulations ($k = 1$) and multitriangulations (convex position), the condition of avoiding $(k+1)$-crossings does not hold for k-pseudotriangulations in general:

Remark 29. There exist k-pseudotriangulations with $(k+1)$-crossings (see Figure 18(b)) as well as $(k+1)$-crossing-free k-alternation-free sets of edges that are not subsets of k-pseudotriangulations (see Figure 18(c)).

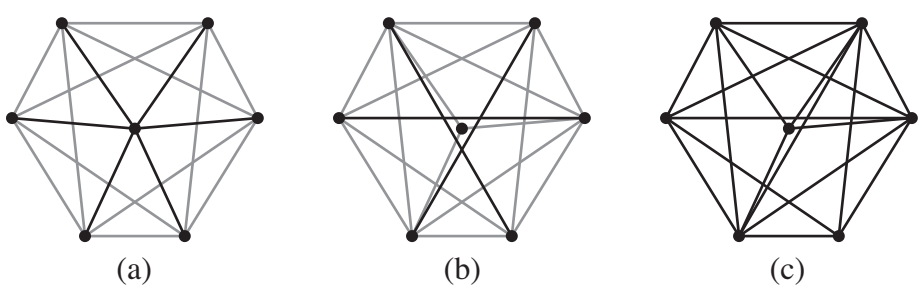

FIGURE 18. (a) A non-2-pointed (but 2-alternation-free) 2-pseudotriangulation. (b) A 2-pseudotriangulation containing a 3-crossing. (c) A 3-crossing-free 2-alternation-free set not contained in a 2-pseudotriangulation.

4.3. **Stars in multipseudotriangulations.** To complete our understanding of the primal of multipseudotriangulations, we need to generalize pseudotriangles of pseudotriangulations and k-stars of k-triangulations: both pseudotriangles and k-stars correspond to pseudolines of the covering pseudoline arrangement.

We keep the notations of the previous section: P is a point set in general position, Λ is a k-pseudotriangulation of P^* and E is the primal set of edges of Λ.

Definition 30. *We call* star *of a pseudoline $\lambda \in \Lambda$ the set of edges*

$$S(\lambda) := \{[p,q] \,|\, p,q \in P,\ p^* \wedge q^* \text{ contact point of } \lambda\}$$

primal to the contact points of λ.

Lemma 31. *For any $\lambda \in \Lambda$, the star $S(\lambda)$ is non-empty.*

Proof. We have to prove that any pseudoline λ of Λ supports at least one contact point. If it is not the case, then λ is also a pseudoline of P^*, and all the $|P|-1$ crossing points of λ with $P^* \smallsetminus \{\lambda\}$ should be crossing points of λ with the arrangement $\Lambda \smallsetminus \{\lambda\}$. This is impossible since $|\Lambda \smallsetminus \{\lambda\}| = |P| - 2k - 1$. □

Similarly to the case of k-triangulations of the n-gon, we say that an edge $[p,q]$ of E is a *k-relevant* (resp. *k-boundary*, resp. *k-irrelevant*) edge if there remain strictly more than (resp. exactly, resp. strictly less than) $k-1$ points of P on each side (resp. one side) of the line (pq). In other words, $p^* \wedge q^*$ is located in the k-kernel (resp. in the intersection of the kth level and the k-kernel, resp. in the first k levels) of the pseudoline arrangement P^*. Thus, the edge $[p,q]$ is contained in 2 (resp. 1, resp. 0) stars of Λ.

The edges of a star $S(\lambda)$ are cyclically ordered by the order of their dual contact points on λ, and thus $S(\lambda)$ forms a (not-necessarily simple) polygonal cycle. For any point q in the plane, let $\sigma_\lambda(q)$ denote the *winding number* of $S(\lambda)$ around q, that is, the number of rounds made by $S(\lambda)$ around the point q (see Figure 19(a)). For example, the winding number of a point in the external face is 0.

We call *k-depth* of a point q the number $\delta^k(q)$ of k-boundary edges of P crossed by any (generic continuous) path from q to the external face, counted positively when passing from the "big" side (the one containing at least k vertices of P) to the "small side" (the one containing $k-1$ vertices of P), and negatively otherwise (see Figure 19(b)). That this number is independent from the path can be seen by mutation. For example, $\delta^1(q)$ is 1 if q is in the convex hull of P and 0 otherwise.

Proposition 32. *Any point q of the plane is covered $\delta^k(q)$ times by the stars $S(\lambda)$, $\lambda \in \Lambda$, of the k-pseudotriangulation Λ of P^*:*

$$\delta^k(q) = \sum_{\lambda \in \Lambda} \sigma_\ell(q).$$

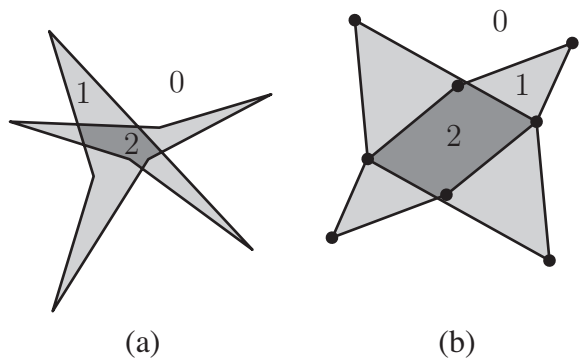

FIGURE 19. (a) Winding number of a star. (b) 2-depth in the point set of Figure 7(a).

The proposition is intuitively clear: let us walk on a continuous path from the external face to the point q. Initially, the winding numbers of all stars of Λ are zero (we start outside all stars of Λ). Then, each time we cross an edge e:

(i) If e is k-irrelevant, it is not contained in any star of Λ, and we do not change the winding numbers of the stars of Λ.
(ii) If e is a k-boundary edge, and if we cross it positively, we increase the winding number of the star S of Λ containing e; if we cross e negatively, we decrease the winding number of S.
(iii) If e is k-relevant, then we decrease the winding number of one star of Λ containing e and increase the winding number of the other star of Λ containing e.

Let us give a formal proof in the dual:

Proof of Proposition 32. Both $\sigma_\lambda(q)$ and $\delta^k(q)$ can be read on the pseudoline q^*:

(i) If $\tau_\lambda(q)$ denotes the number of intersection points between q^* and λ (that is, the number of tangents to $S(\lambda)$ passing through q), then $\sigma_\lambda(q) = (\tau_\lambda(q) - 1)/2$.
(ii) If $\gamma^k(q)$ denotes the number of intersection points between q^* and the first k levels of P^*, then $\delta^k(q) = k - \gamma^k(q)/2$.

The pseudoline q^* has exactly $|P|$ crossings with P^* (since $P^* \cup \{q^*\}$ is an arrangement), which are crossings either with the pseudolines of Λ or with the first k levels of P^*. Hence,

$$|P| = \gamma^k(q) + \sum_{\lambda\in\Lambda} \tau_\lambda(q) = 2k - 2\delta^k(q) + |\Lambda| + 2\sum_{\lambda\in\Lambda} \sigma_\lambda(q),$$

and we get the aforementioned result since $|\Lambda| = |P| - 2k$. $\square$

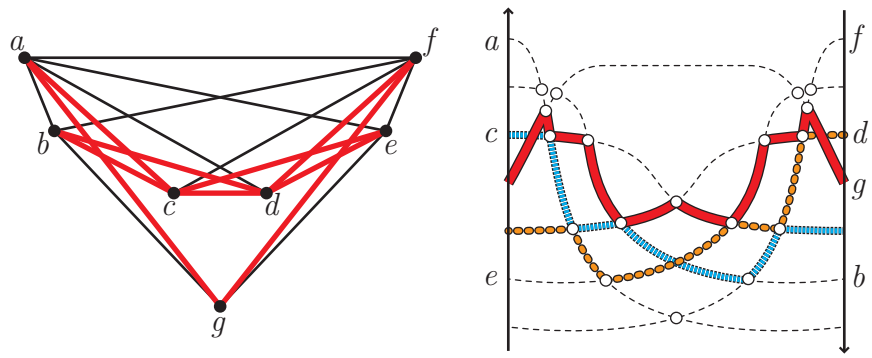

FIGURE 20. A star of a 2-pseudotriangulation of 7 points with 9 corners. This example can be generalized to a 2-pseudotriangulation of n points with $2n-5$ corners.

Remark 33. As a consequence of Proposition 32, we obtain that the k-depth of any point q in any point set P is always non-negative (as a sum of non-negative numbers). It is interesting to notice that Welzl proved in [Wel01] that this non-negativity property is actually equivalent to the Lower Bound Theorem for d-dimensional polytopes with $d+3$ vertices.

A *corner* of the star $S(\lambda)$ is an internal convex angle of it (see Figure 21(a)). In the following proposition, we are interested in the number of corners of $S(\lambda)$. For points in convex position (Section 3.3), the number of corners of a star is always $2k+1$. This is not true anymore in general position as illustrated in Figure 20. The following proposition gives our best bounds on the number of corners of the star of a multipseudotriangulation.

Proposition 34. *The number of corners of a star $S(\lambda)$ of a k-pseudotriangulation of P^* is odd and between $2k+1$ and $2(k-1)|P|+2k+1$.*

Proof. We read convexity of internal angles of $S(\lambda)$ on the preimage $\bar{\lambda}$ of the pseudoline λ under the projection π. Let pqr be an internal angle, let $v = p^* \wedge q^*$ and $w = q^* \wedge r^*$ denote the contact points corresponding to the two edges $[p,q]$ and $[q,r]$ of this angle, and let $\bar{v}$ and $\bar{w}$ denote two consecutive preimages of v and w on $\bar{\lambda}$ (meaning that $\bar{w}$ is located between $\bar{v}$ and $\tau(\bar{v})$). The angle pqr is a corner if and only if $\bar{v}$ and $\bar{w}$ lie on opposite sides of $\bar{\lambda}$, meaning that the other curves touching $\bar{\lambda}$ at $\bar{v}$ and $\bar{w}$ lie on opposite sides, one above and one below $\bar{\lambda}$ (see Figure 21(b)).

In particular, the number $c(\lambda) = c$ of corners of $S(\lambda)$ is the number of opposite consecutive contact points on $\bar{\lambda}$ between two versions $\bar{v}$ and $\tau(\bar{v})$

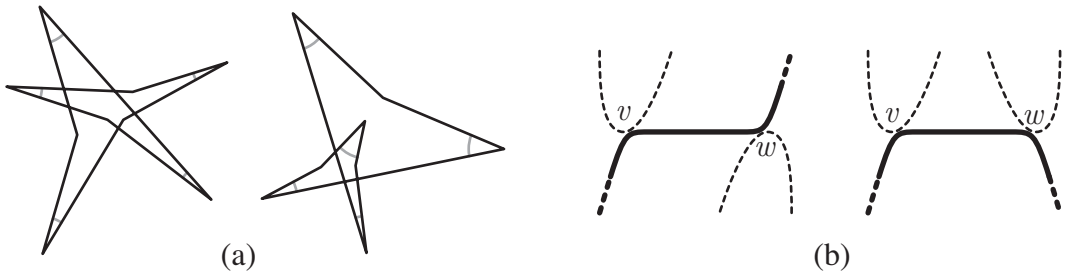

FIGURE 21. (a) Two stars with 5 corners. (b) The two possible configurations of two consecutive contact points on λ: convex (left) and concave (right).

of a contact point v of λ. To see that c is odd, imagine that we are discovering the contact points of λ one by one. The first contact point v that we see corresponds to two opposite contact points $\bar{v}$ and $\tau(\bar{v})$ on $\bar{\lambda}$. Then, at each stage, we insert a new contact point $\bar{w}$ between two old contact points that can be:

(i) either on opposite sides and then we are not changing c;
(ii) or on the same side and we are adding to c either 0 (if $\bar{w}$ is also on the same side) or 2 (if $\bar{w}$ is on the opposite side).

Thus, c remains odd in any case.

To prove the lower bound, we use our witness method. We perturb λ a little bit to obtain a pseudoline μ that passes on the opposite side of each contact point (this is possible since c is odd). This pseudoline μ crosses λ between each pair of opposite contact points and crosses the other pseudolines of Λ exactly as λ does. Thus, μ crosses Λ exactly $|\Lambda| - 1 + c$ times. But since μ is a pseudoline, it has to cross all the pseudolines of P^* at least once. Thus, $|P| \leq |\Lambda| - 1 + c = |P| - 2k - 1 + c$ and $c \geq 2k + 1$.

From this lower bound, we derive automatically the upper bound. Indeed, we know that the number of corners around one point p is at most $\deg(p) - 1$. Consequently,

$$\sum_{p \in P} (\deg(p) - 1) \geq \sum_{\nu \in \Lambda} c(\nu) = c(\lambda) + \sum_{\substack{\nu \in \Lambda \\ \nu \neq \lambda}} c(\nu).$$

The left sum equals $2k(2|P| - 2k - 1) - |P|$ while, according to the previous lower bound, the right one is at least $c + (|P| - 2k - 1)(2k + 1)$. Thus we get $c \leq 2(k-1)|P| + 2k + 1$. $\square$

5. Iterated multipseudotriangulations

By definition, a k-pseudotriangulation of an m-pseudotriangulation of a pseudoline arrangement L is a $(k+m)$-pseudotriangulation of L. In this section, we study these iterated sequences of multipseudotriangulations. In particular, we compare multipseudotriangulations with iterated sequences of 1-pseudotriangulations.

5.1. **Definition and examples.** Let L be a pseudoline arrangement. An *iterated multipseudotriangulation* of L is a sequence $\Lambda_1, \dots, \Lambda_r$ of pseudoline arrangements such that Λ_i is a multipseudotriangulation of Λ_{i-1} for all i (by convention, $\Lambda_0 = L$). We call *signature* of $\Lambda_1, \dots, \Lambda_r$ the sequence $k_1 < \cdots < k_r$ of integers such that Λ_i is a k_i-pseudotriangulation of L for all i. Observe that the assumption that contact points of a pseudoline arrangement L should be contact points of any multipseudotriangulation of L is natural in this setting: iterated multipseudotriangulations correspond to decreasing sequences of sets of crossing points.

A *decomposition* of a multipseudotriangulation Λ of a pseudoline arrangement L is an iterated multipseudotriangulation $\Lambda_1, \dots, \Lambda_r$ of L such that $\Lambda_r = \Lambda$ and $r > 1$. We say that Λ is *decomposable* if such a decomposition exists, and *irreducible* otherwise. The decomposition is *complete* if its signature is $1, 2, \dots, r$.

It is tempting to believe that all multipseudotriangulations are completely decomposable. This would allow to focus only on pseudotriangulations. However, we start by showing that not even all multitriangulations are decomposable. The following example is due to Francisco Santos.

Example 35 (An irreducible 2-triangulation of the 15-gon)**.** We consider the geometric graph T of Figure 22. The edges are:

(i) all the 2-irrelevant and 2-boundary edges of the 15-gon, and
(ii) the five zigzags $Z_a = \{[3a, 3a+6], [3a+6, 3a+1], [3a+1, 3a+5], [3a+5, 3a+2]\}$, for $a \in \{0, 1, 2, 3, 4\}$.

Thus, T has 50 edges and is 3-crossing-free (since the only 2-relevant edges of T that cross a zigzag Z_a are edges of Z_{a-1} and Z_{a+1}). Consequently, T is a 2-triangulation of the 15-gon.

Let us now prove that T^* is irreducible, that is, that T contains no triangulation. Observe first that the edge $[0, 6]$ cannot be an edge of a triangulation contained in T since none of the triangles $06i$, $i \in \{7, \dots, 14\}$, is contained in T. Thus, we are looking for a triangulation contained in $T \smallsetminus \{[0, 6]\}$. Repeating the argument successively for the edges $[1, 6]$,

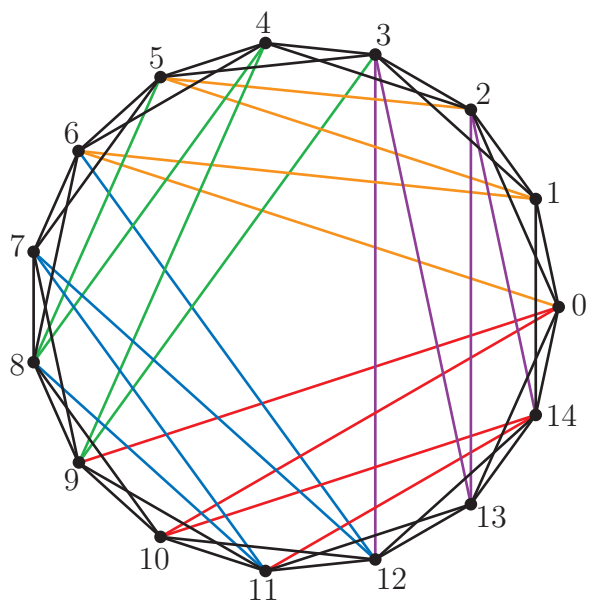

FIGURE 22. An irreducible 2-triangulation of the 15-gon: it contains no triangulation.

$[1,5]$ and $[2,5]$, we prove that the zigzag Z_0 is disjoint from any triangulation contained in T. By symmetry, this proves the irreducibility of T^*.

5.2. **Iterated greedy pseudotriangulations.** Greedy multipseudotriangulations provide interesting examples of iteration of pseudotriangulations. Let L be a pseudoline arrangement, and χ be a cut of L.

Theorem 36. *For any positive integers a and b, $\Gamma_\chi^{a+b}(L) = \Gamma_\chi^b(\Gamma_\chi^a(L))$. Consequently, for any integer k, $\Gamma_\chi^k(L) = \Gamma_\chi^1 \circ \Gamma_\chi^1 \circ \cdots \circ \Gamma_\chi^1(L)$, where $\Gamma_\chi^1(.)$ is iterated k times.*

Proof. Since χ is a cut of L, it is also a cut of $\Gamma_\chi^a(L)$ and thus $\Gamma_\chi^b(\Gamma_\chi^a(L))$ is well defined. Observe also that we can assume that L has no contact point (otherwise, we can open them). Let $n := |L|$ and $m := \binom{n}{2}$.

Let $\chi = \chi_0, \dots, \chi_m = \chi$ be a backward sweep of L. For all i, let v_i denote the vertex of L swept when passing from χ_i to χ_{i+1}, and $i^\square$ denote the integer such that the pseudolines that cross at v_i are the $i^\square$th and $(i^\square+1)$th pseudolines of L on χ_i.

Let $\sigma_0, \dots, \sigma_m$ denote the sequence of permutations corresponding to $\Gamma_\chi^a(L)$ on the sweep $\chi_0, \dots, \chi_m$. In other words, σ_0 is the permutation

$$[1, \dots, a, n-a, n-a-1, \dots, a+2, a+1, n-a+1, \dots, n],$$

whose first a and last a entries are preserved, while its $n-2a$ intermediate entries are inverted. Then, for all i, the permutation σ_{i+1} is obtained from σ_i by sorting its $i^\square$th and $(i^\square+1)$th entries.

Similarly, let $\rho_0, \dots, \rho_m$ and $\omega_0, \dots, \omega_m$ denote the sequences of permutations corresponding to $\Gamma_\chi^{a+b}(L)$ and $\Gamma_\chi^b(\Gamma_\chi^a(L))$ respectively: both ρ_0 and ω_0 equal the permutation whose first and last $a+b$ entries are preserved and whose $n-2a-2b$ intermediate entries are inverted, and:

- ρ_{i+1} is obtained from ρ_i by sorting its $i^\square$th and $(i^\square+1)$th entries;
- if $v_i \notin \Gamma_\chi^a(L)$, then ω_{i+1} is obtained from ω_i by sorting its $i^\square$th and $(i^\square+1)$th entries; otherwise, $\omega_{i+1} = \omega_i$.

We claim that for all i,

(A) all the inversions of ρ_i are also inversions of σ_i: $\rho_i(p) > \rho_i(q)$ implies $\sigma_i(p) > \sigma_i(q)$ for all $1 \le p < q \le n$; and

(B) $\rho_i = \omega_i$.

We prove this claim by induction on i. It is clear for $i=0$. Assume it holds for i and let us prove it for $i+1$. We have two possible situations:

(1) **First case:** $\sigma_i(i^\square) < \sigma_i(i^\square+1)$. Then, $\sigma_{i+1} = \sigma_i$ and $v_i \in \Gamma_\chi^a(L)$. Thus, $\omega_{i+1} = \omega_i$. Furthermore, using Property (A) at rank i, we know that $\rho_i(i^\square) < \rho_i(i^\square+1)$, and thus $\rho_{i+1} = \rho_i$. To summarize, $\sigma_{i+1} = \sigma_i$, $\omega_{i+1} = \omega_i$, and $\rho_{i+1} = \rho_i$, which trivially implies that Properties (A) and (B) remain true.

(2) **Second case:** $\sigma_i(i^\square) > \sigma_i(i^\square+1)$. Then, σ_{i+1} is obtained from σ_i by exchanging the $i^\square$th and $(i^\square+1)$th entries, and $v_i \notin \Gamma_\chi^a(L)$. Consequently, ρ_{i+1} and ω_{i+1} are both obtained from ρ_i and ω_i respectively by sorting their $i^\square$th and $(i^\square+1)$th entries. Thus, Property (B) obviously remains true. As far as Property (A) is concerned, the result is obvious if p and q are different from $i^\square$ and $i^\square+1$. By symmetry, it suffices to prove that for any $p < i^\square$, $\rho_{i+1}(p) > \rho_{i+1}(i^\square)$ implies $\sigma_{i+1}(p) > \sigma_{i+1}(i^\square)$. We have to consider two subcases:

   (a) **First subcase:** $\rho_i(i^\square) < \rho_i(i^\square+1)$. Then $\rho_{i+1} = \rho_i$. Thus, if $p < i^\square$ is such that $\rho_{i+1}(p) > \rho_{i+1}(i^\square)$, then we have $\rho_i(p) > \rho_i(i^\square)$. Consequently, we obtain that $\sigma_{i+1}(p) = \sigma_i(p) > \sigma_i(i^\square) > \sigma_i(i^\square+1) = \sigma_{i+1}(i^\square)$.

   (b) **Second subcase:** $\rho_i(i^\square) > \rho_i(i^\square+1)$. Then ρ_{i+1} is obtained from ρ_i by exchanging its $i^\square$th and $(i^\square+1)$th entries. If $p < i^\square$ is such that $\rho_{i+1}(p) > \rho_{i+1}(i^\square)$, then we have $\rho_i(p) > \rho_i(i^\square+1)$. Consequently, we obtain that $\sigma_{i+1}(p) = \sigma_i(p) > \sigma_i(i^\square+1) = \sigma_{i+1}(i^\square)$.

Obviously, Property (B) of our claim proves the theorem. $\square$

5.3. **Flips in iterated multipseudotriangulations.** Let $\Lambda_1, \dots, \Lambda_r$ be an iterated multipseudotriangulation of a pseudoline arrangement L, with signature $k_1 < \cdots < k_r$. Let v be a contact point of Λ_r (which is not a contact point of L), and let i denote the first integer for which v is a contact point of Λ_i (thus, v is a contact point of Λ_j if and only if $i \le j \le r$). For all $i \le j \le r$, let Λ'_j denote the pseudoline arrangement obtained from Λ_j by flipping v, and let w_j denote the new contact point of Λ'_j. Let j denote the biggest integer such that $w_j = w_i$. There are three possibilities:

(i) If $j = r$, then $\Lambda_1, \dots, \Lambda_{i-1}, \Lambda'_i, \dots, \Lambda'_r$ is an iterated multipseudotriangulation of L. We say that it is obtained from $\Lambda_1, \dots, \Lambda_r$ by a *complete flip* of v .

(ii) If $j < r$, and $w_i = w_j$ is a contact point of Λ_{j+1}, then

$$\Lambda_1, \dots, \Lambda_{i-1}, \Lambda'_i, \dots, \Lambda'_j, \Lambda_{j+1}, \dots, \Lambda_r$$

is an iterated multipseudotriangulation of L. We say that it is obtained from $\Lambda_1, \dots, \Lambda_r$ by a *partial flip* of v.

(iii) If $j < r$, and $w_i = w_j$ is a crossing point in Λ_{j+1}, then we cannot flip v in Λ_i maintaining an iterated multipseudotriangulation of L.

To illustrate these three possible cases, we have labeled on Figure 23 some intersection points of an iterated pseudotriangulation. We have chosen three contact points u, v, w in (b). For $z \in \{u, v, w\}$, we label z' (resp. z'') the crossing point corresponding to z in (b) (resp. in (c)). Observe that:

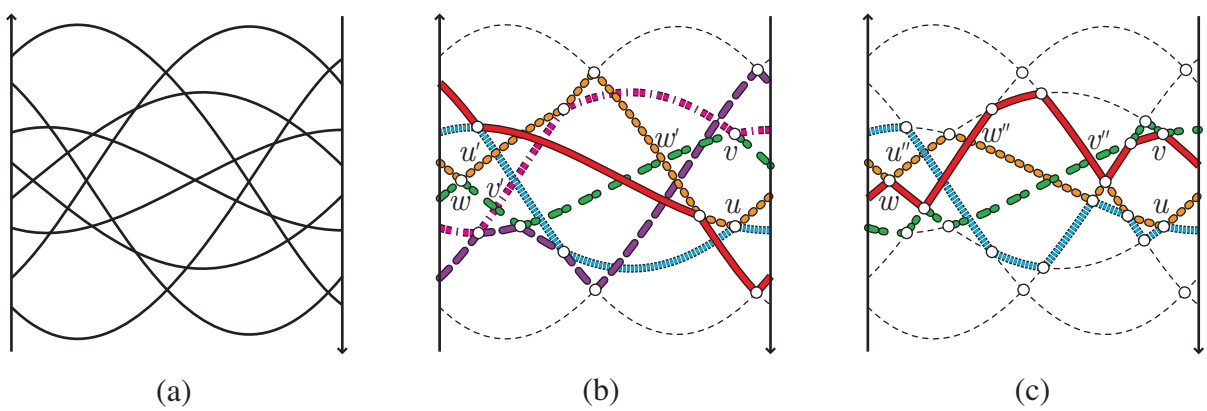

FIGURE 23. An iterated multipseudotriangulation: a pseudoline arrangement L (a), a 1-pseudotriangulation Λ_1 of L (b), and a 1-pseudotriangulation Λ_2 of Λ_1 (c). The contact points u, v, w illustrate the three possible situations for flipping a contact point.

(i) points u' and u'' coincide. Thus we can flip simultaneously point u in (b) and (c) (complete flip);
(ii) points v' is different from v'' but is a contact point in (c). Thus, we can just flip v in (b), without changing (c) and we preserve an iterated pseudotriangulation (partial flip);
(iii) point w' is a crossing point in (c), different from w''. Thus, we cannot flip w in (b) maintaining an iterated pseudotriangulation.

Let $G^{k_1,\dots,k_r}(L)$ be the graph whose vertices are the iterated multipseudotriangulations of L with signature $k_1 < \cdots < k_r$, and whose edges are the pairs of iterated multipseudotriangulations linked by a (complete or partial) flip.

Theorem 37. *The graph of flips $G^{k_1,\dots,k_r}(L)$ is connected.*

To prove this proposition, we need the following lemma:

Lemma 38. *Any intersection point v in the k-kernel of a pseudoline arrangement is a contact point in a k-pseudotriangulation of it.*

Proof. The result holds when $k = 1$. We obtain the general case by iteration. □

Proof of Theorem 37. We prove the result by induction on r (L is fixed). When $r = 1$, we already know that the flip graph is connected. Now, let A_- and A_+ be two iterated multipseudotriangulations of L with signature $k_1 < \cdots < k_r$, that we want to join by flips. Let B_- and B_+ be iterated multipseudotriangulations of L with signature $k_1 < \cdots < k_{r-1}$, and Λ_- and Λ_+ be k_r-pseudotriangulations of L such that $A_- = B_-, \Lambda_-$ and $A_+ = B_+, \Lambda_+$.

By induction, $G^{k_1,\dots,k_{r-1}}(L)$ is connected: let

$$B_- = B_1, B_2, \dots, B_{p-1}, B_p = B_+$$

be a path from B_- to B_+ in $G^{k_1,\dots,k_{r-1}}(L)$. For all j, let v_j be such that B_{j+1} is obtained from B_j by flipping v_j and let w_j be such that B_j is obtained from B_{j+1} by flipping w_j. Let Λ_j be a k_r-pseudotriangulation of L containing the contact points of B_j plus w_j (it exists by Lemma 38), and let $C_j = B_j, \Lambda_j$. Let D_j be the iterated multipseudotriangulation of L obtained from the iterated pseudotriangulation C_j by a partial flip of v_j. Finally, since $G^{k_r}(B_j)$ is connected, there is a path of complete flips from D_{j-1} to C_j.

Merging all these paths, we obtain a global path from A_- to A_+: we transform A_- into C_1 via a path of complete flips; then C_1 into D_1 by the partial flip of v_1; then D_1 into C_2 via a path of complete flips; then C_2 into D_2 by the partial flip of v_2; and so on. □

6. Further topics

We discuss here the extensions in the context of multipseudotriangulations of two known results on pseudotriangulations:

(1) The first one concerns the connection between the greedy pseudotriangulation of a point set and its horizon trees.
(2) The second one extends to arrangements of double pseudolines the definition and properties of multipseudotriangulations.

6.1. **Greedy multipseudotriangulations and horizon graphs.** We have seen in previous sections that the greedy k-pseudotriangulation of a pseudoline arrangement L can be seen as:

(1) the unique source of the graph of increasing flips;
(2) a greedy choice of crossing points given by a sorting network;
(3) a greedy choice of contact points;
(4) an iteration of greedy 1-pseudotriangulations.

In this section, we provide a "pattern avoiding" characterization of the crossing points of the greedy k-pseudotriangulation of L.

Let L be a pseudoline arrangement, and χ be a cut of L. We index by $\ell_1, \dots, \ell_n$ the pseudolines of L in the order in which they cross χ (it is well defined, up to a complete inversion).

We define the *k-upper χ-horizon set* of L to be the set $\mathbb{U}^k_\chi(L)$ of crossing points $\ell_\alpha \wedge \ell_\beta$, with $1 \le \alpha < \beta \le n$, such that there is no $\gamma_1, \dots, \gamma_k$ satisfying $\alpha < \gamma_1 < \cdots < \gamma_k$ and $\ell_\alpha \wedge \ell_{\gamma_i} \preccurlyeq_\chi \ell_\alpha \wedge \ell_\beta$ for all $i \in [k]$. In other words, on each pseudoline ℓ_α of L, the set $\mathbb{U}^k_\chi(L)$ consists of the smallest k crossing points of the form $\ell_\alpha \wedge \ell_\beta$, with $\alpha < \beta$.

Similarly, define the *k-lower χ-horizon set* of L to be the set $\mathbb{L}^k_\chi(L)$ of crossing points $\ell_\alpha \wedge \ell_\beta$, with $1 \le \alpha < \beta \le n$, such that there is no $\delta_1, \dots, \delta_k$ satisfying $\delta_1 < \cdots < \delta_k < \beta$ and $\ell_\beta \wedge \ell_{\delta_i} \preccurlyeq_\chi \ell_\alpha \wedge \ell_\beta$ for all $i \in [k]$. On each pseudoline ℓ_β of L, the set $\mathbb{L}^k_\chi(L)$ consists of the smallest k crossing points of the form $\ell_\alpha \wedge \ell_\beta$, with $\alpha < \beta$.

Finally, we define the set $\mathbb{G}^k_\chi(L)$ to be the set of crossing points $\ell_\alpha \wedge \ell_\beta$, with $1 \le \alpha < \beta \le n$, such that there is no $\gamma_1, \dots, \gamma_k$ and $\delta_1, \dots, \delta_k$ satisfying:

(i) $\alpha < \gamma_1 < \cdots < \gamma_k$, $\delta_1 < \cdots < \delta_k < \beta$, and $\delta_k < \gamma_1$; and
(ii) $\ell_\alpha \wedge \ell_{\gamma_i} \preccurlyeq_\chi \ell_\alpha \wedge \ell_\beta$ and $\ell_\beta \wedge \ell_{\delta_i} \preccurlyeq_\chi \ell_\alpha \wedge \ell_\beta$ for all $i \in [k]$.

Obviously, the sets $\mathbb{U}^k_\chi(L)$ and $\mathbb{L}^k_\chi(L)$ are both contained in $\mathbb{G}^k_\chi(L)$.

Example 39. In Figure 24, we have labeled the vertices of the pseudoline arrangement L of Figure 7(b) with different geometric tags according to their status:

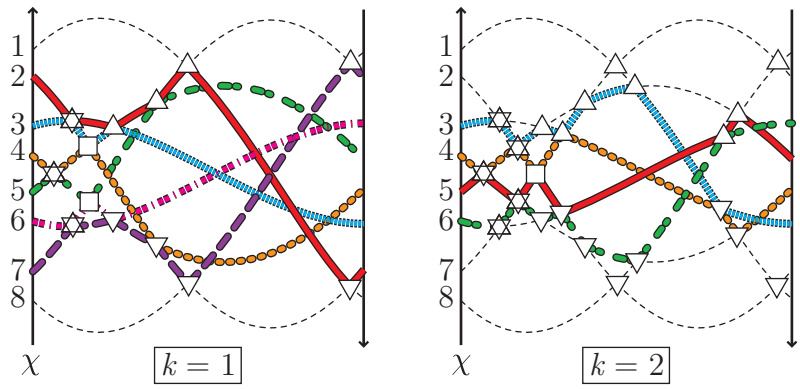

FIGURE 24. The sets $\mathbb{U}^k_\chi(L)$, $\mathbb{L}^k_\chi(L)$, and $\mathbb{G}^k_\chi(L)$ for the pseudoline arrangement of Figure 7(b) and $k \in \{1, 2\}$. The underlying k-pseudotriangulation is the greedy k-pseudotriangulation of L.

($\triangle$) crossing points of the k-upper χ-horizon set $\mathbb{U}^k_\chi(L)$ are represented by up triangles $\triangle$;
($\triangledown$) crossing points of the k-lower χ-horizon set $\mathbb{L}^k_\chi(L)$ are represented by down triangles $\triangledown$;
($\star$) crossing points in both $\mathbb{U}^k_\chi(L)$ and $\mathbb{L}^k_\chi(L)$) are represented by up and down triangles $\star$;
($\square$) crossing points of $\mathbb{G}^k_\chi(L)$ but neither in $\mathbb{U}^k_\chi(L)$, nor in $\mathbb{L}^k_\chi(L)$ are represented by squares $\square$.

Observe that the remaining vertices are exactly the crossing points of the χ-greedy k-pseudotriangulation of L.

Example 40. We consider the arrangement C^*_n of n pseudolines in convex position. Let z be a vertex on the upper hull of its support, $F := \{z' \mid z \preccurlyeq z'\}$ denote the filter generated by z, and χ denote the corresponding cut (see Figure 25). It is easy to check that:

(i) $\mathbb{U}^k_\chi(C^*_n) = \{\ell_\alpha \wedge \ell_\beta \mid 1 \le \alpha \le n \text{ and } \alpha < \beta \le \alpha + k\}$;
(ii) $\mathbb{L}^k_\chi(C^*_n) = \{\ell_\alpha \wedge \ell_\beta \mid 1 \le \alpha \le k \text{ and } \alpha < j \le n\}$; and
(iii) $\mathbb{G}^k_\chi(C^*_n) = \mathbb{U}^k_\chi(\mathcal{C}_n) \cup \mathbb{L}^k_\chi(\mathcal{C}_n)$.

Observe again that the remaining vertices are exactly the crossing points of the χ-greedy k-pseudotriangulation of C^*_n.

Theorem 41 extends this observation to all pseudoline arrangements, using convex position as a starting point for a proof by mutation.

Theorem 41. *For any pseudoline arrangement L with no contact point, and any cut χ of L, the sets $V(\Gamma^k_\chi(L))$ and $\mathbb{G}^k_\chi(L)$ coincide.*

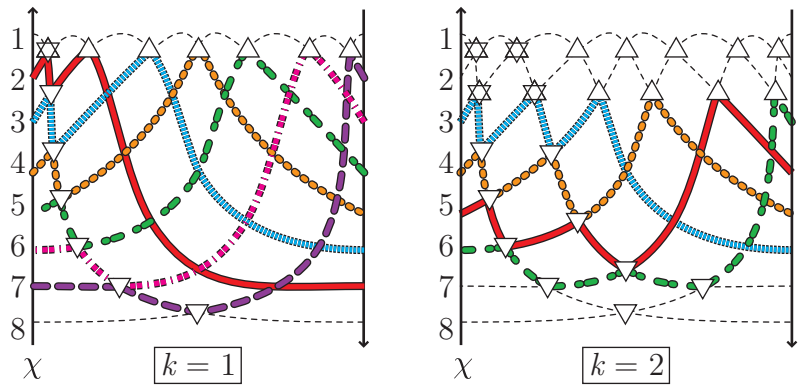

FIGURE 25. The sets $\mathbb{U}^k_\chi(C^*_8)$, $\mathbb{L}^k_\chi(C^*_8)$, and $\mathbb{G}^k_\chi(C^*_8)$ for the arrangement C^*_8 of 8 pseudolines in convex position, and $k \in \{1, 2\}$. The underlying k-pseudotriangulation is the greedy k-pseudotriangulation of C^*_8.

The proof of this theorem works by mutation. A *mutation* is a local transformation of an arrangement L that only inverts one triangular face of L. More precisely, it is a homotopy of arrangements during which only one curve $\ell \in L$ moves, sweeping a single vertex of the remaining arrangement $L \smallsetminus \{\ell\}$ (see Figure 26).

If P is a point set of a topological plane, mutating an empty triangle $p^*q^*r^*$ of P^* by sweeping the vertex $q^* \wedge r^*$ with the pseudoline p^* corresponds in the primal to moving p a little bit such that only the orientation of the triangle pqr changes.

The graph of mutations on pseudoline arrangements is known to be connected: any two pseudoline arrangements (with no contact points

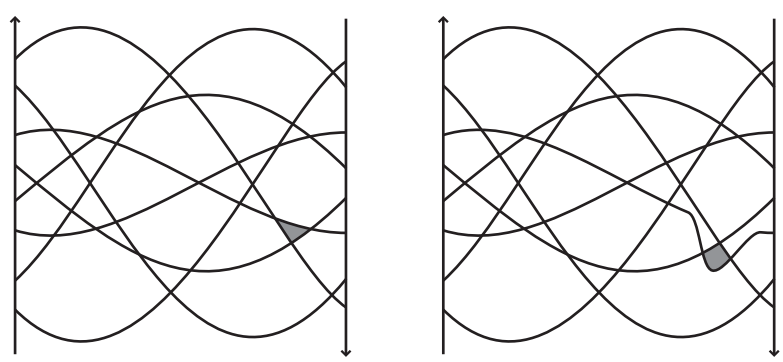

FIGURE 26. A mutation in the arrangement of Figure 7(b).

and the same number of pseudolines) are homotopic via a finite sequence of mutations (followed by a homeomorphism). In fact, one can even avoid mutations of triangles that cross a given cut of L:

Proposition 42. *Let L and L' be two pseudoline arrangements of $\mathcal{M}$ (with no contact points and the same number of pseudolines) and χ be a cut of both of L and L'. There is a finite sequence of mutations of triangles disjoint from χ that transforms L into L'.*

Proof. We prove that any arrangement L of n pseudolines can be transformed into the arrangement C_n^* of n pseudolines in convex position (see Figure 25).

Let $\ell_1, \dots, \ell_n$ denote the pseudolines of L (ordered by their crossings with χ). Let Δ denote the triangle formed by χ, ℓ_1 and ℓ_2. If there is a vertex of the arrangement $L \setminus \{\ell_1, \ell_2\}$ inside Δ, then there is a triangle of the arrangement L inside Δ and adjacent to ℓ_1 or ℓ_2. Mutating this triangle reduces the number of vertices of $L \setminus \{\ell_1, \ell_2\}$ inside Δ such that after some mutations, there is no more vertex inside Δ. If Δ is intersected by pseudolines of $L \setminus \{\ell_1, \ell_2\}$, then there is a triangle inside Δ formed by ℓ_1, ℓ_2 and one of these intersecting pseudolines (the one closest to $\ell_1 \wedge \ell_2$). Mutating this triangle reduces the number of pseudolines intersecting Δ. Thus, after some mutations, Δ is a triangle of the arrangement L.

Repeating these arguments, we prove that for all $i \in \{2, \dots, n-1\}$ and after some mutations, ℓ_i, ℓ_1, ℓ_{i+1} and χ delimit a face of the arrangement L. Thus, one of the two topological disk delimited by χ and ℓ_1 contains no more vertex of L, and the proof is then straightforward by induction. □

Let ∇ be a triangle of L not intersecting χ. Let L' denote the pseudoline arrangement obtained from L by mutating the triangle ∇ into the inverted triangle Δ. Let $a < b < c$ denote the indices of the pseudolines ℓ_a, ℓ_b and ℓ_c that form ∇ and Δ. In ∇, we denote $A = \ell_b \wedge \ell_c$, $B = \ell_a \wedge \ell_c$ and $C = \ell_a \wedge \ell_b$; similarly, in Δ, we denote $D = \ell_b \wedge \ell_c$, $E = \ell_a \wedge \ell_c$ and $F = \ell_a \wedge \ell_b$ (see Figure 27).

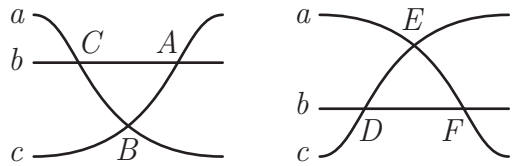

FIGURE 27. A local image of a mutation.

Lemma 43. *With these notations, the following properties hold:*

(i) $B \in \mathbb{U}_\chi^k(L) \Leftrightarrow C \in \mathbb{U}_\chi^{k-1}(L) \Leftrightarrow E \in \mathbb{U}_\chi^{k-1}(L') \Leftrightarrow F \in \mathbb{U}_\chi^k(L')$
$A \in \mathbb{U}_\chi^k(L) \Leftrightarrow D \in \mathbb{U}_\chi^k(L')$
$E \in \mathbb{L}_\chi^k(L') \Leftrightarrow D \in \mathbb{L}_\chi^{k-1}(L') \Leftrightarrow B \in \mathbb{L}_\chi^{k-1}(L) \Leftrightarrow A \in \mathbb{L}_\chi^k(L)$
$F \in \mathbb{L}_\chi^k(L') \Leftrightarrow C \in \mathbb{L}_\chi^k(L)$

(ii) $C \in \mathbb{U}_\chi^k(L) \Rightarrow A \in \mathbb{U}_\chi^k(L)$
$E \in \mathbb{U}_\chi^k(L') \Rightarrow D \in \mathbb{U}_\chi^k(L')$
$D \in \mathbb{L}_\chi^k(L') \Rightarrow F \in \mathbb{L}_\chi^k(L')$
$B \in \mathbb{L}_\chi^k(L) \Rightarrow C \in \mathbb{L}_\chi^k(L)$
$B \in \mathbb{G}_\chi^k(L) \Rightarrow C \in \mathbb{G}_\chi^k(L) \wedge D \in \mathbb{G}_\chi^k(L') \wedge F \in \mathbb{G}_\chi^k(L')$
$E \in \mathbb{G}_\chi^k(L') \Rightarrow D \in \mathbb{G}_\chi^k(L') \wedge C \in \mathbb{G}_\chi^k(L) \wedge A \in \mathbb{G}_\chi^k(L)$

(iii) $C \in \mathbb{G}_\chi^k(L) \Rightarrow A \in \mathbb{U}_\chi^k(L) \vee C \in \mathbb{L}_\chi^k(L)$
$D \in \mathbb{G}_\chi^k(L') \Rightarrow F \in \mathbb{L}_\chi^k(L') \vee D \in \mathbb{U}_\chi^k(L')$
$A \in \mathbb{G}_\chi^k(L) \Rightarrow A \in \mathbb{U}_\chi^k(L) \vee C \in \mathbb{L}_\chi^{k-1}(L)$
$F \in \mathbb{G}_\chi^k(L') \Rightarrow F \in \mathbb{L}_\chi^k(L') \vee D \in \mathbb{U}_\chi^{k-1}(L')$

(iv) $C \in \mathbb{G}_\chi^k(L) \wedge E \notin \mathbb{G}_\chi^k(L') \Rightarrow A \notin \mathbb{G}_\chi^k(L)$
$D \in \mathbb{G}_\chi^k(L') \wedge B \notin \mathbb{G}_\chi^k(L) \Rightarrow F \notin \mathbb{G}_\chi^k(L')$

Proof. By symmetry, it is enough to prove the first line of each of the four points of the lemma.

Properties of point (i) directly come from the definitions. For example, all the assertions of the first line are false if and only if there exist $\gamma_1, \dots, \gamma_{k-1}$ with $a < \gamma_1 < \cdots < \gamma_{k-1}$ and, for all $i \in [k-1]$, $\ell_a \wedge \ell_{\gamma_i} \preccurlyeq_\chi C$ (or equivalently $\ell_a \wedge \ell_{\gamma_i} \preccurlyeq_\chi E$).

We derive point (ii) from the following observation: if $\gamma > b$ and if $\ell_b \wedge \ell_\gamma \preccurlyeq_\chi C$, then $\gamma > a$ and $\ell_a \wedge \ell_\gamma \preccurlyeq_\chi B$.

For point (iii), assume that $A \notin \mathbb{U}_\chi^k(L)$ and $C \notin \mathbb{L}_\chi^k(L)$. Then there exist $\gamma_1, \dots, \gamma_k$ and $\delta_1, \dots, \delta_k$ such that $\delta_1 < \cdots < \delta_k < b < \gamma_1 < \cdots < \gamma_k$ and, for all $i \in [k]$, $\ell_b \wedge \ell_{\gamma_i} \preccurlyeq_\chi A$ (and therefore $\ell_a \wedge \ell_{\gamma_i} \preccurlyeq_\chi C$) and $\ell_b \wedge \ell_{\delta_i} \preccurlyeq_\chi C$. Thus $C \notin \mathbb{G}_\chi^k(L)$.

Finally, assume that $C \in \mathbb{G}_\chi^k(L)$ and $E \notin \mathbb{G}_\chi^k(L')$. Then, there exist $\gamma_1, \dots, \gamma_k$ and $\delta_1, \dots, \delta_k$ such that $a < \gamma_1 < \cdots < \gamma_k$, $\delta_1 < \cdots < \delta_k < c$, $\delta_k < \gamma_1$, and for all $i \in [k]$, $\ell_a \wedge \ell_{\gamma_i} \preccurlyeq_\chi E$ and $\ell_c \wedge \ell_{\delta_i} \preccurlyeq_\chi E$. Since $C \in \mathbb{G}_\chi^k(L)$, we have $\delta_k > b$. Thus $b < \gamma_1 < \cdots < \gamma_k$ and for all $i \in [k]$, $\ell_b \wedge \ell_{\gamma_i} \preccurlyeq_\chi A$ and $\ell_c \wedge \ell_{\delta_i} \preccurlyeq_\chi A$. This implies that $A \notin \mathbb{G}_C^k(L)$. □

We are now ready to establish the proof of Theorem 41:

Proof of Theorem 41. The proof works by mutation. We already observed the result when the pseudoline arrangement is in convex position

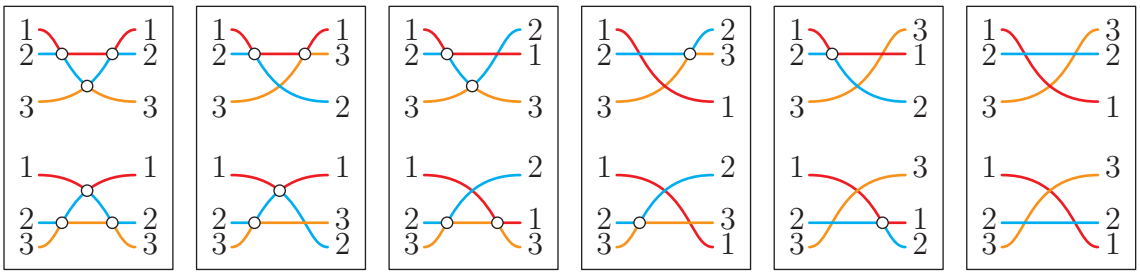

FIGURE 28. The six possible cases for the mutation of the greedy k-pseudotriangulation.

(see Example 40 and Figure 25). Proposition 42 ensures that any pseudoline arrangement can be reached from this convex configuration by mutations of triangles not intersecting χ. Thus, it is sufficient to prove that such a mutation preserves the property.

Assume that L is a pseudoline arrangement and χ is a cut of L, for which the result holds. Let ∇ be a triangle of L not intersecting χ. Let L' denote the pseudoline arrangement obtained from L by mutating the triangle ∇ into the inverted triangle Δ. Let A, B, C and D, E, F denote the vertices of ∇ and Δ as indicated in Figure 27.

If v is a vertex of the arrangement L' different from D, E, F, then:

$$v \in V(\Gamma^k_\chi(L')) \Leftrightarrow v \in V(\Gamma^k_\chi(L)) \Leftrightarrow v \in \mathbb{G}^k_\chi(L) \Leftrightarrow v \in \mathbb{G}^k_\chi(L').$$

Thus, we only have to prove the equivalence when $v \in \{D, E, F\}$. The proof is a (computational) case analysis: using the properties of Lemma 43 as boolean equalities relating the boolean variables defined by "$X \in \mathbb{Y}^p_\chi(L)$" (where $X \in \{A, B, C, D, E, F\}$, $\mathbb{Y} \in \{\mathbb{U}, \mathbb{L}, \mathbb{G}\}$, and $p \in \{k-1, k\}$), we have written a short boolean satisfiability program which affirms that:

(i) either $\{A, B, C\} \subset \mathbb{G}^k_\chi(L)$ and $\{D, E, F\} \subset \mathbb{G}^k_\chi(L')$;
(ii) or $\{A, B, C\} \cap \mathbb{G}^k_\chi(L) = \{A, C\}$ and $\{D, E, F\} \cap \mathbb{G}^k_\chi(L') = \{D, E\}$;
(iii) or $\{A, B, C\} \cap \mathbb{G}^k_\chi(L) = \{B, C\}$ and $\{D, E, F\} \cap \mathbb{G}^k_\chi(L') = \{D, F\}$;
(iv) or $\{A, B, C\} \cap \mathbb{G}^k_\chi(L) = \{A\}$ and $\{D, E, F\} \cap \mathbb{G}^k_\chi(L') = \{D\}$;
(v) or $\{A, B, C\} \cap \mathbb{G}^k_\chi(L) = \{C\}$ and $\{D, E, F\} \cap \mathbb{G}^k_\chi(L') = \{F\}$;
(vi) or $\{A, B, C\} \cap \mathbb{G}^k_\chi(L) = \emptyset$ and $\{D, E, F\} \cap \mathbb{G}^k_\chi(L') = \emptyset$.

It is easy to check that these six cases correspond to sorting the six possible permutations of $\{1, 2, 3\}$ on ∇ and Δ (see Figure 28). Consequently, if $V(\Gamma^k_\chi(L)) \cap \{A, B, C\} = \mathbb{G}^k_\chi(L) \cap \{A, B, C\}$, then we have $V(\Gamma^k_\chi(L')) \cap \{D, E, F\} = \mathbb{G}^k_\chi(L') \cap \{D, E, F\}$, which finishes the proof. $\square$

Let us finish this discussion by recalling the interpretation of the horizon sets when $k = 1$. Let P be a finite point set. For any $p \in P$, let $u(p)$ denote the point q that minimizes the angle (Ox, pq) among all points of P with $y_p < y_q$ (by convention, for the higher point p of P, $u(p) = p$). The *upper horizon tree* of P is the set $\mathbb{U}(P) = \{pu(p) \,|\, p \in P\}$. The *lower horizon tree* $\mathbb{L}(P)$ of P is defined symmetrically. See Figure 29.

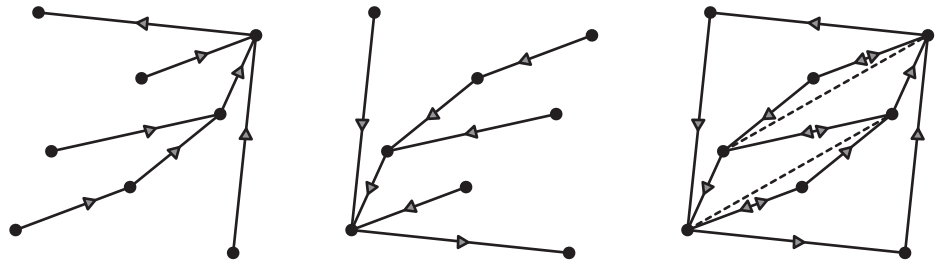

FIGURE 29. The upper horizon tree (left), the lower horizon tree (middle), and the greedy pseudotriangulation (right) of the point set of Figure 7(a). The dashed edges in the greedy pseudotriangulation are not in the horizon trees.

Choosing a cut χ of P^* corresponding to the point at infinity $(-\infty, 0)$ makes coincide primal and dual definitions of horizon sets: we have $\mathbb{U}^1_\chi(P^*) = \mathbb{U}(P)^*$ and $\mathbb{L}^1_\chi(P^*) = \mathbb{L}(P)^*$.

In [Poc97], Pocchiola observed that the set $\mathbb{U}(P) \cup \mathbb{L}(P)$ of edges can be completed into a pseudotriangulation of P just by adding the sources of the faces of P^* intersected by the cut χ. The obtained pseudotriangulation is our χ-greedy 1-pseudotriangulation $\Gamma^1_\chi(P^*)$.

6.2. **Multipseudotriangulations of double pseudoline arrangements.** In this section, we deal with double pseudoline arrangements, *i.e.* duals of sets of disjoint convex bodies. Definitions and properties of multipseudotriangulations naturally extend to these objects.

6.2.1. *Definitions.* A simple closed curve in the Möbius strip can be:

(i) either contractible (homotopic to a point);
(ii) or non separating, or equivalently homotopic to a generator of the fundamental group of $\mathcal{M}$: it is a pseudoline;
(iii) or separating and non-contractible, or equivalently homotopic to the double of a generator of the fundamental group of $\mathcal{M}$: it is called a *double pseudoline* (see Figure 30(a)).

The complement of a double pseudoline ℓ has two connected components: the bounded one is a Möbius strip M_ℓ and the unbounded one is an open cylinder C_ℓ (see Figure 30(a)).

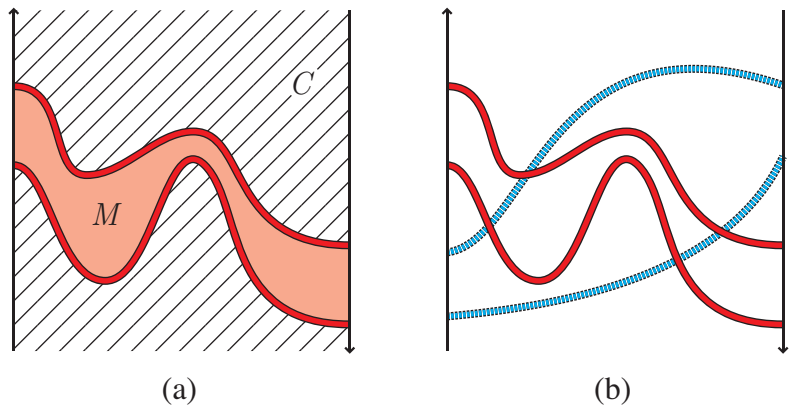

FIGURE 30. (a) A double pseudoline. (b) An arrangement of 2 double pseudolines.

The canonical example of a double pseudoline is the set C^* of tangents to a convex body C of the plane. Observe also that the pth level of a pseudoline arrangement is a double pseudoline. If C is a convex body of the plane, then the Möbius strip M_{C^*} corresponds to lines that pierce C, while C_{C^*} corresponds to lines disjoint from C. If C and C' are two disjoint convex bodies, the two corresponding double pseudolines C^* and C'^* cross four times (see Figure 31 and Figure 32). Each of these four crossings corresponds to one of the four *bitangents* (or *common tangents*) between C and C'.

Definition 44 ([HP09])**.** *A double pseudoline arrangement is a finite set of double pseudolines such that any two of them have exactly four intersection points, cross transversally at these points, and induce a cell*

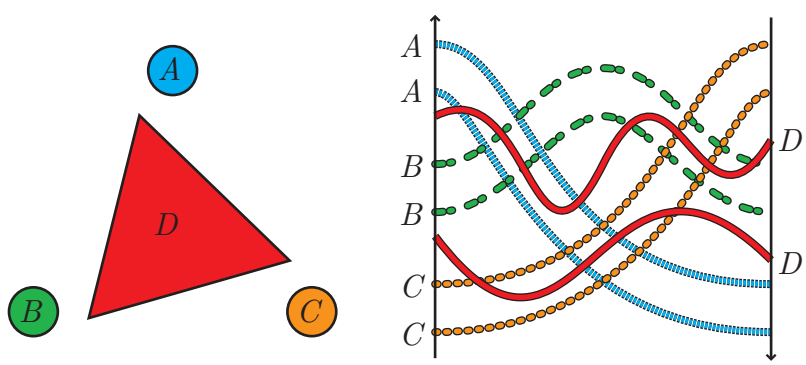

FIGURE 31. A configuration of four disjoint convex bodies and its dual double pseudoline arrangement.

decomposition of the Möbius strip (i.e. the complement of their union is a union of topological disks, together with the external cell).

Given a set Q of disjoint convex bodies in the plane (or in any topological plane), its dual $Q^* := \{C^* \mid C \in Q\}$ is an arrangement of double pseudolines (see Figure 31 and Figure 32). Furthermore, as for pseudoline arrangements, any double pseudoline arrangement can be represented by (*i.e.* is the dual of) a set of disjoint convex bodies in a topological plane [HP09].

In this paper, we only consider *simple* arrangements of double pseudolines. Defining the *support*, the *levels*, and the *kernels* of double pseudoline arrangements as for pseudoline arrangements, we can extend multipseudotriangulations to double pseudoline arrangements (see Figure 33):

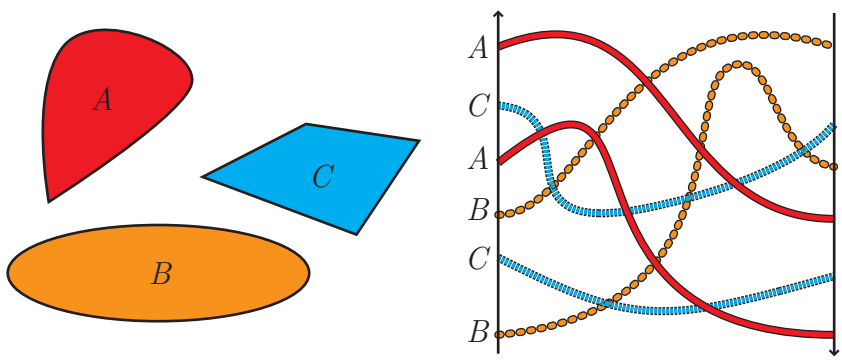

FIGURE 32. A configuration of three disjoint convex bodies and its dual double pseudoline arrangement.

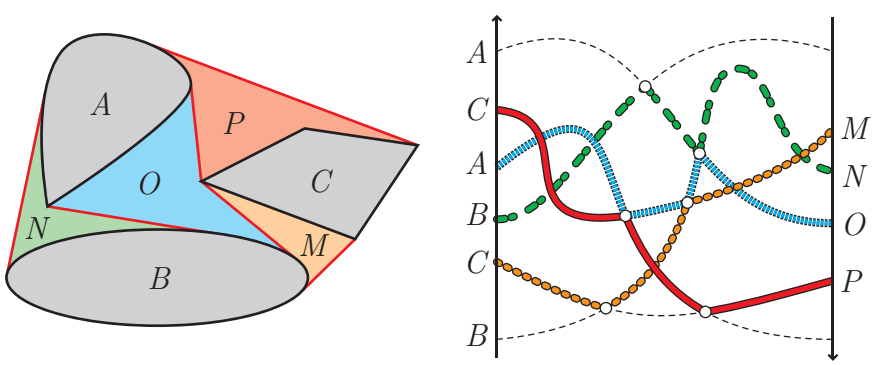

FIGURE 33. A pseudotriangulation of the set of disjoint convex bodies of Figure 32.

Definition 45. *A k-pseudotriangulation of a double pseudoline arrangement L is a pseudoline arrangement supported by the k-kernel of L.*

All the properties related to flips developed in Section 2 apply in this context. In the end of this section, we only revisit the properties of the primal of a multipseudotriangulation of a double pseudoline arrangement.

6.2.2. *Elementary properties.* Let Q be a set disjoint convex bodies in general position in the plane and Q^* be its dual arrangement. Let Λ be a k-pseudotriangulation of Q^*, $V(\Lambda)$ denote all crossing points of Q^* that are not crossing points of Λ, and E denote the corresponding set of bitangents of Q. As in Subsection 4.2, we discuss the properties of the primal configuration E:

Lemma 46. *The set E has $4|Q|k - |Q| - 2k^2 - k$ edges.*

Proof. The number of edges of E is the number of crossing points of Q^* minus the number of crossing points of Λ, *i.e.*

$$\begin{aligned} |E| &= 4\binom{|Q^*|}{2} - \binom{|\Lambda|}{2} = 4\binom{|Q|}{2} - \binom{2|Q| - 2k}{2} \\ &= 4|Q|k - |Q| - 2k^2 - k. \end{aligned}$$

□

We now discuss pointedness. For any convex body C of Q, we arbitrarily choose a point p_C in the interior of C, and we consider the set X_C of all segments between p_C and a sharp boundary point of C. We denote by $X := \bigcup_{C \in Q} X_C$ the set of all these segments.

Lemma 47. *The set $E \cup X$ cannot contain a k-alternation.*

Proof. Let C be a convex body of Q, let q be a sharp point of C and let $F := \{[p_i, q] \mid i \in [2k]\}$ be a set of edges incident to q such that $\{[p_C, q]\} \cup F$ is a k-alternation. We prove that F is not contained in E. Indeed, the dual pseudolines $\{p_i^* \mid i \in [2k]\}$ intersect alternately the double pseudoline C^* between the tangents to C at q (see Figure 34). This ensures the existence of a witness pseudoline ℓ that separates all the contact points $p_i^* \wedge C^*$, while crossing C^* exactly $2k$ times and the other double pseudolines of Q^* exactly has q^* does. (As usual, we obtain it by a perturbation of the pseudoline q^*.) Counting the crossings of ℓ with Q^* and Λ respectively, we obtain:

(i) ℓ crosses Q^* exactly $2|(Q \smallsetminus \{C\})^*| + 2k = 2|Q| + 2k - 2$ times;
(ii) ℓ crosses Λ at least $|\Lambda| = 2|Q| - 2k$ times;
(iii) for each of the points $p_i \wedge q^*$, replacing the crossing point by a contact point removes two crossings with ℓ. □

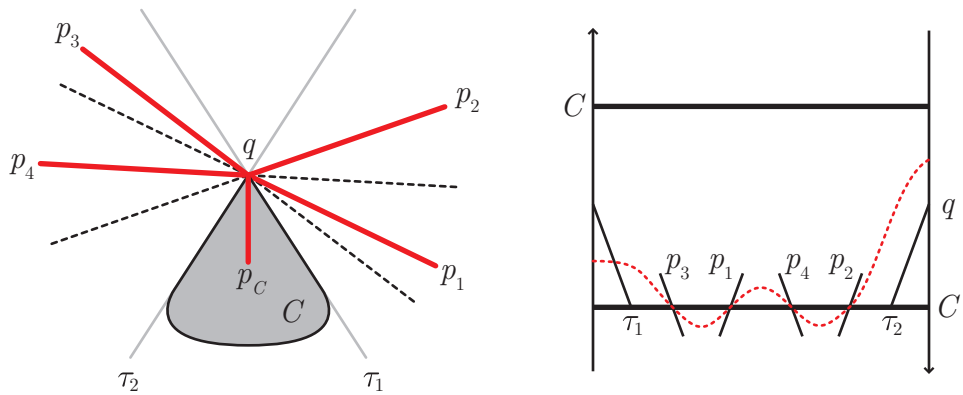

FIGURE 34. A k-alternation at a sharp point.

6.2.3. *Stars.* If λ is a pseudoline of Λ, we call *star* the envelope $S(\lambda)$ of the primal lines of the points of λ. The star $S(\lambda)$ contains:

(i) all bitangents τ between two convex bodies of Q such that τ is a contact point of λ; and
(ii) all convex arcs formed by the tangent points of the lines covered by λ with the convex bodies of Q.

This star is a (non-necessarily simple) closed curve. We again have bounds on the number of *corners* (*i.e.* convex internal angles) of $S(\lambda)$:

Proposition 48. *The number of corners of $S(\lambda)$ is odd and between $2k-1$ and $4k|Q|-2k-1$.*

Proof. In the case of double pseudoline arrangements, corners are even easier to characterize: a bitangent τ between two convex C and C' of Q always defines two corners, one at each extremity. These corners are contained in one of the two stars adjacent to τ. Let λ be a pseudoline with a contact point at τ. In a neighborhood of τ, the pseudoline λ can be contained either in M_{C^*} or in C_{C^*}. In the first case, the star $S(\lambda)$ contains the corner formed by the bitangent τ and the convex C (or possibly, by the bitangent τ and another tangent to C); while in the second case, it does not. (The same observation holds for C'.)

In other words, if the double pseudoline C^* supports a pseudoline λ between two contact points v and w, then one of the three following situations occurs (see Figure 35):

(i) either v and w lie on opposite sides of λ; then exactly one of these contact points lies in M_{C^*}, and $S(\lambda)$ has one corner at C.
(ii) or v and w both lie on M_{C^*}, and $S(\lambda)$ has two corners at C.
(iii) or v and w both lie on C_{C^*}, and $S(\lambda)$ has no corners at C.

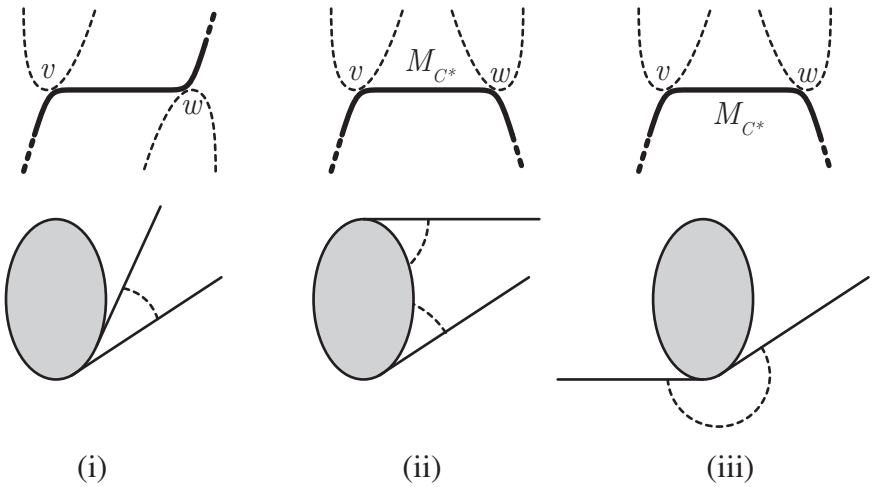

FIGURE 35. The three possible situations for two consecutive contact points on ℓ.

In particular, the number $c = c(\lambda)$ of corners of λ is the number of situations (i) plus twice the number of situations (ii). This proves that c is odd and (at least) bigger than the number of opposite consecutive contact points of the pseudoline λ.

In order to get a lower bound on this number, we construct (as in the proof of Proposition 34) a witness pseudoline μ that crosses λ between each pair of opposite contact points and passes on the opposite side of each contact point. It crosses λ at most c times and $\Lambda \setminus \{\lambda\}$ exactly $|\Lambda| - 1$ times. Moreover, if α is a pseudoline and β is a double pseudoline of $\mathcal{M}$, then either α is contained in M_β and has no crossing with β, or α and β have an even number of crossings. Since μ is a pseudoline and can be contained in at most one Möbius strip M^*_C (for $C \in Q$), the number of crossings of μ with Q^* is at least $2(|Q|-1)$. Thus, we obtain the lower bound $2(|Q| - 1) \leq 2|Q| - 2k - 1 + c$, *i.e.* $c \geq 2k - 1$.

From this lower bound, we obtain the upper bound: the total number of corners is at most twice the number of bitangents:

$$2(4k|Q| - |Q| - 2k^2 - k) \geq \sum_{\mu \in \Lambda} c(\mu) \geq c(\lambda) + (2|Q| - 2k - 1)(2k - 1),$$

and we get $c \leq 4|Q|k - 2k - 1$. □

When $k = 1$, we can even prove that all stars are pseudotriangles. Indeed, since any star has at least 3 corners, the upper bound calculus gives $2(3|Q| - 3) \geq c + 3(2|Q| - 3)$, *i.e.* $c \leq 3$.

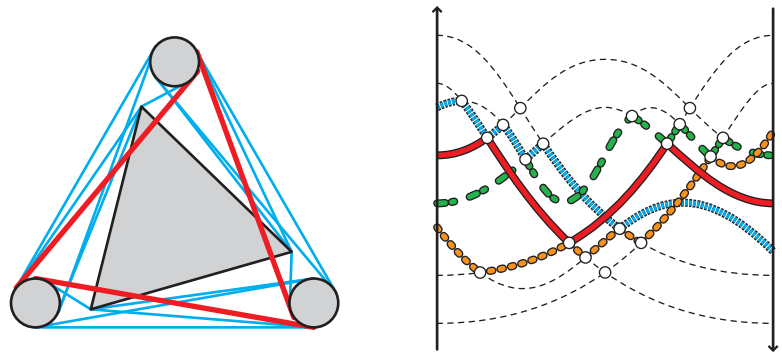

FIGURE 36. A 2-pseudotriangulation of the double pseudoline arrangement of Figure 31. Observe that the bolded red star has only 3 corners.

For general k, observe that contrary to the case of pseudoline arrangements, a star may have $2k-1$ corners (see Figure 36).

Let us now give an analogue of Proposition 32. For any point q in the plane, we denote by $\eta^k(q)$ the number of crossings between q^* and the support of Q^* minus its first k levels. Let $\delta^k(q) := \eta^k(q)/2 - |Q| + k$. For any $\lambda \in \Lambda(U)$ and any point q in the plane, we still denote by $\sigma_\lambda(q)$ the *winding number* of $S(\lambda)$ around q.

Proposition 49. *For any point q of the plane $\delta^k(q) = \sum_{\lambda\in\Lambda} \sigma_\lambda(q)$.*

Proof. Remember that if $\tau_\lambda(q)$ denotes the number of intersection points between q^* and λ, then $\sigma_\lambda(q) = (\tau_\lambda(q) - 1)/2$. Thus, we have

$$\eta^k(q) = \sum_{\lambda\in\Lambda} \tau_\lambda(q) = |\Lambda| + 2\sum_{\lambda\in\Lambda} \sigma_\lambda(q),$$

and we get the result since $|\Lambda| = 2|Q| - 2k$. □

When $k = 1$, it is easy to see that $\delta^1(q)$ is 1 if q is inside the *free space* of the convex hull of Q (*i.e.* in the convex hull of Q, but not in Q), and 0 otherwise. Remember that a *pseudotriangulation* of Q is a pointed set of bitangents that decomposes the free space of the convex hull of Q into pseudotriangles [PV96a]. Propositions 48 and 49 provide, when $k = 1$, the following analogue of Theorem 16:

Theorem 50. *Let Q be a set of disjoint convex bodies (in general position) and Q^* denote its dual arrangement. Then:*

(i) The dual arrangement $T^* := \{\Delta^* \mid \Delta$ pseudotriangle of $T\}$ of a pseudotriangulation T of Q is a 1-pseudotriangulation of Q^*.

(ii) The primal set of edges

$$E := \{[p,q] \,|\, p,q \in P,\ p^* \wedge q^* \text{ is not a crossing point of } \Lambda\}$$

of a 1-pseudotriangulation Λ of Q^* is a pseudotriangulation of Q.

Observe that at least two other arguments are possible to prove (ii):

(1) either comparing the degrees of the flip graphs as in our first proof of Theorem 16;
(2) or checking that all forbidden configurations of the primal (two crossing bitangents, a non-pointed sharp vertex, a non-free bitangent) may not appear in the dual, as in our second proof of Theorem 16.

Let us finish this discussion about stars by interpreting the number $\delta^k(q)$ for general k and for "almost every" point q. For any convex body C of Q, let ∇C denote the intersection of all closed half-planes delimited by a bitangent between two convex bodies of Q, and containing C (see Figure 37). By definition, the bitangents between two convex bodies C and C' of Q coincide with the bitangent of ∇C and $\nabla C'$. Furthermore, the convex bodies ∇C ($C \in Q$) are maximal for this property. We denote $\nabla Q := \{\nabla C \,|\, C \in Q\}$ the set of maximal convex bodies of Q.

For a point q outside ∇Q, the interpretation of $\delta^k(q)$ is similar to the case of points. We call *level* of a bitangent τ the level of the corresponding crossing point in Q^*. Given a point q outside ∇Q, the number $\delta^k(q)$ is the number of bitangents of level k crossed by any

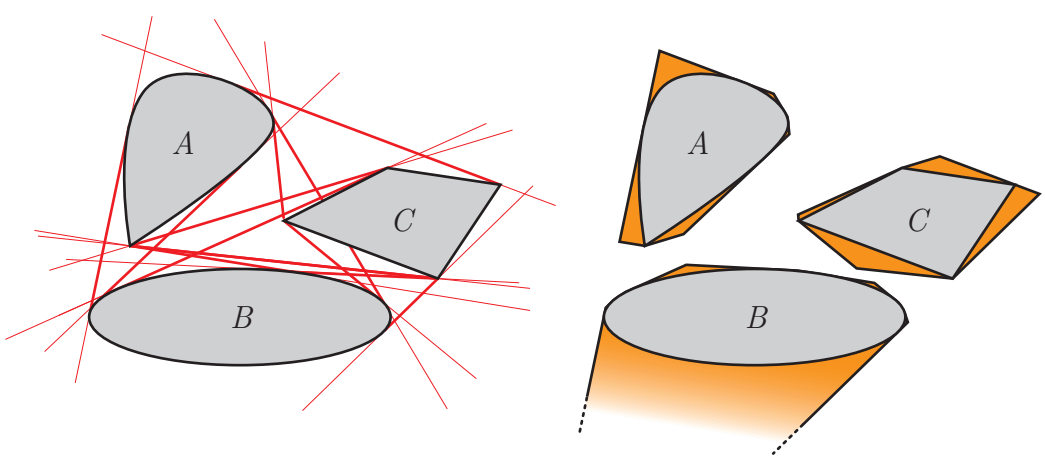

FIGURE 37. The set of all bitangents to the arrangement of convex bodies of Figure 32 and the corresponding maximal convex bodies.

(generic continuous) path from q to the external face (in the complement of ∇Q), counted positively when passing from the "big" side to the "small side", and negatively otherwise.

7. Open questions

We finish by a short presentation of some open questions that have arisen out of this work. Since the submission of this paper, some of these questions were (at least partially) answered in [Stu11, SS12, PS12]. We have decided to keep these questions in the discussion and to refer to the relevant articles in side remarks.

Primal of a multipseudotriangulation. When $k = 1$ or in the case of convex position, primals of k-pseudotriangulations are characterized by simple non-crossing and pointedness conditions. For general k and general position, we know that the primal of a k-pseudotriangulation is k-alternation-free (Lemma 27), but there exist k-pseudotriangulations containing $(k+1)$-crossings as well as $(k+1)$-crossing-free k-alternation-free sets of edges not contained in k-pseudotriangulations (see Figure 18). Thus, we still miss a simple condition to characterize multipseudotriangulations of points in general position:

Question 51. Characterize primals of multipseudotriangulations.

Another question related to the primal of a multipseudotriangulation would be to determine whether for every point set there exists a multipseudotriangulation that looks as simple as possible. When $k = 1$, we know that every point set in general position has:

(i) a pointed pseudotriangulation consisting only of triangles and four-sided pseudotriangles [KKM+03];
(ii) a pointed pseudotriangulation whose maximum degree is at most five [KKM+03].

Question 52. Does every point set in general position have a k-pseudotriangulation with only "simple" stars (resp. only vertices with "little" degree)?

In the previous question, "simple" may be interpreted either as "with exactly $2k+1$ corners" or as "with at most $2k+t$ edges" (for a minimal t). Similarily, "little" means smaller than a constant (as small as possible).

Diameter of the graph of flips. The graph of flips on k-pseudotriangulations of a pseudoline arrangement L is connected, and an easy inductive argument shows that its diameter at is most quadratic in $|L|$. For certain particular cases, even better bounds are known:

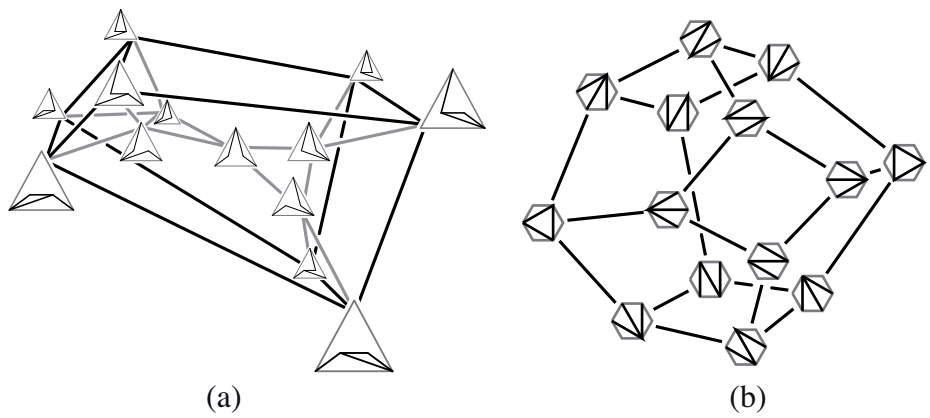

FIGURE 38. (a) The polytope of pointed pseudotriangulations of a set of 5 points. (b) The 6-dimensional associahedron.

(a) The diameter of the graph of flips on pointed pseudotriangulations of a set of n points is at most $O(n \ln n)$ [Ber04].
(b) The graph of flips on the k-triangulations of the n-gon has diameter bounded by $2k(n-2k-1)$ [Nak00, PS09].
(c) For triangulations of the n-gon, the diameter is exactly $2n-10$ [STT88].

Question 53. What is the (asymptotic) diameter of the flip graph?

POLYTOPALITY. Let $\mathcal{S}$ be the support of a pseudoline arrangement. Let $\Delta(\mathcal{S})$ denote the simplicial complex of subsets of contact points of pseudoline arrangements supported by $\mathcal{S}$. Our results ensure that $\Delta(\mathcal{S})$ is an abstract polytope whose ridge graph is the graph of flips (see the discussion in [BKPS06, Subsection 2.2]). When $\mathcal{S}$ is the first kernel of the dual pseudoline arrangement of a point set of the Euclidean plane, it turns out that this abstract polytope can be realized effectively as a polytope of $\mathbb{R}^d$ (where d is the number of flippable edges), which is the polar of the *polytope of pointed pseudotriangulations* of [RSS03]. An example of this polytope is presented in Figure 38(a). When the points are in convex position, this polytope is the *associahedron* (see Figure 38(b)). This leads to the following question:

Question 54. Is $\Delta(\mathcal{S})$ the boundary complex of a polytope?

This question specializes for pseudotriangulations and multitriangulations to the following interesting questions:

(a) Is the graph of flips on pseudotriangulations of a point set polytopal? Since [RSS03] answers positively for Euclidean point sets, this

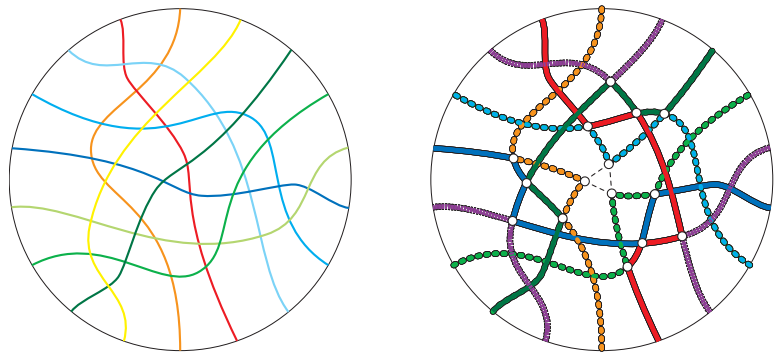

FIGURE 39. A non-stretchable arrangement of 9 pseudolines (left) represented on the projective plane to show its symmetries. Remove the center point to obtain the arrangement in the Möbius strip. A pseudotriangulation of it (right).

question only remains open for non-stretchable arrangements. We have represented in Figure 39 a pseudotriangulation of the smallest non-stretchable simple pseudoline arrangement (the non-Pappus pseudoline arrangement). Since this arrangement is symmetric under the dihedral group D_6, it could be worked out with methods similar to those in [BP09].

(b) Is the graph of flips on k-triangulations of the n-gon polytopal? Jonsson proved that the simplicial complex of $(k+1)$-crossing-free sets of k-relevant diagonals of the n-gon is a combinatorial sphere [Jon03]. However, except for little cases, the question of the polytopality of this complex remains open. We refer to [PS09, BP09] and [Pil10, Section 4.3] for a detailed discussion on this question.

Remark 55. Since the submission of this paper, our knowledge on this question has progressed. In [Stu11], Stump made the connection between the multitriangulations and the type A subword complexes defined in [KM04]. These simplicial complexes are precisely the simplicial complexes $\Delta(\mathcal{S})$ defined above. It implies in particular that

- these simplicial complexes are either topological spheres or topological balls [KM04], and
- Question 54 is the special type A case of Question 6.4 in [KM04].

We close this discussion by mentioning the related construction in [PS12]. It associates to each sorting network $\mathcal{N}$ its so-called *brick polytope* whose vertices correspond to certain pseudoline arrangements with contact points supported by $\mathcal{N}$. For certain well-chosen networks, the brick

polytope specializes to Hohlweg and Lange's realizations of the associahedron [HL07]. This construction answers Question 54 positively for a certain class of supports $\mathcal{S}$. It was moreover extended recently to spherical subword complexes of any finite type in [PS11].

Number of multipseudotriangulations. In his paper [Jon05], Jonsson proved that the number of k-triangulations of the n-gon is equal to the determinant $\det(C_{n-i-j})_{1\leq i,j\leq k}$ (where $C_m = \frac{1}{m+1}\binom{2m}{m}$ denotes the mth Catalan number). This determinant also counts noncrossing k-tuples of Dyck paths of semi-length $n-2k$ (see [PS09] and [Pil10, Section 4.1] for a more detailed discussion). It raises the following question:

Question 56. Find an explicit bijection between Dyck multipaths and multitriangulations.

Our interpretation of multitriangulations in terms of pseudoline arrangements naturally associates to a k-triangulation of the n-gon a set of $n-2k$ lattice paths as follows. Let T be a k-triangulation of the n-gon. For each edge (u,v) of T (with $u<v$), we place a *mirror* at the grid point of coordinates (u,v). This mirror is a double faced mirror parallel to the diagonal $x=y$ so that it reflects a ray coming from $(-\infty,v)$ to a ray going to $(u,+\infty)$, and a ray coming from $(u,-\infty)$ to a ray going to $(+\infty,v)$. Furthermore, for $1\leq i\leq n-2k$, we place a *light beam* at $(-\infty,k-1+i)$ pointing horizontally. We obtain $n-2k$ beams which reflect on the mirrors of T (see Figure 40).

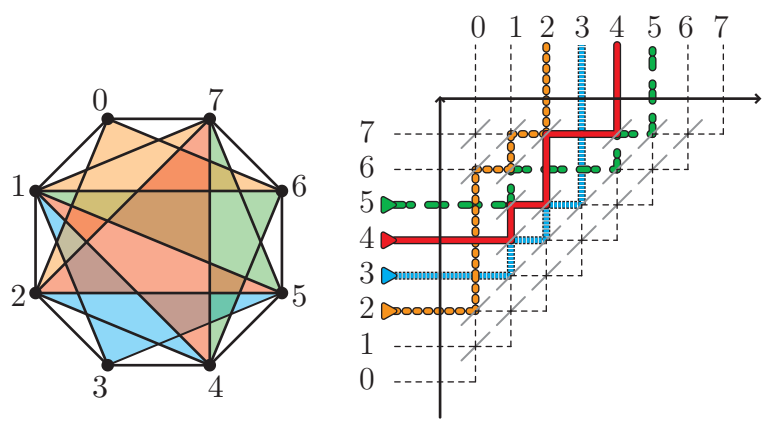

Figure 40. The beam arrangement of a 2-triangulation of the octagon. Each pseudoline is a lattice path.

The results of this paper imply the following properties:

(i) All beams are x- and y-monotone lattice paths.
(ii) The ith beam comes from the direction $(-\infty, k-1+i)$ and goes to the direction $(k-1+i, +\infty)$.
(iii) Each beam reflects exactly $2k+1$ times, and thus, has k vertical segments (plus one vertical half-line) and k horizontal segments (plus one horizontal half-line).
(iv) The beams form a pseudoline arrangement: any two of them cross exactly once.

The ith beam B_i "corresponds" via duality to the k-star S_i of T whose $(k+1)$th vertex is the vertex $k-1+i$ (in other words, the k-star bisected by the line passing through the vertex $k-1+i$ and through the midpoint of $[0, n-1]$). Indeed:

(i) the beam B_i is (by duality) the set of all bisectors of S_i;
(ii) the mirrors which reflect B_i are the edges of S_i; and
(iii) the intersection of two beams B_i and B_j is the common bisector of S_i and S_j.

Observe that instead of k Dyck paths of semi-length $n-2k$, the beam arrangement of a k-triangulation has $n-2k$ beams which all have k horizontal steps.

Remark 57. In their recent paper [SS12], Serrano and Stump also observe the correspondence between multitriangulations and beam arrangements (called "reduced pipe dream" in their paper). Starting from this correspondence, they provide an explicit bijection between multitriangulations and Dyck multipaths. We refer to [Stu11, SS12, Rub10] and the references therein.

Another interesting question concerning the number of multipseudotriangulations would be to determine what point sets give the maximum and minimum number of multipseudotriangulations. For example, when $k = 1$, every point set in general position has at least as many pointed pseudotriangulations as the convex polygon with the same number of points [AAKS04].

Question 58. What point sets have the maximum and minimum number of multipseudotriangulations?

Decomposition of a k-pseudotriangulation. We have seen in Section 5 that certain multipseudotriangulations can be decomposed into iterated multipseudotriangulations. Remember however that there exists irreductible multipseudotriangulations (Example 35).

Question 59. Characterize (completely) decomposable multipseudotriangulations.

The same question can be asked in a more algorithmical flavor:

Question 60. How can we decide algorithmically whether a k-pseudotriangulation is decomposable?

Obviously, a brute-force algorithm is not considered as a good solution. A good way to test efficiency of the answers to Questions 59 and 60 would be to prove/disprove that Example 35 is the minimal irreducible 2-triangulation (or, in other words, that any 2-triangulation of an n-gon, with $n \leq 14$, contains a triangulation).

Remember also that when a multipseudotriangulation is decomposable, the graph of partial flips is not necessarily connected. In particular, finding all decompositions of a multipseudotriangulation cannot be achieved just by searching in the graph of partial flips.

Question 61. How can we enumerate all the decompositions of a multipseudotriangulation?

Computing a k-pseudotriangulation. An initial pseudotriangulation of a set of n points can be computed in $O(n \ln n)$ time, using only the predicate of the chirotope. A similar result would be interesting for k-pseudotriangulations:

Question 62. Compute an initial k-pseudotriangulation of a given (double) pseudoline arrangement, using only its chirotope.

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References

[AAKS04] Oswin Aichholzer, Franz Aurenhammer, Hannes Krasser, and Bettina Speckmann. Convexity minimizes pseudo-triangulations. *Comput. Geom.*, 28(1):3–10, 2004.

[AOSS08] Oswin Aichholzer, David Orden, Francisco Santos, and Bettina Speckmann. On the number of pseudo-triangulations of certain point sets. *J. Combin. Theory Ser. A*, 115(2):254–278, 2008.

[AP03] P. Angelier and M. Pocchiola. A sum of squares theorem for visibility complexes and applications. In B. Aronov, S. Basu, J. Pach, and M. Sharir, editors, *Discrete and Computational Geometry, The Goodman-Pollack Festschrift*, volume 25 of *Algorithms Combin.*, pages 79–139. Springer-Verlag, 2003.

[Ber04] Sergey Bereg. Transforming pseudo-triangulations. *Inform. Process. Lett.*, 90(3):141–145, 2004.

[Ber05] Sergey Bereg. Enumerating pseudo-triangulations in the plane. *Comput. Geom.*, 30(3):207–222, 2005.

[BKPS06] Hervé Brönnimann, Lutz Kettner, Michel Pocchiola, and Jack Snoeyink. Counting and enumerating pointed pseudotriangulations with the greedy flip algorithm. *SIAM J. Comput.*, 36(3):721–739, 2006.

[BP09] Jürgen Bokowski and Vincent Pilaud. On symmetric realizations of the simplicial complex of 3-crossing-free sets of diagonals of the octagon. In *Proc. 25th Canadian Conference on Computational Geometry*, 2009.

[BVS+99] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Günter M. Ziegler. *Oriented matroids*, volume 46 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 1999.

[CP92] Vasilis Capoyleas and János Pach. A Turán-type theorem on chords of a convex polygon. *J. Combin. Theory Ser. B*, 56(1):9–15, 1992.

[dB74] Nicolaas G. de Bruijn. Sorting by means of swappings. *Discrete Math.*, 9:333–339, 1974.

[DKKM01] Andreas W. M. Dress, Michael Klucznik, Jacobus H. Koolen, and Vincent Moulton. $2kn - \binom{2k+1}{2}$: note on extremal combinatorics of cyclic split systems. *Sém. Lothar. Combin.*, 47:Article B47b, 17 pp. (electronic), 2001.

[DKM02] Andreas W. M. Dress, Jacobus H. Koolen, and Vincent Moulton. On line arrangements in the hyperbolic plane. *European J. Combin.*, 23(5):549–557, 2002.

[Goo97] Jacob E. Goodman. Pseudoline arrangements. In *Handbook of discrete and computational geometry*, CRC Press Ser. Discrete Math. Appl., pages 83–109. CRC, Boca Raton, FL, 1997.

[Grü72] Branko Grünbaum. *Arrangements and spreads*. American Mathematical Society Providence, R.I., 1972. *Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 10*.

[HL07] Christophe Hohlweg and Carsten E. M. C. Lange. Realizations of the associahedron and cyclohedron. *Discrete Comput. Geom.*, 37(4):517–543, 2007.

[HOR+05] Ruth Haas, David Orden, Günter Rote, Francisco Santos, Brigitte Servatius, Herman Servatius, Diane Souvaine, Ileana Streinu, and Walter Whiteley. Planar minimally rigid graphs and pseudo-triangulations. *Comput. Geom.*, 31(1-2):31–61, 2005.

[HP07] Luc Habert and Michel Pocchiola. Computing pseudotriangulations via branched coverings. Preprint, `arXiv:1102:0151`. Abbreviated partial version in *20th European Workshop on Comput. Geom. (Seville, 2004)*, pages 111–114, 2007.

[HP09] Luc Habert and Michel Pocchiola. Arrangements of double pseudolines. Preprint, `arXiv:1101:1022`. Abbreviated version in *25th Annual ACM Symposium on Comput. Geom. (Aahrus, 2009)*, pages 314–323, 2009.

[Jon03] Jakob Jonsson. Generalized triangulations of the n-gon. Unpublished manuscript. An abstract was included in *"Topological and Geometric Combinatorics, April 2003"*, Mathematisches Forschungsinstitut Oberwolfach, Report No. 16., 2003.

[Jon05] Jakob Jonsson. Generalized triangulations and diagonal-free subsets of stack polyominoes. *J. Combin. Theory Ser. A*, 112(1):117–142, 2005.

[KKM+03] Lutz Kettner, David Kirkpatrick, Andrea Mantler, Jack Snoeyink, Bettina Speckmann, and Fumihiko Takeuchi. Tight degree bounds for pseudo-triangulations of points. *Comput. Geom.*, 25:3–12, 2003.

[KM04] Allen Knutson and Ezra Miller. Subword complexes in Coxeter groups. *Adv. Math.*, 184(1):161–176, 2004.

[Knu73] Donald E. Knuth. *The art of computer programming. Volume 3.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1973. Sorting and searching, Addison-Wesley Series in Computer Science and Information Processing.

[Knu92] Donald E. Knuth. *Axioms and hulls*, volume 606 of *Lecture Notes in Computer Science*. Springer-Verlag, Berlin, 1992.

[Nak00] Tomoki Nakamigawa. A generalization of diagonal flips in a convex polygon. *Theoret. Comput. Sci.*, 235(2):271–282, 2000.

[Pil10] Vincent Pilaud. *Multitriangulations, pseudotriangulations and some problems of realization of polytopes.* PhD thesis, Université Paris 7 & Universidad de Cantabria, 2010.

[Poc97] Michel Pocchiola. Horizon trees versus pseudo-triangulations. In *13th European Workshop on Comput. Geom. (Würzburg)*. 1997.

[PP09] Vincent Pilaud and Michel Pocchiola. Multipseudotriangulations. Extended abstract in *25th European Workshop on Comput. Geom. (Bruxelles)*, 2009.

[PS09] Vincent Pilaud and Francisco Santos. Multitriangulations as complexes of star polygons. *Discrete Comput. Geom.*, 41(2):284–317, 2009.

[PS11] Vincent Pilaud and Christian Stump. Brick polytopes of spherical subword complexes: A new approach to generalized associahedra. Preprint, `arXiv:1111.3349`, 2011.

[PS12] Vincent Pilaud and Francisco Santos. The brick polytope of a sorting network. *European J. Combin.*, 33(4):632–662, 2012.

[PV94] Michel Pocchiola and Gert Vegter. Order types and visibility types of configurations of disjoint convex plane sets. Technical report, LIENS, Paris, 1994.

[PV96a] Michel Pocchiola and Gert Vegter. Pseudo-triangulations: theory and applications. In *12th Annual ACM Symposium on Comput. Geom. (Philadelphia)*, pages 291–300. 1996.

[PV96b] Michel Pocchiola and Gert Vegter. Topologically sweeping visibility complexes via pseudotriangulations. *Discrete Comput. Geom.*, 16(4):419–453, 1996.

[PV96c] Michel Pocchiola and Gert Vegter. The visibility complex. *Internat. J. Comput. Geom. Appl.*, 6(3):279–308, 1996.

[RSS03] Günter Rote, Francisco Santos, and Ileana Streinu. Expansive motions and the polytope of pointed pseudo-triangulations. In B. Aronov, S. Basu, J. Pach, and M. Sharir, editors, *Discrete and Computational Geometry, The Goodman-Pollack Festschrift*, volume 25 of *Algorithms Combin.*, pages 699–736. Springer, Berlin, 2003.

[RSS08] Günter Rote, Francisco Santos, and Ileana Streinu. Pseudo-triangulations — a survey. In *Surveys on discrete and computational geometry*, volume 453 of *Contemp. Math.*, pages 343–410. Amer. Math. Soc., Providence, RI, 2008.

[Rub10] Martin Rubey. Maximal $0-1$ fillings of moon polyominoes with restricted chain-lengths and RC-graphs. Preprint, `arXiv:1009.3919`, 2010.

[SBG+95] Helmut Salzmann, Dieter Betten, Theo Grundhöfer, Hermann Hähl, Rainer Löwen, and Markus Stroppel. *Compact projective planes*, volume 21 of *de Gruyter Expositions in Mathematics*. Walter de Gruyter & Co., Berlin, 1995.

[SS12] Luis Serrano and Christian Stump. Maximal fillings of moon polyominoes, simplicial complexes, and Schubert polynomials. *Electron. J. Combin.*, 19, 2012.

[Str05] Ileana Streinu. Pseudo-triangulations, rigidity and motion planning. *Discrete Comput. Geom.*, 34(4):587–635, 2005.

[STT88] Daniel D. Sleator, Robert E. Tarjan, and William P. Thurston. Rotation distance, triangulations, and hyperbolic geometry. *J. Amer. Math. Soc.*, 1(3):647–681, 1988.

[Stu11] Christian Stump. A new perspective on k-triangulations. *J. Combin. Theory Ser. A*, 118(6):1794–1800, 2011.

[Wel01] Emo Welzl. Entering and leaving j-facets. *Discrete Comput. Geom.*, 25(3):351–364, 2001.

CNRS & LIX (UMR 7161), École Polytechnique
E-mail address: `vincent.pilaud@lix.polytechnique.fr`
URL: `http://www.lix.polytechnique.fr/~pilaud/`

Institut de Mathmatiques de Jussieu (UMR 7586), Université Pierre et Marie Curie
E-mail address: `pocchiola@math.jussieu.fr`
URL: `http://people.math.jussieu.fr/~pocchiola/`