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Ergodicity and scaling limit of a constrained multivariate Hawkes process

Ban Zheng ^{*} [†] [‡] François Roueff [†] Frédéric Abergel [‡]

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Abstract

We introduce a multivariate Hawkes process with constraints on its conditional density. It is a multivariate point process with conditional intensity similar to that of a multivariate Hawkes process but certain events are forbidden with respect to boundary conditions on a multidimensional *constraint* variable, whose evolution is driven by the point process. We study this process in the special case where the fertility function is exponential so that the process is entirely described by an underlying Markov chain, which includes the constraint variable. Some conditions on the parameters are established to ensure the ergodicity of the chain. Moreover, scaling limits are derived for the integrated point process. This study is primarily motivated by the stochastic modelling of a limit order book for high frequency financial data analysis.

1 Introduction

Since two decades, the high-frequency price dynamics in a *limit order book* has received much attention. On the one hand, an extensive statistical study on the limit order book dynamics and limit order book information content can be found in [9, 26, 39, 19, 30, 10, 25, 14, 41]; on the other hand, many researchers have endeavored to propose different models, including equilibrium model, agent-based model and Markov model, to model the limit order book dynamics, see [36, 38, 17, 15, 1, 16].

More recently point processes have been introduced to financial application. Early works focus on modelling inter-event durations in [23, 24] and the effect of duration on price impact and trade sign autocorrelation in [20]. Dufour and Engle [22] extend these models to allow for the censoring of quotes after a trade by intervening trades. Bauwens and Hautsch [8] provide a comprehensive introduction of the application of point processes in financial time series. Hawkes processes belong to the class of self-exciting point processes, where the intensity is driven by a weighted function of the time distance to previous points of the process. Hawkes processes originate from the literature in seismology and are used to model the occurrence of earthquakes, see [28, 27, 35, 18, 13]. To the best of our knowledge, the first application of Hawkes processes in financial time series is introduced by [11]. Since then, there is a growing literature on the application of Hawkes processes to the modelling of high frequency financial data, see [7, 29, 12,

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31, 5, 2, 34, 21, 6]. Although some nonparametric estimation of the intensity functions has been investigated by Reynaud-Bouret and Schbath [37] and Al Dayri *et al.* [4] in dimension $d = 1$, the majority of Hawkes processes related publications work in the setting of exponential kernel functions for computational reasons. Embrechts *et al.* [21] apply marked multivariate Hawkes process to model the daily financial data. Muni Toke and Pomponio [34] introduce Hawkes processes in the modelling of the *trade-through*. Bacry *et al.* use marked Hawkes processes to reproduce empirically microstructure noise and the stylized facts such as Epps effect and lead-lag in [5] and reach to derive and characterize the exact macroscopic diffusion limit of this model in [6].

In this contribution we wish to model the dynamics of the limit prices in the limit order book. Nevertheless, we remark that the limits evolve according to specific constraints. For instance the best bid/best ask (buy/sell) price's behavior is constrained by a minimal bid-ask spread. This means that, as soon as the minimal bid-ask spread is attained, the best bid price cannot increase until the best ask price goes up. Therefore, in this work, we propose to add a constraint variable to the classical multivariate Hawkes process to model the dynamics of the limit prices. We restrict ourselves in the Markov setting (with exponential fertility functions) and then investigate the ergodicity property and scaling limits of the constrained multivariate Hawkes processes. In contrast, the Markov models proposed in [17, 16, 1] use simple Poisson process and they do not take into account the cross-interaction between different events. Apart from its application to a limit order book, the constrained multivariate Hawkes process can be extended to many domains where the intensities of the process may depend on some constraint variables.

The paper is structured as follows. In Section 2 we introduce the constrained multivariate Hawkes processes and present the main assumptions and notation. Section 3 gives the main results on the geometrical ergodicity and the large scale behavior. The application to a limit order book is presented in Section 4. We have gathered the detailed proofs in Section 5. Some useful technical results are postponed to Appendix A.

2 Main assumptions and notation

2.1 Multivariate Hawkes process with constraints

Let p, q be two positive integers. We introduce a constrained multivariate Hawkes process defined by a couple (\mathbf{S}, \mathbf{N}) , where $\{\mathbf{S}(t), t \geq 0\}$ is a *constraint* process taking its values in $\mathbb{Z}_+^q = \{1, 2, \dots\}^q$ and \mathbf{N} is a p -dimensional multivariate point process. We shall describe the dynamics of this model by the conditional intensity of \mathbf{N} as defined in [13] and an evolution equation for \mathbf{S} .

We assume that \mathbf{S} is right-continuous with left limits and denote by $\mathbf{S}(t-)$ its left limit at time t . The conditional intensity of the multivariate point process \mathbf{N} at time t shall depend on the *constraint* variable $\mathbf{S}(t-)$. For notational compactness we will see \mathbf{N} as a marked point process having arrivals in \mathbb{R} with marks in $\{1, \dots, p\}$. We will denote by $\mathbf{N}_j, j = 1, \dots, p$ the p point processes corresponding to each mark j , namely

$$\mathbf{N} = \sum_{j=1}^p \mathbf{N}_j \otimes \delta_j .$$

In other words, each thinned unmarked process \mathbf{N}_j is given from \mathbf{N} by the formula

$$\mathbf{N}_j(g) = \mathbf{N}(g \otimes 1_{\{j\}}) ,$$

for all non-negative measurable function g defined on \mathbb{R} . Here we use the notation $\mu(g)$ for the integral of g with respect to a measure μ . The *constraint* variable $\mathbf{S}(t-)$ acts on the point process by constraining the set of possible marks for the first event arriving

after time t in \mathbf{N} . The constraint is added in the usual expression of the conditional intensity of a multivariate Hawkes process. More precisely, for all time $t \geq 0$ and any mark $i = 1, \dots, p$, the conditional intensity of \mathbf{N}_i given $\mathbf{N}_{|(-\infty, t)}$ and $\mathbf{S}(t-)$ is set as

$$\boldsymbol{\mu}(t, i) = \begin{cases} 0 & \text{if } \mathbf{S}(t-) \in \mathbf{A}_i \\ \boldsymbol{\mu}_0(i) + \sum_{j=1}^p \int_{u < t} \phi_{i,j}(t-u) \mathbf{N}_j(du) & \text{otherwise,} \end{cases} \quad (2.1)$$

where $\boldsymbol{\mu}_0(i)$ is the immigrant rate of mark i , $\phi_{i,j}$ is the fertility rate for producing a mark i event from a mark j event, and $\mathbf{A}_1, \dots, \mathbf{A}_p$ are products of finite subsets of \mathbb{Z}_+^q , $\mathbf{A}_i = A_i(1) \times \dots \times A_i(q)$.

In turn, an arrival in \mathbf{N} increases the value of the constraint by a value only depending on the mark. More precisely, the *constraint* process \mathbf{S} satisfies the evolution equation, for all $t > u \geq 0$,

$$\mathbf{S}(t) = \mathbf{S}(u) + \mathbf{N}(1_{(u,t]} \otimes \mathbf{J}), \quad (2.2)$$

where \mathbf{J} is defined on $\{1, \dots, p\}$ with values in \mathbb{Z}^q . In other words, at each arrival in \mathbf{N} with mark equal to i , \mathbf{S} jump by an increment given by $\mathbf{J}(i)$.

Note that equations (2.1) and (2.2) are valid for $t \geq 0$ and $t > u \geq 0$ respectively so that the distribution of the process

$$\{(\mathbf{S}(t), \mathbf{N}_j((0, t])), t \geq 0, j = 1, \dots, p\}$$

is defined conditionally to the initial conditions given by $\mathbf{S}(0-)$ and \mathbf{N} restricted to $(-\infty, 0)$. To summarize, this distribution is parameterized by the constrained sets $\mathbf{A}_1, \dots, \mathbf{A}_p$, the immigrants rates $\boldsymbol{\mu}_0(i)$, $i = 1, \dots, p$, the cross-fertility rate functions $\phi_{i,j} : [0, \infty) \rightarrow [0, \infty)$, $i, j = 1, \dots, p$, and the jump increments $\mathbf{J}(i) \in \mathbb{Z}^q$ for $i \in \{1, \dots, p\}$.

Meanwhile, we define an unconstrained multivariate Hawkes process \mathbf{N}' with the same fertility rate and immigrant intensity (but without constraints), that is, with conditional intensity on $[0, \infty)$ satisfying, for $i = 1, \dots, p$,

$$\boldsymbol{\mu}'(t, i) = \boldsymbol{\mu}_0(i) + \sum_{j=1}^p \int_{u \in (-\infty, t)} \phi_{i,j}(t-u) \mathbf{N}'_j(du), \quad t \geq 0,$$

where $\mathbf{N}'_j = \mathbf{N}'(\cdot \otimes 1_{\{j\}})$ for $j = 1, \dots, p$.

The *average fertility* matrix is defined by

$$\aleph = [\alpha_{i,j}]_{i,j=1,\dots,p} \quad \text{with} \quad \alpha_{i,j} = \int \phi_{i,j}, \quad 1 \leq i, j \leq p. \quad (2.3)$$

The following assumption ensures the existence of a stationary version of the (unconstrained) multivariate Hawkes process \mathbf{N}' , see [18, Example 8.3(c)].

Assumption 1. *The spectral radius of \aleph is strictly less than 1.*

Under Assumption 1, we will use the following (well defined) vector

$$\mathbf{u} = (\text{Id}_p - \aleph^T)^{-1} \mathbf{1}_p = \sum_{k \geq 0} \aleph^k \mathbf{1}_p, \quad (2.4)$$

where $\mathbf{1}_p$ denotes p -dimensional ones vector,

$$\mathbf{1}_p = \underbrace{[1 \ \dots \ 1]}_p^T,$$

and Id_p denotes the $p \times p$ identity matrix.

In the following, by convention, we use bold faces symbols only when q may be larger than one. In the case where q is set to 1, we shall write J and S_n in unbold faces since they are scalar valued and, similarly, we write A_i in unbold face for the constraint sets.

For a real valued function w defined on $\{1, \dots, p\}$, we shall use the notation

$$\vec{w} = [w(1), \dots, w(p)]^T .$$

If w is vector-valued, we use the same notation to obtain a matrix. For instance, \vec{J} is a $p \times q$ matrix.

2.2 A very special case

When we study the ergodicity of the process, we investigate the joint ergodicity of the *constraint* variable and of the point process. A very special case is obtained by setting

$$\phi_{i,j} \equiv 0, \quad i, j = 1, \dots, p ,$$

and

$$q = 1 .$$

In this case, we only need to study the stationarity of the *constraint* variable since the intensities are constant. Suppose moreover that J only takes values $+1$ and -1 and the *constraint* sets A_i are all equal to $\{1\}$. Let I_+, I_- be the partition of $\{1, \dots, p\}$ corresponding to the indices where J takes values $+1$ and -1 , respectively. In this case, $\{S(t), t \geq 0\}$ is a birth-death process on \mathbb{Z}_+ with constant birth rate $\mu_+ = \sum_{i \in I_+} \mu_0(i)$ and

constant death rate $\mu_- = \sum_{i \in I_-} \mu_0(i)$. By [3, Corollary 2.5], $(S(t))_{t \geq 0}$ is ergodic if and only if

$$\mu_+ - \mu_- = \vec{J}^T \boldsymbol{\mu}_0 = \sum_{i=1}^p J(i) \mu_0(i) < 0 . \quad (2.5)$$

Here and in the following, A^T denotes the transpose of matrix A .

2.3 The Markov assumption

We shall consider a particular shape of the fertility functions $\phi_{i,j}$ which implies a Markov property for the model. This allows us to obtain very precise results on the ergodicity of (\mathbf{S}, \mathbf{N}) and is also of interest in applications since the model is parameterized by a restricted set of well understood parameters.

From now on, we suppose that there exists $\beta > 0$ such that

$$\phi_{i,j}(u) = \alpha_{i,j} \beta e^{-\beta u}, \quad 1 \leq i, j \leq p .$$

This parameter is the reciprocal of a time : the larger β is, the shorter the dependence persists along the time between successive events. We denote a new process defined in the state space $\mathbf{X} = \mathbb{Z}_+^q \times \mathbb{R}_+^p$ by $\mathbf{X}(t) = (\mathbf{S}(t), \boldsymbol{\lambda}(t))$ where $\boldsymbol{\lambda}(t) = [\lambda(t, 1), \dots, \lambda(t, p)]^T$ with, for all $i \in \{1, \dots, p\}$,

$$\lambda(t, i) = \sum_{j=1}^p \int_{(-\infty, t)} \phi_{i,j}(t-u) \mathbf{N}_j(du) .$$

Proposition 2.1. *The process $\{\mathbf{X}(t), t \geq 0\}$ is a Markov process.*

This property directly follows from the exponential form of the fertility functions $\phi_{i,j}$ (see [32] in the unconstrained case). In the following, we shall describe the dynamics of the process $\{\mathbf{X}(t), t \geq 0\}$ through a discretely sampled version based on the description of the process \mathbf{N} using its hazard rate. This description implies Proposition 2.1. More importantly, in the following, we shall exclusively rely on this discrete description to study the ergodicity of $\{\mathbf{X}(t), t \geq 0\}$.

Let μ_0^0 denote an arbitrary positive constant (say $\mu_0^0 = 1$). The continuous time process $\{\mathbf{X}(t), t \geq 0\}$ is sampled at increasing discrete times $(T_n)_{n \geq 1}$ defined by the jump instants of $\lambda(t)$ and by the arrivals of an independent homogeneous PPP n_0 with intensity μ_0^0 . Observe that $\mathbf{S}(t)$ is constant between two consecutive jump instants. Hence, almost surely, \mathbf{S} does not change at arrivals corresponding to the PPP n_0 . The sample time instants generated by n_0 are artificially added to avoid the periodicity of the embedded chain, see the end of the proof of Proposition 5.3.

Each jump of $\lambda(t)$ corresponds to an arrival in the point process \mathbf{N} . We denote its mark by I_n , which takes values in $\{1, \dots, p\}$, where n is the positive integer such that the corresponding jump instant t equals T_n . For other sample times T_n , which thus correspond to the arrivals of n_0 , we set $I_n = 0$. Hence the marks of the embedded chain will take values in the set $\{0, 1, \dots, p\}$.

Finally, we define $\Delta_1 = T_1$ and, for each $n \geq 2$, $\Delta_n = T_n - T_{n-1}$, $\mathbf{S}_n = \mathbf{S}(T_n)$, $\lambda_n = \lambda(T_n)$ and $\mathbf{X}_n = \mathbf{X}(T_n) = (\mathbf{S}_n, \lambda_n)$. Then, given $(\mathbf{S}_0, \lambda_0)$ and $T_0 = 0$, we can generate the sequence $(\Delta_n, \mathbf{S}_n, \lambda_n)$, $n = 0, 1, \dots$ iteratively as follows.

$$\Delta_{n+1} = \min(\Delta_{n+1}^0, \Delta_{n+1}^1, \dots, \Delta_{n+1}^p),$$

where, given $\mathbf{X}_0, \dots, \mathbf{X}_n$, $\Delta_1, \dots, \Delta_n$ and I_1, \dots, I_n , the conditional distribution of Δ_{n+1}^0 , Δ_{n+1}^1, \dots and Δ_{n+1}^p is that of positive independent variables whose marginal distributions are determined by hazard rates $\text{Hr}_0, \text{Hr}_1, \dots, \text{Hr}_p$ defined for each i and $t \geq 0$ by

$$\text{Hr}_i(t) = \begin{cases} (\mu_0(i) + \lambda_n(i)e^{-\beta t})1_{\mathbf{A}_i^c}(\mathbf{S}_n), & i > 0, \\ \mu_0^0 + \sum_{j=1}^p 1_{\mathbf{A}_j}(\mathbf{S}_n)[\mu_0(j) + \lambda_n(j)e^{-\beta t}], & i = 0. \end{cases} \quad (2.6)$$

Then, the event type at time T_{n+1} is

$$I_{n+1} = \arg \min_{i \in \{0, \dots, p\}} (\Delta_{n+1}^i),$$

the *constraint* variable at time T_{n+1} is given by

$$\mathbf{S}_{n+1} = \mathbf{S}(T_{n+1}) = \mathbf{S}_n + \mathbf{J}_o(I_{n+1}), \quad (2.7)$$

where \mathbf{J}_o is the extension of \mathbf{J} to $\{0, 1, \dots, p\}$ defined by

$$\mathbf{J}_o(i) = \begin{cases} \mathbf{J}(i) & \text{if } i \geq 1 \\ \mathbf{0}_q & \text{if } i = 0. \end{cases} \quad (2.8)$$

and, for all $i \in \{1, \dots, p\}$, the self-excitation intensities at time T_{n+1} are given by

$$\lambda_{n+1}(i) = \lambda(T_{n+1}, i) = \lambda_n(i)e^{-\beta \Delta_{n+1}} + \beta \alpha_{i, I_{n+1}}. \quad (2.9)$$

Observe that, for all $T_n \leq t < T_{n+1}$, the continuous time process is interpolated as

$$\begin{aligned} \mathbf{S}(t) &= \mathbf{S}_n \\ \lambda(t, i) &= \lambda_n(i)e^{-\beta t}. \end{aligned}$$

For any probability distribution ν on $\mathsf{X} = \mathbb{Z}_+^q \times \mathbb{R}_+^p$, we denote by \mathbb{P}^ν the probability corresponding to the initial distribution ν at time $t = 0$. Notation \mathbb{E}^ν corresponds to the corresponding expectation. If ν is a Dirac distribution at point $\mathbf{x} \in \mathsf{X}$, we will simply write $\mathbb{P}^\mathbf{x}$ and $\mathbb{E}^\mathbf{x}$.

The above facts thus imply that $\mathbf{X}_n = (\mathbf{S}_n, \boldsymbol{\lambda}_n)$, $\mathbf{Y}_n = (I_n, \mathbf{S}_n, \boldsymbol{\lambda}_n)$ and $\mathbf{Z}_n = (\Delta_n, I_n, \mathbf{S}_n, \boldsymbol{\lambda}_n)$, defined for $n = 0, 1, \dots$ are Markov chains respectively valued in $\mathsf{X} = \mathbb{Z}_+^q \times \mathbb{R}_+^p$, $\mathsf{Y} = \{0, 1, \dots, p\} \times \mathbb{Z}_+^q \times \mathbb{R}_+^p$ and $\mathsf{Z} = \mathbb{R}_+ \times \{0, \dots, p\} \times \mathbb{Z}_+^q \times \mathbb{R}_+^p$. We shall denote by Q , \tilde{Q} and \bar{Q} the transition kernels of these Markov processes, respectively.

Remark 1. *Observe that, from the description above, the conditional distribution of $(\Delta_1, I_1, \mathbf{S}_1, \boldsymbol{\lambda}_1)$ given $(\Delta_0, I_0, \mathbf{S}_0, \boldsymbol{\lambda}_0)$ does not depend on (Δ_0, I_0) . In particular, the whole path $\{\mathbf{Z}_n, n \geq 1\}$ only depends on the initial condition set on \mathbf{X}_0 and we may write, for all $\mathbf{z} = (\delta, i, \mathbf{x}) \in \mathsf{Z}$ and all $A \in \mathcal{B}(\mathsf{Z})$,*

$$\bar{Q}(\mathbf{z}, A) = \mathbb{P}^\mathbf{x}((\Delta_1, I_1, \mathbf{X}_1) \in A). \quad (2.10)$$

Clearly the same remark holds for \tilde{Q} , namely, $\tilde{Q}(\mathbf{y}, A) = \mathbb{P}^\mathbf{x}((I_1, \mathbf{X}_1) \in A)$ for all $\mathbf{y} = (i, \mathbf{x}) \in \mathsf{Y}$ and $A \in \mathcal{B}(\mathsf{Y})$.

Because the ergodicity of Q will be proved using an induction on the number of constraints q , we need to introduce further Markov chains and transition kernels. To initiate the induction, we define the Markov chain $\{(\tilde{I}_n, \tilde{\boldsymbol{\lambda}}_n), n \geq 0\}$ as the chain valued in $\{0, 1, \dots, p\} \times \mathbb{R}_+^q$ that starts at the same state as $(I_0, \boldsymbol{\lambda}_0)$ but with transition kernel \tilde{Q} , which is defined as \tilde{Q} but with all the \mathbf{A}_i 's replaced by the empty set and without the *constraint* variable. We will call this chain the *unconstrained Hawkes embedded chain*, since it is associated to a classical (unconstrained) multivariate Hawkes point process. This chain corresponds to the case $q = 0$.

Remark 2. *As in the case $q \geq 1$, the conditional distribution of $\{(\tilde{I}_n, \tilde{\boldsymbol{\lambda}}_n), n \geq 1\}$ given the initial condition $(\tilde{I}_0, \tilde{\boldsymbol{\lambda}}_0)$ does not depend on \tilde{I}_0 . Hence we will use the notation \mathbb{E}^ℓ or \mathbb{P}^ℓ to underline this fact, for instance, for any $\mathbf{y} = (i, \ell) \in \{0, 1, \dots, p\} \times \mathbb{R}_+^q$, $j \in \{0, 1, \dots, p\}$ and Borel subset $A \subset \mathbb{R}_+^q$,*

$$\tilde{Q}(\mathbf{y}, \{j\} \times A) = \mathbb{P}^\ell(\tilde{I}_1 = j, \tilde{\boldsymbol{\lambda}}_1 \in A).$$

Furthermore, for any $\mathcal{J} \subseteq \{1, \dots, q\}$, we denote by $\tilde{Q}^{(-\mathcal{J})}$ the transition kernel defined on $\{0, 1, \dots, p\} \times \mathbb{Z}_+^{q-\#\mathcal{J}} \times \mathbb{R}_+^p$ defined as the transition kernel Q but without the *constraint* variable \mathbf{S}_j , $j \in \mathcal{J}$ and their corresponding constraint sets $\mathbf{A}_1(j), \dots, \mathbf{A}_p(j)$. In particular, we have $\tilde{Q}^{(-\emptyset)} = \tilde{Q}$ and $\tilde{Q}^{(-\{1, \dots, q\})} = \tilde{Q}$. Meanwhile, for any $\mathcal{J} \subseteq \{1, \dots, q\}$, we denote by $\tilde{Q}^{(+\mathcal{J})}$ the transition kernel defined on $\{0, 1, \dots, p\} \times \mathbb{Z}_+^{\#\mathcal{J}} \times \mathbb{R}_+^p$ defined as the transition kernel Q with the *constraint* variable \mathbf{S}_j , $j \in \mathcal{J}$ and their corresponding constraints $\mathbf{A}_1(j), \dots, \mathbf{A}_p(j)$. In particular $\tilde{Q}^{(+\{1, \dots, q\})} = \tilde{Q}$ and $\tilde{Q}^{(+\emptyset)} = \tilde{Q}$.

3 Main results

We now present the main results of this work. All the results will rely on the Markov assumption introduced in Section 2.3. Under this Markov framework, in Section 3.1, we show that the kernel Q and \tilde{Q} defined in Section 2.3 are ψ -irreducible and aperiodic. We provide a partial drift condition in Section 3.2 and then prove that \tilde{Q} is V -geometrically ergodic in Section 3.3, where V is unbounded off petite sets. Then we study the ergodicity in the case $q = 1$ in Section 3.4. The general case is presented in Section 3.5. Finally, we determine the scaling limit of the point process in *physical time* in Section 3.6.

3.1 Irreducibility

Because of the constraints sets \mathbf{A}_i and the function \mathbf{J} , the path of the process \mathbf{S} cannot evolve arbitrarily. The following definitions will be useful.

Definition 1. Let m be a positive integer and $\mathbf{s} \in \mathbb{Z}_+^q$, the set of admissible paths $\mathcal{A}_m(\mathbf{s})$ is defined as the set of $(j_1, \dots, j_m) \in \{0, \dots, p\}^m$ such that

$$\mathbf{s} \in \mathbf{A}_{j_1}^c, \mathbf{s} + \mathbf{J}_o(j_1) \in \mathbf{A}_{j_2}^c, \dots, \mathbf{s} + \sum_{n=1}^{m-1} \mathbf{J}_o(j_n) \in \mathbf{A}_{j_m}^c. \quad (3.1)$$

An admissible path $(j_1, \dots, j_m) \in \mathcal{A}_m(\mathbf{s})$ implies that, for any $i = 0, 1, \dots, p$ and $\ell \in (0, \infty)^p$, given $\mathbf{Y}_0 = (i, \mathbf{s}, \ell)$ the conditional probability to have $I_1 = j_1, \dots, I_m = j_m$ (and thus, by (2.7) $\mathbf{S}_k = \mathbf{s} + \sum_{n=1}^k \mathbf{J}_o(j_n)$ for all $k = 1, 2, \dots, m$) is positive.

The main assumption of this section is the following one.

Assumption 2. The fertility matrix \aleph defined in (2.3) is invertible. Moreover, if $q \geq 1$, there exists $\mathbf{s}_o \in \mathbb{Z}_+^q$ such that the following assertion holds. For all integer $K \geq 1$, there exists an integer $m \geq p + 1$ such that, for all $\mathbf{s} \in \{1, \dots, K\}^q$, there exists an admissible path $(j_1, \dots, j_m) \in \mathcal{A}_m(\mathbf{s})$ such that $\mathbf{s} + \sum_{n=1}^m \mathbf{J}_o(j_n) = \mathbf{s}_o$ and $\{j_{m-p+1}, \dots, j_m\} = \{1, \dots, p\}$ (possibly in a different order).

We refer to [33] for the definition of small sets and ψ -irreducibility. The assumption on the invertibility of \aleph will be useful for proving the ψ -irreducibility of the embedded Markov chain. The second assumption says that there exists an admissible path (see Definition 1) with the last p steps containing the p different marks. We will check this condition for the application of the constrained Hawkes process to a limit order book in Section 4. It will be used to establish the existence of small sets for Q , see Proposition 5.3.

We can now state the main result of this section.

Theorem 3.1. Let Q and \tilde{Q} be the transition kernels defined in Section 2.3 on the spaces $\mathbb{Z}_+^q \times \mathbb{R}_+^p$ and $\{0, 1, \dots, p\} \times \mathbb{Z}_+^q \times \mathbb{R}_+^p$, respectively, with $p \geq 1$ and $q \geq 0$. Suppose moreover that Assumption 2 holds. Then the kernels Q and \tilde{Q} are aperiodic and ψ -irreducible. Moreover for all $K \geq 1$ and $M > 0$, $\{1, \dots, K\}^q \times (0, M]^p$ and $\{0, \dots, p\} \times \{1, \dots, K\}^q \times (0, M]^p$ are petite sets for Q and \tilde{Q} , respectively.

The proof is postponed to Section 5.1.

3.2 Partial drift condition

We now derive a partial drift condition that will be useful to obtain a “complete” drift condition on the process $\{\mathbf{X}(t), t \geq 0\}$. In the following result, the drift condition is partial in the sense that it does not control \mathbf{S}_n . It only says that, under Assumption 1, independently of the process \mathbf{S}_n , λ_n should not have large excursions away of a compact set.

Proposition 3.2. Let Q be the kernel defined in Section 2.3 on the space $\mathbb{Z}_+^q \times \mathbb{R}_+^p$ with $p \geq 1$ and $q \geq 0$. Suppose that Assumption 1 holds. Then there exists $\gamma > 0$, $\theta \in (0, 1)$, $M > 0$ and $b > 0$ such that, for all $\mathbf{s} \in \mathbb{Z}_+^q$ and $\ell \in \mathbb{R}_+^p$, we have

$$[Q(1_{\mathbb{Z}_+^q} \otimes V_{1,\gamma})](\mathbf{s}, \ell) \leq \theta V_{1,\gamma}(\ell) + b1_{(0,M]^p}(\ell), \quad (3.2)$$

where

$$V_{1,\gamma}(\ell) = e^{\gamma \mathbf{u}^T \ell}, \quad (3.3)$$

with \mathbf{u} defined as in (2.4).

The proof is postponed to Section 5.2.

3.3 The case $q = 0$

If $q = 0$, the process \mathbf{N} is a standard (unconstrained) multivariate Hawkes process. In this case the process $\{\boldsymbol{\lambda}_n, n \geq 1\}$ is a Markov chain (the *constraint* variable \mathbf{S}_n vanishes) and the “partial” drift condition of Proposition 3.2 becomes a “complete” drift condition for this chain. Moreover, if $q = 0$ in Theorem 3.1, Assumption 2 boils down to assuming that the fertility matrix \aleph defined in (2.3) is invertible. We also note that the sublevel sets of $V_{1,\gamma}$ are small sets by Theorem 3.1. Hence we obtain that $\{\check{\boldsymbol{\lambda}}_n, n \geq 0\}$ is $V_{1,\gamma}$ -geometrically ergodic, see [33, Chapter 15] (and also [33, Theorem 9.1.8] to get the Harris recurrence). In fact the same result holds on the extended chain $\{(\check{I}_n, \check{\boldsymbol{\lambda}}_n), n \geq 0\}$, whose transition kernel has been denoted by \check{Q} in Section 2.3.

Proposition 3.3. *Let \check{Q} be the transition kernel defined in Section 2.3 on the space $\{0, 1, \dots, p\} \times \mathbb{R}_+^p$. Suppose that \aleph is invertible and that Assumption 1 holds. Then there exists $\gamma > 0$ such that \check{Q} is $(1_{\{0, \dots, p\}} \otimes V_{1,\gamma})$ -geometrically ergodic, where $V_{1,\gamma}$ is defined in (3.3).*

Proof. As explained above, Theorem 3.1 and Proposition 3.2 apply to the case $q = 0$ and, as a consequence, $\{\check{\boldsymbol{\lambda}}_n, n \geq 0\}$ is a $V_{1,\gamma}$ -geometrically ergodic. This conclusion also applies for the kernel \check{Q} of the chain $\{(\check{I}_n, \check{\boldsymbol{\lambda}}_n), n \geq 0\}$ with $V_{1,\gamma}$ replaced by $(1_{\{0, \dots, p\}} \otimes V_{1,\gamma})$ because the conditional probability of $(\check{I}_1, \check{\boldsymbol{\ell}}_1)$ given $(\check{I}_0, \check{\boldsymbol{\ell}}_0)$ does not depend on \check{I}_0 . \square

Having proved the ergodicity of \check{Q} , we can establish the following result on its stationary distribution. It will be used to obtain the case $q = 1$.

Corollary 3.4. *Under the same assumptions as Proposition 3.3, denote by $\tilde{\pi}$ the stationary distribution of \check{Q} . Let $\mathbf{w} : \{1, \dots, p\} \rightarrow \mathbb{R}$. Let \mathbf{w}_o be the extension of \mathbf{w} to $\{0, 1, \dots, p\}$ obtained by setting $\mathbf{w}_o(0) = 0$. Then we have*

$$\tilde{\pi}(\mathbf{w}_o \otimes 1_{\mathbb{R}_+^p}) = \frac{\overrightarrow{\mathbf{w}}^T (\text{Id}_p - \aleph)^{-1} \boldsymbol{\mu}_0}{\boldsymbol{\mu}_0^0 + \mathbf{1}_p^T (\text{Id}_p - \aleph)^{-1} \boldsymbol{\mu}_0}. \quad (3.4)$$

The proof is postponed to Section 5.3.

3.4 The case $q = 1$

We shall further denote

$$s^* = 1 + \max \left(\bigcup_{j=1, \dots, p} A_j \right). \quad (3.5)$$

In the case $q = 1$, we need a drift function that applies to $\mathbf{X}_n = (S_n, \boldsymbol{\lambda}_n)$ which is unbounded off petite sets, that is, the drift function must diverge as at least one component of \mathbf{X}_n goes to infinity, while, in Proposition 3.2 the drift function goes to infinity only when one of the components of $\boldsymbol{\lambda}_n$ goes to infinity. To this end we define

$$V_{0,\gamma}(s) = e^{\gamma s}. \quad (3.6)$$

We introduce the following assumption.

Assumption 3. *The following inequality holds*

$$\overrightarrow{\mathbf{J}}^T (\text{Id}_p - \aleph)^{-1} \boldsymbol{\mu}_0 < 0. \quad (3.7)$$

Note that Condition (2.5) corresponds to Assumption 3 in the special case $\aleph = 0$.

Theorem 3.5. *Let $q = 1$ and $p \geq 1$ and suppose that Assumption 1, 2 and 3 hold. Let \tilde{Q} be the kernel defined in Section 2.3 on the space $\{0, 1, \dots, p\} \times \mathbb{Z}_+ \times \mathbb{R}_+^p$ with $p \geq 1$. Then, for all $\gamma_1 > 0$ small enough, there exists $\gamma_0^* > 0$ such that, for all $\gamma_0 \in (0, \gamma_0^*]$, \tilde{Q} is $(1_{\{0, \dots, p\}} \otimes V_{0, \gamma_0} \otimes V_{1, \gamma_1})$ -geometrically ergodic, where V_{0, γ_0} and V_{1, γ_1} are defined in (3.6) and (3.3).*

The proof of Theorem 3.5 is omitted as it is a particular case of Theorem 3.7, which is proved in Section 5.5.

Assumptions 1 and 2 appear to be very mild. Assumption 1 is related to the stability of the underlying unconstrained Hawkes process. Assumption 2 is used to obtain small sets for the chains Q and \tilde{Q} . A natural question is to ask whether Assumption 3 $\vec{J}^T(\text{Id}_p - \aleph)^{-1} \mu_0 < 0$ is sharp. The following theorem partially answers to this question.

Theorem 3.6. *Let $\{(S_n, \lambda_n), n \geq 0\}$ be the Markov chain on the space $\mathbb{Z}_+ \times \mathbb{R}_+^p$ with transition kernel Q defined in Section 2.3. Suppose that Assumptions 1 and 2 hold. Then Q is ψ -irreducible. Moreover the two following assertions holds.*

(i) *If $\psi(\{s^*, s^* + 1, \dots\} \times (0, \infty)^p) = 0$, then Q is $(1_{\{1, \dots, s^*-1\}} \otimes V_{1, \gamma_1})$ -geometrically ergodic, where s^* is defined by (3.5).*

(ii) *Otherwise, if*

$$\vec{J}^T(\text{Id}_p - \aleph)^{-1} \mu_0 > 0, \quad (3.8)$$

then Q is transient.

The proof is postponed to Section 5.4.

Remark 3. *The fact that Q is ψ -irreducible follows from Theorem 3.1. Then, by definition of ν in the proof of Proposition 5.3, Assumption 2 implies that $\psi(\{s_0\} \times (0, \infty)^p) > 0$. Thus if s_0 in Assumption 2 satisfies $s_0 \geq s^*$, then the case (i) does not happen.*

3.5 General case

To obtain a drift function in the general case $q \geq 1$, we shall use the function $V_{0, \gamma}$ defined by (3.6) applied multiplicatively to each component of \mathbf{S}_n and the function V_{1, γ_1} defined by (3.3) applied to intensity λ_n .

We shall see that the proof of the ergodicity for $q \geq 1$ relies on an induction on q , see the details in Section 5.5. For $q = 1$ the induction applies under the simple Assumption 3 because, using Corollary 3.4, it allows us to compute some moment under the stationary distribution of the unconstrained chain (with kernel \tilde{Q}). In general the induction is more involved. We shall need some additional notation. For any set $\mathcal{J} \subseteq \{1, \dots, q\}$, we define $J^{\mathcal{J}} : \{1, \dots, p\} \rightarrow \mathbb{Z}$ by

$$J^{\mathcal{J}}(i) = \sum_{j \in \mathcal{J}} \mathbf{J}_j(i). \quad (3.9)$$

We further denote by $J_o^{\mathcal{J}}$ the extension of $J^{\mathcal{J}}$ on $\{0, \dots, p\}$ defined by $J_o^{\mathcal{J}}(0) = 0$.

Theorem 3.7. *Let $q \geq 1$ and $p \geq 1$, and suppose that Assumption 1 and 2 hold. Let \tilde{Q} be the kernel defined in Section 2.3 on the space $\{0, 1, \dots, p\} \times \mathbb{Z}_+^q \times \mathbb{R}_+^p$. Define, for all $\mathcal{J} \subset \{1, \dots, q\}$, the kernel $\tilde{Q}^{(+\mathcal{J})}$ as in Section 2.3. Check the following conditions in this order :*

for $k = 1, 2, 3, \dots, q$ **do**
for $\mathcal{J} \subset \{1, \dots, q\}$ such that $\#\mathcal{J} = k$ **do**
Check that

$$\left(\vec{\mathcal{J}}^{\mathcal{J}}\right)^T (\text{Id}_p - \aleph)^{-1} \boldsymbol{\mu}_0 < 0. \quad (3.10)$$

for $\mathcal{J}' \subset \mathcal{J}$ with $\mathcal{J}' \notin \{\emptyset, \mathcal{J}\}$ **do**
Check that

$$\tilde{\pi}^{(+\mathcal{J}')} \left[J_o^{\mathcal{J}} \otimes 1_{\mathbb{Z}_+^q \times \mathbb{R}_+^p} \right] < 0. \quad (3.11)$$

end

Then, for all $\gamma_1 > 0$ small enough, there exists $\gamma_0^* > 0$ such that for all $\gamma_0 \in (0, \gamma_0^*]$, $\tilde{Q}^{(+\mathcal{J})}$ is $(1_{\{0, \dots, p\}} \otimes V_{0, \gamma_0}^{\otimes (\#\mathcal{J})} \otimes V_{1, \gamma_1})$ -geometrically ergodic. We denote by $\tilde{\pi}^{(+\mathcal{J})}$ its stationary distribution.

end

end

Then, for all $\gamma_1 > 0$ small enough, there exists $\gamma_0^* > 0$ such that for all $\gamma_0 \in (0, \gamma_0^*]$, \tilde{Q} is $(1_{\{0, \dots, p\}} \otimes V_{0, \gamma_0}^q \otimes V_{1, \gamma_1})$ -geometrically ergodic.

The proof is postponed to the end of Section 5.5.

Remark 4. By Corollary 3.4, Condition (3.10) is equivalent to (3.11) with $\mathcal{J}' = \emptyset$.

Remark 5. Observe that Condition (3.11) makes sense at this step. Indeed, since $\#\mathcal{J} = k$, and $\#\mathcal{J}' < k$, we have already checked that $\tilde{Q}^{(+\mathcal{J}'})$ is geometrically ergodic. Thus the stationary distribution $\tilde{\pi}^{(+\mathcal{J}'})$ is well defined. In practice, one may check this assumption using Monte Carlo simulations, since the law of large number holds in this case (see [33, Theorem 17.1.7]).

3.6 Scaling limit

In the application of this model to the limit order book, an increment of the best bid price or the best ask price at time T_k is specified by the mark $I_k \in \{0, \dots, p\}$, see the details in Section 4. It is interesting to investigate the microscopic behavior of the mid-price (mean of the best bid price and the best ask price) at large scales, see [5] for the non-constrained case.

Suppose that the Assumptions of Theorem 3.7 hold, then \tilde{Q} is $(1_{\{0, \dots, p\}} \otimes V_{0, \gamma_0}^q \otimes V_{1, \gamma_1})$ -geometrically ergodic for some $\gamma_0, \gamma_1 > 0$ and V_{0, γ_0} and V_{1, γ_1} defined in (3.6) and (3.3). Applying the results of [33, Theorem 17.4.2], we obtain a functional central limit theorem (FCLT) in *discrete time* for the chain \tilde{Q} . To obtain a result in (physical) *continuous time*, we actually need a functional CLT for the kernel \overline{Q} defined in Section 2.3. Hence we must first check that \overline{Q} is geometrically ergodic.

Let us define $V_{2, \gamma} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by

$$V_{2, \gamma}(\delta) = e^{\gamma \delta}. \quad (3.12)$$

We provide the ergodicity property of \overline{Q} in the following corollary.

Corollary 3.8. Let $q, p \geq 1$ and \overline{Q} be the kernel defined in Section 2.3 on the space $\mathbb{R}_+ \times \{0, 1, \dots, p\} \times \mathbb{Z}_+^q \times \mathbb{R}_+^p$. Suppose that the Assumptions of Theorem 3.7 hold. Then, for all $\gamma_1 > 0$ small enough, there exists $\gamma_0^* > 0$ such that for all $\gamma_0 \in (0, \gamma_0^*]$ and all $\gamma_2 > 0$ small enough, \overline{Q} is $(V_{2, \gamma_2} \otimes 1_{\{0, \dots, p\}} \otimes V_{0, \gamma_0}^q \otimes V_{1, \gamma_1})$ -geometrically ergodic.

The proof is postponed to Section 5.6.

By Corollary 3.8, \bar{Q} admits a stationary distribution $\bar{\pi}$. We shall use the notation $\bar{\mathbb{E}}$ for the expectation under the stationary distribution.

Let now $g : Z \rightarrow \mathbb{R}$ be such that for all $\gamma_0, \gamma_1, \gamma_2 > 0$,

$$\sup_{\mathbf{z} \in Z} \frac{g(\mathbf{z})}{[V_{2,\gamma_2} \otimes 1_{\{0,\dots,p\}} \otimes V_{0,\gamma_0}^q \otimes V_{1,\gamma_1}](\mathbf{z})} < \infty. \quad (3.13)$$

By Corollary 3.8, under this condition, we have $\bar{\pi}(|g|) < \infty$. Let $\bar{g} = g - \bar{\pi}(g)$. Using Corollary 3.8 and [33, Theorem 17.4.4], for all $\gamma_0, \gamma_1, \gamma_2 > 0$, there exists $R > 0$ such that the Poisson equation $\hat{g} - \bar{Q}(\hat{g}) = \bar{g}$ (see [33, Chapter 17]) admits a solution \hat{g} satisfying the bound $|\hat{g}| \leq R((V_{2,\gamma_2} \otimes 1_{\{0,\dots,p\}} \otimes V_{0,\gamma_0}^q \otimes V_{1,\gamma_1}) + 1)$. Moreover, we have, for all $\mathbf{z} \in Z$,

$$\hat{g}(\mathbf{z}) = \sum_{k=0}^{\infty} \bar{Q}^k(\mathbf{z}, \bar{g}).$$

It also follows that $\bar{\pi}(|\hat{g}|^2) < \infty$. Let us denote the linearly interpolated partial sums of $g(\mathbf{Z}_n)$ by

$$s_n(t, g) = \Sigma_{[nt]}(\bar{g}) + (nt - [nt]) [\Sigma_{[nt]+1}(\bar{g}) - \Sigma_{[nt]}(\bar{g})],$$

where $\Sigma_k(h) = \sum_{j=1}^k h(\mathbf{Z}_j)$. Applying [33, Theorem 17.4.4], if the (nonnegative) constant

$$\sigma_g^2 := \bar{\pi}(\hat{g}^2 - \{\bar{Q}\hat{g}\}^2) \quad (3.14)$$

is strictly positive, we have, for any initial distribution, as $n \rightarrow \infty$,

$$(n\sigma_g^2)^{-1/2} s_n(t, g) \xrightarrow{d} B_t, \quad (3.15)$$

where \xrightarrow{d} denotes the weak convergence in $C([0, 1])$ (the space of continuous functions defined on $[0, 1]$) and B denotes a standard Brownian motion on $[0, 1]$.

Remark 6. Note that σ_g^2 may not be strictly positive. An interesting example is given by $g = 1_{\mathbb{R}_+} \otimes \mathbf{J}_o \otimes 1_{\mathbb{Z}_+^q} \otimes 1_{\mathbb{R}_+^p}$. In this case, we have $\Sigma_n(g) = \mathbf{S}_n - \mathbf{S}_0$ and thus

$$\Sigma_n(\bar{g}) = \mathbf{S}_n - \mathbf{S}_0 - \bar{\mathbb{E}}(\mathbf{S}_n) + \bar{\mathbb{E}}(\mathbf{S}_0) + \sum_{k=1}^n \bar{\mathbb{E}}(\mathbf{J}_o(I_k)) - \bar{\pi}(g),$$

Hence, we get that $\Sigma_n(\bar{g}) = 0_{P_*}(1)$, which implies $\sigma_g^2 = 0$.

We now derive the main result of this section, which determines the scaling limit of the constrained Hawkes process. We need some additional notation before stating the result. Let $\mathbf{w} : \{1, \dots, p\} \rightarrow \mathbb{R}$. Define

$$\mathcal{E}(\mathbf{w}) = \frac{\bar{\mathbb{E}}[\mathbf{w}(I_1)]}{\bar{\mathbb{E}}[\Delta_1]}. \quad (3.16)$$

Further define

$$v(\mathbf{w}) = \sigma_g^2, \quad (3.17)$$

where σ_g^2 is defined as in (3.14) with $g : Z \rightarrow \mathbb{R}$ defined by $g(\mathbf{z}) = \mathbf{w}(i) - \mathcal{E}(\mathbf{w})\delta$ for all $\mathbf{z} = (\delta, i, \mathbf{s}, \ell)$.

Theorem 3.9. Under the assumptions of Theorem 3.7, if $v(\mathbf{w}) > 0$, we have, for any initial condition,

$$T^{-1/2} (\mathbf{N}(1_{[0,tT]} \otimes \mathbf{w}) - t T \mathcal{E}(\mathbf{w})) \xrightarrow{d} (v(\mathbf{w})/\bar{\mathbb{E}}[\Delta_1])^{1/2} B_t. \quad (3.18)$$

The proof is postponed to Section 5.7.

4 Application to a limit order book (LOB)

Let us apply our results in the Markov setting to a simple limit order book where only the two best limits are considered (Best Bid and Best Ask). Figure 1 presents a snapshot of the price evolution in physical time and event time.

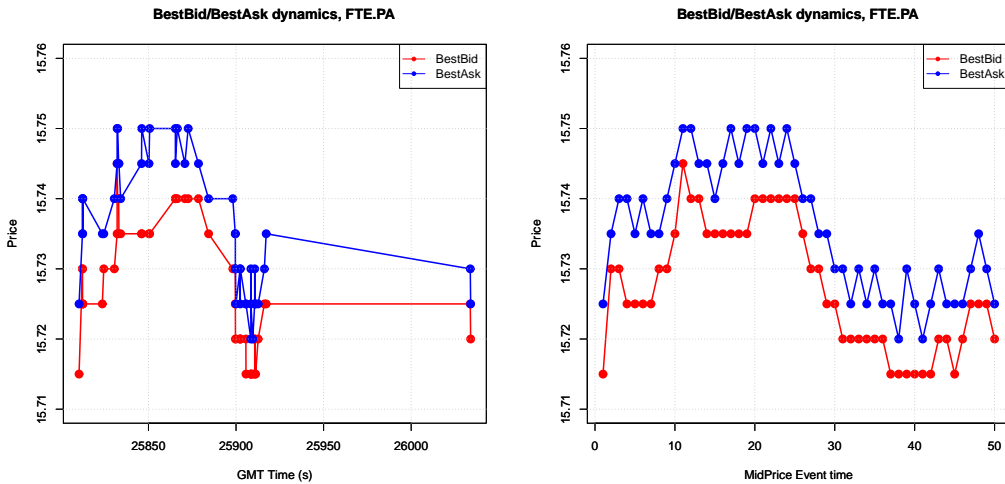


Figure 1: Limit order book of Orange : Evolution of Best Bid and Best Ask prices over a short time window of April 1, 2011. The evolution is displayed in physical time (left) and in event time (right).

In this example, for sake of simplicity, we focus on the following four best limit events :

- *Event 1*: Best Ask price moves upward one tick,
- *Event 2*: Best Ask price moves downward one tick,
- *Event 3*: Best Bid price moves upward one tick,
- *Event 4*: Best Bid price moves downward one tick.

In order to forecast future events based on past observations of the LOB, Hawkes processes seem to be a sensible approach to model the time evolution of these events. However, since the Best Bid has to remain at least one tick below the Best Ask, we need to introduce a constraint. Namely defining the spread as $S = BestAsk - BestBid$ (in ticks), events 2 and 3 do not happen whenever $S = 1$.

Hence we propose the constrained multivariate Hawkes process described in Section 2.1 to model this data. This case corresponds to $p = 4$, $q = 1$ and the constrained sets A_i are defined by

$$A_i = \begin{cases} \emptyset, & i = 1 \text{ or } 4 \\ \{1\}, & i = 2 \text{ or } 3 \end{cases} \quad (4.1)$$

The spread variable S evolves according to (2.2) with a scalar J defined by

$$J(i) = \begin{cases} 1, & i = 1 \text{ or } 4 \\ -1, & i = 2 \text{ or } 3 \end{cases} \quad (4.2)$$

The Markov assumption of Section 2.3 allows us to use a finite set of parameters for this model, namely β , \aleph and μ_0 .

Assumptions 1 and 3 then amount to assumptions on the parameters μ_0 and \aleph . Assumption 2 says that \aleph has to be invertible and requires an additional property which only depends on the constrained sets A_i and J . Hence, to apply Theorem 3.5, we only need to check that this additional property holds. We set $s_o = 2$. For any $K \geq 1$, we set $m = |K - 2| + p$. Then given $s \in \{1, \dots, K\}$ we need to show that there is an admissible path $(j_1, \dots, j_m) \in \mathcal{A}_m(s)$ such that $s + \sum_{k=1}^m J_o(j_k) = 2 (= s_o)$. To see why, let us set, for $1 \leq i \leq |s - 2|$,

$$j_i = \begin{cases} 2 & \text{if } s \geq 2, \\ 1 & \text{if } s = 1. \end{cases}$$

Then, by (4.2), we have $s + \sum_{k=1}^{|s-2|} J(j_k) = 2$. To conclude, we set $j_i = 0$ for $|s-2| < i \leq m-4$

and $j_{m-3} = 1, j_{m-2} = 4, j_{m-1} = 2$ and $j_m = 3$, so that, by (4.2) and (2.8), $s + \sum_{k=1}^m J_o(j_k) = 2$. Moreover, we easily check that (3.1) holds for this choice of (j_1, \dots, j_m) and that $\{j_{m-3}, j_{m-2}, j_{m-1}, j_m\} = \{1, \dots, 4\}$. Hence Assumption 2 holds and Theorem 3.5 applies provided that \aleph is invertible and has spectral radius smaller than 1, and μ_0 satisfies (3.7) with J defined by (4.2). The ergodicity of the underlying chain allows one to perform meaningful statistical analysis of the data. This will be done in a forthcoming paper.

For the moment, we focus on the scaling limit of the mid-price, defined as the middle price between the Best Bid and Best Ask prices. This value is often considered as the continuous time price of the asset. In the framework of our model, the mid price satisfies the following equation

$$P(t) = P(u) + \mathbf{N}(1_{(u,t]} \otimes w) , \quad (4.3)$$

where $w(i)$ takes values $1/2, -1/2, 1/2$ and $-1/2$ for $i = 1, \dots, 4$, respectively. (Note that similar equations hold for the Best Bid and Best Ask prices with different functions w .)

Applying Theorem 3.9, we get the scaling limit of the mid-price in physical time is given by

$$T^{-1/2} (P(tT) - P(0) - t T \mathcal{E}(w)) \xrightarrow{d} (v(w) / \overline{\mathbb{E}}[\Delta_1])^{1/2} B_t ,$$

where $\mathcal{E}(w)$ and $v(w)$ are respectively defined in (3.16) and (3.17), provided that $v(w) > 0$. Although it does not seem easy to check that $v(w) > 0$ (except perhaps by numerical means), we do expect this to be true. Indeed, while the spread behaves as a stationary variable, yielding a vanishing asymptotic variance in the large scale behavior (see Remark 6), the best-bid, best-ask and mid prices behave as co-integrated random walks.

5 Postponed proofs

5.1 Proof of Theorem 3.1

In the following we shall denote

$$\overline{\mu}_0 = \max_j \mu_0(j), \quad \underline{\mu}_0 = \min_j \mu_0(j), \quad (5.1)$$

$$\overline{\alpha} = \max_{i,j} \alpha_{i,j}, \quad \underline{\alpha} = \min_{i,j} \alpha_{i,j}, \quad (5.2)$$

$$\overline{J} = \max_{i=1, \dots, p} \sum_{j=1}^q |\vec{J}_{i,j}|, \quad (5.3)$$

and the j -th column vector of \aleph by \aleph_j , so that $\aleph = (\aleph_1 \ \dots \ \aleph_p)$. To establish the existence of small sets, we need two preliminary results.

Lemma 5.1. *Let $\mathbf{x} = (\mathbf{s}, \boldsymbol{\ell}) \in \mathsf{X}$, $j \in \mathcal{A}_1(\mathbf{s})$ with $j \neq 0$ and $g : \mathsf{X} \rightarrow \mathbb{R}_+$. Then*

$$\mathbb{E}^{\mathbf{x}} [g(\boldsymbol{\lambda}_1) 1_{\{I_1=j\}}] \geq \underline{\mu}_0 \int_0^{+\infty} g(e^{-t\beta} \boldsymbol{\ell} + \beta \aleph_j) e^{-2(\overline{\mu}_0 + \bar{\ell})t} dt, \quad (5.4)$$

where $\bar{\ell} = \max_i \ell(i)$.

Proof. Using (2.9), we get that

$$\mathbb{E}^{\mathbf{x}} [g(\boldsymbol{\lambda}_1) 1_{\{I_1=j\}}] = \mathbb{E}^{\mathbf{x}} \left[g(e^{-\Delta_1^j \beta} \boldsymbol{\ell} + \beta \aleph_j) 1_{\{\Delta_1^j < U_1^j\}} \right],$$

where $U_1^j = \min_{i \neq j} \Delta_1^i$.

Observe that $U_1^j \stackrel{st}{\geq} \text{Exp}(\overline{\mu}_0 + \bar{\ell})$ where $\text{Exp}(z)$ denotes the exponential distribution with mean $\frac{1}{z}$, thus, since Δ_1^j and U_1^j are independent, we have

$$\mathbb{E}^{\mathbf{x}} [g(\boldsymbol{\lambda}_1) 1_{\{I_1=j\}}] \geq \mathbb{E}^{\mathbf{x}} \left[g(e^{-\Delta_1^j \beta} \boldsymbol{\ell} + \beta \aleph_j) e^{-(\overline{\mu}_0 + \bar{\ell})\Delta_1^j} \right].$$

Since $j \in \mathcal{A}_1(\mathbf{s})$, using that the hazard rate of Δ_1^j is bounded between $\underline{\mu}_0$ and $\overline{\mu}_0 + \bar{\ell}$, we see that the density of Δ_1^j is bounded from below by $\underline{\mu}_0 e^{-(\overline{\mu}_0 + \bar{\ell})t}$ on $t \in \mathbb{R}_+$. Hence we get (5.4). \square

Corollary 5.2. *Let m be an integer ≥ 1 , $M \in \mathbb{R}_+$ and $g : \mathsf{X} \rightarrow \mathbb{R}_+$. Let $\mathbf{x} = (\mathbf{s}, \boldsymbol{\ell}) \in \mathsf{X}$ with $\boldsymbol{\ell} \in (0, M]^p$ and $(j_1, \dots, j_m) \in \mathcal{A}_m(\mathbf{s})$ with $j_i \neq 0$ for $i = 1, \dots, m$. Then, we have*

$$\begin{aligned} & \mathbb{E}^{\mathbf{x}} [g(\boldsymbol{\lambda}_m) 1_{\{I_1=j_1, \dots, I_m=j_m\}}] \\ & \geq \underline{\mu}_0^m \int_{\mathbb{R}_+^m} g[e^{-\sum_{i=1}^m t_i \beta} \boldsymbol{\ell} + e^{-\sum_{i=2}^m t_i \beta} \beta \aleph_{j_1} + \dots + \beta \aleph_{j_m}] \\ & \quad \times e^{-C_{m,M} \sum_{i=1}^m t_i} dt_m \dots dt_1, \end{aligned}$$

where $C_{m,M} = 2(\overline{\mu}_0 + M + (m-1)\beta\bar{\alpha})$.

Proof. The case $m = 1$ is given by Lemma 5.1.

To get the result for any $m \geq 2$, we proceed by induction. Let $\mathbf{x} = (\mathbf{s}, \boldsymbol{\ell}) \in \mathsf{X}$ with $\boldsymbol{\ell} \in (0, M]^p$ and $(j_1, \dots, j_m) \in \{1, \dots, p\}^m$, if $(j_1, \dots, j_m) \in \mathcal{A}_m(\mathbf{s})$. We have

$$\begin{aligned} & \mathbb{E}^{\mathbf{x}} [g(\boldsymbol{\lambda}_m) 1_{\{I_1=j_1, \dots, I_m=j_m\}}] \\ & = \mathbb{E}^{\mathbf{x}} [1_{\{I_1=j_1\}} \mathbb{E} [g(\boldsymbol{\lambda}_m) 1_{\{I_2=j_2, \dots, I_m=j_m\}} | (I_1, \mathbf{X}_1)]] . \quad (5.5) \end{aligned}$$

Recall that given (I_1, \mathbf{X}_1) , the distribution of $\{(I_k, \mathbf{X}_k), k \geq 2\}$ does not depend on I_1 , hence, with the Markov property, we obtain that

$$\mathbb{E} [g(\boldsymbol{\lambda}_m) 1_{\{I_2=j_2, \dots, I_m=j_m\}} | (I_1, \mathbf{X}_1)] = \mathbb{E}^{\mathbf{X}_1} [g(\boldsymbol{\lambda}_{m-1}) 1_{\{I_1=j_2, \dots, I_{m-1}=j_m\}}] .$$

Note that, using (2.9) and (5.2), $\{\mathbf{X}_0 = \mathbf{x}\}$ and $\boldsymbol{\ell} \in (0, M]^p$ implies $\boldsymbol{\lambda}_1 \in (0, M + \beta\bar{\alpha}]^p$. Thus, in the event $\{\mathbf{X}_0 = \mathbf{x}\}$, the right-hand side of the last display can be bounded from below by applying the induction hypothesis with (j_1, \dots, j_{m-1}) replaced by (j_2, \dots, j_m) , $\boldsymbol{\ell} = \boldsymbol{\lambda}_1$ and the constant

$$C_{m-1, M + \beta\bar{\alpha}} = 2(\overline{\mu}_0 + M + \beta\bar{\alpha} + (m-2)\beta\bar{\alpha}) = C_{m,M} .$$

Plugging this bound in (5.5), we get that

$$\begin{aligned} & \mathbb{E}^{\mathbf{x}} [g(\boldsymbol{\lambda}_m) 1_{\{I_1=j_1, \dots, I_m=j_m\}}] \\ & \geq \underline{\mu}_0^{m-1} \mathbb{E}^{\mathbf{x}} \left[1_{\{I_1=j_1\}} \int_{\mathbb{R}_+^{m-1}} g[\dots] \times e^{-C_{m,M} \sum_{i=1}^{m-1} t_i} dt_{m-1} \cdots dt_1 \right], \end{aligned} \quad (5.6)$$

where the omitted argument of g reads

$$[\dots] = e^{-\sum_{i=1}^{m-1} t_i \beta} \boldsymbol{\lambda}_1 + e^{-\sum_{i=2}^{m-1} t_i \beta} \beta \aleph_{j_2} + \cdots + \beta \aleph_{j_m}.$$

Using again Lemma 5.1, we have

$$\begin{aligned} \mathbb{E}^{\mathbf{x}} [1_{\{I_1=j_1\}} g[\dots]] & \geq \underline{\mu}_0 \int_0^{+\infty} g[\dots'] e^{-2(\underline{\mu}_0 + \bar{\ell})t} dt \\ & \geq \underline{\mu}_0 \int_0^{+\infty} g[\dots'] e^{-C_{m,M} t} dt, \end{aligned} \quad (5.7)$$

where the omitted argument in the right-hand side reads

$$\begin{aligned} [\dots'] & = e^{-\sum_{i=1}^{m-1} t_i \beta} (e^{-t\beta} \boldsymbol{\ell} + \beta \aleph_{j_1}) + e^{-\sum_{i=2}^{m-1} t_i \beta} \beta \aleph_{j_2} + \cdots + \beta \aleph_{j_m} \\ & = e^{-\sum_{i=1}^{m-1} t_i \beta - t\beta} \boldsymbol{\ell} + e^{-\sum_{i=1}^{m-1} t_i \beta} \beta \aleph_{j_1} + \cdots + \beta \aleph_{j_m}. \end{aligned}$$

Hence we get the result by applying the Tonelli theorem in (5.6) and then using (5.7). \square

Proposition 5.3. *Let Q be the transition kernel defined in Section 2.3 on the space $\mathbb{Z}_+^q \times \mathbb{R}_+^p$ with $p, q \geq 1$. Suppose that Assumption 2 holds. Then there exists a probability measure ν on X such that, for all $K \geq 1$ and $M > 0$, there exists $\epsilon > 0$ and a positive integer m such that $\{1, \dots, K\}^q \times (0, M]^p$ is an (m, ϵ, ν) and $(m+1, \epsilon, \nu)$ -small set for Q .*

Before providing the proof of this result, let us state the following corollary which follows by observing that the conditional distribution of $(I_1, \mathbf{S}_1, \boldsymbol{\lambda}_1)$ given $(I_0, \mathbf{S}_0, \boldsymbol{\lambda}_0)$ does not depend on I_0 .

Corollary 5.4. *Let \tilde{Q} be the transition kernel defined in Section 2.3 on the space $\{0, 1, \dots, p\} \times \mathbb{Z}_+^q \times \mathbb{R}_+^p$ with $p, q \geq 1$. Suppose that Assumption 2 holds. Then there exists a probability measure $\tilde{\nu}$ on X such that, for all $K \geq 1$ and $M > 0$, there exists $\epsilon > 0$ and a positive integer m such that $\{1, \dots, p\} \times \{1, \dots, K\}^q \times (0, M]^p$ is an $(m, \epsilon, \tilde{\nu})$ and $(m+1, \epsilon, \tilde{\nu})$ -small set for \tilde{Q} .*

Proof of Proposition 5.3. Let $K \geq 1$ and $M > 0$ and denote $C = \{1, \dots, K\}^q \times (0, M]^p$. Let $\mathbf{x} = (\mathbf{s}, \boldsymbol{\ell}) \in C$. By Assumption 2, there exists an integer $m > 0$, and an admissible path (j_1, \dots, j_m) such that $\mathbf{s} + \sum_{n=1}^m \mathbf{J}(j_n) = \mathbf{s}_o$ and $\{j_{m-p+1}, \dots, j_m\} = \{1, \dots, p\}$, where $\mathbf{s}_o \in \mathbb{Z}_+^q$ is independent of m, M and K .

Now, for all subsets $A \in \mathcal{B}(\mathbb{Z}_+^q)$ and $B \in \mathcal{B}(\mathbb{R}_+^p)$, we may write

$$\begin{aligned} Q^m(\mathbf{x}, A \times B) & = \mathbb{E}^{\mathbf{x}} [1_A(\mathbf{S}_m) 1_B(\boldsymbol{\lambda}_m)] \\ & \geq \mathbb{E}^{\mathbf{x}} \left[1_A(\mathbf{S}_m) 1_B(\boldsymbol{\lambda}_m) \prod_{i=1}^m 1_{\{I_i=j_i\}} \right] \\ & = 1_A(\mathbf{s}_o) \mathbb{E}^{\mathbf{x}} \left[\prod_{i=1}^{m-p} 1_{\{I_i=j_i\}} \mathbb{E}^{\mathbf{X}^{m-p}} \left[1_B(\boldsymbol{\lambda}_p) \prod_{i=1}^p 1_{\{I_i=j_{m-p+i}\}} \right] \right], \end{aligned} \quad (5.8)$$

where we used that, given \mathbf{Y}_i , $i = 0, 1, \dots, m-p$, the conditional distribution of \mathbf{Y}_j , $j > m-p$ only depends on \mathbf{X}_{m-p} and the Markov property.

Take now $K \geq 1$ and $M > 0$ and let $\mathbf{x} \in \{1, \dots, K\}^q \times (0, M]^p$. By (2.9) and (5.2), under $\{\mathbf{X}_0 = \mathbf{x}\}$, we have $\boldsymbol{\lambda}_{m-p} \in (0, M']^p$ where $M' = M + \bar{\alpha}\beta(m-p)$. Moreover, under $\{\mathbf{X}_0 = \mathbf{x}, I_i = j_i, i = 1, \dots, m-p\}$, the last p -steps path (j_{m-p+1}, \dots, j_m) is admissible, that is, $(j_{m-p+1}, \dots, j_m) \in \mathcal{A}_p(\mathbf{S}_{m-p})$. Applying Corollary 5.2, we get that, under $\{\mathbf{X}_0 = \mathbf{x}, I_i = j_i, i = 1, \dots, m-p\}$,

$$\begin{aligned} \mathbb{E}^{\mathbf{X}_{m-p}} \left[1_B(\boldsymbol{\lambda}_m) \prod_{i=m-p+1}^m 1_{\{I_i=j_i\}} \right] \\ \geq \underline{\mu}_0^p \int_{\mathbb{R}_+^p} 1_B \left(e^{-\sum_{i=1}^p t_i \beta} \boldsymbol{\ell} + e^{-\sum_{i=2}^p t_i \beta} \beta \aleph_{j_1} + \dots + \beta \aleph_{j_p} \right) \\ \times e^{-C_{m,M'} \sum_{i=1}^p t_i} dt_p \dots dt_1, \end{aligned} \quad (5.9)$$

where $C_{m,M'} = 2(\bar{\mu}_0 + M' + (m-1)\bar{\alpha})$. Setting $u = e^{-\beta \sum_{i=1}^p t_i}$, $\boldsymbol{\vartheta} = [\boldsymbol{\vartheta}(1), \dots, \boldsymbol{\vartheta}(p-1)]^T$ where $\boldsymbol{\vartheta}(j) = e^{-\sum_{i=j+1}^p t_i \beta}$ for $j = 1, \dots, p-1$, Inequality (5.9) yields, under $\{\mathbf{X}_0 = \mathbf{x}, I_i = j_i, i = 1, \dots, m-p\}$,

$$\begin{aligned} \mathbb{E}^{\mathbf{X}_{m-p}} \left[1_B(\boldsymbol{\lambda}_m) \prod_{i=m-p+1}^m 1_{\{I_i=j_i\}} \right] \\ \geq \underline{\mu}_0^p \frac{1}{\beta^{p-1}} \int_D 1_B(u\boldsymbol{\ell} + \beta \aleph \boldsymbol{\vartheta}) u^{\frac{C_{m,M'}}{\beta}} d\boldsymbol{\vartheta} du, \end{aligned} \quad (5.10)$$

where $D = \{(u, \boldsymbol{\vartheta}) : 0 < u < \boldsymbol{\vartheta}(1) < \dots < \boldsymbol{\vartheta}(p-1) < 1\}$. Applying Lemma A.1 with $\Gamma = \beta \aleph$ and $\gamma = \frac{C_{m,M'}}{\beta}$ in (5.10), we get that there exists $\epsilon_1 > 0$ and a probability measure ν_1 such that, under $\{\mathbf{X}_0 = \mathbf{x}, I_i = j_i, i = 1, \dots, m-p\}$,

$$\mathbb{E}^{\mathbf{X}_{m-p}} \left[1_B(\boldsymbol{\lambda}_m) \prod_{i=m-p+1}^m 1_{\{I_i=j_i\}} \right] \geq \epsilon_1 \nu_1(B). \quad (5.11)$$

Moreover, it is important to note that ν_1 only depends on β and \aleph .

Using Inequalities (5.8) and (5.11), we get that

$$Q^m(\mathbf{x}, A \times B) \geq \epsilon_1 1_A(\mathbf{s}_o) \nu_1(B) \mathbb{E}^{\mathbf{x}} \left[\prod_{i=1}^{m-p} 1_{\{I_i=j_i\}} \right]. \quad (5.12)$$

Using that $\mathbf{x} \in \{1, \dots, K\}^q \times (0, M]^p$ and Corollary 5.2 with $g = 1$, we further have that

$$\mathbb{E}^{\mathbf{x}} \left[\prod_{i=1}^{m-p} 1_{\{I_i=j_i\}} \right] \geq \underline{\mu}_0^m \int_{\mathbb{R}_+^m} e^{-C_{m,M} \sum_{i=1}^m t_i} dt_m \dots dt_1 = \left(\frac{\underline{\mu}_0}{C_{m,M}} \right)^m.$$

From Inequality (5.12), we get that $Q^m(\mathbf{x}, A \times B) \geq \epsilon \nu(A \times B)$ for some positive constant ϵ not depending on \mathbf{x} and ν is the measure on \mathbf{X} defined by $\nu(A \times B) = 1_A(\mathbf{s}_o) \nu_1(B)$, which does neither depend on M nor K . In other words, C is a (m, ϵ, ν) -small set (see [33]).

To obtain that C is also an $(m+1, \epsilon, \nu)$ -small set (by possibly decreasing ϵ), we simply observe that we can carry out the same proof as above with m replaced by $m+1$ and the sequence of marks (j_1, \dots, j_m) replaced by $(0, j_1, \dots, j_m)$. \square

Proof of Theorem 3.1. In Proposition 5.3, the measure ν does not depend on the set C , which can be chosen to contain any arbitrary point \mathbf{x} in the state space X . Hence the chain is ϕ -irreducible with $\phi = \nu$. Moreover the set C is (m, ϵ, ν) -small and $(m+1, \epsilon, \nu)$ -small; hence, the chain is aperiodic, sse [33, Section 5.4.3]. These arguments hold for the Kernel \tilde{Q} and thus we obtain Theorem 3.1. \square

5.2 Proof of Proposition 3.2

Recall the definition of \mathbf{u} in (2.4). We shall further denote

$$1 \leq \underline{u} = \min_{i=1, \dots, p} \mathbf{u}(i) \leq \bar{u} = \max_{i=1, \dots, p} \mathbf{u}(i) < \infty. \quad (5.13)$$

The inequality $\underline{u} \geq 1$ follows from the fact that \aleph^k has non-negative entries for all $k \geq 1$.

Observe that, given the initial condition $\mathbf{x} = (\mathbf{s}, \boldsymbol{\ell}) \in \mathbb{Z}_+^q \times \mathbb{R}_+^q$, the hazard rate of $\Delta_1 = \min(\Delta_1^i, i = 0, \dots, p)$ reads

$$\text{Hr}(u) = \sum_{i=0}^p \text{Hr}_i(u) = \mu_0^0 + \mathbf{1}_p^T \boldsymbol{\mu}_0 + \mathbf{1}_p^T \boldsymbol{\ell} e^{-\beta u},$$

where we used (2.6). Moreover, for any $i \in \{0, 1, \dots, p\}$, we get that, given Δ_1 , the conditional probability that the first arrival's mark is i is given by

$$\mathbb{P}^{\mathbf{x}}(I_1 = i | \Delta_1) = \text{Hr}_i(\Delta_1) / \text{Hr}(\Delta_1). \quad (5.14)$$

Using (2.9), we have $\boldsymbol{\lambda}_1 = \boldsymbol{\ell} e^{-\beta \Delta_1} + \beta \aleph_{I_1} \mathbf{1}_{\{I_1 \neq 0\}}$, and thus

$$\begin{aligned} & [Q(\mathbf{1}_{\mathbb{Z}_+^q} \otimes V_{1, \gamma})](\mathbf{x}) \\ &= \mathbb{E}^{\mathbf{x}} \left[\exp(\gamma \mathbf{u}^T \boldsymbol{\ell} e^{-\beta \Delta_1} + \gamma \beta \mathbf{u}^T \aleph_{I_1} \mathbf{1}_{\{I_1 \neq 0\}}) \right] \\ &= \mathbb{E}^{\mathbf{x}} \left[\exp(\gamma \mathbf{u}^T \boldsymbol{\ell} e^{-\beta \Delta_1}) \sum_{i=0}^p \mathbb{P}^{\mathbf{x}}(I_1 = i | \Delta_1) e^{\gamma \beta (\mathbf{u}(i) - 1)} \right], \end{aligned}$$

by conditioning on Δ_1 and using that $\aleph^T \mathbf{u} = \mathbf{u} - \mathbf{1}_p$ (see (2.4)) and setting $\mathbf{u}(0) = 1$. Inserting Formula (5.14), we get

$$[Q(\mathbf{1}_{\mathbb{Z}_+^q} \otimes V_{1, \gamma})](\mathbf{x}) = \mathbb{E}^{\mathbf{x}} \left[\sum_{i=0}^p \exp(\gamma \mathbf{u}^T \boldsymbol{\ell} e^{-\beta \Delta_1} + \gamma \beta (\mathbf{u}(i) - 1)) \text{Hr}_i(\Delta_1) / \text{Hr}(\Delta_1) \right].$$

Using the hazard rate of Δ_1 under $\mathbb{P}^{\mathbf{x}}$, the corresponding probability density function is given by $u \mapsto \text{Hr}(u) e^{-\text{Ir}(u)}$ on \mathbb{R}_+ , where

$$\begin{aligned} \text{Ir}_i(t) &= \int_0^t \text{Hr}_i(u) \, du, \quad t \geq 0, \\ \text{Ir}(t) &= \sum_{i=1}^p \text{Ir}^i(t) = \mu_0^0 t + \mathbf{1}_p^T \boldsymbol{\mu}_0 t + \frac{\mathbf{1}_p^T \boldsymbol{\ell}}{\beta} (1 - e^{-\beta t}). \end{aligned} \quad (5.15)$$

Hence we get

$$[Q(\mathbf{1}_{\mathbb{Z}_+^q} \otimes V_{1, \gamma})](\mathbf{x}) = \int_0^\infty e^{\gamma \mathbf{u}^T \boldsymbol{\ell} e^{-\beta u} - \text{Ir}(u)} \left[\sum_{i=0}^p e^{\gamma \beta (\mathbf{u}(i) - 1)} \text{Hr}_i(u) \right] du.$$

Using that $e^{\gamma \beta (\mathbf{u}(0) - 1)} \leq e^{\gamma \beta (\mathbf{u}(i) - 1)}$ for all $i = 1, \dots, p$ and the particular form of $\text{Hr}(0)$ and $\text{Hr}(i)$ in (2.6), we easily obtain that

$$\sum_{i=0}^p e^{\gamma \beta (\mathbf{u}(i) - 1)} \text{Hr}_i(u) \leq \left[\mu_0^0 + \sum_{i=1}^p e^{\gamma \beta (\mathbf{u}(i) - 1)} (\boldsymbol{\mu}_0(i) + \boldsymbol{\ell}(i) e^{-u\beta}) \right].$$

Consequently, we have

$$\begin{aligned} & \frac{1}{V_{1,\gamma}(\boldsymbol{\ell})} [Q(1_{\mathbb{Z}_+^q} \otimes V_{1,\gamma})](\mathbf{x}) \\ & \leq \left(\mu_0^0 + \sum_{i=1}^p e^{\gamma\beta(\mathbf{u}^{(i)}-1)} \boldsymbol{\mu}_0(i) \right) \frac{1}{V_{1,\gamma}(\boldsymbol{\ell})} \int_0^\infty e^{\gamma\mathbf{u}^T \boldsymbol{\ell} e^{-\beta u} - \text{Ir}(u)} \mathrm{d}u \\ & \quad + \left(\sum_{i=1}^p e^{\gamma\beta(\mathbf{u}^{(i)}-1)} \boldsymbol{\ell}(i) \right) \frac{1}{V_{1,\gamma}(\boldsymbol{\ell})} \int_0^\infty e^{\gamma\mathbf{u}^T \boldsymbol{\ell} e^{-\beta u} - \text{Ir}(u) - u\beta} \mathrm{d}u. \end{aligned} \quad (5.16)$$

Using (5.15), we have

$$\begin{aligned} \frac{1}{V_{1,\gamma}(\boldsymbol{\ell})} \int_0^\infty e^{\gamma\mathbf{u}^T \boldsymbol{\ell} e^{-\beta u} - \text{Ir}(u)} \mathrm{d}u &= \int_0^\infty e^{(\gamma\mathbf{u}^T \boldsymbol{\ell} + \frac{\mathbf{1}_p^T \boldsymbol{\ell}}{\beta})(e^{-\beta u} - 1) - (\mu_0^0 + \mathbf{1}_p^T \boldsymbol{\mu}_0)u} \mathrm{d}u \\ &\rightarrow 0 \quad \text{as } \mathbf{1}_p^T \boldsymbol{\ell} \rightarrow \infty, \end{aligned} \quad (5.17)$$

where we used Lemma A.2 with $a = \gamma\mathbf{u}^T \boldsymbol{\ell} + \frac{\mathbf{1}_p^T \boldsymbol{\ell}}{\beta}$ and $\beta' = \mu_0^0 + \mathbf{1}_p^T \boldsymbol{\mu}_0$.

Similarly, applying Lemma A.2 with $\beta' = \beta + \mu_0^0 + \mathbf{1}_p^T \boldsymbol{\mu}_0$, we obtain

$$\begin{aligned} & \frac{1}{V_{1,\gamma}(\boldsymbol{\ell})} \int_0^\infty e^{\gamma\mathbf{u}^T \boldsymbol{\ell} e^{-\beta u} - \text{Ir}(u) - u\beta} \mathrm{d}u \\ &= \int_0^\infty \exp \left((\gamma\mathbf{u}^T \boldsymbol{\ell} + \frac{\mathbf{1}_p^T \boldsymbol{\ell}}{\beta})(e^{-\beta u} - 1) - (\beta + \mu_0^0 + \mathbf{1}_p^T \boldsymbol{\mu}_0)u \right) \mathrm{d}u \\ &\sim \frac{1}{\gamma\beta\mathbf{u}^T \boldsymbol{\ell} + \mathbf{1}_p^T \boldsymbol{\ell}} \quad \text{as } \mathbf{1}_p^T \boldsymbol{\ell} \rightarrow \infty. \end{aligned} \quad (5.18)$$

Therefore, inserting (5.17) and (5.18) in (5.16), we get

$$\limsup_{\mathbf{1}_p^T \boldsymbol{\ell} \rightarrow \infty} \frac{1}{V_{1,\gamma}(\boldsymbol{\ell})} [Q(1_{\mathbb{Z}_+^q} \otimes V_{1,\gamma})](\mathbf{x}) \leq \sup_{\boldsymbol{\ell} \in (0,\infty)^p} \frac{\sum_{i=1}^p e^{\gamma\beta(\mathbf{u}^{(i)}-1)} \boldsymbol{\ell}(i)}{\gamma\beta\mathbf{u}^T \boldsymbol{\ell} + \mathbf{1}_p^T \boldsymbol{\ell}}.$$

Now observe that, using (5.13), we have

$$0 < \underline{u} \leq \mathbf{u}^T \boldsymbol{\ell} / \mathbf{1}_p^T \boldsymbol{\ell} \leq \bar{u} < \infty,$$

and thus, for $\gamma > 0$ small enough, using a Taylor expansion of the exponential function at the origin, we have

$$\begin{aligned} \frac{\sum_{i=1}^p e^{\gamma\beta(\mathbf{u}^{(i)}-1)} \boldsymbol{\ell}(i)}{\beta\gamma\mathbf{u}^T \boldsymbol{\ell} + \mathbf{1}_p^T \boldsymbol{\ell}} &\leq \frac{\mathbf{1}_p^T \boldsymbol{\ell} + \gamma\beta(\mathbf{u}^T \boldsymbol{\ell} - \mathbf{1}_p^T \boldsymbol{\ell}) + C\gamma^2 \mathbf{1}_p^T \boldsymbol{\ell}}{\beta\gamma\mathbf{u}^T \boldsymbol{\ell} + \mathbf{1}_p^T \boldsymbol{\ell}} \\ &\leq 1 - \beta\gamma \frac{\mathbf{1}_p^T \boldsymbol{\ell}}{\beta\gamma\mathbf{u}^T \boldsymbol{\ell} + \mathbf{1}_p^T \boldsymbol{\ell}} + C'\gamma^2, \\ &\leq 1 - \beta\gamma \frac{1}{1 + \beta\gamma\bar{u}} + C'\gamma^2 = 1 - \beta\gamma + O(\gamma^2), \end{aligned}$$

where C, C' do not depend on $\boldsymbol{\ell}$. Hence taking $\gamma_1 > 0$ small enough, we have for all $\gamma \in (0, \gamma_1]$,

$$\theta(\gamma) = \sup_{\boldsymbol{\ell} \in (0,\infty)^p} \frac{\sum_{i=1}^p e^{\gamma\beta(\mathbf{u}^{(i)}-1)} \boldsymbol{\ell}(i)}{\beta\gamma\mathbf{u}^T \boldsymbol{\ell} + \mathbf{1}_p^T \boldsymbol{\ell}} < 1.$$

Hence choosing such a γ and setting $\theta = \theta(\gamma)$, we get (3.2) for $\boldsymbol{\ell}$ out of $(0, M]^p$ provided that M large enough. Since $V_{1,\gamma}$ is bounded on $(0, M]^p$, this achieves the proof.

5.3 Proof of Corollary 3.4

By [5, Theorem (1)], using the definitions and notation of Section 2.1, we have

$$\lim_{T \rightarrow \infty} \frac{1}{T} \mathbf{N}'(1_{[0,T]} \otimes \mathbf{w}) = \overline{\mathbf{w}}^T (\text{Id}_p - \aleph)^{-1} \boldsymbol{\mu}_0 \quad \text{a.s.} \quad (5.19)$$

On the other hand by the law of large numbers for Harris recurrent chains (see [33, Theorem 17.1.7]), we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \mathbf{w}_o(\check{I}_k) = \check{\pi}(\mathbf{w}_o \otimes 1_{\mathbb{R}_+^p}) \quad \text{a.s.}[P_*]. \quad (5.20)$$

Here $[P_*]$ means that the result holds for any initial distribution on $(\check{I}_0, \check{\boldsymbol{\lambda}}_0)$. In particular taking $\check{\boldsymbol{\lambda}}_0 = 0$, we get that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbf{N}'(1_{[0, \check{T}_n]} \otimes \mathbf{w}) = \check{\pi}(\mathbf{w}_o \otimes 1_{\mathbb{R}_+^p}), \quad (5.21)$$

where \check{T}_n is defined as T_n in Section 2.3 but for the unconstrained chain. Note that we used that $\mathbf{w}_o(0) = 0$, so that only the marks in $\{1, \dots, p\}$ are counted in the sum of the left-hand side of (5.20). Observe now that under the event $\{\check{\boldsymbol{\lambda}}_0 = 0\}$, we have

$$\mathfrak{n}_0([0, \check{T}_n]) + \mathbf{N}'(1_{[0, \check{T}_n]} \otimes 1_{\{1, \dots, p\}}) = n.$$

(Recall that \mathfrak{n}_0 is an independent homogeneous PPP which corresponds to the events with marks equal to 0). Since (\check{T}_n) is an increasing sequence, we get that it diverges a.s. and thus, applying (5.19) with $\mathbf{w} = 1_{\{1, \dots, p\}}$, we have, under the event $\{\check{\boldsymbol{\lambda}}_0 = 0\}$,

$$\lim_{n \rightarrow \infty} \frac{1}{\check{T}_n} \mathbf{N}'(1_{[0, \check{T}_n]} \otimes 1_{\{1, \dots, p\}}) = \mathbf{1}_p^T (\text{Id}_p - \aleph)^{-1} \boldsymbol{\mu}_0 \quad \text{a.s.}$$

The last two displays yield, under the event $\{\check{\boldsymbol{\lambda}}_0 = 0\}$,

$$\lim_{n \rightarrow \infty} \frac{\check{T}_n}{n} = \frac{1}{\boldsymbol{\mu}_0^0 + \mathbf{1}_p^T (\text{Id}_p - \aleph)^{-1} \boldsymbol{\mu}_0}.$$

Applying (5.19), (5.21) and the last display, we get (3.4).

5.4 Proof of Theorem 3.6

The case (i) follows from Proposition 3.2.

We next consider the case (ii) so that $\psi(\{s_2\} \times (0, \infty)^p) > 0$ for some $s_2 \geq s^*$. By (3.8) there exists $i \in \{1, \dots, p\}$ such that $J(i) > 0$. Thus, for any $m \geq 1$, the constant sequence (i, \dots, i) of length m belongs to $\mathcal{A}_m(s_2)$ (see Definition 1). It follows that $\psi(\{s_2 + mJ(i)\} \times (0, \infty)^p) > 0$ for all $m \geq 1$. We shall prove that for all $s' \geq 1$ and $M > 0$,

$$\liminf_{s \rightarrow \infty} \inf_{\ell \in (0, M]^p} \mathbb{P}^{(s, \ell)} \left(\inf_{n \geq 1} \mathbf{S}_n > s' \right) > 0. \quad (5.22)$$

Take s' such that $\psi(\{s'\} \times (0, \infty)^p) > 0$. Define $s = s_2 + mJ(i)$ and take m and $M > 0$ large enough so that $\psi(\{s\} \times (0, M)^p) > 0$ and $\mathbb{P}^{(s, \ell)} \left(\inf_{n \geq 1} \mathbf{S}_n > s' \right) > 0$ for all $\ell \in (0, M]^p$. We thus have that the probability that the chain $\{\mathbf{X}_k, k \geq 1\}$ avoids $\{s'\} \times (0, \infty)^p$ starting from anywhere in $\{s\} \times (0, M]^p$ is positive. Since these two sets have positive ψ -measure, we get by [33, Theorem 8.3.6] that Q is transient.

Hence it only remains to show that (5.22) holds. Applying Corollary 3.4 and Condition (3.8), we have

$$\lim_{n \rightarrow \infty} M_n = \infty \quad \text{a.s.}[P_*], \quad \text{with} \quad M_n = \sum_{k=1}^n J_o(\check{I}_k).$$

Hence $U = \inf_{n \geq 1} M_n$ is valued in \mathbb{R} a.s. $[P_*]$ (it equals $-\infty$ with probability 0). Let $\epsilon \in (0, 1)$. Now under the event $\{S_0 = s\}$, we have that $S_n = s + M_n$ as long as S_k does not hit the constraint set $\cup_{i=1}^p A_i$ for $k = 1, \dots, n$.

For all $s \geq s^*$, the event $\{\inf_{n \geq 0} S_n \geq s^*\}$ coincides with the event $\{U + s \geq s^*\}$. Hence, under this latter event we have $\inf_{n \geq 1} S_n = s + U$. Finally, we get that, for all $s \geq s^*$ and all $\ell \in \mathbb{R}_+^p$,

$$\mathbb{P}^{(s, \ell)} \left(\inf_{n \geq 1} S_n > s' \right) \geq \mathbb{P}^\ell (U > (s' \vee s^*) - s). \quad (5.23)$$

(Recall that U only depends on the unconstrained chain $(\check{I}_n, \check{\lambda}_n)$, $n \geq 0$.) Let us denote, for all $k \geq 1$, $U_k = \inf_{n \geq k} M_n$ so that $U = M_1 \wedge \dots \wedge M_k \wedge U_{k+1}$. Since $M_1, \dots, M_k \geq -k\bar{J}$, we have, for all $s_1 > k\bar{J}$,

$$w_{s_1}(\ell) := \mathbb{P}^\ell (U > -s_1) = \mathbb{P}^\ell (U_{k+1} > -s_1) = [\check{Q}^k(w_{s_1})](\ell).$$

Observe that w_{s_1} is a function valued in $[0, 1]$. By Proposition 3.3 and [33, Section 15]), \check{Q} admits a stationary distribution $\tilde{\pi}$ and there are constants $\gamma, C > 0$ and $\theta \in (0, 1)$ not depending on s_1 such that, for all $\ell, \ell_o \in \mathbb{R}_+^p$ and $k \geq 1$,

$$[\check{Q}^k(w_{s_1})](\ell) \geq [\check{Q}^k(w_{s_1})](\ell_o) - C\theta^k (V_{1, \gamma}(\ell) + V_{1, \gamma}(\ell_o)).$$

Now, fix some $\ell_o \in \mathbb{R}_+^p$ and $M > 0$. Choose k large enough so that

$$c = C\theta^k (V_{1, \gamma}(M\mathbf{1}_p) + V_{1, \gamma}(\ell_o)) < 1/2.$$

Then we may choose s_1 large enough so that $s_1 > k\bar{J}$ and

$$[\check{Q}^k(w_{s_1})](\ell_o) = \mathbb{P}^{\ell_o} (U_{k+1} > -s_1) \geq \mathbb{P}^{\ell_o} (U > -s_1) \geq 1/2.$$

The last four displays yield that, for all $\ell \in (0, M]^p$, and all s_1 large enough,

$$\mathbb{P}^\ell (U > -s_1) \geq 1/2 - c > 0.$$

This and (5.23) yields (5.22), which concludes the proof.

5.5 Proof of Theorem 3.7 : induction on q

We have already shown that the chains Q and \tilde{Q} are ψ -irreducible and aperiodic and exhibited petite sets, see Theorem 3.1. To obtain a drift condition on the constrained case $q \geq 1$, we shall reason by induction on the number of constraints q . We shall rely on the transition kernels $\tilde{Q}^{(-\mathcal{J})}$ introduced in Section 2.3 for subsets \mathcal{J} in $\{1, \dots, q\}$. For such a given set \mathcal{J} , we shall further denote by $\{(\check{I}_n, \check{\mathbf{S}}_n, \check{\lambda}_n), n \geq 0\}$ the chain with transition kernel $\tilde{Q}^{(-\mathcal{J})}$ which starts at the same state as $(I_0, \mathbf{S}_0, \lambda_0)$ and, for all $\mathbf{x} \in \mathbb{Z}_+^{q-\#\mathcal{J}} \times \mathbb{R}_+^p$, by $\mathbb{E}^{(-\mathcal{J}), \mathbf{x}}[\dots]$ the expectation under $\{(\check{\mathbf{S}}_0, \check{\lambda}_0) = \mathbf{x}\}$. (The fact that this expectation only depends on \mathbf{x} is explained in Remark 1.)

Proposition 5.5. *Let $p, q \geq 1$ and define the transition kernels Q on the space $\mathbb{Z}_+^q \times \mathbb{R}_+^p$ as in Section 2.3. Suppose that \aleph is invertible and that Assumption 1 holds. For any non-empty $\mathcal{J} \subseteq \{1, \dots, q\}$, let $\{(\check{I}_n, \check{\mathbf{S}}_n, \check{\boldsymbol{\lambda}}_n), n \geq 0\}$ be a Markov chain on the space $\{0, \dots, p\} \times \mathbb{Z}_+^{q-\#\mathcal{J}} \times \mathbb{R}_+^p$ with transition kernel $\tilde{Q}^{(-\mathcal{J})}$ defined in Section 2.3, and suppose that it satisfies the following condition.*

(C) *For all $\gamma_1 > 0$ small enough, there exists $\gamma_0^* > 0$ such that for all $\gamma_0 \in (0, \gamma_0^*]$,*

$$\lim_{m \rightarrow \infty} \sup_{(\mathbf{s}, \boldsymbol{\ell}) \in \mathbb{Z}_+^{q-\#\mathcal{J}} \times \mathbb{R}_+^p} \mathbb{E}^{(-\mathcal{J}), (\mathbf{s}, \boldsymbol{\ell})} \left[e^{\gamma_0 \sum_{k=1}^m \mathbf{w}^{(\check{I}_k)} V_{1, \gamma_1}(\check{\boldsymbol{\lambda}}_m - \boldsymbol{\ell})} \right] = 0, \quad (5.24)$$

where we denoted \mathbf{w} is defined in (5.42).

Then, for all $\gamma_1 > 0$ small enough, there exists $\gamma_0^* > 0$ such that for all $\gamma_0 \in (0, \gamma_0^*]$, there exist $\theta \in (0, 1)$, $b > 0$ and $m \geq 1$, such that for any initial condition $\mathbf{x} \in \mathbf{X}$,

$$Q^m(V_{0, \gamma_0}^{\otimes q} \otimes V_{1, \gamma_1})(\mathbf{x}) \leq \theta (V_{0, \gamma_0}^{\otimes q} \otimes V_{1, \gamma_1})(\mathbf{x}) + b, \quad (5.25)$$

where $V_{1, \gamma}$ and $V_{0, \gamma}$ are defined in (3.3) and (3.6) respectively.

Proof. Take $\mathbf{x} = (\mathbf{s}, \boldsymbol{\ell}) \in \mathbf{X}$. We already have the drift condition (3.2) of Proposition 3.2 (which holds for some $\gamma > 0$ and thus, by Jensen inequality for any $\gamma_1 \in (0, \gamma)$). By induction, for all small enough $\gamma_1 > 0$, there exists $b_1 > 0$ and $\theta_1 > 0$, such that, for all $m \geq 1$,

$$\begin{aligned} Q^m(1_{\mathbb{Z}_+^q} \otimes V_{1, \gamma_1})(\mathbf{x}) &\leq \theta_1^m V_{1, \gamma_1}(\boldsymbol{\ell}) + b_1 (1 + \theta_1 + \dots + \theta_1^m) \\ &\leq \theta_1^m V_{1, \gamma_1}(\boldsymbol{\ell}) + b_1 / (1 - \theta_1). \end{aligned}$$

By (2.7) and (5.3) we have, in the event $\{\mathbf{S}_0 = \mathbf{s}\}$, $V_{0, \gamma_0}^{\otimes q}(\mathbf{S}_1) \leq e^{\gamma_0 \bar{J}} V_{0, \gamma_0}^{\otimes q}(\mathbf{s})$, then we get $V_{0, \gamma_0}^{\otimes q}(\mathbf{S}_m) \leq e^{\gamma_0 \bar{J} m} V_{0, \gamma_0}^{\otimes q}(\mathbf{S}_0)$. Hence, we obtain, for any $\gamma_0 > 0$,

$$\begin{aligned} Q^m(V_{0, \gamma_0}^{\otimes q} \otimes V_{1, \gamma_1})(\mathbf{x}) &\leq e^{\gamma_0 \bar{J} m} V_{0, \gamma_0}^{\otimes q}(\mathbf{s}) [\theta_1^m V_{1, \gamma_1}(\boldsymbol{\ell}) + b_1 / (1 - \theta_1)] \\ &\leq \left(e^{\gamma_0 \bar{J}} \theta_1 \right)^m (V_{0, \gamma_0}^{\otimes q} \otimes V_{1, \gamma_1})(\mathbf{x}) + \frac{b_1 e^{\gamma_0 \bar{J} m}}{1 - \theta_1} V_{0, \gamma_0}^{\otimes q}(\mathbf{s}). \end{aligned} \quad (5.26)$$

Observe that, for a given $\theta_1 < 1$, we can choose $\gamma_0 > 0$ small enough so that $e^{\gamma_0 \bar{J}} \theta_1 < 1$. It follows that, choosing such a γ_0 , for all $m \geq 1$ and $K \geq 1$, the bound (5.26) implies (5.25) for all $\mathbf{x} = (\mathbf{s}, \boldsymbol{\ell}) \in \mathbf{X}$ such that $V_{0, \gamma_0}^{\otimes q}(\mathbf{s}) \leq K$. To address the case $V_{0, \gamma_0}^{\otimes q}(\mathbf{s}) > K$, we shall prove the following assertion.

(D) For all $\gamma_1 > 0$ small enough, there exists $\gamma_0^* > 0$ such that for all $\gamma_0 \in (0, \gamma_0^*]$, there exist $m \geq 1$, $K > 0$, $b \geq 0$ and $\theta \in (0, 1)$ such that (5.25) holds for all $\mathbf{x} = (\mathbf{s}, \boldsymbol{\ell}) \in \mathbf{X}$ satisfying $V_{0, \gamma_0}^{\otimes q}(\mathbf{s}) > K$.

To conclude the proof, we now show that Assertion (D) holds. Let $m \geq 1$ and $\mathbf{x} = (\mathbf{s}, \boldsymbol{\ell}) \in \mathbf{X}$. Observe that, by (2.7), for all $\gamma_0 > 0$, we have, in the event $\{\mathbf{X}_0 = \mathbf{x}\}$,

$$V_{0, \gamma_0}^{\otimes q}(\mathbf{S}_m) = V_{0, \gamma_0}^{\otimes q}(\mathbf{s}) e^{\gamma_0 \sum_{k=1}^m \sum_{j=1}^q \mathbf{J}_j(I_k)} = V_{0, \gamma_0}^{\otimes q}(\mathbf{s}) e^{\gamma_0 \sum_{k=1}^m \mathbf{w}^{(I_k)}}.$$

Hence we have, for all $\mathbf{x} = (\mathbf{s}, \boldsymbol{\ell}) \in \mathbf{X}$,

$$\begin{aligned} Q^m(V_{0, \gamma_0}^{\otimes q} \otimes V_{1, \gamma_1})(\mathbf{x}) &= V_{0, \gamma_0}^{\otimes q}(\mathbf{s}) \mathbb{E}^{\mathbf{x}} \left[e^{\gamma_0 \sum_{k=1}^m \mathbf{w}^{(I_k)}} V_{1, \gamma_1}(\boldsymbol{\lambda}_m) \right] \\ &= (V_{0, \gamma_0}^{\otimes q} \otimes V_{1, \gamma_1})(\mathbf{x}) \mathbb{E}^{\mathbf{x}} \left[e^{\gamma_0 \sum_{k=1}^m \mathbf{w}^{(I_k)}} V_{1, \gamma_1}(\boldsymbol{\lambda}_m - \boldsymbol{\ell}) \right]. \end{aligned} \quad (5.27)$$

Now the idea of the proof relies on the fact that if $V_{0,\gamma_0}^{\otimes q}(\mathbf{S}_0)$ is large enough, then it will take some time for all the components j in a set \mathcal{J} (with cardinal $\#\mathcal{J}$ at least 1) of the multivariate spread to reach their constraint set

$$\mathbf{A}(j) = \bigcup_{i=1,\dots,p} \mathbf{A}_i(j). \quad (5.28)$$

Up to this time the process defined by $\{\mathbf{Y}_n^{(-\mathcal{J})} = (I_n, \mathbf{S}_n^{(-\mathcal{J})}, \boldsymbol{\lambda}_n), n \geq 0\}$, where $\mathbf{S}_n^{(-\mathcal{J})}$ is obtained by removing the j -th component of \mathbf{S}_n for all $j \in \mathcal{J}$, behaves as a Markov chain with kernel $\tilde{Q}^{(-\mathcal{J})}$ (defined as \tilde{Q} but without the j -th constraints with $j \in \mathcal{J}$, see above). More precisely, we use the following coupling argument. For a positive integer m , we denote

$$s_m^*(j) = 1 + m\bar{J} + \max(\mathbf{A}(j)). \quad (5.29)$$

We can then define a Markov chain $\{(\check{I}_n, \check{\mathbf{S}}_n, \check{\boldsymbol{\lambda}}_n), n \geq 0\}$ with transition kernel $\tilde{Q}^{(-\mathcal{J})}$ and such that in the event $\{\mathbf{S}_0(j) \geq s_m^*(j), j \in \mathcal{J}\}$, the sequence $\{(\check{I}_n, \check{\mathbf{S}}_n, \check{\boldsymbol{\lambda}}_n), n = 0, 1, \dots, m\}$ coincides with $\{(I_n, \mathbf{S}_n^{(-\mathcal{J})}, \boldsymbol{\lambda}_n), n = 0, 1, \dots, m\}$. Then, by construction, for all $\mathbf{x} = (\mathbf{s}, \boldsymbol{\ell})$ such that $\mathbf{s}(j) \geq s_m^*(j)$ for all $j \in \mathcal{J}$, denoting $\mathbf{x}' = (\mathbf{s}^{(-\mathcal{J})}, \boldsymbol{\ell})$, we have

$$\mathbb{E}^{\mathbf{x}} \left[e^{\gamma_0 \sum_{k=1}^m \mathbf{w}(I_k)} V_{1,\gamma_1}(\boldsymbol{\lambda}_m - \boldsymbol{\ell}) \right] = \mathbb{E}^{(-\mathcal{J}), \mathbf{x}'} \left[e^{\gamma_0 \sum_{k=1}^m \mathbf{w}(\check{I}_k)} V_{1,\gamma_1}(\boldsymbol{\lambda}_m - \boldsymbol{\ell}) \right]. \quad (5.30)$$

Now, we take $K > 0$ such that $\log K \geq p\gamma_0 \max_{j=1,\dots,q} s_m^*(j)$. We let \mathcal{J} denote the set of all indices $j \in \{1, \dots, q\}$ such that $\mathbf{s}(j) \geq s_m^*(j)$. The set \mathcal{J} is not empty since $V_{0,\gamma_0}^{\otimes q}(\mathbf{s}) > K$ implies $\max(\mathbf{s}) \geq \max(s_m^*)$. Hence, Equality (5.30) applies for all $\mathbf{x} = (\mathbf{s}, \boldsymbol{\ell}) \in \mathbf{X}$ satisfying $V_{0,\gamma_0}^{\otimes q}(\mathbf{s}) > K$. For γ_0^* as in Condition (C) (we can choose it independently of \mathcal{J} since there is a finite number of such subsets), we obtain Assertion (D). \square

Condition (C) of Proposition 5.5 needs to be simplified. A first step in this direction is given by the following lemma.

Lemma 5.6. *Let $\{(I_n, \mathbf{S}_n, \boldsymbol{\lambda}_n), n \geq 0\}$ be the Markov chain on the space $\{0, 1, \dots, p\} \times \mathbb{Z}_+^q \times \mathbb{R}_+^p$ with transition kernel \tilde{Q} defined in Section 2.3. Let $\mathbf{w} : \{0, 1, \dots, p\} \rightarrow \mathbb{R}$. Suppose that the following assertions hold.*

(E-1) *For all $\gamma' > 0$ small enough, there exists $\gamma^* > 0$ such that, for all $\gamma \in (0, \gamma^*]$, \tilde{Q} is $(1_{\{0,1,\dots,p\}} \otimes V_{0,\gamma}^{\otimes q} \otimes V_{1,\gamma'})$ -geometrically ergodic, where $V_{0,\gamma}$ and $V_{1,\gamma'}$ are defined in (3.6) and (3.3) respectively.*

(E-2) *The stationary distribution $\tilde{\pi}$ of \tilde{Q} satisfies $\tilde{\pi} \left[\mathbf{w} \otimes 1_{\mathbb{Z}_+^q \times \mathbb{R}_+^p} \right] < 0$.*

Then, for all $\gamma_1 > 0$ small enough, there are positive constants γ, ρ and γ_0^ such that, for all $\gamma_0 \in (0, \gamma_0^*]$,*

$$\sup_{\mathbf{x} \in \mathbf{X}} \sup_{m \geq 1} \frac{\mathbb{E}^{\mathbf{x}} \left[e^{\gamma_0 \sum_{k=1}^m \{\mathbf{w}(I_k) + \rho\}} (V_{0,\gamma}^{\otimes q} \otimes V_{1,\gamma_1})(\mathbf{X}_m) \right]}{(V_{0,\gamma}^{\otimes q} \otimes V_{1,\gamma_1})(\mathbf{x})} < \infty. \quad (5.31)$$

Proof. Using Assumption (E-1), for any $\gamma' > 0$ small enough, we may choose $\gamma, \gamma_1 \in (0, \gamma^*]$ such that \tilde{Q} is $1_{\{0,\dots,p\}} \otimes V_{0,\gamma}^{\otimes q} \otimes V_{1,\gamma_1}$ -geometrically ergodic. Let $\bar{w} = \max_{0 \leq i \leq p} \mathbf{w}(i)$. Then, for any $\gamma^* > 0$, there $\gamma, \gamma_1 > 0$ small enough, we have for all $j \geq 1$ and $\mathbf{x} = (\mathbf{s}, \boldsymbol{\ell}) \in \mathbf{X}$,

$$\begin{aligned} \mathbb{E}^{\mathbf{x}} \left[e^{\gamma_0 \sum_{k=1}^j \mathbf{w}(I_k)} (V_{0,\gamma}^{\otimes q} \otimes V_{1,\gamma_1})(\mathbf{X}_j) \right] &\leq e^{\gamma_0 \bar{w} j} \mathbb{E}^{\mathbf{x}} \left[(V_{0,\gamma}^{\otimes q} \otimes V_{1,\gamma_1})(\mathbf{X}_j) \right] \\ &\leq C e^{\gamma_0 \bar{w} j} (V_{0,\gamma}^{\otimes q} \otimes V_{1,\gamma_1})(\mathbf{x}), \end{aligned} \quad (5.32)$$

where, in the last inequality, we used (E-1) and thus the constant C only depends on γ, γ_1 . Let now $m, l \geq 1$ and write $m = nl + j$ with $j \in \{0, \dots, l\}$. Then, using (5.32), the Hölder inequality and (5.32) again, we get

$$\begin{aligned}
& \mathbb{E}^{\mathbf{x}} \left[e^{\gamma_0 \sum_{k=1}^m \mathbf{w}(I_k)} (V_{0,\gamma}^{\otimes q} \otimes V_{1,\gamma_1})(\mathbf{X}_m) \right] \\
& \leq C e^{\gamma_0 \bar{w} j} \mathbb{E}^{\mathbf{x}} \left[e^{\gamma_0 \sum_{k=1}^{nl} \mathbf{w}(I_k)} (V_{0,\gamma}^{\otimes q} \otimes V_{1,\gamma_1})(\mathbf{X}_{nl}) \right] \\
& = C e^{\gamma_0 \bar{w} j} \mathbb{E}^{\mathbf{x}} \left[\left\{ \prod_{i=1}^l e^{\gamma_0 \sum_{k=0}^{n-1} \mathbf{w}(I_{kl+i})} \right\} (V_{0,\gamma}^{\otimes q} \otimes V_{1,\gamma_1})(\mathbf{X}_{nl}) \right] \\
& \leq C e^{\gamma_0 \bar{w} j} \left(\prod_{i=1}^l \mathbb{E}^{\mathbf{x}} \left[e^{l \gamma_0 \sum_{k=0}^{n-1} \mathbf{w}(I_{kl+i})} (V_{0,\gamma}^{\otimes q} \otimes V_{1,\gamma_1})(\mathbf{X}_{nl}) \right] \right)^{1/l} \\
& \leq C e^{\gamma_0 \bar{w} j} \left(\prod_{i=1}^l C e^{\gamma_0 \bar{w} (l-i)} \mathbb{E}^{\mathbf{x}} \left[e^{l \gamma_0 \sum_{k=0}^{n-1} \mathbf{w}(I_{kl+i})} (V_{0,\gamma}^{\otimes q} \otimes V_{1,\gamma_1})(\mathbf{X}_{(n-1)l+i}) \right] \right)^{1/l} \\
& \leq C^2 e^{2\gamma_0 \bar{w} l} \left(\prod_{i=1}^l \mathbb{E}^{\mathbf{x}} \left[e^{l \gamma_0 \sum_{k=0}^{n-1} \mathbf{w}(I_{kl+i})} (V_{0,\gamma}^{\otimes q} \otimes V_{1,\gamma_1})(\mathbf{X}_{(n-1)l+i}) \right] \right)^{1/l}. \tag{5.34}
\end{aligned}$$

Applying [33, Theorem 15.0.1], \tilde{Q} has a unique invariant probability measure $\tilde{\pi}$ and there exists $R > 0$ and $\kappa > 0$ such that, for all $i \in \{0, \dots, p\}$, $\mathbf{s} \in \mathbb{Z}_+^q$ and $\boldsymbol{\ell} \in \mathbb{R}_+^p$ and all positive integer m ,

$$\|\tilde{Q}^m((i, \mathbf{s}, \boldsymbol{\ell}), \cdot) - \tilde{\pi}\|_{(\gamma, \gamma_1)} \leq R e^{-\kappa m} V_{0,\gamma}^{\otimes q}(\mathbf{s}) V_{1,\gamma_1}(\boldsymbol{\ell}), \tag{5.35}$$

where for any signed measure ξ on $\{0, \dots, p\} \times \mathbb{Z}_+^q \times \mathbb{R}_+^p$, we set

$$\|\xi\|_{\gamma, \gamma_1} = \sup \{ \xi(g) : \|g / (1_{\{0, \dots, p\}} \otimes V_{0,\gamma}^{\otimes q} \otimes V_{1,\gamma_1})\|_{\infty} \leq 1 \}.$$

Define

$$w^*(i, \mathbf{s}, \boldsymbol{\ell}) = e^{l \gamma_0 \mathbf{w}(i)} V_{0,\gamma}(\mathbf{s}) V_{1,\gamma_1}(\boldsymbol{\ell}),$$

for all $(i, \mathbf{s}, \boldsymbol{\ell}) \in \{0, \dots, p\} \times \mathbb{Z}_+^q \times \mathbb{R}_+^p$. Then

$$\|w^* / (1_{\{0, \dots, p\}} \otimes V_{0,\gamma}^{\otimes q} \otimes V_{1,\gamma_1})\|_{\infty} \leq e^{l \gamma_0 \bar{w}}.$$

Thus, (5.35) yields that, for all $(i, \mathbf{s}, \boldsymbol{\ell}) \in \{0, \dots, p\} \times \mathbb{Z}_+^q \times \mathbb{R}_+^p$,

$$\tilde{Q}^l((i, \mathbf{s}, \boldsymbol{\ell}), w^*) \leq \tilde{\pi}(w^*) + R e^{-\kappa l} e^{l \gamma_0 \bar{w}} V_{0,\gamma}^{\otimes q}(\mathbf{s}) V_{1,\gamma_1}(\boldsymbol{\ell}) \tag{5.36}$$

Now, by dominated convergence, we have, as $a \rightarrow 0$,

$$\tilde{\pi} \left(a^{-1} \left[\exp(a \mathbf{w} \otimes 1_{\mathbb{Z}_+^q \times \mathbb{R}_+^p}) - 1 \right] \right) \rightarrow \tilde{\pi} \left(\mathbf{w} \otimes 1_{\mathbb{Z}_+^q \times \mathbb{R}_+^p} \right),$$

which is negative by Condition (E-2). Hence there exists $a > 0$ such that

$$b := \tilde{\pi} \left(\exp(a \mathbf{w} \otimes 1_{\mathbb{Z}_+^q \times \mathbb{R}_+^p}) \right) < 1. \tag{5.37}$$

Now, by dominated convergence, we may decrease γ^* so that $\gamma, \gamma_1 \in (0, \gamma^*]$ implies $\tilde{\pi}(w^*) \leq b^{1/2}$ for all l and γ_0 satisfying $l \gamma_0 = a$. Then we have for these a and γ, γ_1 , which set the values of R and κ , and for all l and γ_0 satisfying $l \gamma_0 = a$,

$$\tilde{Q}^l((i, \mathbf{s}, \boldsymbol{\ell}), w^*) \leq \left(b^{1/2} + R e^{-\kappa l} e^{a \bar{w}} \right) V_{0,\gamma}^{\otimes q}(\mathbf{s}) V_{1,\gamma_1}(\boldsymbol{\ell}).$$

We now denote by L some integer such that $b^{1/2} + R e^{-\kappa l} e^{a\bar{w}} \leq b^{1/4}$ for all $l \geq L$. Then, provided that $l\gamma_0 = a$ and $l \geq L$, we get

$$\tilde{Q}^l((i, \mathbf{s}, \boldsymbol{\ell}), w^*) \leq b^{1/4} V_{0,\gamma}^{\otimes q}(\mathbf{s}) V_{1,\gamma_1}(\boldsymbol{\ell}).$$

Using this bound, by successive conditioning, we get that, for all $n \geq 1$ and all $1 \leq j \leq l$, and $\mathbf{x} = (\mathbf{s}, \boldsymbol{\ell})$, if $l\gamma_0 = a$,

$$\begin{aligned} & \mathbb{E}^{\mathbf{x}} \left[e^{l\gamma_0 \sum_{k=0}^{n-1} \mathbf{w}(I_{kl+i})} (V_{0,\gamma}^{\otimes q} \otimes V_{1,\gamma_1})(\mathbf{X}_{(n-1)l+i}) \right] \\ & \leq b^{(n-1)/4} \mathbb{E}^{\mathbf{x}} \left[e^{l\gamma_0 \mathbf{w}(I_i)} (V_{0,\gamma}^{\otimes q} \otimes V_{1,\gamma_1})(\mathbf{X}_i) \right] \\ & \leq C e^{a\bar{w}i} b^{(n-1)/4} (V_{0,\gamma}^{\otimes q} \otimes V_{1,\gamma_1})(\mathbf{x}), \end{aligned}$$

where, in the last inequality, we used $e^{l\gamma_0 \mathbf{w}(I_j)} \leq e^{a\bar{w}}$ and Condition (E-1) and thus C is some constant not depending on n . Inserting this in (5.34), we get, for $m = nl + j$ with $j \in \{0, \dots, l\}$ and if $l\gamma_0 = a$ with $l \geq L$,

$$\begin{aligned} & \mathbb{E}^{\mathbf{x}} \left[e^{\gamma_0 \sum_{k=1}^m \mathbf{w}(I_k)} (V_{0,\gamma}^{\otimes q} \otimes V_{1,\gamma_1})(\mathbf{X}_m) \right] \\ & \leq C^3 e^{3a\bar{w}} b^{(n-1)/4} (V_{0,\gamma}^{\otimes q} \otimes V_{1,\gamma_1})(\mathbf{x}) \\ & \leq C^3 e^{3a\bar{w}} b^{-1/2} b^{m/(4l)} (V_{0,\gamma}^{\otimes q} \otimes V_{1,\gamma_1})(\mathbf{x}) \\ & \leq C^3 e^{3a\bar{w}} b^{-1/2} b^{\gamma_0 m/(4a)} (V_{0,\gamma}^{\otimes q} \otimes V_{1,\gamma_1})(\mathbf{x}) \end{aligned} \quad (5.38)$$

Now, this bound holds for any $\gamma_0 = a/l$ with $l \geq L$, where the constants a and b are chosen so that (5.37) holds and L only depends on a, b, γ and γ_1 . This yields (5.31) by taking $\rho > 0$ such that $e^\rho < b^{-1/(4a)}$. Note that we have shown that (5.31) holds for all $\gamma_0 \in (0, a/L]$ but only if a/γ_0 is an integer. Nevertheless, it is easy to shown that this can be extended to all $\gamma_0 \in (0, \gamma_0^*]$ by choosing ρ and γ_0^* adequately. \square

If $q = 1$ in Proposition 5.5, then we only need to prove Condition (C) for the chain $\tilde{Q}^{-\{1\}}$, which then corresponds to $q = 0$, that is $V_{0,\gamma_0}^{\otimes q} \equiv 1$. Hence an immediate consequence of Lemma 5.6 is that if $q = 1$ in Proposition 5.5, Condition (C) is implied by the geometric ergodicity of the unconstrained chain \tilde{Q} (see Section 2.3) and a simple moment condition on the stationary distribution $\tilde{\pi}$ of this chain, see (E-2).

However, for $q \geq 2$, Lemma 5.6 is no longer sufficient to prove Condition (C) of Proposition 5.5 because of the presence of $V_{0,\gamma_0}^{\otimes q}$ in the denominator of (5.31). Again we shall rely on an inductive reasoning to obtain the following result.

Lemma 5.7. *Let $p, q \geq 1$. Let $\{(I_n, \mathbf{S}_n, \boldsymbol{\lambda}_n), n \geq 0\}$ be the Markov chain on the space $\{0, 1, \dots, p\} \times \mathbb{Z}_+^q \times \mathbb{R}_+^p$ with transition kernel \tilde{Q} defined in Section 2.3. For any $\mathcal{J} \subseteq \{1, \dots, q\}$, further define the transition kernel $\tilde{Q}^{(-\mathcal{J})}$ on the space $\{0, \dots, p\} \times \mathbb{Z}_+^{q-\#\mathcal{J}} \times \mathbb{R}_+^p$, as in Section 2.3. Let $\mathbf{w} : \{0, 1, \dots, p\} \rightarrow \mathbb{R}$. Suppose that for any subset \mathcal{J} in $\{1, \dots, q\}$, the transition kernel $\tilde{Q}^{(-\mathcal{J})}$ satisfies Assumptions (E-1) and (E-2). Then for all $\gamma_1 > 0$ small enough, there are positive constants ρ_0 and $\gamma_0^* > 0$ such that, for all $\gamma_0 \in (0, \gamma_0^*]$,*

$$\sup_{(\mathbf{s}, \boldsymbol{\ell}) \in \mathbf{X}} \sup_{m \geq 1} \mathbb{E}^{\mathbf{x}} \left[e^{\gamma_0 \sum_{k=1}^m \{\mathbf{w}(I_k) + \rho_0\}} V_{1,\gamma_1}(\boldsymbol{\lambda}_m - \boldsymbol{\ell}) \right] < \infty. \quad (5.39)$$

Proof. We prove this lemma by induction on q .

For $q = 0$, Lemma 5.6 holds with $V_{0,\gamma}^{\otimes q} \equiv 1$, which implies (5.39) for all $\gamma_1 > 0$ small enough with some ρ_0, γ_0^* only depending on γ_1 . This provides the case $q = 0$ which initiates the induction.

Next we show the result in the case $q \geq 1$ using the induction hypothesis that the result holds for lower values of q .

Define for any $j = 1, \dots, q$, $\mathbf{A}(j)$ as in (5.28) and

$$s^*(j) = 1 + \max(\mathbf{A}(j)) \quad \text{and} \quad \bar{s}^* = \max_{j=1, \dots, q} s^*(j).$$

If $\mathbf{x} = (\mathbf{s}, \ell) \in \mathbf{X}$ is such that

$$\mathbf{s} \in \bar{\mathbf{A}} = \{1, \dots, s^*(1)\} \times \dots \times \{1, \dots, s^*(q)\},$$

then, applying Lemma 5.6, for all $\gamma_1 > 0$ small enough, we can choose $\gamma, \rho, \gamma^* > 0$ such that, for all $\gamma_0 \in (0, \gamma^*]$, there is some constant $C_0 > 0$ such that, for all $m \geq 1$,

$$\begin{aligned} \mathbb{E}^{\mathbf{x}} \left[e^{\gamma_0 \sum_{k=1}^m \mathbf{w}^{(I_k)} V_{1, \gamma_1}(\boldsymbol{\lambda}_m)} \right] &\leq \mathbb{E}^{\mathbf{x}} \left[e^{\gamma_0 \sum_{k=1}^m \mathbf{w}^{(I_k)} (V_{0, \gamma}^{\otimes q} \otimes V_{1, \gamma_1})(\mathbf{X}_m)} \right] \\ &\leq C_0 e^{-\rho \gamma_0 m} (V_{0, \gamma}^{\otimes q} \otimes V_{1, \gamma_1})(\mathbf{x}) \\ &\leq C_0 e^{q \bar{s}^* \gamma} e^{-\rho \gamma_0 m} V_{1, \gamma_1}(\ell). \end{aligned} \quad (5.40)$$

We denote by τ the first hitting time of the set $\bar{\mathbf{A}}$,

$$\tau = \inf \{k \geq 0, \mathbf{S}_k \in \bar{\mathbf{A}}\}$$

and we set $\tau_m = \tau \wedge m$, which is a stopping time with respect to the filtration $\mathcal{F}_k = \sigma(\mathbf{Y}_n, 0 \leq n \leq k)$, $k \geq 0$. We define by \mathcal{J}_m the set of indices j such that $\mathbf{S}_k(j)$ stay away of $\mathbf{A}(j)$ for all $k = 0, 1, \dots, \tau_m$, which is a random set measurable with respect to \mathcal{F}_{τ_m} and is non-empty by definition of τ_m . By definition of τ_m , we have, in the event $\{\tau_m < m\}$, $\mathbf{S}_{\tau_m} \in \bar{\mathbf{A}}$ and thus, by (5.40), setting $C_1 = C_0 e^{q \bar{s}^* \gamma}$,

$$\mathbb{E} \left[e^{\gamma_0 \sum_{k=\tau_m+1}^m \mathbf{w}^{(I_k)} V_{1, \gamma_1}(\boldsymbol{\lambda}_m)} \mid \mathcal{F}_{\tau_m} \right] \leq C_1 e^{-\rho \gamma_0 (m - \tau_m)} V_{1, \gamma_1}(\boldsymbol{\lambda}_{\tau_m}) \quad \text{a.s.} \quad (5.41)$$

On the other hand, up to $n = \tau_m$ the chain \mathbf{X}_n (with transition kernel \tilde{Q}) coincides with the chain with transition kernel $\tilde{Q}^{(-\mathcal{J}_m)}$ that starts at the same state, where \mathcal{J}_m is defined as above. Hence we may write, for all integer $l = 1, 2, \dots, m$, all non-empty subsets \mathcal{J} of $\{1, \dots, q\}$ and $\mathbf{x} = (\mathbf{s}, \ell) \in \mathbf{X}$ satisfying $\mathbf{s} \notin \bar{\mathbf{A}}$,

$$\begin{aligned} &\mathbb{E}^{\mathbf{x}} \left[e^{\gamma_0 \sum_{k=1}^{\tau_m} \mathbf{w}^{(I_k)} e^{-\rho \gamma_0 (m - \tau_m)} V_{1, \gamma_1}(\boldsymbol{\lambda}_{\tau_m})} \mathbf{1}_{\{\tau_m = l\} \cap \{\mathcal{J}_m = \mathcal{J}\}} \right] \\ &= \mathbb{E}^{(-\mathcal{J}), \mathbf{x}} \left[e^{\gamma_0 \sum_{k=1}^l \mathbf{w}^{(\tilde{I}_k)} e^{-\rho \gamma_0 (m - l)} V_{1, \gamma_1}(\tilde{\boldsymbol{\lambda}}_l)} \mathbf{1}_{\{\tau_m = l\} \cap \{\mathcal{J}_m = \mathcal{J}\}} \right] \\ &\leq e^{-\rho \gamma_0 (m - l)} \mathbb{E}^{(-\mathcal{J}), \mathbf{x}} \left[e^{\gamma_0 \sum_{k=1}^l \mathbf{w}^{(\tilde{I}_k)} V_{1, \gamma_1}(\tilde{\boldsymbol{\lambda}}_l)} \right], \end{aligned}$$

where $\{(\tilde{I}_n, \tilde{\mathbf{S}}_n, \tilde{\boldsymbol{\lambda}}_n), n \geq 0\}$ here denotes the chain with transition kernel $\tilde{Q}^{(-\mathcal{J})}$ which starts at the same state as $(I_0, \mathbf{S}_0, \boldsymbol{\lambda}_0)$. Using the induction hypothesis, if $\gamma_1 > 0$ is small enough, there are positive constants γ' and ρ' such that for all $\gamma_0 \in (0, \gamma']$, $\mathbf{x} \in \mathbf{X}$ and $l \geq 1$,

$$\mathbb{E}^{(-\mathcal{J}), \mathbf{x}} \left[e^{\gamma_0 \sum_{k=1}^l \mathbf{w}^{(\tilde{I}_k)} V_{1, \gamma_1}(\tilde{\boldsymbol{\lambda}}_l)} \right] \leq C_2 e^{-\rho' \gamma_0 l} V_{1, \gamma_1}(\ell),$$

for some constant $C_2 > 0$ independent of l and \mathbf{x} . Inserting this bound in the previous display and summing over all $l = 0, 1, \dots, m$ and \mathcal{J} , we get, for all $\gamma_0 \in (0, \gamma']$ and $\mathbf{x} = (\mathbf{s}, \ell) \in \mathbf{X}$ such that $\mathbf{s} \notin \bar{\mathbf{A}}$,

$$\begin{aligned} \mathbb{E}^{\mathbf{x}} \left[e^{\gamma_0 \sum_{k=1}^{\tau_m} \mathbf{w}^{(I_k)} e^{-\rho \gamma_0 (m - \tau_m)} V_{1, \gamma_1}(\boldsymbol{\lambda}_{\tau_m})} \right] &\leq C_3 \sum_{l=0}^m e^{-\rho \gamma_0 (m - l) - \rho' \gamma_0 l} V_{1, \gamma_1}(\ell) \\ &\leq C_3 m e^{-(\rho \wedge \rho') \gamma_0 m} V_{1, \gamma_1}(\ell), \end{aligned}$$

where $C_3 > 0$ is independent of m and \mathbf{x} . This last bound, with (5.41) yields, for all $\gamma_0 \in (0, \gamma' \wedge \gamma^*]$ and $\mathbf{x} = (\mathbf{s}, \boldsymbol{\ell}) \in \mathbf{X}$,

$$\begin{aligned} \mathbb{E}^{\mathbf{x}} \left[e^{\gamma_0 \sum_{k=1}^m \mathbf{w}(I_k)} V_{1, \gamma_1}(\boldsymbol{\lambda}_m) \right] &\leq \mathbb{E}^{\mathbf{x}} \left[e^{\gamma_0 \sum_{k=1}^{\tau_m} \mathbf{w}(I_k)} \mathbb{E} \left[e^{\gamma_0 \sum_{k=\tau_m+1}^m \mathbf{w}(I_k)} V_{1, \gamma_1}(\boldsymbol{\lambda}_m) \mid \mathcal{F}_{\tau_m} \right] \right] \\ &\leq C_1 \mathbb{E}^{\mathbf{x}} \left[e^{\gamma_0 \sum_{k=1}^{\tau_m} \mathbf{w}(I_k)} e^{-\rho(m-\tau_m)} V_{1, \gamma_1}(\boldsymbol{\lambda}_{\tau_m}) \right] \\ &\leq C_4 m e^{-(\rho \wedge \rho') m} V_{1, \gamma_1}(\boldsymbol{\ell}), \end{aligned}$$

where C_4 is some positive constant independent of m and \mathbf{x} . Now, taking $\gamma_0^* = \gamma' \wedge \gamma^*$ $\rho_0 \in (0, \rho \wedge \rho')$, we get (5.39). \square

Theorem 5.8. *Let $p, q \geq 1$ and suppose that Assumptions 1 and 2 hold. Let $\{(I_n, \mathbf{S}_n, \boldsymbol{\lambda}_n), n \geq 0\}$ be the Markov chain on the space $\{0, 1, \dots, p\} \times \mathbb{Z}_+^q \times \mathbb{R}_+^p$ with transition kernel \tilde{Q} defined in Section 2.3. For any $\mathcal{J} \subseteq \{1, \dots, q\}$, further define the transition kernel $\tilde{Q}^{(-\mathcal{J})}$ on the space $\{0, \dots, p\} \times \mathbb{Z}_+^{q-\#\mathcal{J}} \times \mathbb{R}_+^p$, as in Section 2.3. Define*

$$\mathbf{w}(i) = \begin{cases} \sum_{j=1}^q \vec{\mathbf{J}}_{i,j} & \text{for all } i = 1, \dots, p \\ 0 & \text{for } i = 0. \end{cases} \quad (5.42)$$

Suppose that for any subset \mathcal{J} in $\{1, \dots, q\}$, the transition kernel $\tilde{Q}^{(-\mathcal{J})}$ satisfies

(B-1) *For all $\gamma' > 0$ small enough, there exists $\gamma^* > 0$ such that for all $\gamma \in (0, \gamma^*]$, $\tilde{Q}^{(-\mathcal{J})}$ is $(1_{\{0,1,\dots,p\}} \otimes V_{0,\gamma}^{\otimes(q-\#\mathcal{J})} \otimes V_{1,\gamma'})$ -geometrically ergodic.*

(B-2) *The stationary distribution $\tilde{\pi}^{(-\mathcal{J})}$ of $\tilde{Q}^{(-\mathcal{J})}$ satisfies $\tilde{\pi}^{(-\mathcal{J})} \left[\mathbf{w} \otimes 1_{\mathbb{Z}_+^q \times \mathbb{R}_+^p} \right] < 0$.*

Then, for all $\gamma_1 > 0$ small enough, there exists $\gamma_0^* > 0$ such that for all $\gamma_0 \in (0, \gamma_0^*]$, the transition kernel \tilde{Q} is $(1_{\{0,1,\dots,p\}} \otimes V_{0,\gamma_0}^{\otimes q} \otimes V_{1,\gamma_1})$ -geometrically ergodic.

In practice, Theorem 5.8 should be applied by induction, so that only Conditions of the form (B-2) have to be checked out. The case $q = 0$ is treated in Proposition 3.3. We treat the case $q = 1$ in the following section. Then we explain in Section 3.5 how to perform this induction for $q \geq 2$.

Proof of Theorem 5.8. By Conditions (B-1) and (B-2), we may apply Lemma 5.7 to show that Condition (C) of Proposition 5.5 holds. Hence, applying this proposition, we get the drift condition (5.25).

The small sets of Theorem 3.1 are of the form $\{1, \dots, K\}^q \times (0, M]^p$. Observe that the sublevel sets of $V_{0,\gamma_0}^q \otimes V_{1,\gamma_1}$

$$C_{\gamma_0, \gamma_1}(r) = \{ \mathbf{x} \in \mathbb{Z}_+ \times \mathbb{R}_+^p : V_{0,\gamma_0}^q \otimes V_{1,\gamma_1}(\mathbf{x}) \leq r \}, \quad r > 0,$$

are included in such sets. Hence the drift condition (5.25) applies to show that the chain Q is $(V_{0,\gamma_0} \otimes V_{1,\gamma_1})$ -geometrically ergodic, see [33, Chapter 15]. The fact that \tilde{Q} is $(1_{\{0,\dots,p\}} \otimes V_{0,\gamma_0} \otimes V_{1,\gamma_1})$ -geometrically ergodic follows as in the proof of Proposition 3.3. \square

Proof of Theorem 3.7. This result is obtained by induction on q using Theorem 5.8 and Theorem 3.5 to initiate the induction at $q = 1$. \square

5.6 Proof of Corollary 3.8

We first show that for all $D > 0$, $K \geq 1$ and $M > 0$, the set $C = (0, D] \times \{0, \dots, p\} \times \{1, \dots, K\}^q \otimes (0, M]^p$ is a petite set. By Corollary 5.4, there exists $m \geq 1$, $\epsilon > 0$ and a probability measure $\tilde{n}u$ on \mathbf{Y} such that $\{0, \dots, p\} \times \{1, \dots, K\}^q \otimes (0, M]^p$ is an (m, ϵ, \tilde{v})

and $(m+1, \epsilon, \tilde{\nu})$ -small set for \tilde{Q} . For all $\mathbf{z} = (\delta, i, \mathbf{s}, \ell) \in C$, under the initial condition $\mathbf{Z}_0 = \mathbf{z}$, we have $\lambda_{m-1} \in (0, M + (m-1)\bar{\alpha}\beta]^p$. It follows that the density of Δ_m is bounded from below by $c_0 e^{-(m-1)c_1 t}$ over $t \in \mathbb{R}_+$ for some positive constants c_0, c_1 (see the proof of Lemma 5.1). We get that for all Borel sets $A \subset \mathbb{R}_+$ and $B \subset \mathcal{Y}$,

$$\begin{aligned} \bar{Q}^m(\mathbf{z}, A \times B) &\geq c_0 \int_A e^{-c_1 t} dt \tilde{Q}^m(\mathbf{y}, B) \\ &\geq \epsilon(c_0/c_1)\nu(A \times B), \end{aligned}$$

where we used that $\{0, \dots, p\} \times \{1, \dots, K\}^q \otimes (0, M]^p$ is an $(m, \epsilon, \tilde{\nu})$ for \tilde{Q} and ν is the probability measure defined by $\nu(A \times B) = \int_A c_1 e^{-c_1 t} dt \tilde{\nu}(B)$. Hence C is an (m, ϵ, ν) -small set for \bar{Q} . Similarly, we have that is also an $(m+1, \epsilon, \tilde{\nu})$ -small set. Therefore, we conclude that \bar{Q} is ψ -irreducible and aperiodic and that all bounded subsets of \mathcal{Z} are petite sets.

Next we show that the drift condition obtained in Proposition 5.5 for \tilde{Q} extends to \bar{Q} . As explained in Section 3.5, the assumptions of Theorem 3.7 imply that of Theorem 5.8 and thus of Proposition 5.5 (see the proof of Theorem 3.7). Hence, for all $\gamma_1 > 0$ small enough, there exists $\gamma_0^* > 0$, $\theta \in (0, 1)$, $b > 0$ and $m \geq 1$ such that for all $\gamma_0 \in (0, \gamma_0^*]$ such that for any initial condition $\mathbf{x} \in \mathcal{X}$, we have

$$\left[\tilde{Q}^m(V_{0, \gamma_0} \otimes V_{1, \gamma_1}) \right] (\mathbf{x}) \leq \theta (V_{0, \gamma_0} \otimes V_{1, \gamma_1})(\mathbf{x}) + b.$$

By the Cauchy-Schwartz inequality and using that for all $\mathbf{x} \in \mathcal{X}$, Δ_m is stochastically smaller than an exponential distribution with mean $(\mathbf{1}_p^T \boldsymbol{\mu}_0)^{-1}$ (see the proof of Lemma 5.1), we have

$$\begin{aligned} &\bar{Q}^m(V_{2, \gamma_2} \otimes \mathbf{1}_{\{0, \dots, p\}} \otimes V_{0, \gamma_0/2} \otimes V_{1, \gamma_1/2})(\Delta, i, \mathbf{x}) \\ &\leq \left(\mathbb{E}^{\mathbf{x}} [V_{2, 2\gamma_2}(\Delta_m)] \right)^{1/2} \left(\left[\bar{Q}^m(\mathbf{1}_{\{0, \dots, p\}} \otimes V_{0, \gamma_0} \otimes V_{1, \gamma_1}) \right] (\Delta, i, \mathbf{x}) \right)^{1/2} \\ &\leq \left(\int_0^{+\infty} (\mathbf{1}_p^T \boldsymbol{\mu}_0) e^{2\gamma_2 r} e^{-\mathbf{1}_p^T \boldsymbol{\mu}_0 r} dr \right)^{1/2} (\theta (V_{0, \gamma_0} \otimes V_{1, \gamma_1})(\mathbf{x}) + b)^{1/2}. \end{aligned}$$

Provided that $\gamma_2 < \mathbf{1}_p^T \boldsymbol{\mu}_0/2$, we thus get

$$\begin{aligned} &\bar{Q}^m(V_{2, \gamma_2} \otimes \mathbf{1}_{\{0, \dots, p\}} \otimes V_{0, \gamma_0/2} \otimes V_{1, \gamma_1/2})(\Delta, i, \mathbf{x}) \\ &\leq (1 - 2\gamma_2/\mathbf{1}_p^T \boldsymbol{\mu}_0)^{-1/2} \left[\theta^{1/2} (V_{0, \gamma_0/2} \otimes V_{1, \gamma_1/2})(\mathbf{x}) + b^{1/2} \right]. \end{aligned}$$

Hence, for $\gamma_2 > 0$ small enough, we have, for some $\theta' \in (0, 1)$ and $b' > 0$,

$$\bar{Q}^m(V_{2, \gamma_2} \otimes \mathbf{1}_{\{0, \dots, p\}} \otimes V_{0, \gamma_0/2} \otimes V_{1, \gamma_1/2})(\Delta, i, \mathbf{x}) \leq \theta' (V_{0, \gamma_0/2} \otimes V_{1, \gamma_1/2})(\mathbf{x}) + b'.$$

Hence we obtain a drift condition with an unbounded off petite set function $V_{2, \gamma_2} \otimes \mathbf{1}_{\{0, \dots, p\}} \otimes V_{0, \gamma_0/2} \otimes V_{1, \gamma_1/2}$. The geometric ergodicity follows (see [33, Section 15]).

5.7 Proof of Theorem (3.9)

We need the following lemmas to complete the proof for this theorem. Denote the number of arrivals on $[0, u]$ by $\mathbf{N}_u = \mathbf{N}(\mathbf{1}_{(0, u]} \otimes \mathbf{1}_{\{0, \dots, p\}})$ and the time interval between the last arrival before u and u by $\Gamma_u = u - \sum_{k=1}^{\mathbf{N}_u} \Delta_k$.

Lemma 5.9. *We have*

$$\lim_{u \rightarrow \infty} \frac{\mathbf{N}_u}{u} \rightarrow \frac{1}{\overline{\mathbb{E}}(\Delta_1)}, \text{ a.s. } [P_*].$$

Proof. By definition of \mathbf{N}_u , we have

$$\sum_{k=1}^{\mathbf{N}_u} \Delta_k \leq u \leq \sum_{k=1}^{\mathbf{N}_u+1} \Delta_k.$$

Since \overline{Q} is positive Harris, using [33, Theorem 17.3.2], we get

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \Delta_k = \overline{\mathbb{E}}(\Delta_1), \text{ a.s. } [P_*].$$

We deduce that

$$\lim_{u \rightarrow +\infty} \frac{u}{\mathbf{N}_u} = \lim_{u \rightarrow +\infty} \frac{\sum_{k=1}^{\mathbf{N}_u} \Delta_k}{\mathbf{N}_u} = \lim_{u \rightarrow +\infty} \frac{\sum_{k=1}^{\mathbf{N}_u+1} \Delta_k}{\mathbf{N}_u} = \overline{\mathbb{E}}(\Delta_1), \text{ a.s. } [P_*].$$

Hence the result. \square

Lemma 5.10. *For all $T \in \mathbb{R}_+$, we have*

$$\sup_{\tau \in [0,1]} \Gamma_{\tau T}^{\mathbf{x}} = o_{P_*}(\sqrt{T}), \text{ a.s. } [P_*].$$

Proof. Let us define

$$n_T = \left\lceil \frac{T}{\overline{\mathbb{E}}(\Delta_1)} \right\rceil. \quad (5.43)$$

By definition of Γ_u , for all $\tau \in [0, 1]$, we have that $\Gamma_{\tau T} \leq \max_{k=1, \dots, \mathbf{N}_{\tau T}+1} \Delta_k$.

For any $\tau \in [0, 1]$, $\epsilon > 0$ and $T \in \mathbb{R}_+$, we get, for all $\mathbf{x} \in \mathbf{X}$,

$$\mathbb{P}(\Gamma_{\tau T} \geq \epsilon \sqrt{T}) \leq \mathbb{P}^{\mathbf{x}} \left(\max_{k=1, \dots, 2n_T} \Delta_k \geq \epsilon \sqrt{T} \right) + \mathbb{P}^{\mathbf{x}}(\mathbf{N}_T + 1 > 2n_T).$$

Using Lemma 5.9, we have $\lim_{T \rightarrow \infty} \mathbb{P}^{\mathbf{x}}(\mathbf{N}_T + 1 > 2n_T) = 0$. Observe that we have, for all

$k \geq 1$, $\Delta_k \stackrel{st}{\leq} E_k$ where $\{E_i\}_{i \geq 1}$ are the inter-arrivals of a PPP with intensity $1_p^T \ell$. We get, for all $\mathbf{x} \in \mathbf{X}$,

$$\lim_{T \rightarrow \infty} \mathbb{P}^{\mathbf{x}} \left(\max_{k=1, \dots, 2n_T} \Delta_k \geq \epsilon \sqrt{T} \right) \leq \lim_{T \rightarrow \infty} \mathbb{P}^{\mathbf{x}} \left(\max_{k=1, \dots, 2n_T} E_k \geq \epsilon \sqrt{T} \right) = 0,$$

which concludes the proof. \square

We now provide a proof of Theorem 3.9 using Lemmas 5.9 and 5.10.

Proof of Theorem 3.9. For all $t \in [0, 1]$, $T \in \mathbb{R}_+$ and $\mathbf{x} \in \mathbf{X}$, we have

$$(\mathbf{N}^{\mathbf{x}}(1_{[0,tT]} \otimes \mathbf{w}) - tT E(\mathbf{w})) = \sum_{k=1}^{\mathbf{N}_{tT}^{\mathbf{x}}} (\mathbf{w}(I_k) - E(\mathbf{w})\Delta_k) - E(\mathbf{w})\Gamma_{tT}^{\mathbf{x}}.$$

We also observe that, defining n_T as in (5.43),

$$\sum_{k=1}^{\mathbf{N}_{tT}^{\mathbf{x}}} (\mathbf{w}(I_k) - E(\mathbf{w})\Delta_k) = \Sigma_{n_T} \left(\frac{\mathbf{N}_{tT}^{\mathbf{x}}}{n_T}, g \right) = s_{n_T} \left(\frac{\mathbf{N}_{tT}^{\mathbf{x}}}{n_T}, g \right),$$

where we used that, by definition of $E(\mathbf{w})$, $\bar{\pi}(g) = 0$ and that \mathbf{N}_{tT} is an integer. Hence by Lemma 5.10, we get

$$(\mathbf{N}^{\mathbf{x}}(1_{[0,tT]} \otimes \mathbf{w}) - tT E(\mathbf{w})) = s_{n_T} \left(\frac{\mathbf{N}_{tT}^{\mathbf{x}}}{n_T}, g \right) + o_{P_*}(\sqrt{T}). \quad (5.44)$$

Now, by Lemma 5.9, we have, for all $t \in [0, 1]$, $\mathbf{N}_{tT}^{\mathbf{x}}/n_T \rightarrow t$ as $T \rightarrow \infty$ a.s. $[P_*]$. By the Dini theorem, this convergence holds uniformly over $t \in [0, 1]$. Applying [40, Theorem 13.2.1], this convergence and the one in (3.15) implies that, for any initial distribution,

$$(n_T \sigma_g^2)^{-1/2} s_{n_T} \left(\frac{\mathbf{N}_{tT}^{\mathbf{x}}}{n_T}, g \right) \xrightarrow{d} B_t.$$

This, (5.44), $\sigma_g^2 = v(\mathbf{w})$ and the fact that $n_T/T \rightarrow 1/\bar{\mathbb{E}}[\Delta_1]$, we get (3.18), which achieves the proof. \square

6 Conclusion

In this paper we have introduced and studied constrained multivariate Hawkes processes. The constraints are expressed using a multidimensional *constraint* variable, whose evolution is driven by the point process. Under the markov setting (exponential fertility functions), we have proven that the underlying Markov chain is V -geometrically ergodic under some conditions on the parameters. A converse result leading to the transience of the chain in the case where an univariate constraint variable is used ($q = 1$) illustrates the sharpness of our conditions. Moreover, in the general case, we used a functional central limit theorem applying to the chain to derive the scaling limit of the integrated point process in physical time. Finally we have briefly explained how the constrained multivariate Hawkes process can be applied to model the dynamics of a limit order book. The scaling limit of the mid-price can be deduced from our findings.

We only presented the case of the order book of one asset with only two limits. This is clearly not a restriction. Multi-assets limit order books can be considered, yielding multivariate boundary conditions (hence $q \geq 2$). In a forthcoming paper, we will use this model to provide an empirical study from real data and discuss the potential application of this model to the dynamics of limit order books. We believe that the applicability of the constrained multivariate Hawkes process extends well beyond financial applications, as it could model the motion of an object on a discrete net with some boundary conditions.

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A Additional technical lemmas

The following result is used in the proof of Proposition 5.3.

Lemma A.1. *Let $p \geq 1$, and Γ is a $p \times p$ invertible matrix. Then there exists a probability measure ν such that, for all $M > 0$, $\gamma > 0$, there exists $\epsilon > 0$ such that for all $g : \mathbb{R}^p \rightarrow \mathbb{R}_+$ and $\ell \in (0, M]^p$,*

$$\int_D g(u\ell + \Gamma\vartheta) u^\gamma \mathrm{d}u \mathrm{d}\vartheta \geq \epsilon \int g \mathrm{d}\nu,$$

where $D = \{(u, \vartheta) \in (0, 1] \times (0, 1]^p, 0 < u < \vartheta(1) < \dots < \vartheta(p)\}$.

Proof. Setting $\omega = u\ell + \Gamma\vartheta$, we have

$$\int_D g(u\ell + \Gamma\vartheta)u^\gamma dud\vartheta = \frac{1}{\det \Gamma} \int g(\omega)u^\gamma 1_D((u, \Gamma^{-1}\omega - u\Gamma^{-1}\ell))dud\omega .$$

We choose any $0 < \eta < 1/(p+1)$ so that the open set $B_\eta = \{\vartheta \in (0, 1]^p : \eta < \vartheta(1), \eta + \vartheta(1) < \vartheta(2), \dots, \eta + \vartheta(p-1) < \vartheta(p), \vartheta(p) < 1 - \eta\}$ is not empty.

Denoting $\Gamma B_\eta = \{\Gamma\vartheta : \vartheta \in B_\eta\}$ and setting

$$\eta' = \frac{1}{2}\eta \min \left(1, \left(\max_{i=1, \dots, p} \sup_{\ell \in (0, M]^p} |[\Gamma^{-1}\ell](i)| \right)^{-1} \right)$$

we have that for all $u \in (0, \eta']$, $\ell \in (0, M]^p$ and $\omega \in \Gamma B_\eta$, $[u, \Gamma^{-1}\omega - u\Gamma^{-1}\ell] \in D$. We obtain that

$$\begin{aligned} \int_D g(u\ell + \Gamma\vartheta)u^\gamma dud\vartheta &\geq \frac{1}{\det \Gamma} \int_{\Gamma B_\eta} g(\omega) \left(\int_0^{\eta'} u^\gamma du \right) d\omega \\ &= \frac{\mu^{Leb}(\Gamma B_\eta)(\eta')^{\gamma+1}}{(\gamma+1)\det \Gamma} \int g d\nu , \end{aligned}$$

where μ^{Leb} is Lebesgue measure, ν is the uniform probability measure on ΓB_η . \square

The following result is used in the proof of Proposition 3.2.

Lemma A.2. *Let $\beta > 0$ and $\beta' > 0$. Then, as $a \rightarrow \infty$,*

$$\int_0^\infty e^{a(e^{-\beta t} - 1) - \beta' t} dt \rightarrow 0 . \quad (\text{A.1})$$

If moreover $\beta' > \beta$, then we have the following asymptotic equivalence as $a \rightarrow \infty$,

$$\int_0^\infty e^{a(e^{-\beta t} - 1) - \beta' t} dt \sim \frac{1}{a\beta} . \quad (\text{A.2})$$

Proof. Setting $\vartheta = e^{-\beta t}$, we get

$$\int_0^\infty e^{a(e^{-\beta t} - 1) - \beta' t} dt = \frac{1}{\beta} \int_0^1 e^{a(\vartheta - 1)} \vartheta^{\frac{\beta'}{\beta} - 1} d\vartheta .$$

Letting $a \rightarrow \infty$, by dominated convergence, we get (A.1).

Now we set $\omega = -a(\vartheta - 1)$ and obtain that

$$\frac{1}{\beta} \int_0^1 e^{a(\vartheta - 1)} \vartheta^{\frac{\beta'}{\beta} - 1} d\vartheta = \frac{1}{a\beta} \int_0^a e^{-\omega} \left(1 - \frac{\omega}{a}\right)^{\frac{\beta'}{\beta} - 1} d\omega .$$

Letting $a \rightarrow \infty$, we obtain (A.2) by dominated convergence. \square

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