C_∞-structure on the cohomology of free 2-nilpotent Lie algebra

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2-Step Nilpotent Lie algebra and C_∞-Algebras

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Abstract. We consider the free 2-step nilpotent graded Lie algebra and its cohomology ring. The homotopy transfer induces a homotopy commutative algebra on its cohomology ring which we describe.

1. Homotopy algebras

The homotopy associative algebras, or A_∞-algebras were introduced by Jim Stasheff in the 1960's as a tool in algebraic topology for studying 'group-like' spaces. Homotpy algebras received a new attention and further development in the 1990's after the discovery of their relevence into a multitude of topics in algebraic geometry, symplectic and contact geometry, knot theory, moduli spaces, deformation theory...

Definition 1.1. (A_∞-algebra) A homotopy associative algebra, or A_∞-algebra, over $\mathbb{K}$ is a $\mathbb{Z}$-graded vector space $A = \bigoplus_{i\in\mathbb{Z}} A^i$ endowed with a family of graded mappings (operations)

$$m_n : A^{\otimes n} \to A, \qquad \deg(m_n) = 2 - n \qquad n \geqslant 1$$

satisfying the Stasheff identities $\mathbf{SI}(\mathbf{n})$ for $n \geqslant 1$

$$\mathbf{SI}(\mathbf{n}) : \quad \sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(Id^{\otimes r} \otimes m_s \otimes Id^{\otimes t}) = 0 \qquad r \geqslant 0,\, t \geqslant 0,\, s \geqslant 1$$

where the sum runs over all decompositions $n = r + s + t$. Throughout the text we assume the Koszul sign convention $(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y)$.

A morphism of two A_∞-algebras A and B is a family of graded maps$f_n : A^{\otimes n} \to B$ for $n \geqslant 1$ with $\deg f_n = 1 - n$ such that the following conditions hold

$$\sum_{r+s+t=n} (-1)^{r+st} f_{r+1+t}(Id^{\otimes r} \otimes m_s \otimes Id^{\otimes r}) = \sum_{1\leqslant r\leqslant n} (-1)^S m_r(f_{i_1} \otimes f_{i_2} \otimes \ldots \otimes f_{i_r})$$

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where the sum is on all decompositions $i_1+\ldots+i_r=n$ and the sign $(-1)^S$ on RHS is determined by $S=(r-1)(i_1-1)+(r-2)(i_2-1)+\ldots+2(i_{r-2}-1)+(i_{r-1}-1)$. The morphism f is a *quasi-isomorphism of A_∞-algebras* if f_1 is a quasi-isomorphism. It is strict if $f_i=0$ for all $i\neq 1$. The identity morphism of A is the strict morphism f such that f_1 is the identity of A.

We define the shuffle product $Sh_{p,q}: A^{\otimes p}\otimes A^{\otimes q}\to A^{\otimes p+q}$ throughout the expression

$$(a_1\otimes\ldots\otimes a_p)\,\Delta\,(a_{p+1}\otimes\ldots\otimes a_{p+q})=\sum_{\sigma\in Sh_{p,q}} sgn(\sigma)\,a_{\sigma^{-1}(1)}\otimes\ldots\otimes a_{\sigma^{-1}(p+q)}$$

where the sum runs over all (p,q)-shuffles $Sh_{p,q}$, i.e., over all permutations $\sigma\in S_{p+q}$ such that $\sigma(1)<\sigma(2)<\ldots<\sigma(p)$ and $\sigma(p+1)<\sigma(p+2)<\ldots<\sigma(p+q)$.

Definition 1.2. (C_∞-algebra [**9**]) A homotopy commutative algebra, or C_∞-algebra, is an A_∞-algebra $\{A,m_n\}$ such that each operation m_n vanishes on non-trivial shuffles

$$m_n\left((a_1\otimes\ldots\otimes a_p)\,\Delta\,(a_{p+1}\otimes\ldots\otimes a_n)\right)=0\ ,\qquad 1\leqslant p\leqslant n-1\ . \tag{1.1}$$

In particular for m_2 we have $m_2(a\otimes b\pm b\otimes a)=0$, so a C_∞-algebra such that $m_n=0$ for $n\geqslant 3$ is a (super-)commutative DGA.

A morphism of C_∞-algebras is a morphism of A_∞-algebras vanishing on non-trivial shuffles $f_n\left((a_1\otimes\ldots\otimes a_p)\,\Delta\,(a_{p+1}\otimes\ldots\otimes a_n)\right)=0\ ,\ 1\leqslant p\leqslant n-1\ .$

2. Homotopy Transfer Theorem

Lemma 2.1. *Every cochain complex (A,d) of vector spaces over a field $\mathbb{K}$ has its cohomology $H^\bullet(A)$ as a deformation retract.*

One can always choose a vector space decomposition of the cochain complex (A,d) such that $A^n\cong B^n\oplus H^n\oplus B^{n+1}$ where H^n is the cohomology and B^n is the space of coboundaries, $B^n=dA^{n-1}$. We choose a homotopy $h: A^n\to A^{n-1}$ which identifies B^n with its copy in A^{n-1} and is 0 on $H^n\oplus B^{n+1}$. The projection p to the cohomology and the cocycle-choosing inclusion i given by $A^n \underset{i}{\overset{p}{\rightleftarrows}} H^n$ are chain homomorphisms, satisfying the additional *side conditions*

$$hh=0,\qquad hi=0,\qquad ph=0\ .$$

With these choices done the complex $(H^\bullet(A),0)$ is a deformation retract of (A,d)

$$h\circlearrowright (A,d)\underset{i}{\overset{p}{\rightleftarrows}}(H^\bullet(A),0)\ \ ,\qquad pi=Id_{H^\bullet(A)}\ ,\qquad ip-Id_A=dh+hd\ .$$

Let now (A,d,μ) be a DGA, i.e., A is endowed with an associative product μ compatible with d. The cochain complexes (A,d) and its contraction $H^\bullet(A)$ are homotopy equivalent, but the associative structure is not stable under homotopy equivalence. However the associative structure on A can be transferred to an A_∞-structure on a homotopy equivalent complex, a particular interesting complex being the deformation retract $H^\bullet(A)$. For a friendly introduction to homotopy transfer

theorems in much boarder context we send the reader to the textbook [**12**], see chapter 9.

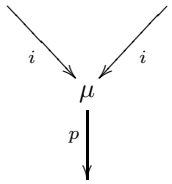
THEOREM 2.1 (Kadeishvili [**9**]). *Let (A, d, μ) be a (commutative) DGA over a field $\mathbb{K}$. There exists a A_∞-algebra (C_∞-algebra) structure on the cohomology $H^\bullet(A)$ and a $A_\infty(C_\infty)$-quasi-isomorphism*

$$f_i : (\otimes^i H^\bullet(A), \{m_i\}) \to (A, \{d, \mu, 0, 0, \ldots\})$$

such that the inclusion $f_1 = i : H^\bullet(A) \to A$ is a cocycle-choosing homomorphism of cochain complexes. The differential m_1 on $H^\bullet(A)$ is zero ($m_1 = 0$) and m_2 is the strictly associative operation induced by the multiplication on A. The resulting structure is unique up to quasi-isomorphism.

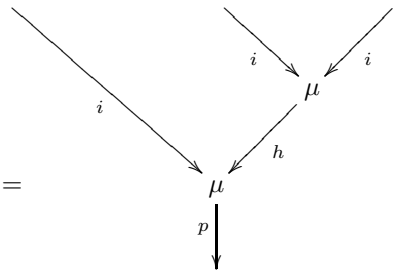
Kontsevich and Soibelman [**10**] gave an explicit expressions for the higher operations of the induced A_∞-structure as sums over decorated planar binary trees with one root where all leaves are decorated by the inclusion i, the root by the projection p the vertices by the product μ of the (commutative) DGA (A, d, μ) and the internal edges by the homotopy h. The C_∞-structure implies additional symmetries on trees.
We will make use of the graphic representation for the binary operation on $H^\bullet(A)$

$m_2(x, y) := p\mu(i(x), i(y))$ or $m_2 =$



and the ternary one $m_3(x, y, z) = p\mu(i(x), h\mu(i(y), i(z))) - p\mu(h\mu(i(x), i(y)), i(z))$ being the sum of two planar binary trees with three leaves

$m_3 =$



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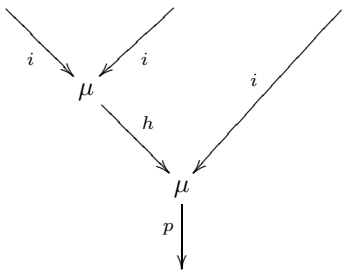

3. Homology and cohomology of Lie algebra $\mathfrak{g}$

A non-minimal projective(in fact free) resolution of the trivial $U\mathfrak{g}$-module $\mathbb{K}$, $C(\mathfrak{g}) \xrightarrow{\epsilon} \mathbb{K}$ is given by the standard Chevalley-Eilenberg chain complex $C_\bullet(\mathfrak{g}) =$

$(U\mathfrak{g}\otimes_{\mathbb{K}}\wedge^p\mathfrak{g}, d_p)$ with differential maps

$$d_p(u\otimes x_1\wedge\ldots\wedge x_p)=\sum_i(-1)^{i+1}ux_i\otimes x_1\wedge\ldots\wedge\hat{x}_i\wedge\ldots\wedge x_p$$

$$+\sum_{i<j}(-1)^{i+j}u\otimes[x_i,x_j]\wedge x_1\wedge\ldots\wedge\hat{x}_i\wedge\ldots\wedge\hat{x}_j\wedge\ldots\wedge x_p \tag{3.1}$$

The homologies $H_n(\mathfrak{g},\mathbb{K})$ of the Lie algebra $\mathfrak{g}$ with trivial coefficients are given by the homologies of the derived complex $\mathbb{K}\otimes_{U\mathfrak{g}}C_\bullet(\mathfrak{g})$

$$\mathrm{Tor}_n^{U\mathfrak{g}}(\mathbb{K},\mathbb{K})\cong H_n(\mathbb{K}\otimes_{U\mathfrak{g}}C_\bullet(\mathfrak{g}))=H_n(\mathfrak{g},\mathbb{K})\ .$$

The complex $\mathbb{K}\otimes_{U\mathfrak{g}}C_\bullet(\mathfrak{g})$ is the chain complex with degrees $\bigwedge^\bullet\mathfrak{g}=\mathbb{K}\otimes_{U\mathfrak{g}}U\mathfrak{g}\otimes\bigwedge^\bullet\mathfrak{g}$ and differentials $\partial_p:=id\otimes_{U\mathfrak{g}}d_p:\bigwedge^p\mathfrak{g}\to\bigwedge^{p-1}\mathfrak{g}$ induced by the extension as coderivation of the Lie bracket $\partial_2:=-[\,\cdot\,,\,\cdot\,]:\bigwedge^2\mathfrak{g}\to\mathfrak{g}$.

The dual cochain complex $\mathrm{Hom}_{U\mathfrak{g}}(C(\mathfrak{g}),\mathbb{K})=(\bigwedge^\bullet\mathfrak{g}^*,\delta)$ has coboundary map $\delta^p:\bigwedge^p\mathfrak{g}^*\to\bigwedge^{p+1}\mathfrak{g}^*$ (being transposed to the differential ∂_{p+1}) which is the extension as derivation of the dualization of the Lie bracket $\delta^1:=[\,\cdot\,,\,\cdot\,]^*:\mathfrak{g}^*\to\bigwedge^2\mathfrak{g}^*$. One calculates the cohomologies[1] of the Lie algebra $\mathfrak{g}$ as

$$\mathrm{Ext}^n_{U\mathfrak{g}}(\mathbb{K},\mathbb{K})\cong H^n(\mathrm{Hom}_{U\mathfrak{g}}(C(\mathfrak{g}),\mathbb{K}))=H^n(\mathfrak{g},\mathbb{K})\ . \tag{3.2}$$

Hence the algebra $(\bigwedge^\bullet\mathfrak{g}^*,\delta)$ equipped with δ is a *(super)commutative* DGA and the Yoneda algebra $\mathrm{Ext}^\bullet_{U\mathfrak{g}}(\mathbb{K},\mathbb{K})=\bigoplus_n\mathrm{Ext}^n_{U\mathfrak{g}}(\mathbb{K},\mathbb{K})$ has structure of commutative associative algebra. Moreover due to the Kadeishvili theorem the Yoneda algebra $\mathrm{Ext}^\bullet_{U\mathfrak{g}}(\mathbb{K},\mathbb{K})=H^\bullet(\mathfrak{g},\mathbb{K})$ is a C_∞-algebra which stems from the homotopy transfer of the wedge product $\wedge$ on cohomology classes

$$H^i(\mathfrak{g},\mathbb{K})\wedge H^j(\mathfrak{g},\mathbb{K})\to H^{i+j}(\mathfrak{g},\mathbb{K})\ .$$

4. Abelian Lie algebra $\mathfrak{h}=V$

Let us take as a basic example the abelian Lie algebra $\mathfrak{h}$, that is, the free nilpotent Lie algebra generated by a finite dimensional vector space V of degree 1. The Lie bracket of $\mathfrak{h}$ is trivial $[V,V]=0$. According to Poincaré-Birkhoff-Witt theorem the universal enveloping algebra of the abelian Lie algebra $\mathfrak{h}=V$ is isomorphic to the symmetric algebra

$$U(\mathfrak{h})\cong S(V)\ .$$

The Chevalley-Eilenberg complex $C_\bullet(\mathfrak{h})=S(V)\otimes_{\mathbb{K}}\Lambda^\bullet V$ yields the resolution of the trivial $U(\mathfrak{h})$-module $\mathbb{K}$

$$0\to S(V)\otimes\Lambda^{\dim V}V\to S(V)\otimes\Lambda^{\dim V-1}V\to\cdots$$

$$\cdots\to S(V)\otimes\Lambda^2V\to S(V)\otimes V\to S(V)\to\mathbb{K}\to 0\ . \tag{4.1}$$

The derived complex $\mathbb{K}\otimes_{U\mathfrak{h}}C(\mathfrak{h})$ has zero differential and the Chevalley-Eilenberg resolution turns out to be minimal(which is not the case in general)

$$H_n(\mathfrak{h},\mathbb{K})\cong H_n(\mathbb{K}\otimes_{U\mathfrak{h}}C(\mathfrak{h}))\cong\Lambda^nV\ .$$

[1] In the presence of metric one has $\delta:=\partial^*$(see below)

The Chevalley-Eilenberg resolution coincides with the Koszul complex $K(A) = A \otimes (A^!)^*$ of the symmetric algebra $A = S(V)$. The Koszul dual algebra of the symmetric algebra is the exteriour algebra $S(V)^! = \Lambda V^*$. A quadratic algebra is said to be a Koszul algebra when its Koszul complex $K_\bullet(A) = A \otimes (A^!_\bullet)^*$ is acyclic everywhere except in degree 0(where its homology is $\mathbb{K}$). Then the Koszul complex yields a minimal projective (in fact free) resolution by (left) A-modules of the trivial A-module $\mathbb{K}$

$$K(A) \xrightarrow{\epsilon} \mathbb{K} \to 0 \ .$$

In particular the resolution 4.1 is the same as the the resolution by the Koszul complex $K_n(S(V)) = S(V) \otimes \Lambda^n V^*$ thus the algebra $S(V)$ is *Koszul algebra*. One has another equvalent definition of Koszul algebra based on the following proposition;

PROPOSITION 4.1. *A finitely generated quadratic algebra A is Koszul iff its Yoneda algebra* $\mathrm{Ext}_A(\mathbb{K},\mathbb{K})$ *is generated in degree 1. One has then* $\mathrm{Ext}_A(\mathbb{K},\mathbb{K}) \cong A^!$.

Indeed the Yoneda algebra $\mathrm{Ext}_{S(V)}(\mathbb{K},\mathbb{K})$ of the symmetric algebra $S(V)$ is just the exteriour algebra

$$\mathrm{Ext}^n_{S(V)}(\mathbb{K},\mathbb{K}) = (\mathrm{Tor}^{S(V)}_n(\mathbb{K},\mathbb{K}))^* = \Lambda^n V^*$$

which is obviously generated by V^*, *i.e.,* in degree 1, by the wedge product. Through the homotopy transfer the Yoneda algebra $\mathrm{Ext}_{S(V)}(\mathbb{K},\mathbb{K})$ inherits a C_∞-structure but it is easy to show (by degree preserving argument) that the latter C_∞-algebra is formal, i.e., all higher multiplications are trivial, $m_n = 0$ for $n \neq 2$.

5. Homology of Free 2-nilpotent algebra $\mathfrak{g} = V \oplus \Lambda^2 V$

Let $\mathfrak{g}$ be the free graded 2-step nilpotent Lie algebra generated by vector space V in degree 1, $\mathfrak{g} = V \oplus [V,V]$. In other words we consider the graded Lie algebra $\mathfrak{g}$ with Lie bracket

$$[u,v] = \begin{cases} u \wedge v & \in \Lambda^2 V \\ 0 & \text{otherwise} \end{cases}$$

We denote the Universal Enveloping Algebra(UEA) $U\mathfrak{g}$ by PS and refer to it as *parastatistics algebra*. [2] Throughout this note we will consider the generators space V to be an ordinary vector space V which corresponds to a parafermionic algebra $PS(V) = U\mathfrak{g}$. The case of a $\mathbb{Z}_2$-space of generators $V = V_0 \oplus V_1$, that is, $PS(V)$ is the Universal Enveloping Algebra of a Lie super-algebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$(which would include the parabosonic algebras) will be treated elsewhere. More on parastatistics algebras and their application to combinatorics the reader could find in the articles [**5, 11**].

[2]Such cubic algebras arise through the exchange relations between the operators in a quantization procedure intoduced by H. S. Green [**7**] for particles obeying more general statistics than Bose-Einstein or Fermi-Dirac, coined parabosons and parafermions.

The parastatistics algebra $PS(V)$ generated by a finite dimensional vector space V in degree 1 is the positively graded algebra

$$PS(V) := U\mathfrak{g} = U(V \oplus \bigwedge^2 V) = T(V)/([[V,V],V]) \ .$$

We shall write simply PS when the space of generators V is clear from the context.

The homologies $H_n(\mathfrak{g},\mathbb{K})$ of the 2-nilpotent Lie algebra $\mathfrak{g}$ are the homologies of the chain complex

$$\bigwedge^n \mathfrak{g} = \bigwedge^n (V \oplus \bigwedge^2 V) = \bigoplus_{s+r=n} \bigwedge^s (\bigwedge^2 V) \otimes \bigwedge^r (V)$$

with differentials $\partial_n : \bigwedge^s(\bigwedge^2 V) \otimes \bigwedge^r(V) \to \bigwedge^{s+1}(\bigwedge^2 V) \otimes \bigwedge^{r-2}(V)$ are given by

$$\begin{aligned} \partial_n : \ & e_{i_1 j_1} \wedge \ldots \wedge e_{i_s j_s} \ \otimes \ e_1 \wedge \ldots \wedge e_r \ \mapsto \\ & \sum_{i<j} (-1)^{i+j} e_{ij} \wedge e_{i_1 j_1} \wedge \ldots \wedge e_{i_s j_s} \ \otimes \ e_1 \wedge \ldots \wedge \hat{e_i} \wedge \ldots \wedge \hat{e_j} \wedge \ldots \wedge e_r \ . \end{aligned}$$

The differential ∂ identifies a pair of degree 1 generators $e_i, e_j \in V$ with one degree 2 generator $e_{ij} := (e_i \wedge e_j) = [e_i, e_j] \in \Lambda^2 V$.

The cohomologies $H^n(\mathfrak{g},\mathbb{K})$ arise from the dualized complex with coboundary map $\delta^n : \bigwedge^n \mathfrak{g}^* \to \bigwedge^{n+1} \mathfrak{g}^*$ which is transposed to the differential ∂_{n+1}

$$\begin{aligned} (5.1)\ \delta^n : \ & e^*_{i_1 j_1} \wedge \ldots \wedge e^*_{i_s j_s} \ \otimes \ e^*_{l_1} \wedge \ldots \wedge e^*_{l_r} \ \mapsto \\ & \sum_{k=1}^{s} \sum_{i_k<j_k} (-1)^{i+j} e^*_{i_1 j_1} \wedge \ldots \wedge \hat{e}^*_{i_k j_k} \wedge \ldots \wedge e^*_{i_s j_s} \ \otimes \ e^*_{i_k} \wedge e^*_{j_k} \wedge e^*_{l_1} \wedge \ldots \wedge \ldots \wedge e^*_{l_r} \ . \end{aligned}$$

In the presence of metric g one has an identification $V \overset{g}{\cong} V^*$, and $\bigwedge^\bullet \mathfrak{g} \overset{g}{\cong} \bigwedge^\bullet \mathfrak{g}^*$. The adjoint operator $\partial_n^* : \bigwedge^n \mathfrak{g} \to \bigwedge^{n+1} \mathfrak{g}$ is defined by $g(\partial_n^* v, w) = g(v, \partial_{n+1} w)$. One can show that independently of the metric g chosen the action of ∂_n^* takes the form

$$\begin{aligned} (5.2)\ \partial_n^* : \ & e_{i_1 j_1} \wedge \ldots \wedge e_{i_s j_s} \ \otimes \ e_{l_1} \wedge \ldots \wedge e_{l_r} \ \mapsto \\ & \sum_{k=1}^{s} \sum_{i_k<j_k} (-1)^{i+j} e_{i_1 j_1} \wedge \ldots \wedge \hat{e}_{i_k j_k} \wedge \ldots \wedge e_{i_s j_s} \ \otimes \ e_{i_k} \wedge e_{j_k} \wedge e_{l_1} \wedge \ldots \wedge \ldots \wedge e_{l_r} \ . \end{aligned}$$

We will see in the following that after the identification $\bigwedge^\bullet \mathfrak{g} \overset{g}{\cong} \bigwedge^\bullet \mathfrak{g}^*$ the map $\partial^* \overset{g}{=} \delta$ will play the role of homotopy for the chain complex $(\bigwedge^\bullet \mathfrak{g}, \partial_\bullet)$, and vice versa the boundary map $\partial \overset{g}{=} \delta^*$ is a homotopy for the cochain complex $(\bigwedge^\bullet \mathfrak{g}^*, \delta^\bullet)$.

The complexes $(\bigwedge^n \mathfrak{g}, \partial_n, \partial_n^*)$ and $(\bigwedge^n \mathfrak{g}^*, \delta^n, \delta^{*n})$ are bigraded by two different degrees; the homological degree $n := r + s$ counting the number of Lie algebra generators and the tensor degree $t := 2s + r$ also called weight. The cohomologies $H^n(\mathfrak{g},\mathbb{K})$ can have components of different weight t, $H^n(\mathfrak{g},\mathbb{K}) = \bigoplus_t H^n(\mathfrak{g},\mathbb{K})_t$ and the weight t is in fact the *Adams grading* on Yoneda algebra $\mathrm{Ext}^n_{U\mathfrak{g}}(\mathbb{K},\mathbb{K})_t$ [**13**]. The differential and the homotopy, $\delta = \partial^*$ and $\partial = \delta^*$ do not alter the weight t, but raise and lower the homological degree n.

The operations m_k in homotopy algebra are bigraded by homological and Adams gradings of bidegree $(k,t) = (2-k,0)$. The bi-grading impose the vanishing of many higher products.

5.1. Homology of $\mathfrak{g}$ as a $GL(V)$-module. A Schur module V_λ is an irreducible polynomial $GL(V)$-module labelled by a Young diagram λ. The basis of a Schur module V_λ is in bijection with semistandard Young tableaux which are fillings of the Young diagram λ with the numbers of the set $\{1,\dots,\dim V\}$. The action of the linear group $GL(V)$ on the space V of the generators of the Lie algebra $\mathfrak{g}$ induces a $GL(V)$-action on the universal enveloping algebra $PS = U\mathfrak{g} \cong S(V\oplus\Lambda^2 V)$ and on the space $\bigwedge^\bullet \mathfrak{g} \cong \bigwedge^\bullet (V\oplus\bigwedge^2 V)$.

The maps ∂ and ∂^* both commute with the $GL(V)$-action. It follows that the homology and cohomology carry structure of $GL(V)$-modules hence can be decomposed into irreducibles.

The *Laplacian* $\Delta = \oplus_{n\geqslant 0}\Delta_n$ is defined to be the self-adjoint operator

$$\Delta_n = \partial_{n+1}\partial_{n+1}^* + \partial_n^*\partial_n \in \mathrm{End}(\bigwedge^n \mathfrak{g}) \ .$$

Its kernel is a complete set of representatives for the homology classes in $H_p(\mathfrak{g},\mathbb{K})$

$$\ker \Delta_n \cong H_n(\mathfrak{g},\mathbb{K}) \ .$$

The decomposition of the $GL(V)$-module $H_n(\mathfrak{g},\mathbb{K})$ into irreducible polynomial representations V_λ is given by the following theorem;

THEOREM 5.1 (Józefiak and Weyman [**8**], Sigg [**14**]). *The homology $H_\bullet(\mathfrak{g},\mathbb{K})$ of the 2-nilpotent Lie algebra $\mathfrak{g} = V\oplus\bigwedge^2 V$ decomposes into irreducible $GL(V)$-modules*

$$H_n(\mathfrak{g},\mathbb{K}) = H_n(\bigwedge^\bullet \mathfrak{g},\partial) \cong \mathrm{Tor}_n^{PS}(\mathbb{K},\mathbb{K})(V) \cong \bigoplus_{\lambda:\lambda=\lambda'} V_\lambda \tag{5.3}$$

where the sum is over self-conjugate Young diagrams λ such that $n = \frac{1}{2}(|\lambda| + r(\lambda))$.

5.2. Homological interpretation of the Littlewood formula. We recall the beautiful result of Józefiak and Weyman [**8**] giving a representation-theoretic interpretation of the Littlewood formula

$$\prod_i (1-x_i)\prod_{i<j}(1-x_ix_j) = \sum_{\lambda:\lambda=\lambda'} (-1)^{\frac{1}{2}(|\lambda|+r(\lambda))} s_\lambda(x) \ . \tag{5.4}$$

Here the sum is over the self-dual Young diagrams λ, $s_\lambda(x)$ stands for the Schur function and $r(\lambda)$ stands the rank of λ which is the number of diagonal boxes in λ.

One knows that for the graded algebra PS there exists a minimal resolution by projective modules

$$P_\bullet : \qquad 0\to P_d\to\cdots\to P_n\to\cdots\to P_2\to P_1\to P_0\xrightarrow{\epsilon}\mathbb{K}\to 0 \ . \tag{5.5}$$

Here the length d of the resolution is the projective dimension of the algebra PS which is $d = \frac{\dim V(\dim V+1)}{2}$. Since PS is positively graded and, in the category of positively graded modules over connected locally finite graded algebras, projective module is the same as free module [**4**], we have $P_n \cong PS\otimes E_n$ where E_n are finite

dimensional vector spaces. Thus we deal with a minimal resolution of $\mathbb{K}$ by free PS-modules and the minimality implies that the derived complex $\mathbb{K} \otimes_{PS} P_\bullet$ has vanishing differentials, *i.e.*, $\mathrm{Tor}^{PS}_\bullet(\mathbb{K},\mathbb{K}) = H_\bullet(\mathbb{K} \otimes_{PS} P_\bullet) = \mathbb{K} \otimes_{PS} P_\bullet$. Then the multiplicity spaces $E_n = \mathrm{Tor}^{PS}_n(\mathbb{K},\mathbb{K})$ are fixed by Theorem 5.1 thus the data $H_n(\mathfrak{g},\mathbb{K}) = \mathrm{Tor}^{PS}_n(\mathbb{K},\mathbb{K})$ encodes the minimal free resolution $P_\bullet$ (cf. (5.5)) which is unique(up to isomorphism).

The Euler characteristics of $P_\bullet$ implies an identity about the $GL(V)$-characters

$$ch\, PS(V)\,.\, ch\left(\bigoplus_{\lambda:\lambda=\lambda'} (-1)^{\frac{1}{2}(|\lambda|+r(\lambda))} V_\lambda\right) = 1\ .$$

The character of a Schur module V_λ is the Schur function, $ch\, V_\lambda = s_\lambda(x)$. Due to the Poincaré-Birkhoff-Witt theorem $ch\, PS(V) = ch\, S(V \oplus \bigwedge^2 V)$ thus the identity reads

$$\prod_i \frac{1}{(1-x_i)} \prod_{i<j} \frac{1}{(1-x_i x_j)} \sum_{\lambda:\lambda=\lambda'} (-1)^{\frac{1}{2}(|\lambda|+r(\lambda))} s_\lambda(x) = 1\ .$$

But the latter identity is nothing but rewriting of the Littlewood identity (5.4). The moral is that the Littlewood identity reflects a homological property of the algebra PS, namely the above particular structure of the minimal projective (free) resolution of $\mathbb{K}$ by PS-modules.

5.3. $\mathrm{Ext}^\bullet_{PS}(\mathbb{K},\mathbb{K})$ as C_∞-algebra.

THEOREM 5.2. *The cohomology $H^\bullet(\mathfrak{g},\mathbb{K}) \cong \mathrm{Ext}^\bullet_{PS}(\mathbb{K},\mathbb{K})$ of the free 2-nilpotent graded Lie algebra $\mathfrak{g} = V \otimes \bigwedge^2 V$ is a homotopy commutative algebra which is generated in degree 1 (i.e., in $H^1(\mathfrak{g},\mathbb{K})$) by the operations m_2 and m_3.*

PROOF. We start by choosing a metric g on the vector space V and an orthonormal basis $g(e_i,e_j) = \delta_{ij}$. The choice induces a metric on $\bigwedge^\bullet \mathfrak{g} \overset{g}{\cong} \bigwedge^\bullet \mathfrak{g}^*$.

The isomorphisms $V \cong V^*$ and $\mathrm{Tor}^{PS}_n(\mathbb{K},\mathbb{K}) \cong \mathrm{Ext}^n_{PS}(\mathbb{K},\mathbb{K})$ and the theorem 5.1 imply the decomposition of $H^\bullet(\mathfrak{g},\mathbb{K})$ into irreducible $GL(V)$-modules

$$H^n(\mathfrak{g},\mathbb{K}) \cong H^n(\bigwedge \mathfrak{g}^*, \delta) \cong \mathrm{Ext}^n_{PS}(\mathbb{K},\mathbb{K}) \cong \bigoplus_{\lambda:\lambda=\lambda'} V_\lambda$$

where the sum is over self-conjugate diagrams λ such that $n = \frac{1}{2}(|\lambda| + r(\lambda))$.

The adjoint of the boundary map ∂, $\delta \overset{g}{:=} \partial^*$ is the differential in the DGA $(\bigwedge \mathfrak{g}^*, \delta)$ while $\delta^* \overset{g}{:=} \partial$ plays the role of a homotopy. In view of lemma 2.1 we have the cohomology $H^\bullet(\bigwedge^\bullet \mathfrak{g}^*, \delta^\bullet)$ as deformation retract of the complex $(\bigwedge^\bullet \mathfrak{g}^*, \delta^\bullet)$,

$$pi = Id_{H^\bullet(\bigwedge^\bullet \mathfrak{g}^*)}\ , \qquad ip - Id_{\bigwedge^\bullet \mathfrak{g}^*} = \delta\delta^* + \delta^*\delta\ , \qquad \delta^* \overset{g}{=} \partial\ .$$

Here the projection p identifies the subspace $\ker\delta \cap \ker\delta^*$ with $H^\bullet(\bigwedge^\bullet \mathfrak{g}^*)$, which is the orthogonal complement of the space of the coboundaries $\mathrm{im}\delta$. The cocycle-choosing homomorphism i is Id on $H^\bullet(\bigwedge^\bullet \mathfrak{g}^*)$ and zero on coboundaries.

We apply the Kadeishvili homotopy transfer Theorem 2.1 for the commutative DGA $(\bigwedge^\bullet \mathfrak{g}^*, \mu, \delta^\bullet)$ and its deformation retract $H^\bullet(\bigwedge^\bullet \mathfrak{g}^*) \cong H^\bullet(\mathfrak{g},\mathbb{K})$ and conclude that the cohomology $H^\bullet(\mathfrak{g},\mathbb{K})$ is a C_∞-algebra.

The Kontsevich and Soibelman tree representations of the operations m_n provide explicit expressions. Let us take μ to be the super-commutative product $\wedge$ on the DGA $(\bigwedge^\bullet \mathfrak{g}^*, \delta^\bullet)$. The projection p maps onto the Schur modules V_λ with self-conjugated Young diagram $\lambda = \lambda'$.

The binary operation on the generators $e_i \in H^1(\mathfrak{g}, \mathbb{K})$ is trivial, one gets

$$m_2(e_i, e_j) = p(e_i \wedge e_j) = 0 \qquad p(V_{(1^2)}) = 0.$$

Hence $H^\bullet(\mathfrak{g}, \mathbb{K})$ could not be generated in $H^1(\mathfrak{g}, \mathbb{K})$ as algebra with product m_2.

The ternary operation m_3 restricted to $H^1(\mathfrak{g}, \mathbb{K})$ is nontrivial, indeed one has

$$\begin{aligned} m_3(e_i, e_j, e_k) = p\left\{e_i \wedge \partial(e_j \wedge e_k) - \partial(e_i \wedge e_j) \wedge e_k\right\} &= p\left\{e_{ij} \wedge e_k - e_i \wedge e_{jk}\right\} \\ = p\left\{(e_{ij} \wedge e_k + e_{jk} \wedge e_i + e_{ki} \wedge e_j) - e_{ki} \wedge e_j\right\} &= e_{ik} \wedge e_j \in H^2(\mathfrak{g}, \mathbb{K}) \end{aligned}$$

The completely antisymmetric combination in the brackets $(\ldots)$ spans the Schur module $V_{(1^3)}$, $p(e_{ij} \wedge e_k + e_{jk} \wedge e_i + e_{ki} \wedge e_j) = 0$ yields a Jacobi-type identity. The monomials $e_{ij} \wedge e_k$ modulo $V_{(1^3)}$ span a Schur module $V_{(2,1)} \in H^2(\mathfrak{g}, \mathbb{K})$ with basis in bijection with the semistandard Young tableaux

$$e_{ik} \wedge e_j \leftrightarrow \begin{array}{|c|c|}\hline i & j \\ \hline k \\ \cline{1-1}\end{array} \qquad \text{and} \qquad e_{ij} \wedge e_k \leftrightarrow \begin{array}{|c|c|}\hline i & k \\ \hline j \\ \cline{1-1}\end{array}\,.$$

We check the symmetry condition on ternary operation m_3 in C_∞-algebra; indeed m_3 vanishes on the (signed) shuffles $Sh_{1,2}$

$$m_3(e_i \,\Delta\, e_j \otimes e_k) = m_3(e_i, e_j, e_k) - m_3(e_j, e_i, e_k) + m_3(e_j, e_k, e_i) = 0.$$

Similarly one gets $m_3(e_i \otimes e_j \,\Delta\, e_k) = 0$ on shuffles $Sh_{2,1}$.

On the level of Schur modules the ternary operation glues three fundamental $GL(V)$-representations $V_\square$ into a Schur module $V_{(2,1)}$. By iteration of the process of gluing boxes we generate all elementary hooks $V_k := V_{(k+1,1^k)}$,

$$\begin{aligned} m_3(V_\square, V_\square, V_\square) &= V_{\begin{array}{|c|c|}\hline \\ \hline \\ \cline{1-1}\end{array}} \\ m_3\left(V_\square, V_{\begin{array}{|c|c|}\hline \\ \hline \\ \cline{1-1}\end{array}}, V_\square\right) &= V_{\begin{array}{|c|c|c|}\hline \\ \hline \\ \cline{1-1}\end{array}} \\ &\ldots \\ m_3(V_0, V_k, V_0) &= V_{k+1}\ . \end{aligned}$$

In our context the more convenient notation for Young diagrams is due to Frobenius: $\lambda := (a_1, \ldots, a_r | b_1, \ldots b_r)$ stands for a diagram λ with a_i boxes in the i-th row on the right of the diagonal, and with b_i boxes in the i-th column below the diagonal and the rank $r = r(\lambda)$ is the number of boxes on the diagonal.

For self-dual diagrams $\lambda = \lambda'$, i.e., $a_i = b_i$ we set $V_{a_1,\ldots,a_r} := V_{(a_1,\ldots,a_r|a_1,\ldots a_r)}$ when $a_1 > a_2 > \ldots > a_r \geqslant 0$ (and set the convention $V_{a_1,\ldots,a_r} := 0$ otherwise). Any two elementary hooks V_{a_1} and V_{a_2} can be glued together by the binary operation m_2, the decomposition of $m_2(V_{a_1}, V_{a_2}) \cong m_2(V_{a_2}, V_{a_1})$ is given by

$$m_2(V_{a_1}, V_{a_2}) = V_{a_1,a_2} \oplus (\bigoplus_{i=1}^{a_2} V_{a_1+i,a_2-i}) \qquad a_1 \geqslant a_2$$

where the "leading" term V_{a_1,a_2} has the diagram with minimal height. Hence any m_2-bracketing of the hooks $V_{a_1}, V_{a_2} \ldots, V_{a_r}$ yields[3] a sum of $GL(V)$-modules

$$m_2(\ldots m_2(m_2(V_{a_1}, V_{a_2}), V_{a_3}), \ldots, V_{a_r}) = V_{a_1,\ldots,a_r} \oplus \ldots$$

whose module with minimal height is precisely $V_{a_1,\ldots,a_r}$. We conclude that all elements in the C_∞-algebra $H^\bullet(\mathfrak{g}, \mathbb{K})$ can be generated in $H^1(\mathfrak{g}, \mathbb{K})$ by m_2 and m_3. □

One could draw a parallel between the latter theorem for the cubic algebra PS and the proposition 4.1 for Koszul algebra; in both cases the Yoneda algebra $\mathrm{Ext}^\bullet_{PS}(\mathbb{K}, \mathbb{K})$ is generated only in $\mathrm{Ext}^1_{PS}(\mathbb{K}, \mathbb{K})$. Although we have the notion of N-Koszul algebras for the N-homogeneous algebras [**2, 3**], it turns out that the cubic algebra PS is not 3-Koszul, beside the exceptional case when $\dim V = 2$. Instead the algebra $PS = U\mathfrak{g}$ falls in the class of *Artin-Schelter-regular algebras* [**1**], being an UEA of positively graded Lie algebra (for a proof see [**6**]). The parallel between the quadratic Koszul algebra $S(V)$ and the cubic AS-regular regular algebra $PS(V)$ suggests that the C_∞-algebra $\mathrm{Ext}^\bullet_{PS}(\mathbb{K}, \mathbb{K})$ is a generalization of a Koszul dual algebra of PS in the realm of the homotopy algebras, an idea that has been put forward in [**13**].

The analogy would be complete if we had the following conjectural proposition.

Proposition 5.1. *The cohomology $H^\bullet(\mathfrak{g}, \mathbb{K}) \cong \mathrm{Ext}^\bullet_{PS}(\mathbb{K}, \mathbb{K})$ of the free 2-nilpotent graded Lie algebra $\mathfrak{g} = V \otimes \bigwedge^2 V$ can be endowed with a structure of C_∞-algebra having trivial higher multiplications $m_k = 0$, $k \geqslant 4$.*

So far we were able to prove this conjecture only in dimensions $\dim V \leqslant 3$. Our proof rests entirely on the bigrading $(2 - k, 0)$ of the multiplication m_k by homological and tensor degree in the C_∞-algebra $\mathrm{Ext}^\bullet_{PS}(\mathbb{K}, \mathbb{K})$. The bigrading arguments work only for $\dim V = 2$ and $\dim V = 3$ thus for a complete proof the conjecture would need more refined methods.

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[3]The operation m_2 is associative thus the result does not depend on the choice of the bracketing.

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