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► **To cite this version:**

Michel Dubois-Violette, Todor Popov. C_∞ -structure on the cohomology of free 2-nilpotent Lie algebra. Publications de l'Institut Mathématique., 2013, 94 (108), pp.99-109. 10.2298/PIM1308099D . hal-00777930

HAL Id: hal-00777930

<https://hal.science/hal-00777930>

Submitted on 18 Jan 2013

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2-Step Nilpotent Lie algebra and C_∞ -Algebras

Michel Dubois-Violette and Todor Popov

Communicated by

ABSTRACT. We consider the free 2-step nilpotent graded Lie algebra and its cohomology ring. The homotopy transfer induces a homotopy commutative algebra on its cohomology ring which we describe.

1. Homotopy algebras

The homotopy associative algebras, or A_∞ -algebras were introduced by Jim Stasheff in the 1960's as a tool in algebraic topology for studying 'group-like' spaces. Homotopy algebras received a new attention and further development in the 1990's after the discovery of their relevance into a multitude of topics in algebraic geometry, symplectic and contact geometry, knot theory, moduli spaces, deformation theory...

DEFINITION 1.1. (A_∞ -algebra) A homotopy associative algebra, or A_∞ -algebra, over \mathbb{K} is a \mathbb{Z} -graded vector space $A = \bigoplus_{i \in \mathbb{Z}} A^i$ endowed with a family of graded mappings (operations)

$$m_n : A^{\otimes n} \rightarrow A, \quad \deg(m_n) = 2 - n \quad n \geq 1$$

satisfying the Stasheff identities **SI**(\mathbf{n}) for $n \geq 1$

$$\mathbf{SI}(\mathbf{n}) : \sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(Id^{\otimes r} \otimes m_s \otimes Id^{\otimes t}) = 0 \quad r \geq 0, t \geq 0, s \geq 1$$

where the sum runs over all decompositions $n = r + s + t$. Throughout the text we assume the Koszul sign convention $(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y)$.

A morphism of two A_∞ -algebras A and B is a family of graded maps $f_n : A^{\otimes n} \rightarrow B$ for $n \geq 1$ with $\deg f_n = 1 - n$ such that the following conditions hold

$$\sum_{r+s+t=n} (-1)^{r+st} f_{r+1+t}(Id^{\otimes r} \otimes m_s \otimes Id^{\otimes t}) = \sum_{1 \leq r \leq n} (-1)^S m_r(f_{i_1} \otimes f_{i_2} \otimes \dots \otimes f_{i_r})$$

2010 *Mathematics Subject Classification.* Primary 17B35, 17B56; Secondary 18G10, 17D98.
Partially supported by ...

where the sum is on all decompositions $i_1 + \dots + i_r = n$ and the sign $(-1)^S$ on RHS is determined by $S = (r-1)(i_1-1) + (r-2)(i_2-1) + \dots + 2(i_{r-2}-1) + (i_{r-1}-1)$. The morphism f is a *quasi-isomorphism of A_∞ -algebras* if f_1 is a quasi-isomorphism. It is strict if $f_i = 0$ for all $i \neq 1$. The identity morphism of A is the strict morphism f such that f_1 is the identity of A .

We define the shuffle product $Sh_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \rightarrow A^{\otimes p+q}$ throughout the expression

$$(a_1 \otimes \dots \otimes a_p) \Delta (a_{p+1} \otimes \dots \otimes a_{p+q}) = \sum_{\sigma \in Sh_{p,q}} sgn(\sigma) a_{\sigma^{-1}(1)} \otimes \dots \otimes a_{\sigma^{-1}(p+q)}$$

where the sum runs over all (p, q) -shuffles $Sh_{p,q}$, i.e., over all permutations $\sigma \in S_{p+q}$ such that $\sigma(1) < \sigma(2) < \dots < \sigma(p)$ and $\sigma(p+1) < \sigma(p+2) < \dots < \sigma(p+q)$.

DEFINITION 1.2. (C_∞ -algebra [9]) A homotopy commutative algebra, or C_∞ -algebra, is an A_∞ -algebra $\{A, m_n\}$ such that each operation m_n vanishes on non-trivial shuffles

$$(1.1) \quad m_n((a_1 \otimes \dots \otimes a_p) \Delta (a_{p+1} \otimes \dots \otimes a_n)) = 0, \quad 1 \leq p \leq n-1.$$

In particular for m_2 we have $m_2(a \otimes b \pm b \otimes a) = 0$, so a C_∞ -algebra such that $m_n = 0$ for $n \geq 3$ is a (super-)commutative DGA.

A morphism of C_∞ -algebras is a morphism of A_∞ -algebras vanishing on non-trivial shuffles $f_n((a_1 \otimes \dots \otimes a_p) \Delta (a_{p+1} \otimes \dots \otimes a_n)) = 0$, $1 \leq p \leq n-1$.

2. Homotopy Transfer Theorem

LEMMA 2.1. *Every cochain complex (A, d) of vector spaces over a field \mathbb{K} has its cohomology $H^\bullet(A)$ as a deformation retract.*

One can always choose a vector space decomposition of the cochain complex (A, d) such that $A^n \cong B^n \oplus H^n \oplus B^{n+1}$ where H^n is the cohomology and B^n is the space of coboundaries, $B^n = dA^{n-1}$. We choose a homotopy $h : A^n \rightarrow A^{n-1}$ which identifies B^n with its copy in A^{n-1} and is 0 on $H^n \oplus B^{n+1}$. The projection p to the cohomology and the cocycle-choosing inclusion i given by $A^n \begin{matrix} \xrightarrow{p} \\ \xleftarrow{i} \end{matrix} H^n$ are chain homomorphisms, satisfying the additional *side conditions*

$$hh = 0, \quad hi = 0, \quad ph = 0.$$

With these choices done the complex $(H^\bullet(A), 0)$ is a deformation retract of (A, d)

$$h \circlearrowleft (A, d) \begin{matrix} \xrightarrow{p} \\ \xleftarrow{i} \end{matrix} (H^\bullet(A), 0), \quad pi = Id_{H^\bullet(A)}, \quad ip - Id_A = dh + hd.$$

Let now (A, d, μ) be a DGA, i.e., A is endowed with an associative product μ compatible with d . The cochain complexes (A, d) and its contraction $H^\bullet(A)$ are homotopy equivalent, but the associative structure is not stable under homotopy equivalence. However the associative structure on A can be transferred to an A_∞ -structure on a homotopy equivalent complex, a particular interesting complex being the deformation retract $H^\bullet(A)$. For a friendly introduction to homotopy transfer

theorems in much boarder context we send the reader to the textbook [12], see chapter 9.

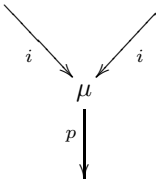
THEOREM 2.1 (Kadeishvili [9]). *Let (A, d, μ) be a (commutative) DGA over a field \mathbb{K} . There exists a A_∞ -algebra (C_∞ -algebra) structure on the cohomology $H^\bullet(A)$ and a $A_\infty(C_\infty)$ -quasi-isomorphism*

$$f_i : (\otimes^i H^\bullet(A), \{m_i\}) \rightarrow (A, \{d, \mu, 0, 0, \dots\})$$

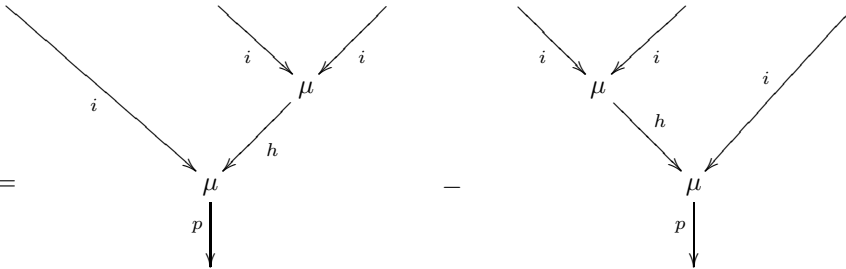
such that the inclusion $f_1 = i : H^\bullet(A) \rightarrow A$ is a cocycle-choosing homomorphism of cochain complexes. The differential m_1 on $H^\bullet(A)$ is zero ($m_1 = 0$) and m_2 is the strictly associative operation induced by the multiplication on A . The resulting structure is unique up to quasi-isomorphism.

Kontsevich and Soibelman [10] gave an explicit expressions for the higher operations of the induced A_∞ -structure as sums over decorated planar binary trees with one root where all leaves are decorated by the inclusion i , the root by the projection p the vertices by the product μ of the (commutative) DGA (A, d, μ) and the internal edges by the homotopy h . The C_∞ -structure implies additional symmetries on trees.

We will make use of the graphic representation for the binary operation on $H^\bullet(A)$

$$m_2(x, y) := p\mu(i(x), i(y)) \quad \text{or} \quad m_2 =$$


and the ternary one $m_3(x, y, z) = p\mu(i(x), h\mu(i(y), i(z))) - p\mu(h\mu(i(x), i(y)), i(z))$ being the sum of two planar binary trees with three leaves

$$m_3 =$$


3. Homology and cohomology of Lie algebra \mathfrak{g}

A non-minimal projective(in fact free) resolution of the trivial $U\mathfrak{g}$ -module \mathbb{K} , $C(\mathfrak{g}) \xrightarrow{\epsilon} \mathbb{K}$ is given by the standard Chevalley-Eilenberg chain complex $C_\bullet(\mathfrak{g}) =$

$(U\mathfrak{g} \otimes_{\mathbb{K}} \wedge^p \mathfrak{g}, d_p)$ with differential maps

$$(3.1) \quad \begin{aligned} d_p(u \otimes x_1 \wedge \dots \wedge x_p) &= \sum_i (-1)^{i+1} u x_i \otimes x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge x_p \\ &+ \sum_{i < j} (-1)^{i+j} u [x_i, x_j] \wedge x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge \hat{x}_j \wedge \dots \wedge x_p \end{aligned}$$

The homologies $H_n(\mathfrak{g}, \mathbb{K})$ of the Lie algebra \mathfrak{g} with trivial coefficients are given by the homologies of the derived complex $\mathbb{K} \otimes_{U\mathfrak{g}} C_{\bullet}(\mathfrak{g})$

$$\mathrm{Tor}_n^{U\mathfrak{g}}(\mathbb{K}, \mathbb{K}) \cong H_n(\mathbb{K} \otimes_{U\mathfrak{g}} C_{\bullet}(\mathfrak{g})) = H_n(\mathfrak{g}, \mathbb{K}) .$$

The complex $\mathbb{K} \otimes_{U\mathfrak{g}} C_{\bullet}(\mathfrak{g})$ is the chain complex with degrees $\wedge^{\bullet} \mathfrak{g} = \mathbb{K} \otimes_{U\mathfrak{g}} U\mathfrak{g} \otimes \wedge^{\bullet} \mathfrak{g}$ and differentials $\partial_p := id \otimes_{U\mathfrak{g}} d_p : \wedge^p \mathfrak{g} \rightarrow \wedge^{p-1} \mathfrak{g}$ induced by the extension as coderivation of the Lie bracket $\partial_2 := -[\cdot, \cdot] : \wedge^2 \mathfrak{g} \rightarrow \mathfrak{g}$.

The dual cochain complex $\mathrm{Hom}_{U\mathfrak{g}}(C(\mathfrak{g}), \mathbb{K}) = (\wedge^{\bullet} \mathfrak{g}^*, \delta)$ has coboundary map $\delta^p : \wedge^p \mathfrak{g}^* \rightarrow \wedge^{p+1} \mathfrak{g}^*$ (being transposed to the differential ∂_{p+1}) which is the extension as derivation of the dualization of the Lie bracket $\delta^1 := [\cdot, \cdot]^* : \mathfrak{g}^* \rightarrow \wedge^2 \mathfrak{g}^*$. One calculates the cohomologies¹ of the Lie algebra \mathfrak{g} as

$$(3.2) \quad \mathrm{Ext}_{U\mathfrak{g}}^n(\mathbb{K}, \mathbb{K}) \cong H^n(\mathrm{Hom}_{U\mathfrak{g}}(C(\mathfrak{g}), \mathbb{K})) = H^n(\mathfrak{g}, \mathbb{K}) .$$

Hence the algebra $(\wedge^{\bullet} \mathfrak{g}^*, \delta)$ equipped with δ is a (*super*)commutative DGA and the Yoneda algebra $\mathrm{Ext}_{U\mathfrak{g}}^{\bullet}(\mathbb{K}, \mathbb{K}) = \bigoplus_n \mathrm{Ext}_{U\mathfrak{g}}^n(\mathbb{K}, \mathbb{K})$ has structure of commutative associative algebra. Moreover due to the Kadeishvili theorem the Yoneda algebra $\mathrm{Ext}_{U\mathfrak{g}}^{\bullet}(\mathbb{K}, \mathbb{K}) = H^{\bullet}(\mathfrak{g}, \mathbb{K})$ is a C_{∞} -algebra which stems from the homotopy transfer of the wedge product \wedge on cohomology classes

$$H^i(\mathfrak{g}, \mathbb{K}) \wedge H^j(\mathfrak{g}, \mathbb{K}) \rightarrow H^{i+j}(\mathfrak{g}, \mathbb{K}) .$$

4. Abelian Lie algebra $\mathfrak{h} = V$

Let us take as a basic example the abelian Lie algebra \mathfrak{h} , that is, the free nilpotent Lie algebra generated by a finite dimensional vector space V of degree 1. The Lie bracket of \mathfrak{h} is trivial $[V, V] = 0$. According to Poincaré-Birkhoff-Witt theorem the universal enveloping algebra of the abelian Lie algebra $\mathfrak{h} = V$ is isomorphic to the symmetric algebra

$$U(\mathfrak{h}) \cong S(V) .$$

The Chevalley-Eilenberg complex $C_{\bullet}(\mathfrak{h}) = S(V) \otimes_{\mathbb{K}} \wedge^{\bullet} V$ yields the resolution of the trivial $U(\mathfrak{h})$ -module \mathbb{K}

$$(4.1) \quad \begin{aligned} 0 &\rightarrow S(V) \otimes \wedge^{\dim V} V \rightarrow S(V) \otimes \wedge^{\dim V - 1} V \rightarrow \dots \\ \dots &\rightarrow S(V) \otimes \wedge^2 V \rightarrow S(V) \otimes V \rightarrow S(V) \rightarrow \mathbb{K} \rightarrow 0 . \end{aligned}$$

The derived complex $\mathbb{K} \otimes_{U\mathfrak{h}} C(\mathfrak{h})$ has zero differential and the Chevalley-Eilenberg resolution turns out to be minimal(which is not the case in general)

$$H_n(\mathfrak{h}, \mathbb{K}) \cong H_n(\mathbb{K} \otimes_{U\mathfrak{h}} C(\mathfrak{h})) \cong \wedge^n V .$$

¹In the presence of metric one has $\delta := \partial^*$ (see below)

The Chevalley-Eilenberg resolution coincides with the Koszul complex $K(A) = A \otimes (A^!)^*$ of the symmetric algebra $A = S(V)$. The Koszul dual algebra of the symmetric algebra is the exterior algebra $S(V)^! = \Lambda V^*$. A quadratic algebra is said to be a Koszul algebra when its Koszul complex $K_\bullet(A) = A \otimes (A^!)^*$ is acyclic everywhere except in degree 0 (where its homology is \mathbb{K}). Then the Koszul complex yields a minimal projective (in fact free) resolution by (left) A -modules of the trivial A -module \mathbb{K}

$$K(A) \xrightarrow{\epsilon} \mathbb{K} \rightarrow 0 .$$

In particular the resolution 4.1 is the same as the the resolution by the Koszul complex $K_n(S(V)) = S(V) \otimes \Lambda^n V^*$ thus the algebra $S(V)$ is *Koszul algebra*. One has another equivalent definition of Koszul algebra based on the following proposition;

PROPOSITION 4.1. *A finitely generated quadratic algebra A is Koszul iff its Yoneda algebra $\text{Ext}_A(\mathbb{K}, \mathbb{K})$ is generated in degree 1. One has then $\text{Ext}_A(\mathbb{K}, \mathbb{K}) \cong A^!$.*

Indeed the Yoneda algebra $\text{Ext}_{S(V)}(\mathbb{K}, \mathbb{K})$ of the symmetric algebra $S(V)$ is just the exterior algebra

$$\text{Ext}_{S(V)}^n(\mathbb{K}, \mathbb{K}) = (\text{Tor}_n^{S(V)}(\mathbb{K}, \mathbb{K}))^* = \Lambda^n V^*$$

which is obviously generated by V^* , *i.e.*, in degree 1, by the wedge product. Through the homotopy transfer the Yoneda algebra $\text{Ext}_{S(V)}(\mathbb{K}, \mathbb{K})$ inherits a C_∞ -structure but it is easy to show (by degree preserving argument) that the latter C_∞ -algebra is formal, *i.e.*, all higher multiplications are trivial, $m_n = 0$ for $n \neq 2$.

5. Homology of Free 2-nilpotent algebra $\mathfrak{g} = V \oplus \Lambda^2 V$

Let \mathfrak{g} be the free graded 2-step nilpotent Lie algebra generated by vector space V in degree 1, $\mathfrak{g} = V \oplus [V, V]$. In other words we consider the graded Lie algebra \mathfrak{g} with Lie bracket

$$[u, v] = \begin{cases} u \wedge v & \in \Lambda^2 V \\ 0 & \text{otherwise} \end{cases}$$

We denote the Universal Enveloping Algebra(UEA) $U\mathfrak{g}$ by PS and refer to it as *parastatistics algebra*.² Throughout this note we will consider the generators space V to be an ordinary vector space V which corresponds to a parafermionic algebra $PS(V) = U\mathfrak{g}$. The case of a \mathbb{Z}_2 -space of generators $V = V_0 \oplus V_1$, that is, $PS(V)$ is the Universal Enveloping Algebra of a Lie super-algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ (which would include the parabosonic algebras) will be treated elsewhere. More on parastatistics algebras and their application to combinatorics the reader could find in the articles [5, 11].

²Such cubic algebras arise through the exchange relations between the operators in a quantization procedure introduced by H. S. Green [7] for particles obeying more general statistics than Bose-Einstein or Fermi-Dirac, coined parabosons and parafermions.

The parastatistics algebra $PS(V)$ generated by a finite dimensional vector space V in degree 1 is the positively graded algebra

$$PS(V) := U\mathfrak{g} = U(V \oplus \bigwedge^2 V) = T(V)/(\llbracket [V, V], V \rrbracket).$$

We shall write simply PS when the space of generators V is clear from the context.

The homologies $H_n(\mathfrak{g}, \mathbb{K})$ of the 2-nilpotent Lie algebra \mathfrak{g} are the homologies of the chain complex

$$\bigwedge^n \mathfrak{g} = \bigwedge^n (V \oplus \bigwedge^2 V) = \bigoplus_{s+r=n} \bigwedge^s (\bigwedge^2 V) \otimes \bigwedge^r (V)$$

with differentials $\partial_n : \bigwedge^s (\bigwedge^2 V) \otimes \bigwedge^r (V) \rightarrow \bigwedge^{s+1} (\bigwedge^2 V) \otimes \bigwedge^{r-2} (V)$ are given by

$$\begin{aligned} \partial_n : e_{i_1 j_1} \wedge \dots \wedge e_{i_s j_s} \otimes e_1 \wedge \dots \wedge e_r &\mapsto \\ \sum_{i < j} (-1)^{i+j} e_{ij} \wedge e_{i_1 j_1} \wedge \dots \wedge e_{i_s j_s} \otimes e_1 \wedge \dots \wedge \hat{e}_i \wedge \dots \wedge \hat{e}_j \wedge \dots \wedge e_r. \end{aligned}$$

The differential ∂ identifies a pair of degree 1 generators $e_i, e_j \in V$ with one degree 2 generator $e_{ij} := (e_i \wedge e_j) = [e_i, e_j] \in \bigwedge^2 V$.

The cohomologies $H^n(\mathfrak{g}, \mathbb{K})$ arise from the dualized complex with coboundary map $\delta^n : \bigwedge^n \mathfrak{g}^* \rightarrow \bigwedge^{n+1} \mathfrak{g}^*$ which is transposed to the differential ∂_{n+1}

$$\begin{aligned} (5.1) \quad \delta^n : e_{i_1 j_1}^* \wedge \dots \wedge e_{i_s j_s}^* \otimes e_{l_1}^* \wedge \dots \wedge e_{l_r}^* &\mapsto \\ \sum_{k=1}^s \sum_{i_k < j_k} (-1)^{i+j} e_{i_1 j_1}^* \wedge \dots \wedge \hat{e}_{i_k j_k}^* \wedge \dots \wedge e_{i_s j_s}^* \otimes e_{i_k}^* \wedge e_{j_k}^* \wedge e_{l_1}^* \wedge \dots \wedge \dots \wedge e_{l_r}^*. \end{aligned}$$

In the presence of metric g one has an identification $V \stackrel{g}{\cong} V^*$, and $\bigwedge^\bullet \mathfrak{g} \stackrel{g}{\cong} \bigwedge^\bullet \mathfrak{g}^*$. The adjoint operator $\partial_n^* : \bigwedge^n \mathfrak{g} \rightarrow \bigwedge^{n+1} \mathfrak{g}$ is defined by $g(\partial_n^* v, w) = g(v, \partial_{n+1} w)$. One can show that independently of the metric g chosen the action of ∂_n^* takes the form

$$\begin{aligned} (5.2) \quad \partial_n^* : e_{i_1 j_1} \wedge \dots \wedge e_{i_s j_s} \otimes e_{l_1} \wedge \dots \wedge e_{l_r} &\mapsto \\ \sum_{k=1}^s \sum_{i_k < j_k} (-1)^{i+j} e_{i_1 j_1} \wedge \dots \wedge \hat{e}_{i_k j_k} \wedge \dots \wedge e_{i_s j_s} \otimes e_{i_k} \wedge e_{j_k} \wedge e_{l_1} \wedge \dots \wedge \dots \wedge e_{l_r}. \end{aligned}$$

We will see in the following that after the identification $\bigwedge^\bullet \mathfrak{g} \stackrel{g}{\cong} \bigwedge^\bullet \mathfrak{g}^*$ the map $\partial^* \stackrel{g}{=} \delta$ will play the role of homotopy for the chain complex $(\bigwedge^\bullet \mathfrak{g}, \partial_\bullet)$, and vice versa the boundary map $\partial \stackrel{g}{=} \delta^*$ is a homotopy for the cochain complex $(\bigwedge^\bullet \mathfrak{g}^*, \delta^\bullet)$.

The complexes $(\bigwedge^n \mathfrak{g}, \partial_n, \partial_n^*)$ and $(\bigwedge^n \mathfrak{g}^*, \delta^n, \delta^{*n})$ are bigraded by two different degrees; the homological degree $n := r + s$ counting the number of Lie algebra generators and the tensor degree $t := 2s + r$ also called weight. The cohomologies $H^n(\mathfrak{g}, \mathbb{K})$ can have components of different weight t , $H^n(\mathfrak{g}, \mathbb{K}) = \bigoplus_t H^n(\mathfrak{g}, \mathbb{K})_t$ and the weight t is in fact the *Adams grading* on Yoneda algebra $\text{Ext}_{U\mathfrak{g}}^n(\mathbb{K}, \mathbb{K})_t$ [13]. The differential and the homotopy, $\delta = \partial^*$ and $\partial = \delta^*$ do not alter the weight t , but raise and lower the homological degree n .

The operations m_k in homotopy algebra are bigraded by homological and Adams gradings of bidegree $(k, t) = (2 - k, 0)$. The bi-grading impose the vanishing of many higher products.

5.1. Homology of \mathfrak{g} as a $GL(V)$ -module. A Schur module V_λ is an irreducible polynomial $GL(V)$ -module labelled by a Young diagram λ . The basis of a Schur module V_λ is in bijection with semistandard Young tableaux which are fillings of the Young diagram λ with the numbers of the set $\{1, \dots, \dim V\}$. The action of the linear group $GL(V)$ on the space V of the generators of the Lie algebra \mathfrak{g} induces a $GL(V)$ -action on the universal enveloping algebra $PS = U\mathfrak{g} \cong S(V \oplus \Lambda^2 V)$ and on the space $\bigwedge^\bullet \mathfrak{g} \cong \bigwedge^\bullet (V \oplus \Lambda^2 V)$.

The maps ∂ and ∂^* both commute with the $GL(V)$ -action. It follows that the homology and cohomology carry structure of $GL(V)$ -modules hence can be decomposed into irreducibles.

The *Laplacian* $\Delta = \bigoplus_{n \geq 0} \Delta_n$ is defined to be the self-adjoint operator

$$\Delta_n = \partial_{n+1} \partial_{n+1}^* + \partial_n^* \partial_n \in \text{End}(\bigwedge^n \mathfrak{g}) .$$

Its kernel is a complete set of representatives for the homology classes in $H_p(\mathfrak{g}, \mathbb{K})$

$$\ker \Delta_n \cong H_n(\mathfrak{g}, \mathbb{K}) .$$

The decomposition of the $GL(V)$ -module $H_n(\mathfrak{g}, \mathbb{K})$ into irreducible polynomial representations V_λ is given by the following theorem;

THEOREM 5.1 (Józefiak and Weyman [8], Sigg [14]). *The homology $H_\bullet(\mathfrak{g}, \mathbb{K})$ of the 2-nilpotent Lie algebra $\mathfrak{g} = V \oplus \Lambda^2 V$ decomposes into irreducible $GL(V)$ -modules*

$$(5.3) \quad H_n(\mathfrak{g}, \mathbb{K}) = H_n(\bigwedge^\bullet \mathfrak{g}, \partial) \cong \text{Tor}_n^{PS}(\mathbb{K}, \mathbb{K})(V) \cong \bigoplus_{\lambda: \lambda = \lambda'} V_\lambda$$

where the sum is over self-conjugate Young diagrams λ such that $n = \frac{1}{2}(|\lambda| + r(\lambda))$.

5.2. Homological interpretation of the Littlewood formula. We recall the beautiful result of Józefiak and Weyman [8] giving a representation-theoretic interpretation of the Littlewood formula

$$(5.4) \quad \prod_i (1 - x_i) \prod_{i < j} (1 - x_i x_j) = \sum_{\lambda: \lambda = \lambda'} (-1)^{\frac{1}{2}(|\lambda| + r(\lambda))} s_\lambda(x) .$$

Here the sum is over the self-dual Young diagrams λ , $s_\lambda(x)$ stands for the Schur function and $r(\lambda)$ stands the rank of λ which is the number of diagonal boxes in λ .

One knows that for the graded algebra PS there exists a minimal resolution by projective modules

$$(5.5) \quad P_\bullet : \quad 0 \rightarrow P_d \rightarrow \dots \rightarrow P_n \rightarrow \dots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \xrightarrow{\epsilon} \mathbb{K} \rightarrow 0 .$$

Here the length d of the resolution is the projective dimension of the algebra PS which is $d = \frac{\dim V(\dim V + 1)}{2}$. Since PS is positively graded and, in the category of positively graded modules over connected locally finite graded algebras, projective module is the same as free module [4], we have $P_n \cong PS \otimes E_n$ where E_n are finite

dimensional vector spaces. Thus we deal with a minimal resolution of \mathbb{K} by free PS -modules and the minimality implies that the derived complex $\mathbb{K} \otimes_{PS} P_\bullet$ has vanishing differentials, *i.e.*, $\text{Tor}_\bullet^{PS}(\mathbb{K}, \mathbb{K}) = H_\bullet(\mathbb{K} \otimes_{PS} P_\bullet) = \mathbb{K} \otimes_{PS} P_\bullet$. Then the multiplicity spaces $E_n = \text{Tor}_n^{PS}(\mathbb{K}, \mathbb{K})$ are fixed by Theorem 5.1 thus the data $H_n(\mathfrak{g}, \mathbb{K}) = \text{Tor}_n^{PS}(\mathbb{K}, \mathbb{K})$ encodes the minimal free resolution P_\bullet (cf. (5.5)) which is unique (up to isomorphism).

The Euler characteristics of P_\bullet implies an identity about the $GL(V)$ -characters

$$ch PS(V) \cdot ch \left(\bigoplus_{\lambda: \lambda = \lambda'} (-1)^{\frac{1}{2}(|\lambda| + r(\lambda))} V_\lambda \right) = 1 .$$

The character of a Schur module V_λ is the Schur function, $ch V_\lambda = s_\lambda(x)$. Due to the Poincaré-Birkhoff-Witt theorem $ch PS(V) = ch S(V \oplus \Lambda^2 V)$ thus the identity reads

$$\prod_i \frac{1}{(1-x_i)} \prod_{i < j} \frac{1}{(1-x_i x_j)} \sum_{\lambda: \lambda = \lambda'} (-1)^{\frac{1}{2}(|\lambda| + r(\lambda))} s_\lambda(x) = 1 .$$

But the latter identity is nothing but rewriting of the Littlewood identity (5.4). The moral is that the Littlewood identity reflects a homological property of the algebra PS , namely the above particular structure of the minimal projective (free) resolution of \mathbb{K} by PS -modules.

5.3. $\text{Ext}_{PS}^\bullet(\mathbb{K}, \mathbb{K})$ as C_∞ -algebra.

THEOREM 5.2. *The cohomology $H^\bullet(\mathfrak{g}, \mathbb{K}) \cong \text{Ext}_{PS}^\bullet(\mathbb{K}, \mathbb{K})$ of the free 2-nilpotent graded Lie algebra $\mathfrak{g} = V \otimes \Lambda^2 V$ is a homotopy commutative algebra which is generated in degree 1 (*i.e.*, in $H^1(\mathfrak{g}, \mathbb{K})$) by the operations m_2 and m_3 .*

PROOF. We start by choosing a metric g on the vector space V and an orthonormal basis $g(e_i, e_j) = \delta_{ij}$. The choice induces a metric on $\Lambda^\bullet \mathfrak{g} \stackrel{g}{\cong} \Lambda^\bullet \mathfrak{g}^*$.

The isomorphisms $V \cong V^*$ and $\text{Tor}_n^{PS}(\mathbb{K}, \mathbb{K}) \cong \text{Ext}_{PS}^n(\mathbb{K}, \mathbb{K})$ and the theorem 5.1 imply the decomposition of $H^\bullet(\mathfrak{g}, \mathbb{K})$ into irreducible $GL(V)$ -modules

$$H^n(\mathfrak{g}, \mathbb{K}) \cong H^n(\Lambda^\bullet \mathfrak{g}^*, \delta) \cong \text{Ext}_{PS}^n(\mathbb{K}, \mathbb{K}) \cong \bigoplus_{\lambda: \lambda = \lambda'} V_\lambda$$

where the sum is over self-conjugate diagrams λ such that $n = \frac{1}{2}(|\lambda| + r(\lambda))$.

The adjoint of the boundary map ∂ , $\delta \stackrel{g}{:=} \partial^*$ is the differential in the DGA $(\Lambda^\bullet \mathfrak{g}^*, \delta)$ while $\delta^* \stackrel{g}{:=} \partial$ plays the role of a homotopy. In view of lemma 2.1 we have the cohomology $H^\bullet(\Lambda^\bullet \mathfrak{g}^*, \delta^\bullet)$ as deformation retract of the complex $(\Lambda^\bullet \mathfrak{g}^*, \delta^\bullet)$,

$$pi = Id_{H^\bullet(\Lambda^\bullet \mathfrak{g}^*)} , \quad ip - Id_{\Lambda^\bullet \mathfrak{g}^*} = \delta \delta^* + \delta^* \delta , \quad \delta^* \stackrel{g}{=} \partial .$$

Here the projection p identifies the subspace $\ker \delta \cap \ker \delta^*$ with $H^\bullet(\Lambda^\bullet \mathfrak{g}^*)$, which is the orthogonal complement of the space of the coboundaries $\text{im} \delta$. The cocycle-choosing homomorphism i is Id on $H^\bullet(\Lambda^\bullet \mathfrak{g}^*)$ and zero on coboundaries.

We apply the Kadeishvili homotopy transfer Theorem 2.1 for the commutative DGA $(\Lambda^\bullet \mathfrak{g}^*, \mu, \delta^\bullet)$ and its deformation retract $H^\bullet(\Lambda^\bullet \mathfrak{g}^*) \cong H^\bullet(\mathfrak{g}, \mathbb{K})$ and conclude that the cohomology $H^\bullet(\mathfrak{g}, \mathbb{K})$ is a C_∞ -algebra.

The Kontsevich and Soibelman tree representations of the operations m_n provide explicit expressions. Let us take μ to be the super-commutative product \wedge on the DGA $(\wedge^\bullet \mathfrak{g}^*, \delta^\bullet)$. The projection p maps onto the Schur modules V_λ with self-conjugated Young diagram $\lambda = \lambda'$.

The binary operation on the generators $e_i \in H^1(\mathfrak{g}, \mathbb{K})$ is trivial, one gets

$$m_2(e_i, e_j) = p(e_i \wedge e_j) = 0 \quad p(V_{(1^2)}) = 0.$$

Hence $H^\bullet(\mathfrak{g}, \mathbb{K})$ could not be generated in $H^1(\mathfrak{g}, \mathbb{K})$ as algebra with product m_2 .

The ternary operation m_3 restricted to $H^1(\mathfrak{g}, \mathbb{K})$ is nontrivial, indeed one has

$$\begin{aligned} m_3(e_i, e_j, e_k) &= p\{e_i \wedge \partial(e_j \wedge e_k) - \partial(e_i \wedge e_j) \wedge e_k\} = p\{e_{ij} \wedge e_k - e_i \wedge e_{jk}\} \\ &= p\{(e_{ij} \wedge e_k + e_{jk} \wedge e_i + e_{ki} \wedge e_j) - e_{ki} \wedge e_j\} = e_{ik} \wedge e_j \in H^2(\mathfrak{g}, \mathbb{K}) \end{aligned}$$

The completely antisymmetric combination in the brackets (...) spans the Schur module $V_{(1^3)}$, $p(e_{ij} \wedge e_k + e_{jk} \wedge e_i + e_{ki} \wedge e_j) = 0$ yields a Jacobi-type identity. The monomials $e_{ij} \wedge e_k$ modulo $V_{(1^3)}$ span a Schur module $V_{(2,1)} \in H^2(\mathfrak{g}, \mathbb{K})$ with basis in bijection with the semistandard Young tableaux

$$e_{ik} \wedge e_j \leftrightarrow \begin{array}{|c|c|} \hline i & j \\ \hline k & \\ \hline \end{array} \quad \text{and} \quad e_{ij} \wedge e_k \leftrightarrow \begin{array}{|c|c|} \hline i & k \\ \hline j & \\ \hline \end{array}.$$

We check the symmetry condition on ternary operation m_3 in C_∞ -algebra; indeed m_3 vanishes on the (signed) shuffles $Sh_{1,2}$

$$m_3(e_i \Delta e_j \otimes e_k) = m_3(e_i, e_j, e_k) - m_3(e_j, e_i, e_k) + m_3(e_j, e_k, e_i) = 0.$$

Similarly one gets $m_3(e_i \otimes e_j \Delta e_k) = 0$ on shuffles $Sh_{2,1}$.

On the level of Schur modules the ternary operation glues three fundamental $GL(V)$ -representations V_\square into a Schur module $V_{(2,1)}$. By iteration of the process of gluing boxes we generate all elementary hooks $V_k := V_{(k+1, 1^k)}$,

$$\begin{aligned} m_3(V_\square, V_\square, V_\square) &= V_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} \\ m_3\left(V_\square, V_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}}, V_\square\right) &= V_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array}} \\ &\dots \\ m_3(V_0, V_k, V_0) &= V_{k+1}. \end{aligned}$$

In our context the more convenient notation for Young diagrams is due to Frobenius: $\lambda := (a_1, \dots, a_r | b_1, \dots, b_r)$ stands for a diagram λ with a_i boxes in the i -th row on the right of the diagonal, and with b_i boxes in the i -th column below the diagonal and the rank $r = r(\lambda)$ is the number of boxes on the diagonal.

For self-dual diagrams $\lambda = \lambda'$, i.e., $a_i = b_i$ we set $V_{a_1, \dots, a_r} := V_{(a_1, \dots, a_r | a_1, \dots, a_r)}$ when $a_1 > a_2 > \dots > a_r \geq 0$ (and set the convention $V_{a_1, \dots, a_r} := 0$ otherwise). Any two elementary hooks V_{a_1} and V_{a_2} can be glued together by the binary operation m_2 , the decomposition of $m_2(V_{a_1}, V_{a_2}) \cong m_2(V_{a_2}, V_{a_1})$ is given by

$$m_2(V_{a_1}, V_{a_2}) = V_{a_1, a_2} \oplus \left(\bigoplus_{i=1}^{a_2} V_{a_1+i, a_2-i} \right) \quad a_1 \geq a_2$$

where the “leading” term V_{a_1, a_2} has the diagram with minimal height. Hence any m_2 -bracketing of the hooks $V_{a_1}, V_{a_2}, \dots, V_{a_r}$ yields³ a sum of $GL(V)$ -modules

$$m_2(\dots m_2(m_2(V_{a_1}, V_{a_2}), V_{a_3}), \dots, V_{a_r}) = V_{a_1, \dots, a_r} \oplus \dots$$

whose module with minimal height is precisely V_{a_1, \dots, a_r} . We conclude that all elements in the C_∞ -algebra $H^\bullet(\mathfrak{g}, \mathbb{K})$ can be generated in $H^1(\mathfrak{g}, \mathbb{K})$ by m_2 and m_3 . \square

One could draw a parallel between the latter theorem for the cubic algebra PS and the proposition 4.1 for Koszul algebra; in both cases the Yoneda algebra $\text{Ext}_{PS}^\bullet(\mathbb{K}, \mathbb{K})$ is generated only in $\text{Ext}_{PS}^1(\mathbb{K}, \mathbb{K})$. Although we have the notion of N -Koszul algebras for the N -homogeneous algebras [2, 3], it turns out that the cubic algebra PS is not 3-Koszul, beside the exceptional case when $\dim V = 2$. Instead the algebra $PS = U\mathfrak{g}$ falls in the class of *Artin-Schelter-regular algebras* [1], being an UEA of positively graded Lie algebra (for a proof see [6]). The parallel between the quadratic Koszul algebra $S(V)$ and the cubic AS-regular algebra $PS(V)$ suggests that the C_∞ -algebra $\text{Ext}_{PS}^\bullet(\mathbb{K}, \mathbb{K})$ is a generalization of a Koszul dual algebra of PS in the realm of the homotopy algebras, an idea that has been put forward in [13].

The analogy would be complete if we had the following conjectural proposition.

PROPOSITION 5.1. *The cohomology $H^\bullet(\mathfrak{g}, \mathbb{K}) \cong \text{Ext}_{PS}^\bullet(\mathbb{K}, \mathbb{K})$ of the free 2-nilpotent graded Lie algebra $\mathfrak{g} = V \otimes \bigwedge^2 V$ can be endowed with a structure of C_∞ -algebra having trivial higher multiplications $m_k = 0$, $k \geq 4$.*

So far we were able to prove this conjecture only in dimensions $\dim V \leq 3$. Our proof rests entirely on the bigrading $(2 - k, 0)$ of the multiplication m_k by homological and tensor degree in the C_∞ -algebra $\text{Ext}_{PS}^\bullet(\mathbb{K}, \mathbb{K})$. The bigrading arguments work only for $\dim V = 2$ and $\dim V = 3$ thus for a complete proof the conjecture would need more refined methods.

Acknowledgements. We are grateful to Jean-Louis Loday for many enlightening discussions and his encouraging interest. Todor Popov thanks the Serbian hosts for the warm hospitality, the financial support and for the stimulating atmosphere during the conference in Zlatibor.

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³The operation m_2 is associative thus the result does not depend on the choice of the bracketing.

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