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Michel Dubois-Violette, Todor Popov

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2-Step Nilpotent Lie algebra and C_∞ -Algebras

Michel Dubois-Violette and Todor Popov

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ABSTRACT. We consider the free 2-step nilpotent graded Lie algebra and its cohomology ring. The homotopy transfer induces a homotopy commutative algebra on its cohomology ring which we describe.

1. Homotopy algebras

The homotopy associative algebras, or A_∞ -algebras were introduced by Jim Stasheff in the 1960's as a tool in algebraic topology for studying 'group-like' spaces. Homotopy algebras received a new attention and further development in the 1990's after the discovery of their relevance into a multitude of topics in algebraic geometry, symplectic and contact geometry, knot theory, moduli spaces, deformation theory...

DEFINITION 1.1. (A_∞ -algebra) A homotopy associative algebra, or A_∞ -algebra, over \mathbb{K} is a \mathbb{Z} -graded vector space $A = \bigoplus_{i \in \mathbb{Z}} A^i$ endowed with a family of graded mappings (operations)

$$m_n : A^{\otimes n} \rightarrow A, \quad \deg(m_n) = 2 - n \quad n \geq 1$$

satisfying the Stasheff identities **SI(n)** for $n \geq 1$

$$\mathbf{SI}(n) : \sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(Id^{\otimes r} \otimes m_s \otimes Id^{\otimes t}) = 0 \quad r \geq 0, t \geq 0, s \geq 1$$

where the sum runs over all decompositions $n = r + s + t$. Throughout the text we assume the Koszul sign convention $(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y)$.

A morphism of two A_∞ -algebras A and B is a family of graded maps $f_n : A^{\otimes n} \rightarrow B$ for $n \geq 1$ with $\deg f_n = 1 - n$ such that the following conditions hold

$$\sum_{r+s+t=n} (-1)^{r+st} f_{r+1+t}(Id^{\otimes r} \otimes m_s \otimes Id^{\otimes t}) = \sum_{1 \leq r \leq n} (-1)^S m_r(f_{i_1} \otimes f_{i_2} \otimes \dots \otimes f_{i_r})$$

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where the sum is on all decompositions $i_1 + \dots + i_r = n$ and the sign $(-1)^S$ on RHS is determined by $S = (r-1)(i_1-1) + (r-2)(i_2-1) + \dots + 2(i_{r-2}-1) + (i_{r-1}-1)$. The morphism f is a *quasi-isomorphism of A_∞ -algebras* if f_1 is a quasi-isomorphism. It is strict if $f_i = 0$ for all $i \neq 1$. The identity morphism of A is the strict morphism f such that f_1 is the identity of A .

We define the shuffle product $Sh_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \rightarrow A^{\otimes p+q}$ throughout the expression

$$(a_1 \otimes \dots \otimes a_p) \Delta (a_{p+1} \otimes \dots \otimes a_{p+q}) = \sum_{\sigma \in Sh_{p,q}} sgn(\sigma) a_{\sigma^{-1}(1)} \otimes \dots \otimes a_{\sigma^{-1}(p+q)}$$

where the sum runs over all (p, q) -shuffles $Sh_{p,q}$, i.e., over all permutations $\sigma \in S_{p+q}$ such that $\sigma(1) < \sigma(2) < \dots < \sigma(p)$ and $\sigma(p+1) < \sigma(p+2) < \dots < \sigma(p+q)$.

DEFINITION 1.2. (C_∞ -algebra [9]) A homotopy commutative algebra, or C_∞ -algebra, is an A_∞ -algebra $\{A, m_n\}$ such that each operation m_n vanishes on non-trivial shuffles

$$(1.1) \quad m_n((a_1 \otimes \dots \otimes a_p) \Delta (a_{p+1} \otimes \dots \otimes a_n)) = 0, \quad 1 \leq p \leq n-1.$$

In particular for m_2 we have $m_2(a \otimes b \pm b \otimes a) = 0$, so a C_∞ -algebra such that $m_n = 0$ for $n \geq 3$ is a (super-)commutative DGA.

A morphism of C_∞ -algebras is a morphism of A_∞ -algebras vanishing on non-trivial shuffles $f_n((a_1 \otimes \dots \otimes a_p) \Delta (a_{p+1} \otimes \dots \otimes a_n)) = 0$, $1 \leq p \leq n-1$.

2. Homotopy Transfer Theorem

LEMMA 2.1. *Every cochain complex (A, d) of vector spaces over a field \mathbb{K} has its cohomology $H^\bullet(A)$ as a deformation retract.*

One can always choose a vector space decomposition of the cochain complex (A, d) such that $A^n \cong B^n \oplus H^n \oplus B^{n+1}$ where H^n is the cohomology and B^n is the space of coboundaries, $B^n = dA^{n-1}$. We choose a homotopy $h : A^n \rightarrow A^{n-1}$ which identifies B^n with its copy in A^{n-1} and is 0 on $H^n \oplus B^{n+1}$. The projection p to the cohomology and the cocycle-choosing inclusion i given by $A^n \begin{matrix} \xrightarrow{p} \\ \xleftarrow{i} \end{matrix} H^n$ are chain homomorphisms, satisfying the additional *side conditions*

$$hh = 0, \quad hi = 0, \quad ph = 0.$$

With these choices done the complex $(H^\bullet(A), 0)$ is a deformation retract of (A, d)

$$h \circlearrowleft (A, d) \begin{matrix} \xrightarrow{p} \\ \xleftarrow{i} \end{matrix} (H^\bullet(A), 0), \quad pi = Id_{H^\bullet(A)}, \quad ip - Id_A = dh + hd.$$

Let now (A, d, μ) be a DGA, i.e., A is endowed with an associative product μ compatible with d . The cochain complexes (A, d) and its contraction $H^\bullet(A)$ are homotopy equivalent, but the associative structure is not stable under homotopy equivalence. However the associative structure on A can be transferred to an A_∞ -structure on a homotopy equivalent complex, a particular interesting complex being the deformation retract $H^\bullet(A)$. For a friendly introduction to homotopy transfer

theorems in much boarder context we send the reader to the textbook [12], see chapter 9.

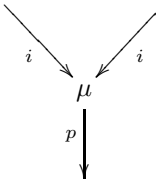
THEOREM 2.1 (Kadeishvili [9]). *Let (A, d, μ) be a (commutative) DGA over a field \mathbb{K} . There exists a A_∞ -algebra (C_∞ -algebra) structure on the cohomology $H^\bullet(A)$ and a $A_\infty(C_\infty)$ -quasi-isomorphism*

$$f_i : (\otimes^i H^\bullet(A), \{m_i\}) \rightarrow (A, \{d, \mu, 0, 0, \dots\})$$

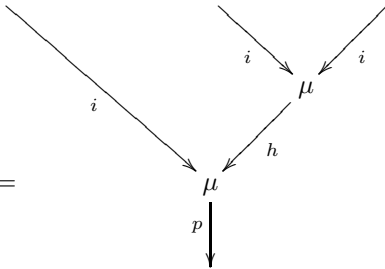
such that the inclusion $f_1 = i : H^\bullet(A) \rightarrow A$ is a cocycle-choosing homomorphism of cochain complexes. The differential m_1 on $H^\bullet(A)$ is zero ($m_1 = 0$) and m_2 is the strictly associative operation induced by the multiplication on A . The resulting structure is unique up to quasi-isomorphism.

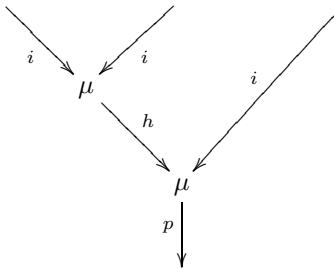
Kontsevich and Soibelman [10] gave an explicit expressions for the higher operations of the induced A_∞ -structure as sums over decorated planar binary trees with one root where all leaves are decorated by the inclusion i , the root by the projection p the vertices by the product μ of the (commutative) DGA (A, d, μ) and the internal edges by the homotopy h . The C_∞ -structure implies additional symmetries on trees.

We will make use of the graphic representation for the binary operation on $H^\bullet(A)$

$$m_2(x, y) := p\mu(i(x), i(y)) \quad \text{or} \quad m_2 =$$


and the ternary one $m_3(x, y, z) = p\mu(i(x), h\mu(i(y), i(z))) - p\mu(h\mu(i(x), i(y)), i(z))$ being the sum of two planar binary trees with three leaves

$$m_3 =$$


$$-$$


3. Homology and cohomology of Lie algebra \mathfrak{g}

A non-minimal projective(in fact free) resolution of the trivial $U\mathfrak{g}$ -module \mathbb{K} , $C(\mathfrak{g}) \xrightarrow{\epsilon} \mathbb{K}$ is given by the standard Chevalley-Eilenberg chain complex $C_\bullet(\mathfrak{g}) =$

$(U\mathfrak{g} \otimes_{\mathbb{K}} \wedge^p \mathfrak{g}, d_p)$ with differential maps

$$(3.1) \quad \begin{aligned} d_p(u \otimes x_1 \wedge \dots \wedge x_p) &= \sum_i (-1)^{i+1} u x_i \otimes x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge x_p \\ &+ \sum_{i < j} (-1)^{i+j} u [x_i, x_j] \wedge x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge \hat{x}_j \wedge \dots \wedge x_p \end{aligned}$$

The homologies $H_n(\mathfrak{g}, \mathbb{K})$ of the Lie algebra \mathfrak{g} with trivial coefficients are given by the homologies of the derived complex $\mathbb{K} \otimes_{U\mathfrak{g}} C_{\bullet}(\mathfrak{g})$

$$\mathrm{Tor}_n^{U\mathfrak{g}}(\mathbb{K}, \mathbb{K}) \cong H_n(\mathbb{K} \otimes_{U\mathfrak{g}} C_{\bullet}(\mathfrak{g})) = H_n(\mathfrak{g}, \mathbb{K}) .$$

The complex $\mathbb{K} \otimes_{U\mathfrak{g}} C_{\bullet}(\mathfrak{g})$ is the chain complex with degrees $\wedge^{\bullet} \mathfrak{g} = \mathbb{K} \otimes_{U\mathfrak{g}} U\mathfrak{g} \otimes \wedge^{\bullet} \mathfrak{g}$ and differentials $\partial_p := id \otimes_{U\mathfrak{g}} d_p : \wedge^p \mathfrak{g} \rightarrow \wedge^{p-1} \mathfrak{g}$ induced by the extension as coderivation of the Lie bracket $\partial_2 := -[\cdot, \cdot] : \wedge^2 \mathfrak{g} \rightarrow \mathfrak{g}$.

The dual cochain complex $\mathrm{Hom}_{U\mathfrak{g}}(C(\mathfrak{g}), \mathbb{K}) = (\wedge^{\bullet} \mathfrak{g}^*, \delta)$ has coboundary map $\delta^p : \wedge^p \mathfrak{g}^* \rightarrow \wedge^{p+1} \mathfrak{g}^*$ (being transposed to the differential ∂_{p+1}) which is the extension as derivation of the dualization of the Lie bracket $\delta^1 := [\cdot, \cdot]^* : \mathfrak{g}^* \rightarrow \wedge^2 \mathfrak{g}^*$. One calculates the cohomologies¹ of the Lie algebra \mathfrak{g} as

$$(3.2) \quad \mathrm{Ext}_{U\mathfrak{g}}^n(\mathbb{K}, \mathbb{K}) \cong H^n(\mathrm{Hom}_{U\mathfrak{g}}(C(\mathfrak{g}), \mathbb{K})) = H^n(\mathfrak{g}, \mathbb{K}) .$$

Hence the algebra $(\wedge^{\bullet} \mathfrak{g}^*, \delta)$ equipped with δ is a (*super*)commutative DGA and the Yoneda algebra $\mathrm{Ext}_{U\mathfrak{g}}^{\bullet}(\mathbb{K}, \mathbb{K}) = \bigoplus_n \mathrm{Ext}_{U\mathfrak{g}}^n(\mathbb{K}, \mathbb{K})$ has structure of commutative associative algebra. Moreover due to the Kadeishvili theorem the Yoneda algebra $\mathrm{Ext}_{U\mathfrak{g}}^{\bullet}(\mathbb{K}, \mathbb{K}) = H^{\bullet}(\mathfrak{g}, \mathbb{K})$ is a C_{∞} -algebra which stems from the homotopy transfer of the wedge product \wedge on cohomology classes

$$H^i(\mathfrak{g}, \mathbb{K}) \wedge H^j(\mathfrak{g}, \mathbb{K}) \rightarrow H^{i+j}(\mathfrak{g}, \mathbb{K}) .$$

4. Abelian Lie algebra $\mathfrak{h} = V$

Let us take as a basic example the abelian Lie algebra \mathfrak{h} , that is, the free nilpotent Lie algebra generated by a finite dimensional vector space V of degree 1. The Lie bracket of \mathfrak{h} is trivial $[V, V] = 0$. According to Poincaré-Birkhoff-Witt theorem the universal enveloping algebra of the abelian Lie algebra $\mathfrak{h} = V$ is isomorphic to the symmetric algebra

$$U(\mathfrak{h}) \cong S(V) .$$

The Chevalley-Eilenberg complex $C_{\bullet}(\mathfrak{h}) = S(V) \otimes_{\mathbb{K}} \wedge^{\bullet} V$ yields the resolution of the trivial $U(\mathfrak{h})$ -module \mathbb{K}

$$(4.1) \quad \begin{aligned} 0 &\rightarrow S(V) \otimes \wedge^{\dim V} V \rightarrow S(V) \otimes \wedge^{\dim V - 1} V \rightarrow \dots \\ \dots &\rightarrow S(V) \otimes \wedge^2 V \rightarrow S(V) \otimes V \rightarrow S(V) \rightarrow \mathbb{K} \rightarrow 0 . \end{aligned}$$

The derived complex $\mathbb{K} \otimes_{U\mathfrak{h}} C(\mathfrak{h})$ has zero differential and the Chevalley-Eilenberg resolution turns out to be minimal(which is not the case in general)

$$H_n(\mathfrak{h}, \mathbb{K}) \cong H_n(\mathbb{K} \otimes_{U\mathfrak{h}} C(\mathfrak{h})) \cong \wedge^n V .$$

¹In the presence of metric one has $\delta := \partial^*$ (see below)

The Chevalley-Eilenberg resolution coincides with the Koszul complex $K(A) = A \otimes (A^!)^*$ of the symmetric algebra $A = S(V)$. The Koszul dual algebra of the symmetric algebra is the exterior algebra $S(V)^! = \Lambda V^*$. A quadratic algebra is said to be a Koszul algebra when its Koszul complex $K_\bullet(A) = A \otimes (A^!)^*$ is acyclic everywhere except in degree 0 (where its homology is \mathbb{K}). Then the Koszul complex yields a minimal projective (in fact free) resolution by (left) A -modules of the trivial A -module \mathbb{K}

$$K(A) \xrightarrow{\epsilon} \mathbb{K} \rightarrow 0 .$$

In particular the resolution 4.1 is the same as the the resolution by the Koszul complex $K_n(S(V)) = S(V) \otimes \Lambda^n V^*$ thus the algebra $S(V)$ is *Koszul algebra*. One has another equivalent definition of Koszul algebra based on the following proposition;

PROPOSITION 4.1. *A finitely generated quadratic algebra A is Koszul iff its Yoneda algebra $\text{Ext}_A(\mathbb{K}, \mathbb{K})$ is generated in degree 1. One has then $\text{Ext}_A(\mathbb{K}, \mathbb{K}) \cong A^!$.*

Indeed the Yoneda algebra $\text{Ext}_{S(V)}(\mathbb{K}, \mathbb{K})$ of the symmetric algebra $S(V)$ is just the exterior algebra

$$\text{Ext}_{S(V)}^n(\mathbb{K}, \mathbb{K}) = (\text{Tor}_n^{S(V)}(\mathbb{K}, \mathbb{K}))^* = \Lambda^n V^*$$

which is obviously generated by V^* , *i.e.*, in degree 1, by the wedge product. Through the homotopy transfer the Yoneda algebra $\text{Ext}_{S(V)}(\mathbb{K}, \mathbb{K})$ inherits a C_∞ -structure but it is easy to show (by degree preserving argument) that the latter C_∞ -algebra is formal, *i.e.*, all higher multiplications are trivial, $m_n = 0$ for $n \neq 2$.

5. Homology of Free 2-nilpotent algebra $\mathfrak{g} = V \oplus \Lambda^2 V$

Let \mathfrak{g} be the free graded 2-step nilpotent Lie algebra generated by vector space V in degree 1, $\mathfrak{g} = V \oplus [V, V]$. In other words we consider the graded Lie algebra \mathfrak{g} with Lie bracket

$$[u, v] = \begin{cases} u \wedge v & \in \Lambda^2 V \\ 0 & \text{otherwise} \end{cases}$$

We denote the Universal Enveloping Algebra(UEA) $U\mathfrak{g}$ by PS and refer to it as *parastatistics algebra*.² Throughout this note we will consider the generators space V to be an ordinary vector space V which corresponds to a parafermionic algebra $PS(V) = U\mathfrak{g}$. The case of a \mathbb{Z}_2 -space of generators $V = V_0 \oplus V_1$, that is, $PS(V)$ is the Universal Enveloping Algebra of a Lie super-algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ (which would include the parabosonic algebras) will be treated elsewhere. More on parastatistics algebras and their application to combinatorics the reader could find in the articles [5, 11].

²Such cubic algebras arise through the exchange relations between the operators in a quantization procedure introduced by H. S. Green [7] for particles obeying more general statistics than Bose-Einstein or Fermi-Dirac, coined parabosons and parafermions.

The parastatistics algebra $PS(V)$ generated by a finite dimensional vector space V in degree 1 is the positively graded algebra

$$PS(V) := U\mathfrak{g} = U(V \oplus \bigwedge^2 V) = T(V)/(\llbracket [V, V], V \rrbracket).$$

We shall write simply PS when the space of generators V is clear from the context.

The homologies $H_n(\mathfrak{g}, \mathbb{K})$ of the 2-nilpotent Lie algebra \mathfrak{g} are the homologies of the chain complex

$$\bigwedge^n \mathfrak{g} = \bigwedge^n (V \oplus \bigwedge^2 V) = \bigoplus_{s+r=n} \bigwedge^s (\bigwedge^2 V) \otimes \bigwedge^r (V)$$

with differentials $\partial_n : \bigwedge^s (\bigwedge^2 V) \otimes \bigwedge^r (V) \rightarrow \bigwedge^{s+1} (\bigwedge^2 V) \otimes \bigwedge^{r-2} (V)$ are given by

$$\begin{aligned} \partial_n : e_{i_1 j_1} \wedge \dots \wedge e_{i_s j_s} \otimes e_1 \wedge \dots \wedge e_r &\mapsto \\ \sum_{i < j} (-1)^{i+j} e_{ij} \wedge e_{i_1 j_1} \wedge \dots \wedge e_{i_s j_s} \otimes e_1 \wedge \dots \wedge \hat{e}_i \wedge \dots \wedge \hat{e}_j \wedge \dots \wedge e_r. \end{aligned}$$

The differential ∂ identifies a pair of degree 1 generators $e_i, e_j \in V$ with one degree 2 generator $e_{ij} := (e_i \wedge e_j) = [e_i, e_j] \in \bigwedge^2 V$.

The cohomologies $H^n(\mathfrak{g}, \mathbb{K})$ arise from the dualized complex with coboundary map $\delta^n : \bigwedge^n \mathfrak{g}^* \rightarrow \bigwedge^{n+1} \mathfrak{g}^*$ which is transposed to the differential ∂_{n+1}

$$\begin{aligned} (5.1) \quad \delta^n : e_{i_1 j_1}^* \wedge \dots \wedge e_{i_s j_s}^* \otimes e_{l_1}^* \wedge \dots \wedge e_{l_r}^* &\mapsto \\ \sum_{k=1}^s \sum_{i_k < j_k} (-1)^{i+j} e_{i_1 j_1}^* \wedge \dots \wedge \hat{e}_{i_k j_k}^* \wedge \dots \wedge e_{i_s j_s}^* \otimes e_{i_k}^* \wedge e_{j_k}^* \wedge e_{l_1}^* \wedge \dots \wedge \dots \wedge e_{l_r}^*. \end{aligned}$$

In the presence of metric g one has an identification $V \stackrel{g}{\cong} V^*$, and $\bigwedge^\bullet \mathfrak{g} \stackrel{g}{\cong} \bigwedge^\bullet \mathfrak{g}^*$. The adjoint operator $\partial_n^* : \bigwedge^n \mathfrak{g} \rightarrow \bigwedge^{n+1} \mathfrak{g}$ is defined by $g(\partial_n^* v, w) = g(v, \partial_{n+1} w)$. One can show that independently of the metric g chosen the action of ∂_n^* takes the form

$$\begin{aligned} (5.2) \quad \partial_n^* : e_{i_1 j_1} \wedge \dots \wedge e_{i_s j_s} \otimes e_{l_1} \wedge \dots \wedge e_{l_r} &\mapsto \\ \sum_{k=1}^s \sum_{i_k < j_k} (-1)^{i+j} e_{i_1 j_1} \wedge \dots \wedge \hat{e}_{i_k j_k} \wedge \dots \wedge e_{i_s j_s} \otimes e_{i_k} \wedge e_{j_k} \wedge e_{l_1} \wedge \dots \wedge \dots \wedge e_{l_r}. \end{aligned}$$

We will see in the following that after the identification $\bigwedge^\bullet \mathfrak{g} \stackrel{g}{\cong} \bigwedge^\bullet \mathfrak{g}^*$ the map $\partial^* \stackrel{g}{=} \delta$ will play the role of homotopy for the chain complex $(\bigwedge^\bullet \mathfrak{g}, \partial_\bullet)$, and vice versa the boundary map $\partial \stackrel{g}{=} \delta^*$ is a homotopy for the cochain complex $(\bigwedge^\bullet \mathfrak{g}^*, \delta^\bullet)$.

The complexes $(\bigwedge^n \mathfrak{g}, \partial_n, \partial_n^*)$ and $(\bigwedge^n \mathfrak{g}^*, \delta^n, \delta^{*n})$ are bigraded by two different degrees; the homological degree $n := r + s$ counting the number of Lie algebra generators and the tensor degree $t := 2s + r$ also called weight. The cohomologies $H^n(\mathfrak{g}, \mathbb{K})$ can have components of different weight t , $H^n(\mathfrak{g}, \mathbb{K}) = \bigoplus_t H^n(\mathfrak{g}, \mathbb{K})_t$ and the weight t is in fact the *Adams grading* on Yoneda algebra $\text{Ext}_{U\mathfrak{g}}^n(\mathbb{K}, \mathbb{K})_t$ [13]. The differential and the homotopy, $\delta = \partial^*$ and $\partial = \delta^*$ do not alter the weight t , but raise and lower the homological degree n .

The operations m_k in homotopy algebra are bigraded by homological and Adams gradings of bidegree $(k, t) = (2 - k, 0)$. The bi-grading impose the vanishing of many higher products.

5.1. Homology of \mathfrak{g} as a $GL(V)$ -module. A Schur module V_λ is an irreducible polynomial $GL(V)$ -module labelled by a Young diagram λ . The basis of a Schur module V_λ is in bijection with semistandard Young tableaux which are fillings of the Young diagram λ with the numbers of the set $\{1, \dots, \dim V\}$. The action of the linear group $GL(V)$ on the space V of the generators of the Lie algebra \mathfrak{g} induces a $GL(V)$ -action on the universal enveloping algebra $PS = U\mathfrak{g} \cong S(V \oplus \Lambda^2 V)$ and on the space $\bigwedge^\bullet \mathfrak{g} \cong \bigwedge^\bullet (V \oplus \Lambda^2 V)$.

The maps ∂ and ∂^* both commute with the $GL(V)$ -action. It follows that the homology and cohomology carry structure of $GL(V)$ -modules hence can be decomposed into irreducibles.

The *Laplacian* $\Delta = \bigoplus_{n \geq 0} \Delta_n$ is defined to be the self-adjoint operator

$$\Delta_n = \partial_{n+1} \partial_{n+1}^* + \partial_n^* \partial_n \in \text{End}(\bigwedge^n \mathfrak{g}) .$$

Its kernel is a complete set of representatives for the homology classes in $H_p(\mathfrak{g}, \mathbb{K})$

$$\ker \Delta_n \cong H_n(\mathfrak{g}, \mathbb{K}) .$$

The decomposition of the $GL(V)$ -module $H_n(\mathfrak{g}, \mathbb{K})$ into irreducible polynomial representations V_λ is given by the following theorem;

THEOREM 5.1 (Józefiak and Weyman [8], Sigg [14]). *The homology $H_\bullet(\mathfrak{g}, \mathbb{K})$ of the 2-nilpotent Lie algebra $\mathfrak{g} = V \oplus \Lambda^2 V$ decomposes into irreducible $GL(V)$ -modules*

$$(5.3) \quad H_n(\mathfrak{g}, \mathbb{K}) = H_n(\bigwedge^\bullet \mathfrak{g}, \partial) \cong \text{Tor}_n^{PS}(\mathbb{K}, \mathbb{K})(V) \cong \bigoplus_{\lambda: \lambda = \lambda'} V_\lambda$$

where the sum is over self-conjugate Young diagrams λ such that $n = \frac{1}{2}(|\lambda| + r(\lambda))$.

5.2. Homological interpretation of the Littlewood formula. We recall the beautiful result of Józefiak and Weyman [8] giving a representation-theoretic interpretation of the Littlewood formula

$$(5.4) \quad \prod_i (1 - x_i) \prod_{i < j} (1 - x_i x_j) = \sum_{\lambda: \lambda = \lambda'} (-1)^{\frac{1}{2}(|\lambda| + r(\lambda))} s_\lambda(x) .$$

Here the sum is over the self-dual Young diagrams λ , $s_\lambda(x)$ stands for the Schur function and $r(\lambda)$ stands the rank of λ which is the number of diagonal boxes in λ .

One knows that for the graded algebra PS there exists a minimal resolution by projective modules

$$(5.5) \quad P_\bullet : \quad 0 \rightarrow P_d \rightarrow \dots \rightarrow P_n \rightarrow \dots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \xrightarrow{\epsilon} \mathbb{K} \rightarrow 0 .$$

Here the length d of the resolution is the projective dimension of the algebra PS which is $d = \frac{\dim V(\dim V + 1)}{2}$. Since PS is positively graded and, in the category of positively graded modules over connected locally finite graded algebras, projective module is the same as free module [4], we have $P_n \cong PS \otimes E_n$ where E_n are finite

dimensional vector spaces. Thus we deal with a minimal resolution of \mathbb{K} by free PS -modules and the minimality implies that the derived complex $\mathbb{K} \otimes_{PS} P_\bullet$ has vanishing differentials, *i.e.*, $\text{Tor}_\bullet^{PS}(\mathbb{K}, \mathbb{K}) = H_\bullet(\mathbb{K} \otimes_{PS} P_\bullet) = \mathbb{K} \otimes_{PS} P_\bullet$. Then the multiplicity spaces $E_n = \text{Tor}_n^{PS}(\mathbb{K}, \mathbb{K})$ are fixed by Theorem 5.1 thus the data $H_n(\mathfrak{g}, \mathbb{K}) = \text{Tor}_n^{PS}(\mathbb{K}, \mathbb{K})$ encodes the minimal free resolution P_\bullet (cf. (5.5)) which is unique (up to isomorphism).

The Euler characteristics of P_\bullet implies an identity about the $GL(V)$ -characters

$$ch PS(V) \cdot ch \left(\bigoplus_{\lambda: \lambda = \lambda'} (-1)^{\frac{1}{2}(|\lambda| + r(\lambda))} V_\lambda \right) = 1 .$$

The character of a Schur module V_λ is the Schur function, $ch V_\lambda = s_\lambda(x)$. Due to the Poincaré-Birkhoff-Witt theorem $ch PS(V) = ch S(V \oplus \Lambda^2 V)$ thus the identity reads

$$\prod_i \frac{1}{(1-x_i)} \prod_{i < j} \frac{1}{(1-x_i x_j)} \sum_{\lambda: \lambda = \lambda'} (-1)^{\frac{1}{2}(|\lambda| + r(\lambda))} s_\lambda(x) = 1 .$$

But the latter identity is nothing but rewriting of the Littlewood identity (5.4). The moral is that the Littlewood identity reflects a homological property of the algebra PS , namely the above particular structure of the minimal projective (free) resolution of \mathbb{K} by PS -modules.

5.3. $\text{Ext}_{PS}^\bullet(\mathbb{K}, \mathbb{K})$ as C_∞ -algebra.

THEOREM 5.2. *The cohomology $H^\bullet(\mathfrak{g}, \mathbb{K}) \cong \text{Ext}_{PS}^\bullet(\mathbb{K}, \mathbb{K})$ of the free 2-nilpotent graded Lie algebra $\mathfrak{g} = V \otimes \Lambda^2 V$ is a homotopy commutative algebra which is generated in degree 1 (*i.e.*, in $H^1(\mathfrak{g}, \mathbb{K})$) by the operations m_2 and m_3 .*

PROOF. We start by choosing a metric g on the vector space V and an orthonormal basis $g(e_i, e_j) = \delta_{ij}$. The choice induces a metric on $\Lambda^\bullet \mathfrak{g} \stackrel{g}{\cong} \Lambda^\bullet \mathfrak{g}^*$.

The isomorphisms $V \cong V^*$ and $\text{Tor}_n^{PS}(\mathbb{K}, \mathbb{K}) \cong \text{Ext}_{PS}^n(\mathbb{K}, \mathbb{K})$ and the theorem 5.1 imply the decomposition of $H^\bullet(\mathfrak{g}, \mathbb{K})$ into irreducible $GL(V)$ -modules

$$H^n(\mathfrak{g}, \mathbb{K}) \cong H^n(\Lambda^\bullet \mathfrak{g}^*, \delta) \cong \text{Ext}_{PS}^n(\mathbb{K}, \mathbb{K}) \cong \bigoplus_{\lambda: \lambda = \lambda'} V_\lambda$$

where the sum is over self-conjugate diagrams λ such that $n = \frac{1}{2}(|\lambda| + r(\lambda))$.

The adjoint of the boundary map ∂ , $\delta \stackrel{g}{:=} \partial^*$ is the differential in the DGA $(\Lambda^\bullet \mathfrak{g}^*, \delta)$ while $\delta^* \stackrel{g}{:=} \partial$ plays the role of a homotopy. In view of lemma 2.1 we have the cohomology $H^\bullet(\Lambda^\bullet \mathfrak{g}^*, \delta^\bullet)$ as deformation retract of the complex $(\Lambda^\bullet \mathfrak{g}^*, \delta^\bullet)$,

$$pi = Id_{H^\bullet(\Lambda^\bullet \mathfrak{g}^*)} , \quad ip - Id_{\Lambda^\bullet \mathfrak{g}^*} = \delta \delta^* + \delta^* \delta , \quad \delta^* \stackrel{g}{=} \partial .$$

Here the projection p identifies the subspace $\ker \delta \cap \ker \delta^*$ with $H^\bullet(\Lambda^\bullet \mathfrak{g}^*)$, which is the orthogonal complement of the space of the coboundaries $\text{im} \delta$. The cocycle-choosing homomorphism i is Id on $H^\bullet(\Lambda^\bullet \mathfrak{g}^*)$ and zero on coboundaries.

We apply the Kadeishvili homotopy transfer Theorem 2.1 for the commutative DGA $(\Lambda^\bullet \mathfrak{g}^*, \mu, \delta^\bullet)$ and its deformation retract $H^\bullet(\Lambda^\bullet \mathfrak{g}^*) \cong H^\bullet(\mathfrak{g}, \mathbb{K})$ and conclude that the cohomology $H^\bullet(\mathfrak{g}, \mathbb{K})$ is a C_∞ -algebra.

The Kontsevich and Soibelman tree representations of the operations m_n provide explicit expressions. Let us take μ to be the super-commutative product \wedge on the DGA $(\wedge^\bullet \mathfrak{g}^*, \delta^\bullet)$. The projection p maps onto the Schur modules V_λ with self-conjugated Young diagram $\lambda = \lambda'$.

The binary operation on the generators $e_i \in H^1(\mathfrak{g}, \mathbb{K})$ is trivial, one gets

$$m_2(e_i, e_j) = p(e_i \wedge e_j) = 0 \quad p(V_{(1^2)}) = 0.$$

Hence $H^\bullet(\mathfrak{g}, \mathbb{K})$ could not be generated in $H^1(\mathfrak{g}, \mathbb{K})$ as algebra with product m_2 .

The ternary operation m_3 restricted to $H^1(\mathfrak{g}, \mathbb{K})$ is nontrivial, indeed one has

$$\begin{aligned} m_3(e_i, e_j, e_k) &= p\{e_i \wedge \partial(e_j \wedge e_k) - \partial(e_i \wedge e_j) \wedge e_k\} = p\{e_{ij} \wedge e_k - e_i \wedge e_{jk}\} \\ &= p\{(e_{ij} \wedge e_k + e_{jk} \wedge e_i + e_{ki} \wedge e_j) - e_{ki} \wedge e_j\} = e_{ik} \wedge e_j \in H^2(\mathfrak{g}, \mathbb{K}) \end{aligned}$$

The completely antisymmetric combination in the brackets (...) spans the Schur module $V_{(1^3)}$, $p(e_{ij} \wedge e_k + e_{jk} \wedge e_i + e_{ki} \wedge e_j) = 0$ yields a Jacobi-type identity. The monomials $e_{ij} \wedge e_k$ modulo $V_{(1^3)}$ span a Schur module $V_{(2,1)} \in H^2(\mathfrak{g}, \mathbb{K})$ with basis in bijection with the semistandard Young tableaux

$$e_{ik} \wedge e_j \leftrightarrow \begin{array}{|c|c|} \hline i & j \\ \hline k & \\ \hline \end{array} \quad \text{and} \quad e_{ij} \wedge e_k \leftrightarrow \begin{array}{|c|c|} \hline i & k \\ \hline j & \\ \hline \end{array}.$$

We check the symmetry condition on ternary operation m_3 in C_∞ -algebra; indeed m_3 vanishes on the (signed) shuffles $Sh_{1,2}$

$$m_3(e_i \Delta e_j \otimes e_k) = m_3(e_i, e_j, e_k) - m_3(e_j, e_i, e_k) + m_3(e_j, e_k, e_i) = 0.$$

Similarly one gets $m_3(e_i \otimes e_j \Delta e_k) = 0$ on shuffles $Sh_{2,1}$.

On the level of Schur modules the ternary operation glues three fundamental $GL(V)$ -representations V_\square into a Schur module $V_{(2,1)}$. By iteration of the process of gluing boxes we generate all elementary hooks $V_k := V_{(k+1, 1^k)}$,

$$\begin{aligned} m_3(V_\square, V_\square, V_\square) &= V_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}} \\ m_3\left(V_\square, V_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}}, V_\square\right) &= V_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array}} \\ &\dots \\ m_3(V_0, V_k, V_0) &= V_{k+1}. \end{aligned}$$

In our context the more convenient notation for Young diagrams is due to Frobenius: $\lambda := (a_1, \dots, a_r | b_1, \dots, b_r)$ stands for a diagram λ with a_i boxes in the i -th row on the right of the diagonal, and with b_i boxes in the i -th column below the diagonal and the rank $r = r(\lambda)$ is the number of boxes on the diagonal.

For self-dual diagrams $\lambda = \lambda'$, i.e., $a_i = b_i$ we set $V_{a_1, \dots, a_r} := V_{(a_1, \dots, a_r | a_1, \dots, a_r)}$ when $a_1 > a_2 > \dots > a_r \geq 0$ (and set the convention $V_{a_1, \dots, a_r} := 0$ otherwise). Any two elementary hooks V_{a_1} and V_{a_2} can be glued together by the binary operation m_2 , the decomposition of $m_2(V_{a_1}, V_{a_2}) \cong m_2(V_{a_2}, V_{a_1})$ is given by

$$m_2(V_{a_1}, V_{a_2}) = V_{a_1, a_2} \oplus \left(\bigoplus_{i=1}^{a_2} V_{a_1+i, a_2-i} \right) \quad a_1 \geq a_2$$

where the “leading” term V_{a_1, a_2} has the diagram with minimal height. Hence any m_2 -bracketing of the hooks $V_{a_1}, V_{a_2}, \dots, V_{a_r}$ yields³ a sum of $GL(V)$ -modules

$$m_2(\dots m_2(m_2(V_{a_1}, V_{a_2}), V_{a_3}), \dots, V_{a_r}) = V_{a_1, \dots, a_r} \oplus \dots$$

whose module with minimal height is precisely V_{a_1, \dots, a_r} . We conclude that all elements in the C_∞ -algebra $H^\bullet(\mathfrak{g}, \mathbb{K})$ can be generated in $H^1(\mathfrak{g}, \mathbb{K})$ by m_2 and m_3 . \square

One could draw a parallel between the latter theorem for the cubic algebra PS and the proposition 4.1 for Koszul algebra; in both cases the Yoneda algebra $\text{Ext}_{PS}^\bullet(\mathbb{K}, \mathbb{K})$ is generated only in $\text{Ext}_{PS}^1(\mathbb{K}, \mathbb{K})$. Although we have the notion of N -Koszul algebras for the N -homogeneous algebras [2, 3], it turns out that the cubic algebra PS is not 3-Koszul, beside the exceptional case when $\dim V = 2$. Instead the algebra $PS = U\mathfrak{g}$ falls in the class of *Artin-Schelter-regular algebras* [1], being an UEA of positively graded Lie algebra (for a proof see [6]). The parallel between the quadratic Koszul algebra $S(V)$ and the cubic AS-regular algebra $PS(V)$ suggests that the C_∞ -algebra $\text{Ext}_{PS}^\bullet(\mathbb{K}, \mathbb{K})$ is a generalization of a Koszul dual algebra of PS in the realm of the homotopy algebras, an idea that has been put forward in [13].

The analogy would be complete if we had the following conjectural proposition.

PROPOSITION 5.1. *The cohomology $H^\bullet(\mathfrak{g}, \mathbb{K}) \cong \text{Ext}_{PS}^\bullet(\mathbb{K}, \mathbb{K})$ of the free 2-nilpotent graded Lie algebra $\mathfrak{g} = V \otimes \bigwedge^2 V$ can be endowed with a structure of C_∞ -algebra having trivial higher multiplications $m_k = 0$, $k \geq 4$.*

So far we were able to prove this conjecture only in dimensions $\dim V \leq 3$. Our proof rests entirely on the bigrading $(2 - k, 0)$ of the multiplication m_k by homological and tensor degree in the C_∞ -algebra $\text{Ext}_{PS}^\bullet(\mathbb{K}, \mathbb{K})$. The bigrading arguments work only for $\dim V = 2$ and $\dim V = 3$ thus for a complete proof the conjecture would need more refined methods.

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³The operation m_2 is associative thus the result does not depend on the choice of the bracketing.

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LABORATOIRE DE PHYSIQUE THEORIQUE, UMR 8627, UNIVERSITE PARIS XI, BATIMENT 210,
F-91 405 ORSAY CEDEX, FRANCE

E-mail address: Michel.Dubois-Violette@th.u-psud.fr

THEORETICAL PHYSICS DIVISION, INSTITUTE FOR NUCLEAR RESEARCH AND NUCLEAR ENERGY,
BULGARIAN ACADEMY OF SCIENCES, SOFIA, BG-1784, BULGARIA

E-mail address: tpopov@inrne.bas.bg