



# $C_\infty$ -structure on the cohomology of free 2-nilpotent Lie algebra

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# 2-Step Nilpotent Lie algebra and $C_\infty$ -Algebras

Michel Dubois-Violette and Todor Popov

*Communicated by*

ABSTRACT. We consider the free 2-step nilpotent graded Lie algebra and its cohomology ring. The homotopy transfer induces a homotopy commutative algebra on its cohomology ring which we describe.

## 1. Homotopy algebras

The homotopy associative algebras, or  $A_\infty$ -algebras were introduced by Jim Stasheff in the 1960's as a tool in algebraic topology for studying 'group-like' spaces. Homotopy algebras received a new attention and further development in the 1990's after the discovery of their relevance into a multitude of topics in algebraic geometry, symplectic and contact geometry, knot theory, moduli spaces, deformation theory...

DEFINITION 1.1. ( $A_\infty$ -algebra) A homotopy associative algebra, or  $A_\infty$ -algebra, over  $\mathbb{K}$  is a  $\mathbb{Z}$ -graded vector space  $A = \bigoplus_{i \in \mathbb{Z}} A^i$  endowed with a family of graded mappings (operations)

$$m_n : A^{\otimes n} \rightarrow A, \quad \deg(m_n) = 2 - n \quad n \geq 1$$

satisfying the Stasheff identities **SI**(**n**) for  $n \geq 1$

$$\mathbf{SI}(\mathbf{n}) : \sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(Id^{\otimes r} \otimes m_s \otimes Id^{\otimes t}) = 0 \quad r \geq 0, t \geq 0, s \geq 1$$

where the sum runs over all decompositions  $n = r + s + t$ . Throughout the text we assume the Koszul sign convention  $(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y)$ .

A morphism of two  $A_\infty$ -algebras  $A$  and  $B$  is a family of graded maps  $f_n : A^{\otimes n} \rightarrow B$  for  $n \geq 1$  with  $\deg f_n = 1 - n$  such that the following conditions hold

$$\sum_{r+s+t=n} (-1)^{r+st} f_{r+1+t}(Id^{\otimes r} \otimes m_s \otimes Id^{\otimes t}) = \sum_{1 \leq r \leq n} (-1)^S m_r(f_{i_1} \otimes f_{i_2} \otimes \dots \otimes f_{i_r})$$

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where the sum is on all decompositions  $i_1 + \dots + i_r = n$  and the sign  $(-1)^S$  on RHS is determined by  $S = (r-1)(i_1-1) + (r-2)(i_2-1) + \dots + 2(i_{r-2}-1) + (i_{r-1}-1)$ . The morphism  $f$  is a *quasi-isomorphism of  $A_\infty$ -algebras* if  $f_1$  is a quasi-isomorphism. It is strict if  $f_i = 0$  for all  $i \neq 1$ . The identity morphism of  $A$  is the strict morphism  $f$  such that  $f_1$  is the identity of  $A$ .

We define the shuffle product  $Sh_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \rightarrow A^{\otimes p+q}$  throughout the expression

$$(a_1 \otimes \dots \otimes a_p) \Delta (a_{p+1} \otimes \dots \otimes a_{p+q}) = \sum_{\sigma \in Sh_{p,q}} sgn(\sigma) a_{\sigma^{-1}(1)} \otimes \dots \otimes a_{\sigma^{-1}(p+q)}$$

where the sum runs over all  $(p, q)$ -shuffles  $Sh_{p,q}$ , i.e., over all permutations  $\sigma \in S_{p+q}$  such that  $\sigma(1) < \sigma(2) < \dots < \sigma(p)$  and  $\sigma(p+1) < \sigma(p+2) < \dots < \sigma(p+q)$ .

DEFINITION 1.2. ( $C_\infty$ -algebra [9]) A homotopy commutative algebra, or  $C_\infty$ -algebra, is an  $A_\infty$ -algebra  $\{A, m_n\}$  such that each operation  $m_n$  vanishes on non-trivial shuffles

$$(1.1) \quad m_n((a_1 \otimes \dots \otimes a_p) \Delta (a_{p+1} \otimes \dots \otimes a_n)) = 0, \quad 1 \leq p \leq n-1.$$

In particular for  $m_2$  we have  $m_2(a \otimes b \pm b \otimes a) = 0$ , so a  $C_\infty$ -algebra such that  $m_n = 0$  for  $n \geq 3$  is a (super-)commutative DGA.

A morphism of  $C_\infty$ -algebras is a morphism of  $A_\infty$ -algebras vanishing on non-trivial shuffles  $f_n((a_1 \otimes \dots \otimes a_p) \Delta (a_{p+1} \otimes \dots \otimes a_n)) = 0$ ,  $1 \leq p \leq n-1$ .

## 2. Homotopy Transfer Theorem

LEMMA 2.1. *Every cochain complex  $(A, d)$  of vector spaces over a field  $\mathbb{K}$  has its cohomology  $H^\bullet(A)$  as a deformation retract.*

One can always choose a vector space decomposition of the cochain complex  $(A, d)$  such that  $A^n \cong B^n \oplus H^n \oplus B^{n+1}$  where  $H^n$  is the cohomology and  $B^n$  is the space of coboundaries,  $B^n = dA^{n-1}$ . We choose a homotopy  $h : A^n \rightarrow A^{n-1}$  which identifies  $B^n$  with its copy in  $A^{n-1}$  and is 0 on  $H^n \oplus B^{n+1}$ . The projection  $p$  to the cohomology and the cocycle-choosing inclusion  $i$  given by  $A^n \xrightleftharpoons[i]{p} H^n$  are chain homomorphisms, satisfying the additional *side conditions*

$$hh = 0, \quad hi = 0, \quad ph = 0.$$

With these choices done the complex  $(H^\bullet(A), 0)$  is a deformation retract of  $(A, d)$

$$h \circlearrowleft (A, d) \xrightleftharpoons[i]{p} (H^\bullet(A), 0), \quad pi = Id_{H^\bullet(A)}, \quad ip - Id_A = dh + hd.$$

Let now  $(A, d, \mu)$  be a DGA, i.e.,  $A$  is endowed with an associative product  $\mu$  compatible with  $d$ . The cochain complexes  $(A, d)$  and its contraction  $H^\bullet(A)$  are homotopy equivalent, but the associative structure is not stable under homotopy equivalence. However the associative structure on  $A$  can be transferred to an  $A_\infty$ -structure on a homotopy equivalent complex, a particular interesting complex being the deformation retract  $H^\bullet(A)$ . For a friendly introduction to homotopy transfer

theorems in much boarder context we send the reader to the textbook [12], see chapter 9.

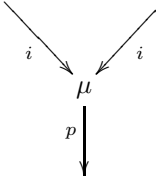
**THEOREM 2.1** (Kadeishvili [9]). *Let  $(A, d, \mu)$  be a (commutative) DGA over a field  $\mathbb{K}$ . There exists a  $A_\infty$ -algebra ( $C_\infty$ -algebra) structure on the cohomology  $H^\bullet(A)$  and a  $A_\infty(C_\infty)$ -quasi-isomorphism*

$$f_i : (\otimes^i H^\bullet(A), \{m_i\}) \rightarrow (A, \{d, \mu, 0, 0, \dots\})$$

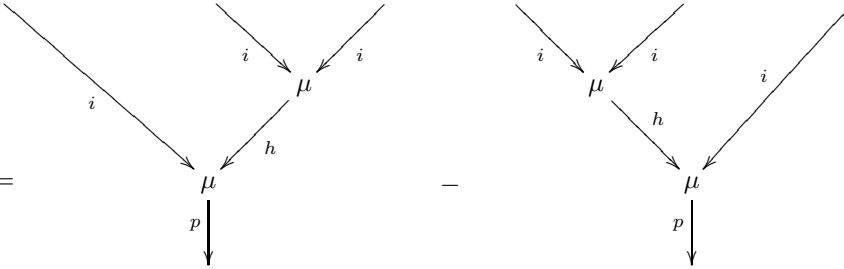
*such that the inclusion  $f_1 = i : H^\bullet(A) \rightarrow A$  is a cocycle-choosing homomorphism of cochain complexes. The differential  $m_1$  on  $H^\bullet(A)$  is zero ( $m_1 = 0$ ) and  $m_2$  is the strictly associative operation induced by the multiplication on  $A$ . The resulting structure is unique up to quasi-isomorphism.*

Kontsevich and Soibelman [10] gave an explicit expressions for the higher operations of the induced  $A_\infty$ -structure as sums over decorated planar binary trees with one root where all leaves are decorated by the inclusion  $i$ , the root by the projection  $p$  the vertices by the product  $\mu$  of the (commutative) DGA  $(A, d, \mu)$  and the internal edges by the homotopy  $h$ . The  $C_\infty$ -structure implies additional symmetries on trees.

We will make use of the graphic representation for the binary operation on  $H^\bullet(A)$

$$m_2(x, y) := p\mu(i(x), i(y)) \quad \text{or} \quad m_2 =$$


and the ternary one  $m_3(x, y, z) = p\mu(i(x), h\mu(i(y), i(z))) - p\mu(h\mu(i(x), i(y)), i(z))$  being the sum of two planar binary trees with three leaves

$$m_3 =$$


### 3. Homology and cohomology of Lie algebra $\mathfrak{g}$

A non-minimal projective(in fact free) resolution of the trivial  $U\mathfrak{g}$ -module  $\mathbb{K}$ ,  $C(\mathfrak{g}) \xrightarrow{\epsilon} \mathbb{K}$  is given by the standard Chevalley-Eilenberg chain complex  $C_\bullet(\mathfrak{g}) =$

$(U\mathfrak{g} \otimes_{\mathbb{K}} \wedge^p \mathfrak{g}, d_p)$  with differential maps

$$(3.1) \quad \begin{aligned} d_p(u \otimes x_1 \wedge \dots \wedge x_p) &= \sum_i (-1)^{i+1} u x_i \otimes x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge x_p \\ &+ \sum_{i < j} (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge \hat{x}_j \wedge \dots \wedge x_p \end{aligned}$$

The homologies  $H_n(\mathfrak{g}, \mathbb{K})$  of the Lie algebra  $\mathfrak{g}$  with trivial coefficients are given by the homologies of the derived complex  $\mathbb{K} \otimes_{U\mathfrak{g}} C_{\bullet}(\mathfrak{g})$

$$\mathrm{Tor}_n^{U\mathfrak{g}}(\mathbb{K}, \mathbb{K}) \cong H_n(\mathbb{K} \otimes_{U\mathfrak{g}} C_{\bullet}(\mathfrak{g})) = H_n(\mathfrak{g}, \mathbb{K}) .$$

The complex  $\mathbb{K} \otimes_{U\mathfrak{g}} C_{\bullet}(\mathfrak{g})$  is the chain complex with degrees  $\bigwedge^{\bullet} \mathfrak{g} = \mathbb{K} \otimes_{U\mathfrak{g}} U\mathfrak{g} \otimes \bigwedge^{\bullet} \mathfrak{g}$  and differentials  $\partial_p := id \otimes_{U\mathfrak{g}} d_p : \bigwedge^p \mathfrak{g} \rightarrow \bigwedge^{p-1} \mathfrak{g}$  induced by the extension as coderivation of the Lie bracket  $\partial_2 := -[\cdot, \cdot] : \bigwedge^2 \mathfrak{g} \rightarrow \mathfrak{g}$ .

The dual cochain complex  $\mathrm{Hom}_{U\mathfrak{g}}(C(\mathfrak{g}), \mathbb{K}) = (\bigwedge^{\bullet} \mathfrak{g}^*, \delta)$  has coboundary map  $\delta^p : \bigwedge^p \mathfrak{g}^* \rightarrow \bigwedge^{p+1} \mathfrak{g}^*$  (being transposed to the differential  $\partial_{p+1}$ ) which is the extension as derivation of the dualization of the Lie bracket  $\delta^1 := [\cdot, \cdot]^* : \mathfrak{g}^* \rightarrow \bigwedge^2 \mathfrak{g}^*$ . One calculates the cohomologies<sup>1</sup> of the Lie algebra  $\mathfrak{g}$  as

$$(3.2) \quad \mathrm{Ext}_{U\mathfrak{g}}^n(\mathbb{K}, \mathbb{K}) \cong H^n(\mathrm{Hom}_{U\mathfrak{g}}(C(\mathfrak{g}), \mathbb{K})) = H^n(\mathfrak{g}, \mathbb{K}) .$$

Hence the algebra  $(\bigwedge^{\bullet} \mathfrak{g}^*, \delta)$  equipped with  $\delta$  is a (*super*)commutative DGA and the Yoneda algebra  $\mathrm{Ext}_{U\mathfrak{g}}^{\bullet}(\mathbb{K}, \mathbb{K}) = \bigoplus_n \mathrm{Ext}_{U\mathfrak{g}}^n(\mathbb{K}, \mathbb{K})$  has structure of commutative associative algebra. Moreover due to the Kadeishvili theorem the Yoneda algebra  $\mathrm{Ext}_{U\mathfrak{g}}^{\bullet}(\mathbb{K}, \mathbb{K}) = H^{\bullet}(\mathfrak{g}, \mathbb{K})$  is a  $C_{\infty}$ -algebra which stems from the homotopy transfer of the wedge product  $\wedge$  on cohomology classes

$$H^i(\mathfrak{g}, \mathbb{K}) \wedge H^j(\mathfrak{g}, \mathbb{K}) \rightarrow H^{i+j}(\mathfrak{g}, \mathbb{K}) .$$

#### 4. Abelian Lie algebra $\mathfrak{h} = V$

Let us take as a basic example the abelian Lie algebra  $\mathfrak{h}$ , that is, the free nilpotent Lie algebra generated by a finite dimensional vector space  $V$  of degree 1. The Lie bracket of  $\mathfrak{h}$  is trivial  $[V, V] = 0$ . According to Poincaré-Birkhoff-Witt theorem the universal enveloping algebra of the abelian Lie algebra  $\mathfrak{h} = V$  is isomorphic to the symmetric algebra

$$U(\mathfrak{h}) \cong S(V) .$$

The Chevalley-Eilenberg complex  $C_{\bullet}(\mathfrak{h}) = S(V) \otimes_{\mathbb{K}} \Lambda^{\bullet} V$  yields the resolution of the trivial  $U(\mathfrak{h})$ -module  $\mathbb{K}$

$$(4.1) \quad \begin{aligned} 0 &\rightarrow S(V) \otimes \Lambda^{\dim V} V \rightarrow S(V) \otimes \Lambda^{\dim V - 1} V \rightarrow \dots \\ \dots &\rightarrow S(V) \otimes \Lambda^2 V \rightarrow S(V) \otimes V \rightarrow S(V) \rightarrow \mathbb{K} \rightarrow 0 . \end{aligned}$$

The derived complex  $\mathbb{K} \otimes_{U\mathfrak{h}} C(\mathfrak{h})$  has zero differential and the Chevalley-Eilenberg resolution turns out to be minimal (which is not the case in general)

$$H_n(\mathfrak{h}, \mathbb{K}) \cong H_n(\mathbb{K} \otimes_{U\mathfrak{h}} C(\mathfrak{h})) \cong \Lambda^n V .$$

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<sup>1</sup>In the presence of metric one has  $\delta := \partial^*$  (see below)

The Chevalley-Eilenberg resolution coincides with the Koszul complex  $K(A) = A \otimes (A^!)^*$  of the symmetric algebra  $A = S(V)$ . The Koszul dual algebra of the symmetric algebra is the exterior algebra  $S(V)^! = \Lambda V^*$ . A quadratic algebra is said to be a Koszul algebra when its Koszul complex  $K_\bullet(A) = A \otimes (A^!_\bullet)^*$  is acyclic everywhere except in degree 0 (where its homology is  $\mathbb{K}$ ). Then the Koszul complex yields a minimal projective (in fact free) resolution by (left)  $A$ -modules of the trivial  $A$ -module  $\mathbb{K}$

$$K(A) \xrightarrow{\epsilon} \mathbb{K} \rightarrow 0.$$

In particular the resolution 4.1 is the same as the resolution by the Koszul complex  $K_n(S(V)) = S(V) \otimes \Lambda^n V^*$  thus the algebra  $S(V)$  is *Koszul algebra*. One has another equivalent definition of Koszul algebra based on the following proposition;

**PROPOSITION 4.1.** *A finitely generated quadratic algebra  $A$  is Koszul iff its Yoneda algebra  $\text{Ext}_A(\mathbb{K}, \mathbb{K})$  is generated in degree 1. One has then  $\text{Ext}_A(\mathbb{K}, \mathbb{K}) \cong A^!$ .*

Indeed the Yoneda algebra  $\text{Ext}_{S(V)}(\mathbb{K}, \mathbb{K})$  of the symmetric algebra  $S(V)$  is just the exterior algebra

$$\text{Ext}_{S(V)}^n(\mathbb{K}, \mathbb{K}) = (\text{Tor}_n^{S(V)}(\mathbb{K}, \mathbb{K}))^* = \Lambda^n V^*$$

which is obviously generated by  $V^*$ , i.e., in degree 1, by the wedge product. Through the homotopy transfer the Yoneda algebra  $\text{Ext}_{S(V)}(\mathbb{K}, \mathbb{K})$  inherits a  $C_\infty$ -structure but it is easy to show (by degree preserving argument) that the latter  $C_\infty$ -algebra is formal, i.e., all higher multiplications are trivial,  $m_n = 0$  for  $n \neq 2$ .

## 5. Homology of Free 2-nilpotent algebra $\mathfrak{g} = V \oplus \Lambda^2 V$

Let  $\mathfrak{g}$  be the free graded 2-step nilpotent Lie algebra generated by vector space  $V$  in degree 1,  $\mathfrak{g} = V \oplus [V, V]$ . In other words we consider the graded Lie algebra  $\mathfrak{g}$  with Lie bracket

$$[u, v] = \begin{cases} u \wedge v & \in \Lambda^2 V \\ 0 & \text{otherwise} \end{cases}$$

We denote the Universal Enveloping Algebra(UEA)  $U\mathfrak{g}$  by  $PS$  and refer to it as *parastatistics algebra*.<sup>2</sup> Throughout this note we will consider the generators space  $V$  to be an ordinary vector space  $V$  which corresponds to a parafermionic algebra  $PS(V) = U\mathfrak{g}$ . The case of a  $\mathbb{Z}_2$ -space of generators  $V = V_0 \oplus V_1$ , that is,  $PS(V)$  is the Universal Enveloping Algebra of a Lie super-algebra  $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$  (which would include the parabosonic algebras) will be treated elsewhere. More on parastatistics algebras and their application to combinatorics the reader could find in the articles [5, 11].

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<sup>2</sup>Such cubic algebras arise through the exchange relations between the operators in a quantization procedure introduced by H. S. Green [7] for particles obeying more general statistics than Bose-Einstein or Fermi-Dirac, coined parabosons and parafermions.

The parastatistics algebra  $PS(V)$  generated by a finite dimensional vector space  $V$  in degree 1 is the positively graded algebra

$$PS(V) := U\mathfrak{g} = U(V \oplus \bigwedge^2 V) = T(V)/([V, V], V) .$$

We shall write simply  $PS$  when the space of generators  $V$  is clear from the context.

The homologies  $H_n(\mathfrak{g}, \mathbb{K})$  of the 2-nilpotent Lie algebra  $\mathfrak{g}$  are the homologies of the chain complex

$$\bigwedge^n \mathfrak{g} = \bigwedge^n (V \oplus \bigwedge^2 V) = \bigoplus_{s+r=n} \bigwedge^s (\bigwedge^2 V) \otimes \bigwedge^r (V)$$

with differentials  $\partial_n : \bigwedge^s (\bigwedge^2 V) \otimes \bigwedge^r (V) \rightarrow \bigwedge^{s+1} (\bigwedge^2 V) \otimes \bigwedge^{r-2} (V)$  are given by

$$\begin{aligned} \partial_n : e_{i_1 j_1} \wedge \dots \wedge e_{i_s j_s} \otimes e_1 \wedge \dots \wedge e_r &\mapsto \\ \sum_{i < j} (-1)^{i+j} e_{ij} \wedge e_{i_1 j_1} \wedge \dots \wedge e_{i_s j_s} \otimes e_1 \wedge \dots \wedge \hat{e}_i \wedge \dots \wedge \hat{e}_j \wedge \dots \wedge e_r . \end{aligned}$$

The differential  $\partial$  identifies a pair of degree 1 generators  $e_i, e_j \in V$  with one degree 2 generator  $e_{ij} := (e_i \wedge e_j) = [e_i, e_j] \in \bigwedge^2 V$ .

The cohomologies  $H^n(\mathfrak{g}, \mathbb{K})$  arise from the dualized complex with coboundary map  $\delta^n : \bigwedge^n \mathfrak{g}^* \rightarrow \bigwedge^{n+1} \mathfrak{g}^*$  which is transposed to the differential  $\partial_{n+1}$

$$\begin{aligned} (5.1) \quad \delta^n : e_{i_1 j_1}^* \wedge \dots \wedge e_{i_s j_s}^* \otimes e_{l_1}^* \wedge \dots \wedge e_{l_r}^* &\mapsto \\ \sum_{k=1}^s \sum_{i_k < j_k} (-1)^{i+j} e_{i_1 j_1}^* \wedge \dots \wedge \hat{e}_{i_k j_k}^* \wedge \dots \wedge e_{i_s j_s}^* \otimes e_{i_k}^* \wedge e_{j_k}^* \wedge e_{l_1}^* \wedge \dots \wedge \dots \wedge e_{l_r}^* . \end{aligned}$$

In the presence of metric  $g$  one has an identification  $V \xrightarrow{g} V^*$ , and  $\bigwedge^\bullet \mathfrak{g} \xrightarrow{g} \bigwedge^\bullet \mathfrak{g}^*$ . The adjoint operator  $\partial_n^* : \bigwedge^n \mathfrak{g} \rightarrow \bigwedge^{n+1} \mathfrak{g}$  is defined by  $g(\partial_n^* v, w) = g(v, \partial_{n+1} w)$ . One can show that independently of the metric  $g$  chosen the action of  $\partial_n^*$  takes the form

$$\begin{aligned} (5.2) \quad \partial_n^* : e_{i_1 j_1} \wedge \dots \wedge e_{i_s j_s} \otimes e_{l_1} \wedge \dots \wedge e_{l_r} &\mapsto \\ \sum_{k=1}^s \sum_{i_k < j_k} (-1)^{i+j} e_{i_1 j_1} \wedge \dots \wedge \hat{e}_{i_k j_k} \wedge \dots \wedge e_{i_s j_s} \otimes e_{i_k} \wedge e_{j_k} \wedge e_{l_1} \wedge \dots \wedge \dots \wedge e_{l_r} . \end{aligned}$$

We will see in the following that after the identification  $\bigwedge^\bullet \mathfrak{g} \xrightarrow{g} \bigwedge^\bullet \mathfrak{g}^*$  the map  $\partial^* \xrightarrow{g} \delta$  will play the role of homotopy for the chain complex  $(\bigwedge^\bullet \mathfrak{g}, \partial_\bullet)$ , and vice versa the boundary map  $\partial \xrightarrow{g} \delta^*$  is a homotopy for the cochain complex  $(\bigwedge^\bullet \mathfrak{g}^*, \delta^\bullet)$ .

The complexes  $(\bigwedge^n \mathfrak{g}, \partial_n, \partial_n^*)$  and  $(\bigwedge^n \mathfrak{g}^*, \delta^n, \delta^{*n})$  are bigraded by two different degrees; the homological degree  $n := r + s$  counting the number of Lie algebra generators and the tensor degree  $t := 2s + r$  also called weight. The cohomologies  $H^n(\mathfrak{g}, \mathbb{K})$  can have components of different weight  $t$ ,  $H^n(\mathfrak{g}, \mathbb{K}) = \bigoplus_t H^n(\mathfrak{g}, \mathbb{K})_t$  and the weight  $t$  is in fact the *Adams grading* on Yoneda algebra  $\text{Ext}_{U\mathfrak{g}}^n(\mathbb{K}, \mathbb{K})_t$  [13]. The differential and the homotopy,  $\delta = \partial^*$  and  $\partial = \delta^*$  do not alter the weight  $t$ , but raise and lower the homological degree  $n$ .

The operations  $m_k$  in homotopy algebra are bigraded by homological and Adams gradings of bidegree  $(k, t) = (2 - k, 0)$ . The bi-grading impose the vanishing of many higher products.

**5.1. Homology of  $\mathfrak{g}$  as a  $GL(V)$ -module.** A Schur module  $V_\lambda$  is an irreducible polynomial  $GL(V)$ -module labelled by a Young diagram  $\lambda$ . The basis of a Schur module  $V_\lambda$  is in bijection with semistandard Young tableaux which are fillings of the Young diagram  $\lambda$  with the numbers of the set  $\{1, \dots, \dim V\}$ . The action of the linear group  $GL(V)$  on the space  $V$  of the generators of the Lie algebra  $\mathfrak{g}$  induces a  $GL(V)$ -action on the universal enveloping algebra  $PS = U\mathfrak{g} \cong S(V \oplus \Lambda^2 V)$  and on the space  $\bigwedge^\bullet \mathfrak{g} \cong \bigwedge^\bullet (V \oplus \Lambda^2 V)$ .

The maps  $\partial$  and  $\partial^*$  both commute with the  $GL(V)$ -action. It follows that the homology and cohomology carry structure of  $GL(V)$ -modules hence can be decomposed into irreducibles.

The *Laplacian*  $\Delta = \oplus_{n \geq 0} \Delta_n$  is defined to be the self-adjoint operator

$$\Delta_n = \partial_{n+1} \partial_{n+1}^* + \partial_n^* \partial_n \in \text{End}(\bigwedge^n \mathfrak{g}) .$$

Its kernel is a complete set of representatives for the homology classes in  $H_p(\mathfrak{g}, \mathbb{K})$

$$\ker \Delta_n \cong H_n(\mathfrak{g}, \mathbb{K}) .$$

The decomposition of the  $GL(V)$ -module  $H_n(\mathfrak{g}, \mathbb{K})$  into irreducible polynomial representations  $V_\lambda$  is given by the following theorem;

**THEOREM 5.1** (Józefiak and Weyman [8], Sigg [14]). *The homology  $H_\bullet(\mathfrak{g}, \mathbb{K})$  of the 2-nilpotent Lie algebra  $\mathfrak{g} = V \oplus \Lambda^2 V$  decomposes into irreducible  $GL(V)$ -modules*

$$(5.3) \quad H_n(\mathfrak{g}, \mathbb{K}) = H_n(\bigwedge^\bullet \mathfrak{g}, \partial) \cong \text{Tor}_n^{PS}(\mathbb{K}, \mathbb{K})(V) \cong \bigoplus_{\lambda: \lambda = \lambda'} V_\lambda$$

where the sum is over self-conjugate Young diagrams  $\lambda$  such that  $n = \frac{1}{2}(|\lambda| + r(\lambda))$ .

**5.2. Homological interpretation of the Littlewood formula.** We recall the beautiful result of Józefiak and Weyman [8] giving a representation-theoretic interpretation of the Littlewood formula

$$(5.4) \quad \prod_i (1 - x_i) \prod_{i < j} (1 - x_i x_j) = \sum_{\lambda: \lambda = \lambda'} (-1)^{\frac{1}{2}(|\lambda| + r(\lambda))} s_\lambda(x) .$$

Here the sum is over the self-dual Young diagrams  $\lambda$ ,  $s_\lambda(x)$  stands for the Schur function and  $r(\lambda)$  stands the rank of  $\lambda$  which is the number of diagonal boxes in  $\lambda$ .

One knows that for the graded algebra  $PS$  there exists a minimal resolution by projective modules

$$(5.5) \quad P_\bullet : \quad 0 \rightarrow P_d \rightarrow \dots \rightarrow P_n \rightarrow \dots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \xrightarrow{\epsilon} \mathbb{K} \rightarrow 0 .$$

Here the length  $d$  of the resolution is the projective dimension of the algebra  $PS$  which is  $d = \frac{\dim V(\dim V + 1)}{2}$ . Since  $PS$  is positively graded and, in the category of positively graded modules over connected locally finite graded algebras, projective module is the same as free module [4], we have  $P_n \cong PS \otimes E_n$  where  $E_n$  are finite

dimensional vector spaces. Thus we deal with a minimal resolution of  $\mathbb{K}$  by free  $PS$ -modules and the minimality implies that the derived complex  $\mathbb{K} \otimes_{PS} P_\bullet$  has vanishing differentials, *i.e.*,  $\text{Tor}_\bullet^{PS}(\mathbb{K}, \mathbb{K}) = H_\bullet(\mathbb{K} \otimes_{PS} P_\bullet) = \mathbb{K} \otimes_{PS} P_\bullet$ . Then the multiplicity spaces  $E_n = \text{Tor}_n^{PS}(\mathbb{K}, \mathbb{K})$  are fixed by Theorem 5.1 thus the data  $H_n(\mathfrak{g}, \mathbb{K}) = \text{Tor}_n^{PS}(\mathbb{K}, \mathbb{K})$  encodes the minimal free resolution  $P_\bullet$  (cf. (5.5)) which is unique (up to isomorphism).

The Euler characteristics of  $P_\bullet$  implies an identity about the  $GL(V)$ -characters

$$ch PS(V) \cdot ch \left( \bigoplus_{\lambda: \lambda = \lambda'} (-1)^{\frac{1}{2}(|\lambda| + r(\lambda))} V_\lambda \right) = 1 .$$

The character of a Schur module  $V_\lambda$  is the Schur function,  $ch V_\lambda = s_\lambda(x)$ . Due to the Poincaré-Birkhoff-Witt theorem  $ch PS(V) = ch S(V \oplus \bigwedge^2 V)$  thus the identity reads

$$\prod_i \frac{1}{(1-x_i)} \prod_{i < j} \frac{1}{(1-x_i x_j)} \sum_{\lambda: \lambda = \lambda'} (-1)^{\frac{1}{2}(|\lambda| + r(\lambda))} s_\lambda(x) = 1 .$$

But the latter identity is nothing but rewriting of the Littlewood identity (5.4). The moral is that the Littlewood identity reflects a homological property of the algebra  $PS$ , namely the above particular structure of the minimal projective (free) resolution of  $\mathbb{K}$  by  $PS$ -modules.

### 5.3. $\text{Ext}_{PS}^\bullet(\mathbb{K}, \mathbb{K})$ as $C_\infty$ -algebra.

**THEOREM 5.2.** *The cohomology  $H^\bullet(\mathfrak{g}, \mathbb{K}) \cong \text{Ext}_{PS}^\bullet(\mathbb{K}, \mathbb{K})$  of the free 2-nilpotent graded Lie algebra  $\mathfrak{g} = V \otimes \bigwedge^2 V$  is a homotopy commutative algebra which is generated in degree 1 (*i.e.*, in  $H^1(\mathfrak{g}, \mathbb{K})$ ) by the operations  $m_2$  and  $m_3$ .*

**PROOF.** We start by choosing a metric  $g$  on the vector space  $V$  and an orthonormal basis  $g(e_i, e_j) = \delta_{ij}$ . The choice induces a metric on  $\bigwedge^\bullet \mathfrak{g} \stackrel{g}{\cong} \bigwedge^\bullet \mathfrak{g}^*$ .

The isomorphisms  $V \cong V^*$  and  $\text{Tor}_n^{PS}(\mathbb{K}, \mathbb{K}) \cong \text{Ext}_{PS}^n(\mathbb{K}, \mathbb{K})$  and the theorem 5.1 imply the decomposition of  $H^\bullet(\mathfrak{g}, \mathbb{K})$  into irreducible  $GL(V)$ -modules

$$H^n(\mathfrak{g}, \mathbb{K}) \cong H^n(\bigwedge \mathfrak{g}^*, \delta) \cong \text{Ext}_{PS}^n(\mathbb{K}, \mathbb{K}) \cong \bigoplus_{\lambda: \lambda = \lambda'} V_\lambda$$

where the sum is over self-conjugate diagrams  $\lambda$  such that  $n = \frac{1}{2}(|\lambda| + r(\lambda))$ .

The adjoint of the boundary map  $\partial$ ,  $\delta \stackrel{g}{:=} \partial^*$  is the differential in the DGA  $(\bigwedge \mathfrak{g}^*, \delta)$  while  $\delta^* \stackrel{g}{:=} \partial$  plays the role of a homotopy. In view of lemma 2.1 we have the cohomology  $H^\bullet(\bigwedge^\bullet \mathfrak{g}^*, \delta^\bullet)$  as deformation retract of the complex  $(\bigwedge^\bullet \mathfrak{g}^*, \delta^\bullet)$ ,

$$pi = Id_{H^\bullet(\bigwedge^\bullet \mathfrak{g}^*)} , \quad ip - Id_{\bigwedge^\bullet \mathfrak{g}^*} = \delta \delta^* + \delta^* \delta , \quad \delta^* \stackrel{g}{=} \partial .$$

Here the projection  $p$  identifies the subspace  $\ker \delta \cap \ker \delta^*$  with  $H^\bullet(\bigwedge^\bullet \mathfrak{g}^*)$ , which is the orthogonal complement of the space of the coboundaries  $\text{im} \delta$ . The cocycle-choosing homomorphism  $i$  is  $Id$  on  $H^\bullet(\bigwedge^\bullet \mathfrak{g}^*)$  and zero on coboundaries.

We apply the Kadeishvili homotopy transfer Theorem 2.1 for the commutative DGA  $(\bigwedge^\bullet \mathfrak{g}^*, \mu, \delta^\bullet)$  and its deformation retract  $H^\bullet(\bigwedge^\bullet \mathfrak{g}^*) \cong H^\bullet(\mathfrak{g}, \mathbb{K})$  and conclude that the cohomology  $H^\bullet(\mathfrak{g}, \mathbb{K})$  is a  $C_\infty$ -algebra.

The Kontsevich and Soibelman tree representations of the operations  $m_n$  provide explicit expressions. Let us take  $\mu$  to be the super-commutative product  $\wedge$  on the DGA  $(\wedge^\bullet \mathfrak{g}^*, \delta^\bullet)$ . The projection  $p$  maps onto the Schur modules  $V_\lambda$  with self-conjugated Young diagram  $\lambda = \lambda'$ .

The binary operation on the generators  $e_i \in H^1(\mathfrak{g}, \mathbb{K})$  is trivial, one gets

$$m_2(e_i, e_j) = p(e_i \wedge e_j) = 0 \quad p(V_{(1^2)}) = 0.$$

Hence  $H^\bullet(\mathfrak{g}, \mathbb{K})$  could not be generated in  $H^1(\mathfrak{g}, \mathbb{K})$  as algebra with product  $m_2$ .

The ternary operation  $m_3$  restricted to  $H^1(\mathfrak{g}, \mathbb{K})$  is nontrivial, indeed one has

$$\begin{aligned} m_3(e_i, e_j, e_k) &= p\{e_i \wedge \partial(e_j \wedge e_k) - \partial(e_i \wedge e_j) \wedge e_k\} = p\{e_{ij} \wedge e_k - e_i \wedge e_{jk}\} \\ &= p\{(e_{ij} \wedge e_k + e_{jk} \wedge e_i + e_{ki} \wedge e_j) - e_{ki} \wedge e_j\} = e_{ik} \wedge e_j \in H^2(\mathfrak{g}, \mathbb{K}) \end{aligned}$$

The completely antisymmetric combination in the brackets (...) spans the Schur module  $V_{(1^3)}$ ,  $p(e_{ij} \wedge e_k + e_{jk} \wedge e_i + e_{ki} \wedge e_j) = 0$  yields a Jacobi-type identity. The monomials  $e_{ij} \wedge e_k$  modulo  $V_{(1^3)}$  span a Schur module  $V_{(2,1)} \in H^2(\mathfrak{g}, \mathbb{K})$  with basis in bijection with the semistandard Young tableaux

$$e_{ik} \wedge e_j \leftrightarrow \begin{array}{|c|c|} \hline i & j \\ \hline k & \\ \hline \end{array} \quad \text{and} \quad e_{ij} \wedge e_k \leftrightarrow \begin{array}{|c|c|} \hline i & k \\ \hline j & \\ \hline \end{array}.$$

We check the symmetry condition on ternary operation  $m_3$  in  $C_\infty$ -algebra; indeed  $m_3$  vanishes on the (signed) shuffles  $Sh_{1,2}$

$$m_3(e_i \Delta e_j \otimes e_k) = m_3(e_i, e_j, e_k) - m_3(e_j, e_i, e_k) + m_3(e_j, e_k, e_i) = 0.$$

Similarly one gets  $m_3(e_i \otimes e_j \Delta e_k) = 0$  on shuffles  $Sh_{2,1}$ .

On the level of Schur modules the ternary operation glues three fundamental  $GL(V)$ -representations  $V_\square$  into a Schur module  $V_{(2,1)}$ . By iteration of the process of gluing boxes we generate all elementary hooks  $V_k := V_{(k+1, 1^k)}$ ,

$$\begin{aligned} m_3(V_\square, V_\square, V_\square) &= V_{\begin{array}{|c|c|} \hline & \\ \hline \end{array}} \\ m_3\left(V_\square, V_{\begin{array}{|c|c|} \hline & \\ \hline \end{array}}, V_\square\right) &= V_{\begin{array}{|c|c|c|} \hline & & \\ \hline \end{array}} \\ &\dots \\ m_3(V_0, V_k, V_0) &= V_{k+1}. \end{aligned}$$

In our context the more convenient notation for Young diagrams is due to Frobenius:  $\lambda := (a_1, \dots, a_r | b_1, \dots, b_r)$  stands for a diagram  $\lambda$  with  $a_i$  boxes in the  $i$ -th row on the right of the diagonal, and with  $b_i$  boxes in the  $i$ -th column below the diagonal and the rank  $r = r(\lambda)$  is the number of boxes on the diagonal.

For self-dual diagrams  $\lambda = \lambda'$ , i.e.,  $a_i = b_i$  we set  $V_{a_1, \dots, a_r} := V_{(a_1, \dots, a_r | a_1, \dots, a_r)}$  when  $a_1 > a_2 > \dots > a_r \geq 0$  (and set the convention  $V_{a_1, \dots, a_r} := 0$  otherwise). Any two elementary hooks  $V_{a_1}$  and  $V_{a_2}$  can be glued together by the binary operation  $m_2$ , the decomposition of  $m_2(V_{a_1}, V_{a_2}) \cong m_2(V_{a_2}, V_{a_1})$  is given by

$$m_2(V_{a_1}, V_{a_2}) = V_{a_1, a_2} \oplus \left( \bigoplus_{i=1}^{a_2} V_{a_1+i, a_2-i} \right) \quad a_1 \geq a_2$$

where the “leading” term  $V_{a_1, a_2}$  has the diagram with minimal height. Hence any  $m_2$ -bracketing of the hooks  $V_{a_1}, V_{a_2}, \dots, V_{a_r}$  yields<sup>3</sup> a sum of  $GL(V)$ -modules

$$m_2(\dots m_2(m_2(V_{a_1}, V_{a_2}), V_{a_3}), \dots, V_{a_r}) = V_{a_1, \dots, a_r} \oplus \dots$$

whose module with minimal height is precisely  $V_{a_1, \dots, a_r}$ . We conclude that all elements in the  $C_\infty$ -algebra  $H^\bullet(\mathfrak{g}, \mathbb{K})$  can be generated in  $H^1(\mathfrak{g}, \mathbb{K})$  by  $m_2$  and  $m_3$ .  $\square$

One could draw a parallel between the latter theorem for the cubic algebra  $PS$  and the proposition 4.1 for Koszul algebra; in both cases the Yoneda algebra  $\text{Ext}_{PS}^\bullet(\mathbb{K}, \mathbb{K})$  is generated only in  $\text{Ext}_{PS}^1(\mathbb{K}, \mathbb{K})$ . Although we have the notion of  $N$ -Koszul algebras for the  $N$ -homogeneous algebras [2, 3], it turns out that the cubic algebra  $PS$  is not 3-Koszul, beside the exceptional case when  $\dim V = 2$ . Instead the algebra  $PS = U\mathfrak{g}$  falls in the class of *Artin-Schelter-regular algebras* [1], being an UEA of positively graded Lie algebra (for a proof see [6]). The parallel between the quadratic Koszul algebra  $S(V)$  and the cubic AS-regular algebra  $PS(V)$  suggests that the  $C_\infty$ -algebra  $\text{Ext}_{PS}^\bullet(\mathbb{K}, \mathbb{K})$  is a generalization of a Koszul dual algebra of  $PS$  in the realm of the homotopy algebras, an idea that has been put forward in [13].

The analogy would be complete if we had the following conjectural proposition.

**PROPOSITION 5.1.** *The cohomology  $H^\bullet(\mathfrak{g}, \mathbb{K}) \cong \text{Ext}_{PS}^\bullet(\mathbb{K}, \mathbb{K})$  of the free 2-nilpotent graded Lie algebra  $\mathfrak{g} = V \otimes \bigwedge^2 V$  can be endowed with a structure of  $C_\infty$ -algebra having trivial higher multiplications  $m_k = 0$ ,  $k \geq 4$ .*

So far we were able to prove this conjecture only in dimensions  $\dim V \leq 3$ . Our proof rests entirely on the bigrading  $(2 - k, 0)$  of the multiplication  $m_k$  by homological and tensor degree in the  $C_\infty$ -algebra  $\text{Ext}_{PS}^\bullet(\mathbb{K}, \mathbb{K})$ . The bigrading arguments work only for  $\dim V = 2$  and  $\dim V = 3$  thus for a complete proof the conjecture would need more refined methods.

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## References

1. M. Artin, W.F. Schelter, *Graded algebras of global dimension 3*, Adv. Math. **66** (1987), 171-216.
2. R. Berger, *Koszulity for nonquadratic algebras*, J. Algebra **239** (2001), 705-734.
3. R. Berger, M. Dubois-Violette, M. Wambst, *Homogeneous algebras*, J. Algebra **261**(2003) 172-185.
4. H. Cartan, *Homologie et cohomologie d' une algèbre graduée*, Séminaire Henri Cartan, **11** (1958), 1-20.
5. M. Dubois-Violette, T. Popov, *Homogeneous Algebras, Statistics and Combinatorics*, LMP **61** (2002), 159-170.

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<sup>3</sup>The operation  $m_2$  is associative thus the result does not depend on the choice of the bracketing.

6. G. Floystad, J.E. Vayne, *Artin-Schelter regular algebras of dimension five*, Algebra, Geometry, and Mathematical Physics, Banach Center Publications, **93** (2011), 19-39.
7. H. S. Green, *A Generalized Method of Field Quantization*, Phys. Rev. **90** (1953), 270-273.
8. T. Józefiak, J. Weyman, *Representation-theoretic interpretation of a formula of D. E. Littlewood*, Math. Proc. Cambridge Phil. Soc., **103** (1988), 193-196.
9. T. Kadeishvili, *The  $A_\infty$ -algebra Structure and Cohomology of Hochschild and Harrison*, Proc. of Tbil. Math. Inst. **91** (1988), 19-27.
10. M. Kontsevich, Y. Soibelman, *Deformations of algebras over operads and the Deligne conjecture*, Math. Phys. Stud., **21** (2000), Kluwer Acad. Publ., Dordrecht, 255-307.
11. J.-L. Loday and T. Popov, *Parastatistics Algebra, Young Tableaux and the Super Plactic Monoid*, Int. J. Geom. Meth. in Mod. Phys. **5** (2008), 1295-1314.
12. J.-L. Loday, B. Vallette, *Algebraic Operads*, Grundlehren Math. Wiss. **346**, Springer, Heidelberg, 2012.
13. D.-M. Lu, J. H. Palmieri, Q.-S. Wu, J. J. Zhang,  *$A_\infty$ -algebras for ring theorists*, Algebra Colloquium **11** (2004), 91-128.
14. S. Sigg, *Laplacian and homology of free 2-step nilpotent Lie algebras*, J. Algebra **185** (1996), 144-161.

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