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ON THE NUMBER OF SIMPLE ARRANGEMENTS OF FIVE DOUBLE PSEUDOLINES

JULIEN FERTÉ, VINCENT PILAUD, AND MICHEL POCCHIOLA

Abstract. We describe an incremental algorithm to enumerate the isomorphism classes of double pseudoline arrangements. The correction of our algorithm is based on the connectedness under mutations of the spaces of one-extensions of double pseudoline arrangements, proved in this paper. Counting results derived from an implementation of our algorithm are also reported.

1. Introduction

An *arrangement of double pseudolines* is a finite set of separating simple closed curves embedded in a real two-dimensional projective plane such that any two curves have exactly four intersection points, cross transversally at these points, and induce a cell decomposition of their underlying projective plane. Two arrangements of double pseudolines are said to be *isomorphic* if one is the image of the other by a homeomorphism of their underlying projective planes. There is a unique isomorphism class of arrangements of two double pseudolines and Figure 1 depicts representatives of the thirteen isomorphism classes of simple arrangements of three double pseudolines where, as usual, a simple arrangement is an arrangement where no three curves meet at the same point.

In this paper we prove that the one-extension spaces of double pseudoline arrangements are connected under mutations. This connectedness result yields an incremental enumeration algorithm that takes as input the class of isomorphism classes of arrangements of n double pseudolines and returns as output the class of isomorphism classes of arrangements of $n+1$ double pseudolines. The key feature of this enumeration algorithm is that its working space is proportional to the size of the input and *not* to the size of the output; it turns out to be sufficient in practice for the enumeration of simple arrangements of five double pseudolines using a relatively modest amount of process time: we establish in this paper that their number is 181 403 533.

Double pseudoline arrangements have been introduced recently by Luc Habert and the third author of the paper as a combinatorial abstraction of finite families of disjoint convex bodies of real two-dimensional projective geometries, in connection with their study of primitive operations for efficient visibility graph algorithms for planar convex shapes. More precisely they have shown that the class of isomorphism classes of double pseudoline arrangements coincides with the class of isomorphism classes of dual arrangements of finite families of pairwise disjoint convex bodies of projective geometries, and they have exploited this latter result to give an axiomatic characterization of the class of *chirotopes* of finite families of pairwise disjoint oriented convex bodies of projective geometries very similar to classical axiomatic characterizations of the class of chirotopes of finite families of points of projective geometries known under the generic name of (rank

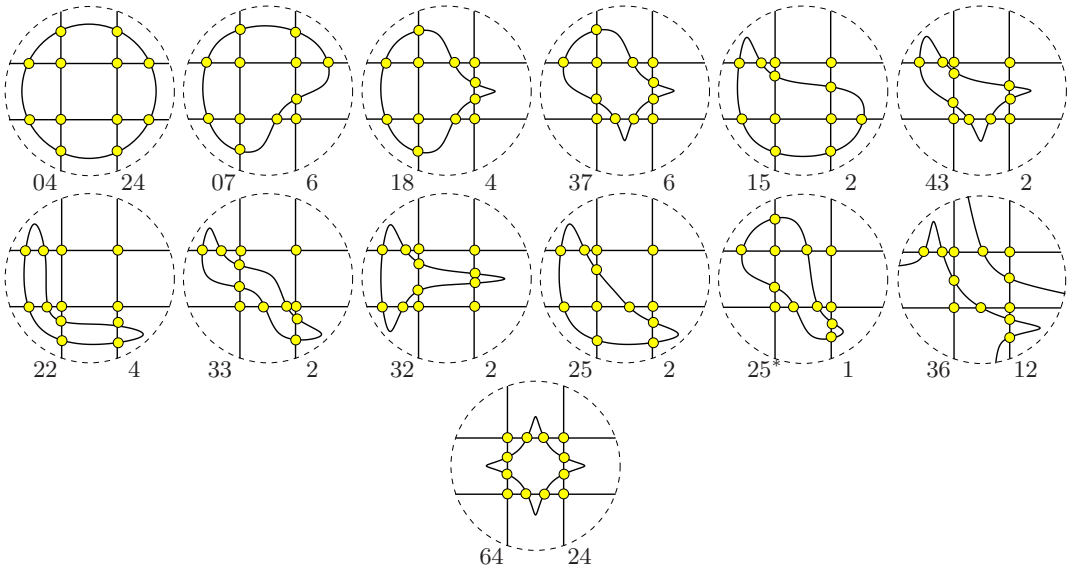

FIGURE 1. Representatives of the thirteen isomorphism classes of simple arrangements of three double pseudolines. Each arrangement is labelled at its left bottom corner with the 2-sequence of its numbers of two-cells of size 2 and 3, and at its right bottom corner with the order of its automorphism group.

three) oriented matroids. (The *chirotope* of a finite family of pairwise disjoint oriented convex bodies of a projective geometry is defined as the map that assigns to each triple of convex bodies the set of relative positions of the bodies of the triple with respect to the lines of the projective geometry—the primitive operations to which we referred above consist precisely in the evaluation of the chirotope of the input configuration to the visibility graph algorithm.) These results provide our main motivation to enumerate double pseudoline arrangements (cf. [14]).

The paper is organized as follows. In Section 2 we recall definitions and structural properties of arrangements of double pseudolines that on one hand motivate the paper and on the other hand are used subsequently in the technical developments. These structural properties are *the axiomatic characterization of the class of isomorphim classes of double pseudoline arrangements in terms of chirotopes*, *the connectedness of mutation graphs*, and the so-called *Pumping Lemma* and *Geometric Representation Theorem*. Still in Section 2 we establish an enhanced version of the *Pumping Lemma* using the *Geometric Representation Theorem*. In Section 3 we prove the connectedness result mentioned above using the enhanced version of the *Pumping Lemma*. In Section 4 we describe our incremental algorithm; in particular we explain how to add a double pseudoline to an arrangement of double pseudolines, a procedure that we believe to be of independant interest. In Section 5 we report some of the counting results for simple as well as non-simple arrangements of double pseudolines that we have obtained from our implementation of the incremental algorithm. (Counting results for the subclass of the so-called Möbius arrangements are also reported; this subclass captures exactly the class

of chirotopes of affine configurations of disjoint convex bodies.) Finally in Section 6 we conclude by a short series of questions and possible developments suggested by this research. Throughout the paper we assume the reader to be familiar with the basic terminology of pseudoline arrangements [12, 3, 4, 13, 9].

2. Preliminaries

2.1. **Projective arrangements of double pseudolines.** Let $\mathcal{P}$ be a real two-dimensional projective plane. In our drawings we will represent it by a circular diagram with antipodal boundary points identified. A simple closed curve in $\mathcal{P}$ is a *pseudoline* if it is non-separating (or equivalently, non-contractible), and a *double pseudoline* otherwise. The complement of a pseudoline γ has one connected component: a topological disk $\mathcal{D}_\gamma$. The complement of a double pseudoline γ has two connected components: a Möbius strip $\mathcal{M}_\gamma$ and a topological disk $\mathcal{D}_\gamma$. Observe that a pseudoline avoiding a double pseudoline is necessarily included in the Möbius strip surrounded by the double pseudoline (Fig. 2).

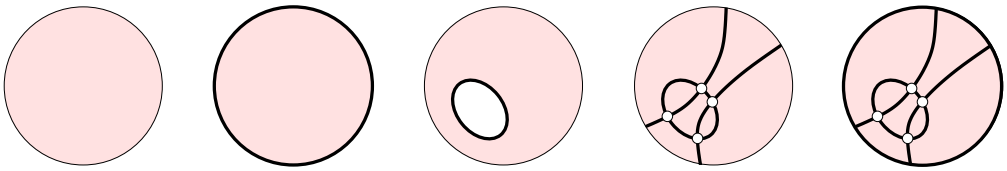

Figure 2. A projective plane represented by a circular diagram with antipodal points identified, a pseudoline, a double pseudoline, an arrangement of two double pseudolines, and a one-marked arrangement of two double pseudolines.

An *arrangement of double pseudolines* in $\mathcal{P}$ is a finite set of double pseudolines such that any two double pseudolines have exactly four intersection points, cross transversally at these points, and induce a cell decomposition of $\mathcal{P}$ (Fig. 2). We call it *full* if the intersection of the topological disks surrounded by its double pseudolines is empty. The *order* of an arrangement is its number of curves. As usual a *simple* arrangement is an arrangement where no three curves meet at the same point. A *marked* arrangement of double pseudolines is an arrangement of double pseudolines augmented with an arrangement of pseudolines such that any pseudoline avoids one double pseudoline of the arrangement and intersects the other ones in exactly two points. A *T-marked* arrangement of double pseudolines is an arrangement of double pseudolines augmented with an arrangement of pseudolines such that any pseudoline intersects one double pseudoline of the arrangement in exactly one point and intersects the other ones in exactly two points.

Two arrangements are *isomorphic* if there is a homeomorphism of their underlying projective planes that sends one arrangement onto the other. The reader will easily check that there is a unique isomorphism class of arrangements of two double pseudolines, a unique isomorphism class of one-marked arrangements of two double pseudolines, and (less easily) that there are thirteen isomorphism classes of simple arrangements of three double pseudolines of which only one is full. They are depicted in Figure 1. In this paper, we are interested in enumerating isomorphism classes of arrangements of double pseudolines. The number of isomorphism classes of arrangements of order n will be

denoted by p_n, the number of isomorphism classes of full arrangements of order n will be denoted by t_n, and the number of isomorphism classes of one-marked arrangements of order n will be denoted by d_n. The corresponding number for simple arrangements will be denoted by p_n^S, t_n^S and d_n^S.. Thus $p_2 = p_2^S = d_2^S = 1$, $t_2 = t_2^S = 0$, $p_3^S = 13$ and $t_3^S = 1$.

2.2. **Chirotopes.** An *indexed oriented arrangement* is an arrangement whose curves are oriented and one-to-one indexed with some indexing set. The *chirotope* of an indexed oriented arrangement is the application that assigns to each triple of indices the isomorphism class of the subarrangement indexed by this triple.

As for pseudoline arrangements, the isomorphism class of an indexed oriented arrangement of double pseudolines only depends on its chirotope. Furthermore, given an application χ that assigns to each triple of indices the isomorphism class of an oriented arrangement of double pseudolines indexed by this triple, the two following properties are equivalent

(1) χ is the chirotope of an indexed oriented arrangement,
(2) the restriction of χ to the set of triples of any subset of at most five indices is the chirotope of an indexed oriented arrangement [14].

This result, called the *Axiomatization Theorem* for double pseudoline arrangements, provides a strong motivation for enumerating arrangements of at most five double pseudolines. In particular the number ρ_n of chirotopes on n double pseudolines (on a given indexing set) is given by the sum

$$\sum_{k\neq 0} \frac{n!2^n}{k} g_n(k) \tag{1}$$

where $g_n(k)$ is the number of arrangements of n double pseudolines with automorphism groups of order k; thus $\sum_{k\neq 0} g_n(k) = p_n$. The corresponding numbers for simple arrangements will be denoted by ρ_n^S and $g_n^S(k)$. For example we read from the data of Figure 1 that $g_3^S(1) = 1$, $g_3^S(2) = 5$, $g_3^S(4) = 2$, $g_3^S(6) = 2$, $g_3^S(12) = 1$, $g_3^S(24) = 1$ and that $g_3^S(k) = 0$ for $k \notin \{1, 2, 4, 6, 12, 24\}$; consequently there are $\rho_3^S = 214$ simple chirotopes on three double pseudolines.

2.3. **Möbius arrangements.** Let $\mathcal{M}$ be a Möbius strip and let $\overline{\mathcal{M}} = \mathcal{M} \cup \{\infty\}$ be its one-point compactification. An arrangement of double pseudolines in $\mathcal{M}$ is an arrangement of double pseudolines in $\overline{\mathcal{M}}$ with the property that the intersection of the topological disks surrounded by the double pseudolines of the arrangement is nonempty and contains the point at infinity ∞. An indexed oriented Möbius arrangement is a Möbius arrangement whose double pseudolines are one-to-one indexed and oriented; the arrangement is called acyclic if the orientations of the double pseudolines are coherent, in the sense that the double pseudolines are homotopic as oriented curves[1].

As for projective double pseudoline arrangements (i) two Möbius arrangements are called isomorphic if one is the image of the other by a homeomorphism of their underlying Möbius strips, and (ii) the chirotope of an indexed oriented Möbius arrangement

[1]In other words the curves are oriented according to the choice of a generator of the (infinite cyclic) fundamental group of the underlying Möbius strip.

is defined as the map that assigns to each triple of indices the isomorphism class of the (indexed and oriented) subarrangement indexed by this triple. The *Axiomatization Theorem* for projective arrangements extends word for word for Möbius arrangements as well as acyclic Möbius arrangements.

We denote by q_n the number of isomorphism classes of Möbius arrangements of order n, by r_n the number of isomorphism classes of acyclically oriented Möbius arrangements of order n, and by τ_n the number of isomorphism classes of indexed oriented acyclic Möbius arrangements of order n. The number τ_n of acyclic Möbius chirotopes on n double pseudolines is given by the sum

$$\sum_{k\neq 0} \frac{n!2}{k} h_n(k) \tag{2}$$

where $h_n(k)$ is the number of Möbius arrangements of n double pseudolines with automorphism groups of order k. The corresponding numbers for simple arrangements will be denoted by q_n^S, r_n^S, τ_n^S, and $h_n^S(k)$.

2.4. **Mutations.** A *mutation* is a local transformation of an arrangement Γ that only destroys, or creates, or inverts a fan of Γ; more precisely, it is a homotopy of arrangements in which only one curve γ moves, reaching or leaving or first reaching and then leaving a single vertex of the remaining arrangement $\Gamma \setminus \{\gamma\}$ (Fig. 3).

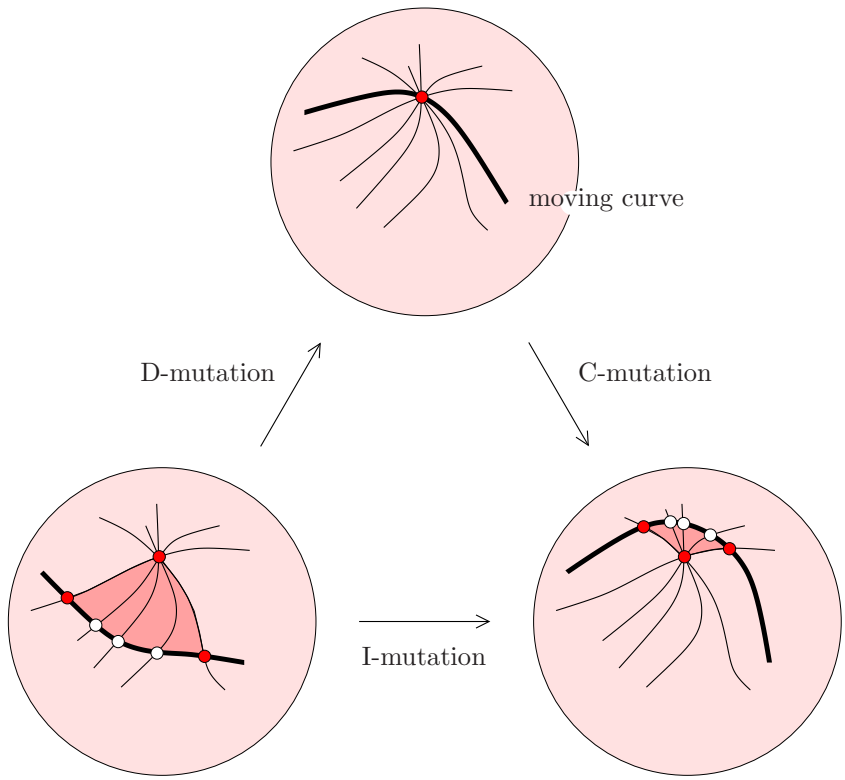

FIGURE 3. A mutation destroys, creates, or inverts a fan supported by the moving curve. (D-,C- and I- for destroying, creating and inverting.)

As for pseudoline arrangements iterating this local transformation is in fact sufficient to obtain all possible arrangements of given order: *any two arrangements of the same order are homotopic via a finite sequence of mutations followed by an isotopy; in other words the graph of mutations on isomorphism classes of arrangements of given order is connected.* The graph of mutations on simple arrangements of three double pseudolines is depicted in Figure 4.

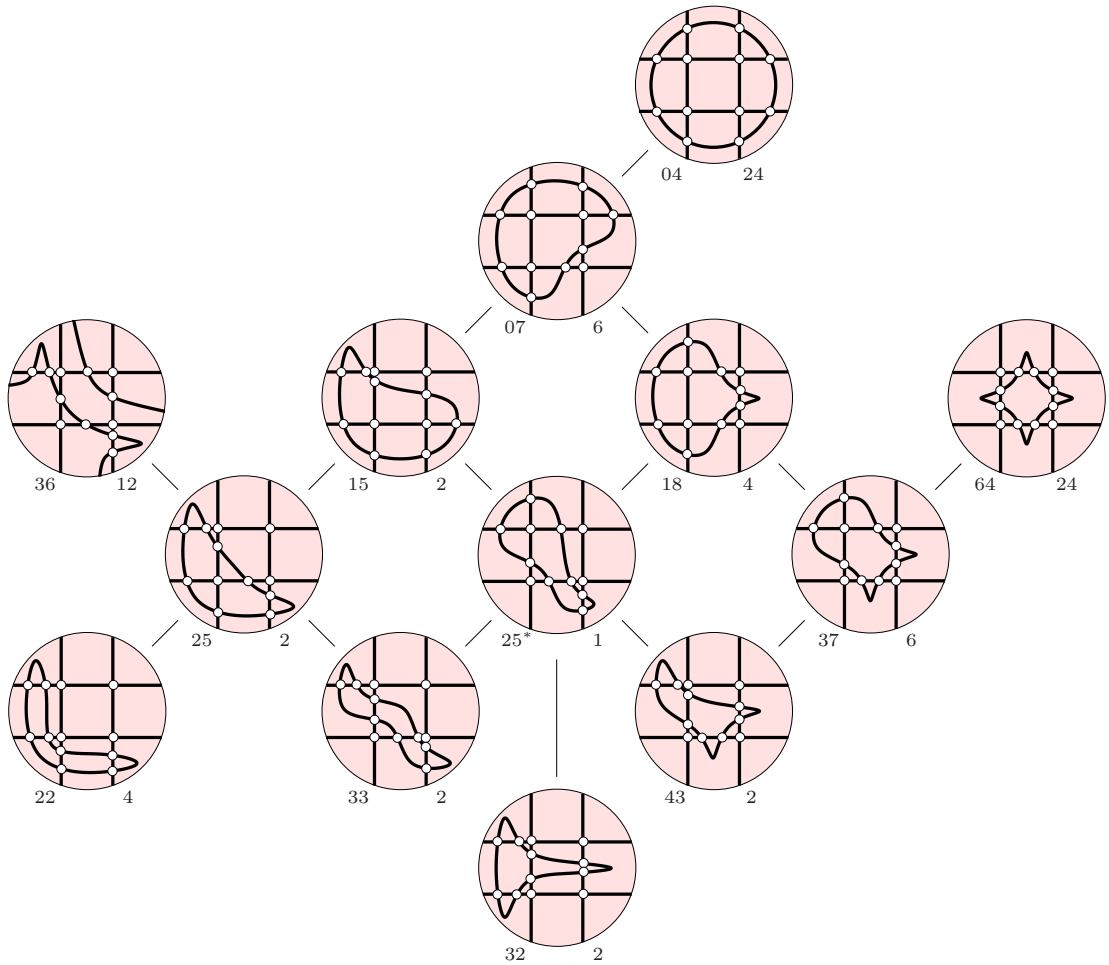

FIGURE 4. The graph of mutations on simple arrangements of three double pseudolines.

This connectedness result is proved in [14] by reduction to the case of arrangements of pseudolines. The argument consists in mutating each double pseudoline of the arrangement until it becomes *thin* (that is, until there remains no vertex of the arrangement inside its enclosed Möbius strip), and to observe that arrangements of thin double pseudolines behave exactly as simple pseudoline arrangements. The fact that an arrangement can be mutated until all its double pseudolines become thin is ensured by the following crucial lemma:

Lemma 1 (Pumping Lemma [14]). *Let Γ be a simple arrangement of double pseudolines and let γ be a distinguished double pseudoline of Γ. Assume that there is a vertex of Γ lying in the interior of the Möbius strip $\mathcal{M}_\gamma$ bounded by γ. Then there is a triangular face of Γ supported by γ and included in $\mathcal{M}_\gamma$.* □

Observe that the Pumping Lemma allows to turn any arrangement of double pseudolines into a marked arrangement with any prescribed numbers of pseudolines missing the double pseudolines (Fig. 5).

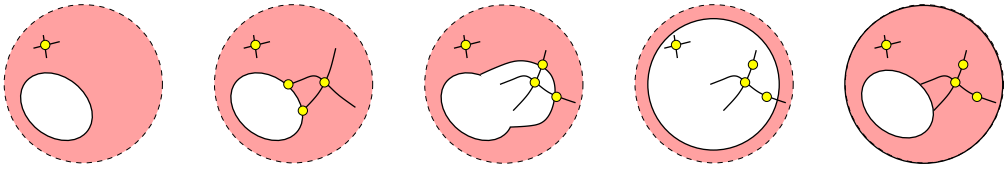

FIGURE 5. Pumping vertices out of the Möbius strip of a double pseudoline and turning an arrangement into a marked one.

From the connectedness property of mutation graphs it ensues an enumeration algorithm that simply consists in traversing the graph of mutations starting from any given arrangement. This algorithm is sufficient in practice for the enumeration of (not necessarily simple) arrangements of three or four double pseudolines but already fails for simple arrangements of five double pseudolines because of random-access memory limitations.

In order to go a little bit further (and particularly, to enumerate arrangements of five double pseudolines), we show that the mutation graph on the space of one-extensions of any given arrangement is connected. From this result we derive a simple enumeration algorithm for the class of arrangements of order $n+1$ based on the traversal of the graphs of mutations of the one-extension spaces of the arrangements of order n whose working space is only proportional to the number of isomorphism classes of double pseudoline arrangements of order n times the space of encoding of an arrangement. It turns out that this incremental algorithm is sufficient in practice for the enumeration of simple arrangements of five double pseudolines.

2.5. **Geometric representation theorem.** A (real two-dimensional) *projective geometry* is a topological projective point-line incidence geometry $(\mathcal{P}, \mathcal{L})$ whose point space $\mathcal{P}$ is a projective plane and whose line space $\mathcal{L}$ is a subspace of the space of pseudolines of $\mathcal{P}$; as usual the dual of a point p of a projective geometry is denoted p^* and is defined as its set of incident lines.

The *duality principle* for projective geometries asserts that the dual $(\mathcal{L}, \mathcal{P}^*)$ of a projective geometry $(\mathcal{P}, \mathcal{L})$ is still a projective geometry, i.e., $\mathcal{L}$ is a projective plane and $\mathcal{P}^*$ is a subspace of the space of pseudolines of $\mathcal{L}$; in particular the dual of a finite point set is an arrangement of pseudolines. The *Geometric Representation Theorem* for pseudoline arrangements asserts that the converse is true: *any arrangement of pseudolines is isomorphic to the dual arrangement of a finite set of points of a projective geometry.* (This is an easy consequence of the duality principle for projective geometries combined with

the embedabbility of any arrangement of pseudolines in the line space of a projective geometry [10], see the discussion in [14].)

The *Geometric Representation Theorem* for pseudoline arrangements has the following extension for marked arrangements of double pseudolines: *first, the dual of a convex body of a projective geometry—defined as its set of tangent lines, i.e., the set of lines touching the body but not its interior—is a double pseudoline in the line space of the projective geometry; second, the dual arrangement of a finite set of pairwise disjoint marked convex bodies is a marked arrangement of double pseudolines; third, any marked arrangement of double pseudolines is isomorphic to the dual arrangement of a finite set of pairwise disjoint marked convex bodies of a projective geometry* [14]. In other words: *the class of isomorphism classes of marked arrangements of double pseudolines coincides with the class of isomorphism classes of dual arrangements of finite families of disjoint marked convex bodies of projective geometries.* Similarly the *Geometric Representation Theorem* for marked Möbius arrangements asserts that their class coincides with the class of dual arrangements of finite families of disjoint marked convex bodies of affine geometries [14].

From this result, it is easy, using continuous motions, to derive the following extension of the Pumping Lemma.

Lemma 2. *Let Γ be an arrangement of double pseudolines and let $\gamma \in \Gamma$. Assume that there is a vertex of Γ lying in the interior of the Möbius strip $\mathcal{M}_\gamma$ bounded by γ and that the vertices of Γ supported by γ are simple (i.e., have degree four). Then there is at least two fans of Γ included in $\mathcal{M}_\gamma$ and supported by γ.* □

This enhanced version of the Pumping Lemma allows to mutate any double pseudoline until it becomes thin while keeping fixed any one of its points. Replacing the resulting thin double pseudoline by one of its core pseudoline and putting back in the arrangement the mutated double pseudoline we get a T-marked version of the initial arrangement with one pseudoline touching the arrangement at the point kept fixed (Fig. 6).

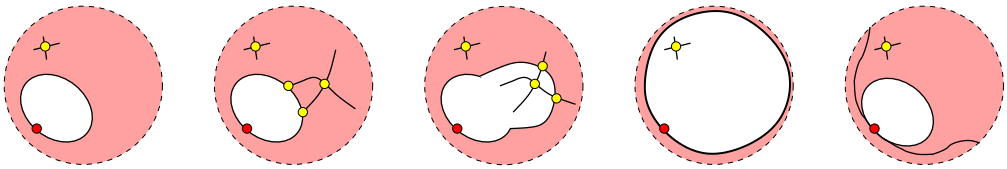

FIGURE 6. Pumping vertices out of the Möbius strip of a double pseudoline while keeping fixed one of the points of the double pseudoline and turning an arrangement into a T-marked one.

By a repeated application of this T-marking operation one can turn any double pseudoline arrangement into a T-marked version with any prescribed pattern of touching points. In particular if we choose as set of touching points the set of pairs (γ, x) where γ ranges over the set of double pseudolines of the arrangement Γ and where x ranges over the set V_γ of vertices of the arrangement lying on γ we obtain a *linearization* of the arrangement Γ, i.e., an arrangement of pseudolines $\ell(\gamma, x)$, $x \in V_\gamma$, $\gamma \in \Gamma$, such that (1) the sub-arrangement of $\ell(\gamma, x)$, $x \in V_\gamma$, is a cyclic arrangement with a distinguished central cell $\mathcal{C}_\gamma$ (the one including the topological disk $\mathcal{D}_\gamma$), (2) the family of boundaries of the $\mathcal{C}_\gamma$, $\gamma \in \Gamma$, is an arrangement of double pseudolines isomorphic to Γ (Fig. 7).

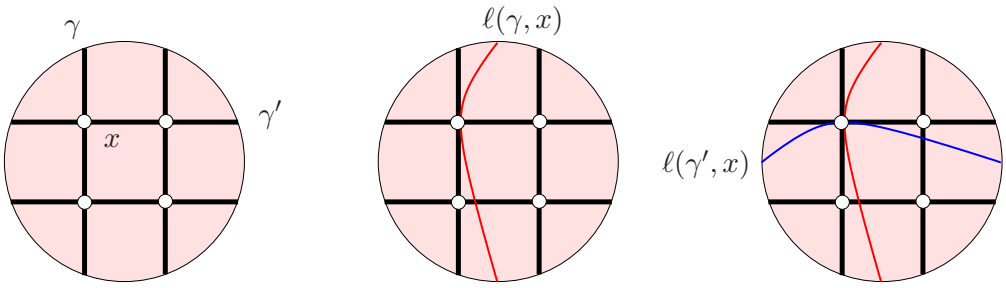

FIGURE 7. Two steps of the linearization process of an arrangement of two double pseudolines.

If we think of our double pseudoline arrangement Γ as the dual of a configuration of smooth convex bodies Δ, this linearization process corresponds to insert a point $p(\gamma, x)$ on the intersection between x and the convex body δ corresponding to γ and to replace δ by the convex hull C_δ of the $p(\gamma, x)$, $x \in \gamma$, with respect to any line avoiding δ. The dual arrangement of the convex bodies C_δ is then isomorphic to the arrangement Γ. Concatening this linearization algorithm with any algorithmic version of the Geometric Representation Theorem for pseudoline arrangements (e.g. [1]) we thus get an algorithmic version of the Geometric Representation Theorem for double pseudoline arrangements (Fig. 8).

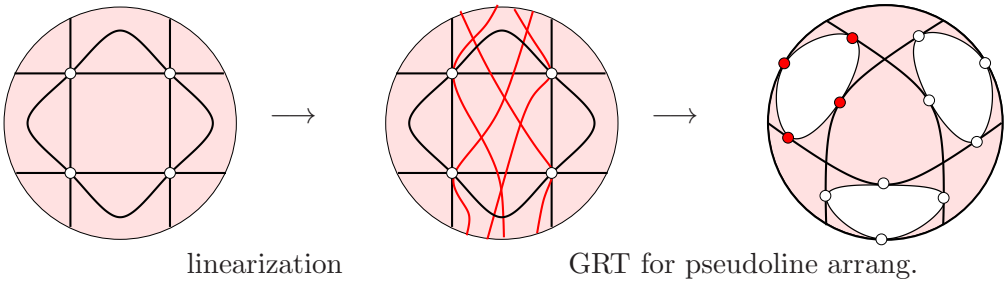

FIGURE 8. Concatening the linearization process of an arrangement with a constructive version of the Geometric Representation Theorem for pseudoline arrangements yields to a constructive version of the Geometric Representation Theorem for double pseudoline arrangements.

The enhanced version of the Pumping Lemma will be used both in our enumeration algorithm—to insert a double pseudoline in a given double pseudoline arrangement—and in the proof of the connectedness of the space of one-extensions of a given arrangement, proof to which we now come.

3. Connectedness of the spaces of one-extensions

A one-extension of an arrangement of n double pseudolines Γ is an arrangement of $n+1$ double pseudolines Γ' of which Γ is a subarrangement. The double pseudoline of Γ' not in Γ is called the one-extension element.

For example Figure 9 depicts representatives of the twenty-three simple (isomorphism classes of) one-extensions of an arrangement of two double pseudolines, with the curved double pseudoline in the role of the one-element extension.

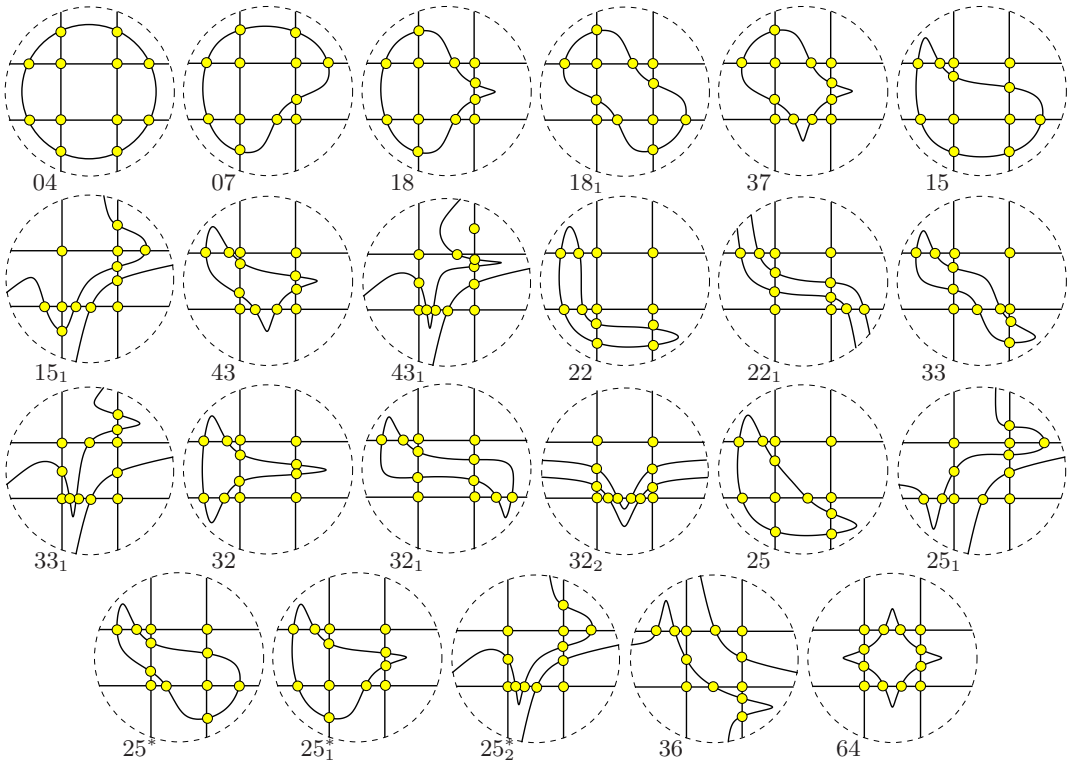

FIGURE 9. Representatives of the twenty-three isomorphism classes of simple one-extensions of an arrangement of two double pseudolines with the curved double pseudoline in the role of the one-extension element.

Theorem 3. *Let Γ' and Γ'' be two one-extensions of an arrangement of double pseudolines Γ. Then Γ' and Γ'' are homotopic via a finite sequence of mutations followed by an isotopy during which the only moving curves are the one-extension elements of Γ' and Γ''.*

Proof. Let γ be a double pseudoline of Γ and let γ' and γ'' be the one-extension elements of Γ' and Γ'', respectively. Using continuous motions—thanks to the *Geometric Representation Theorem*—one can easily reduce the analysis to the following case

(1) γ' is a *thin* double pseudoline in Γ', i.e., a double pseudoline whose Möbius strip is free of vertices of Γ;
(2) γ and γ' are *touching* at σ': by this we mean that σ' is one the two 2-cells of size two of the subarrangement $\{\gamma, \gamma'\}$ and that σ' is also a 2-cell of the whole arrangement Γ';
(3) similarly γ'' is a thin double pseudoline in Γ'', and γ and γ' are touching at σ'';
(4) γ' and γ'' coincide in the disk $\mathcal{D}_\gamma$, and $\sigma' = \sigma''$;

(5) $\mathcal{M}_{\gamma'}$ is a tubular neighbourhood of a pseudoline γ'_* with the property that γ'_* intersects any double pseudoline of Γ in exactly two points, the intersection points being transversal; similarly $\mathcal{M}_{\gamma''}$ is a tubular neighbourhood of a pseudoline γ''_* with the property that γ''_* intersects any double pseudoline of Γ in exactly two points, the intersection points being transversal;
(6) γ'_* and γ''_* coincide in $\mathcal{D}_\gamma$, and intersect finitely many in $\mathcal{M}_\gamma$, the intersection points being transversal;
(7) $\mathcal{M}_{\gamma'} \cup \mathcal{M}_{\gamma''}$ is a tubular neighbourhood of $\gamma'_* \cup \gamma''_*$.

Let $\Gamma'_* = \Gamma \cup \{\gamma'_*\}$ and, similarly, let $\Gamma''_* = \Gamma \cup \{\gamma''_*\}$. It should be clear to the reader that the proof of our theorem boils down to show that the mixed arrangements Γ'_* and Γ''_* are homotopic via a finite sequence of mutations during which the only moving curves are the pseudolines γ'_* and γ''_*. This is precisely here that we are going to use Lemma 2. But before using this lemma we define a γ-curve as a connected component of the trace on $\mathcal{M}_\gamma$ of a double pseudoline of $\Gamma \setminus \{\gamma\}$, we observe that the pseudoline γ'_* intersects a γ-curve in at most one point (necessarily a transversal intersection point), and similarly that the pseudoline γ''_* intersects a γ-curve in at most one point. Let now τ be a copy of γ and let Γ_τ be the symmetric difference between Γ and $\{\gamma, \tau\}$. If τ is thin in Γ_τ we are done modulo an isotopy. Otherwise Lemma 2 asserts that there exists a fan Δ of the arrangement Γ_τ supported by τ, included in $\mathcal{M}_\tau$, and missing the 2-cell σ'. Modulo a finite sequence of mutations in Γ'_τ or in Γ''_τ or in both Γ'_τ and in Γ''_τ during which the only moving curves are γ'_* or γ''_* or both γ'_* and γ''_* one can assume that Δ misses γ'_* and γ''_*. We then perform a mutation of Δ in Γ_τ with τ in the role of the moving curve—during this process we do not touch at σ'. By repeated application of this process we arrive at the situation where τ is thin in Γ_τ and where γ'_* and γ''_* are included in $\mathcal{M}_\tau$, excepted their common part in $\mathcal{D}_\gamma$—which is included in $\mathcal{D}_\tau$. At this point we are done modulo an isotopy. □

In other words Theorem 3 asserts that the space of one-extensions of an arrangement of double pseudolines is connected under mutations. Figure 10 depicts the graph of mutations on simple one-extensions of an arrangement of two double pseudolines. Similarly one can define a k-extension of an arrangement of n double pseudolines Γ as an arrangement of $n + k$ double pseudolines Γ' of which Γ is a subarrangement and show that the space of k-extensions of an arrangement of double pseudolines is connected under mutations. Finally we mention that similar results hold for marked arrangements and their k-extensions.

4. The incremental algorithm

4.1. **Description.** A *pointed* arrangement is an arrangement with a distinguished double pseudoline. We always use the notation $A^\bullet$ for a pointed arrangement and A for its non-pointed version (and similarly for sets of pointed arrangements). We also use the notation $\mathrm{Sub}(A)$ for the set of subarrangements of an arrangement A.

Let $\mathcal{A}_n$ denote the set of isomorphism classes of arrangements of n double pseudolines, and p_n be its cardinality. Our algorithm enumerates $\mathcal{A}_n$ from an enumeration $a_1, a_2, \ldots, a_{p_{n-1}}$ of $\mathcal{A}_{n-1}$, by mutating an added distinguished double pseudoline to the a_i.

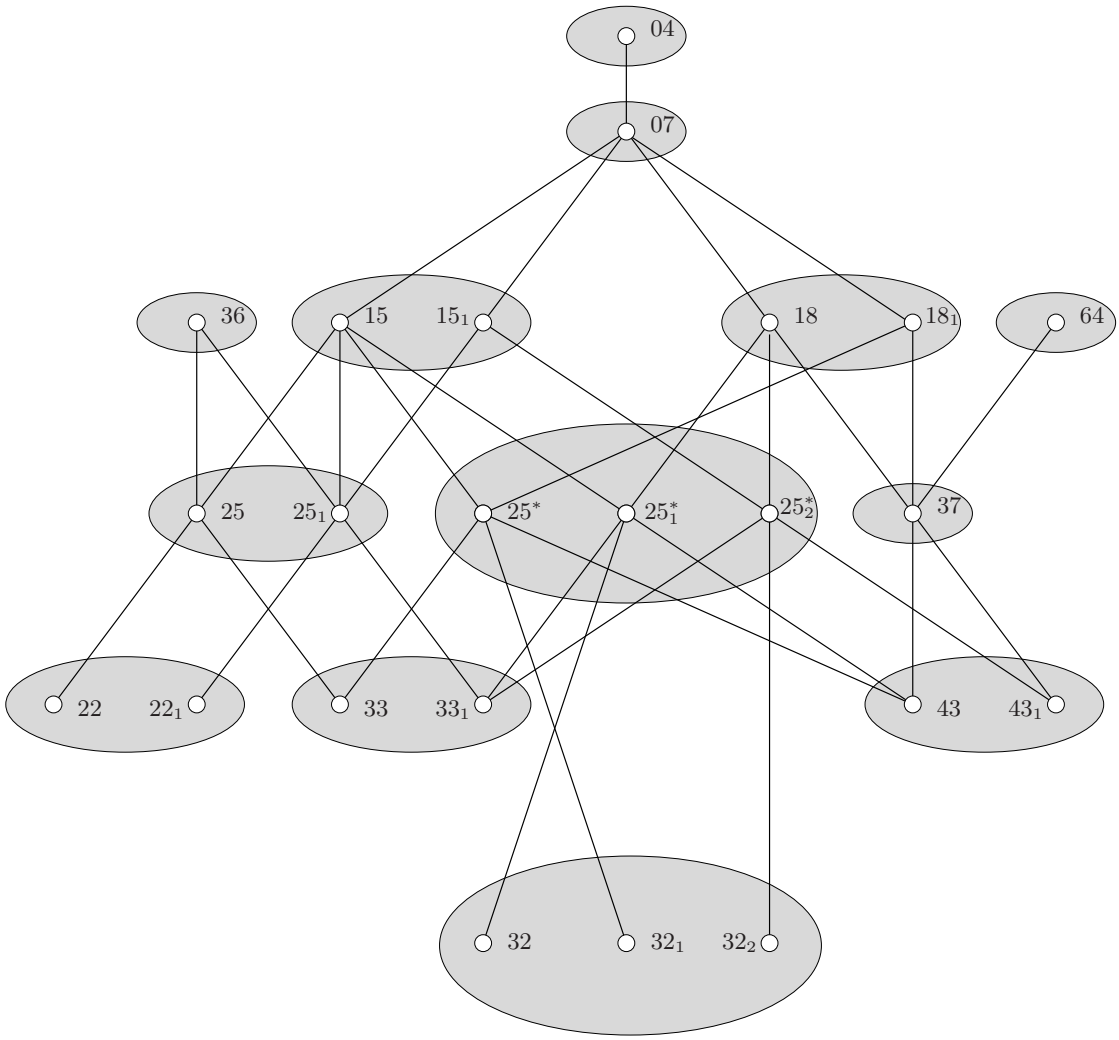

FIGURE 10. The graph of mutations on the simple one-extensions of an arrangement of two double pseudolines.

For each $i \in \{1, \dots, p_{n-1}\}$, the algorithm enumerates the subset $S_i^\bullet$ of arrangements of $\mathcal{A}_n^\bullet$ containing a_i as a subarrangement, by mutations of a distinguished added double pseudoline. From the set S_i, it selects the subset R_i of arrangements with no subarrangements in $\{a_1, \dots, a_{i-1}\}$. In other words, R_i is the subset of arrangements of $\mathcal{A}_n$ whose first subarrangement, in the sequence $a_1, \dots, a_{p_{n-1}}$, is a_i. Thus, $\mathcal{A}_n$ is the disjoint union of the R_i.

An alternative approach[2] for counting arrangements is to enumerate the subsets $S_i^\bullet$ and to compute, for each arrangement $A^\bullet$ of $S_i^\bullet$, the number $\sigma(A^\bullet)$ of double pseudolines

[2]We thank Luc Habert for this suggestion.

Algorithm 1 Incremental enumeration

Require: $\mathcal{A}_{n-1} = \{a_1, \ldots, a_{p_{n-1}}\}$.
Ensure: $\mathcal{A}_n$.

```
for i from 1 to p_{n-1} do
  A• ← add a pointed double pseudoline to a_i.
  if Sub(A) ∩ {a_1, ..., a_{i-1}} = ∅ then
    write A.
  end if
  Q• ← [A•]. S• ← {A•}.
  while Q• ≠ ∅ do
    A• ← pop Q•.
    T ← list the fans of A• supported by its pointed double pseudoline.
    for t ∈ T do
      B• ← mutate the fan t in A•.
      if B• ∉ S• then
        if Sub(B) ∩ {a_1, ..., a_{i-1}} = ∅ and B ∉ S then
          write B.
        end if
        push(B•, Q•). S• ← S• ∪ {B•}.
      end if
    end for
  end while
end for
```

α of A such that A pointed at α is isomorphic to $A^\bullet$. Then

$$p_n = \frac{1}{n} \sum_{i=1}^{p_{n-1}} \sum_{A^\bullet \in S_i^\bullet} \sigma(A^\bullet).$$

The main advantage of this version is to reduce the number of accesses to the data base of arrangements of order $n-1$; an advantage which turns to be crucial in case the data base does not fit in main memory. However, it only counts p_n and can not provide a data base for $\mathcal{A}_n$.

4.2. **Adding a double pseudoline.** One of the important steps of the incremental method is to add a double pseudoline to an initial arrangement. Our method uses three steps (see Fig. 11):

(1) *duplicate a double pseudoline*: we choose one arbitrary double pseudoline γ, duplicate it, drawing a new double pseudoline γ' completely included in the Möbius strip $\mathcal{M}_\gamma$ and we denote R any rectangle delimited by γ and γ'.
(2) *flatten*: we pump the double pseudoline γ' such that no vertex of the arrangement lies in the Möbius strip $\mathcal{M}_{\gamma'}$. During this process, we do not touch the rectangle R.
(3) *add four crossings*: we replace the rectangle R by four crossings between γ and γ'.

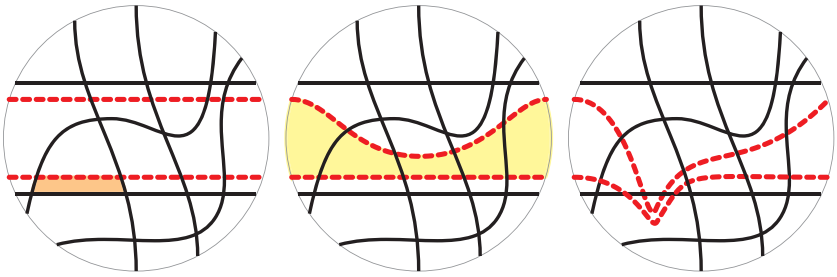

FIGURE 11. Three steps to insert a double pseudoline in a double pseudoline arrangement: duplicate a double pseudoline (a), flatten it (b) and add four crossings (c).

If we think of our double pseudoline arrangement as the dual of a configuration of convex bodies, this method corresponds to: (1) choosing one convex C and drawing a new convex C' inside C; (2) flattening the convex C' until it becomes almost a single point, maintaining it almost in contact with the boundary of C; and (3) moving C' outside C.

4.3. **Encoding an arrangement.** In order to manipulate arrangements, one can encode it in several different ways. We used two different encodings (one for an easy manipulation of it and one for short storage):

(1) *flag representation*: a *flag* of an arrangement Γ is a triple (v, e, f) consisting of a vertex v, an edge e and a face f of Γ, such that $v \in e \subset f$. The three involutions $\sigma_0, \sigma_1, \sigma_2$, that change the vertex, the edge and the face of a flag (cf. Fig. 12), completely determine Γ (see, e.g., [11]). This representation is convenient to perform all the necessary elementary operations we need to perform on arrangements (such as mutations, extensions, etc.).

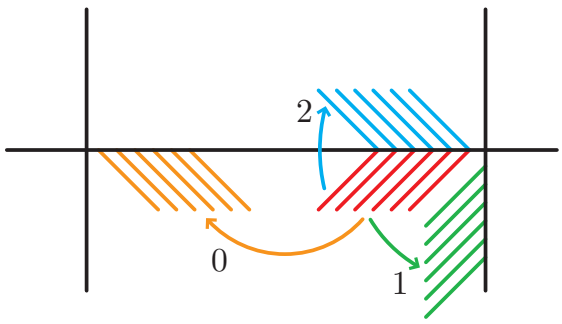

FIGURE 12. The three involutions σ_0, σ_1 and σ_2.

(2) *encoding*: In the enumeration process, once we have finished to manipulate an arrangement Γ, we still have to remember it and to test whether the new arrangements we find afterwards are or not isomorphic to Γ. For this, we compute

another representation of Γ, which is shorter than the flag representation, and which allows a quick isomorphism test.

We first associate to each flag $\phi = (v, e, f)$ of the arrangement Γ a word w_ϕ constructed as follows. Let γ_1 be the double pseudoline containing e, and, for all $2 \leq p \leq n$, let γ_p be the pth double pseudoline crossed by γ_1 on a walk starting at ϕ and oriented by $\sigma_0\sigma_1\sigma_2\sigma_1$. This walk also defines a starting flag ϕ_i for each γ_i. We walk successively on $\gamma_1, \ldots, \gamma_n$, starting from ϕ_i and in the direction given by $\sigma_0\sigma_1\sigma_2\sigma_1$, and index the vertices by $1, 2, \ldots, V$ in the order of appearance. For all i, let w_i denote the word formed by reading the indices of the vertices of γ_i starting from ϕ_i. The word w_ϕ is the concatenation of $w_1, w_2, \ldots, w_n$. Finally, we associate to the arrangement Γ the lexicographically smallest word among the w_ϕ where ϕ ranges over all the flags of Γ.

5. Results

We have implemented this algorithm in the `C++` programming language. The documentation (as well as the source code) of this implementation is available upon request to the authors.

5.1. **Numbers of arrangements and numbers of chirotopes.** This implementation provided us with the following table for the values of p_n^S, t_n^S, p_n, t_n, d_n^S, d_n, ρ_n^S, ρ_n, q_n^S, q_n, τ_n^S, and τ_n.

n	0	1	2	3	4	5
p_n^S	1	1	1	13	6 570	181 403 533
t_n^S	0	0	0	1	615	NC
p_n	1	1	1	46	153 528	NC
t_n	0	0	0	5	18 648	NC
d_n^S	0	1	1	67	355 153	NC
d_n	0	1	1	398	NC	NC
ρ_n^S	1	1	1	214	2 415 112	NC
ρ_n	1	1	1	1 086	58 266 120	NC
q_n^S	1	1	1	16	11 502	238 834 187
r_n^S	1	1	1	22	22 620	NC
q_n	1	1	1	59	245 351	NC
r_n	1	1	1	92	488 303	NC
τ_n^S	1	1	1	118	541 820	NC
τ_n	1	1	1	531	11 715 138	NC

Table 1. Numbers of arrangements and numbers of chirotopes.

Apart the trivial cases of $n = 0, 1, 2$ only the values of $p_3^S, t_3^S, q_3^S, r_3^S, q_4^S, r_4^S$ and τ_3^S were known previously. The comparaison of the data of the lines q_n and r_n confirms the intuition that very few Möbius arrangements are isotopic to their mirror image.

5.2. **Mixed arrangements.** In our presentation, we concentrated on the enumeration of double pseudoline arrangements or, equivalently by the *Geometric Representation Theorem*, on the enumeration of dual arrangements of configurations of pairwise disjoint convex bodies of projective (or affine) geometries. It is also interesting to enumerate dual arrangements of configurations of disjoint points and convex bodies. This class is captured by the notion of *mixed arrangements* and can be enumerated using the same algorithm.

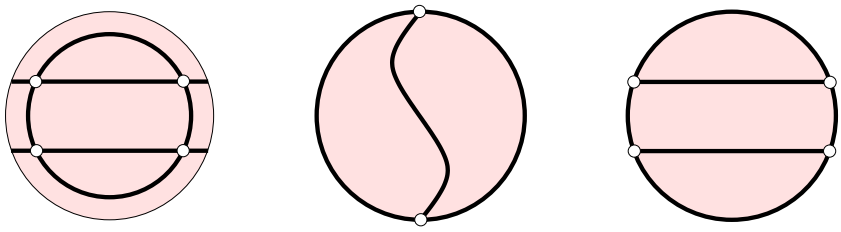

FIGURE 13. The three mixed arrangements of size two.

A *mixed arrangement* in the projective plane $\mathcal{P}$ is a finite set of pseudolines and double pseudolines in $\mathcal{P}$ such that (i) any two pseudolines have a unique intersection point; (ii) a pseudoline and a double pseudoline have exactly two intersection points and cross transversally at these points; and (iii) any two double pseudolines have exactly four intersection points, cross transversally at these points, and induce a cell decomposition of $\mathcal{P}$ (Fig. 13). The *Geometric Representation Theorem* for mixed arrangements in a projective plane asserts that their class coincides with the class of dual arrangements of configurations of disjoint points and convex bodies of projective geometries [14]. A *mixed arrangement* in the Möbius strip $\mathcal{M}$ is a mixed arrangement in the projective plane $\overline{\mathcal{M}}$ with the property that the intersection of the topological disks surrounded by the pseudolines and double pseudolines of the arrangement is nonempty and contains the point at infinity. The *Geometric Representation Theorem* for mixed arrangements in a Möbius strip asserts that their class coincides with the class of dual arrangements of configurations of disjoint points and convex bodies of affine geometries [14].

Our incremental enumeration algorithm for double pseudoline arrangements extends easily to mixed arrangements in the projective plane and in the Möbius strip. Our implementation provided us with the following four tables for the numbers of isomorphism classes of mixed arrangements composed of n pseudolines and m double pseudolines in the projective plane or in the Möbius strip. The sequences of $p^S_{n,0}$, $q^S_{n,0}$, $p_{n,0}$ and $q_{n,0}$ appear in the database of The On-Line Encyclopedia of Integer Sequences[3] under the code numbers A006248, A006247, A063800, and A063854 respectively.

[3] `http://www.research.att.com/~njas/sequences/A006248`

$p^S_{n,m}$	0	1	2	3	4	5
0	1	1	1	13	6 570	181 403 533
1	1	1	4	626	4 822 394	
2	1	2	48	86 715		
3	1	5	1 329			
4	1	25	80 253			
5	1	302				
6	4	9 194				
7	11	556 298				
8	135					
9	4 382					
10	312 356					

TABLE 2. Simple projective mixed arrangements.

$q^S_{n,m}$	0	1	2	3	4	5
0	1	1	1	16	11 502	238 834 187
1	1	1	7	1 499	9 186 477	
2	1	3	140	245 222		
3	1	13	5 589			
4	2	122	416 569			
5	3	2 445				
6	16	102 413				
7	135	7 862 130				
8	3 315					
9	158 830					

TABLE 3. Simple Möbius mixed arrangements.

$p_{n,m}/q_{n,m}$	0	1	2	3	4
0	1/1	1/1	1/1	46/59	153 528/245 351
1	1/1	1/1	9/17	6 998/15 649	
2	1/1	3/5	265/799		
3	1/1	16/45	18 532/74 559		
4	2/3	159/832			
5	4/11	4 671/37 461			
6	17/93	342 294/NC			
7	143/2 121				
8	4 890/122 508				
9	461 053/NC				

TABLE 4. Projective/Möbius mixed arrangements.

5.3. **Automorphism groups.** Our implementation provided us with the following values for the sequences $(k, g_n^S(k))$, $k \in \{u \mid g_n^S(u) \neq 0\}$, in the case $n = 2, 3$, and 4

$$(8, 1)$$

$$(1, 1)(2, 5)(4, 2)(6, 2)(12, 1)(24, 2)$$

$$(1, 6\,042)(2, 466)(3, 15)(4, 26)(6, 14)(8, 1)(12, 3)(16, 1)(24, 2)$$

from which we have deduced that $\rho_2^S = 1$, $\rho_3^S = 214$ and $\rho_4^S = 2\,415\,112$. The corresponding sequences for not necessarily simple projective arrangements, simple Möbius arrangements, and not necessarily simple Möbius arrangements are in order

$$(8, 1)$$

$$(1, 10)(2, 20)(4, 5)(6, 7)(12, 1)(24, 3)$$

$$(1, 150\,042)(2, 3\,288)(3, 58)(4, 84)(6, 45)(8, 2)(12, 5)(16, 1)(24, 3),$$

$$(4, 1)$$

$$(1, 6)(2, 6)(4, 2)(6, 2)$$

$$(1, 11\,088)(2, 387)(3, 6)(4, 12)(6, 7)(8, 2),$$

and

$$(4, 1)$$

$$(1, 33)(2, 20)(4, 3)(6, 3)$$

$$(1, 242\,815), (2, 2\,466), (3, 16), (4, 38), (6, 13), (8, 3).$$

These data confirm the intuition that most of the arrangements have trivial automorphism groups, excepted in the basic case of (projective or Möbius) arrangements of three double pseudolines. For example only 528 of the 6570 simple projective arrangements of four double pseudolines have a non trivial automorphism group. The two simple projective arrangements of four double pseudolines with automorphism groups of maximal order (i.e., 24) are depicted in the Figure 14.

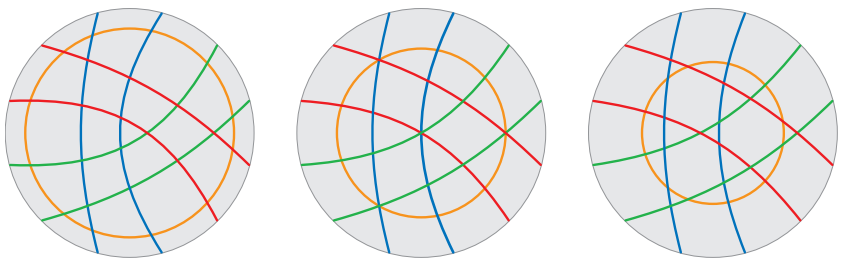

FIGURE 14. The three projective arrangements of four double pseudolines whose automorphism groups have the maximal size 24.

5.4. **Running time.** Let us briefly comment on running time. Observe that our algorithm can be parallelized very easily (separating each enumeration of S_i and R_i, for $i \in \{1, \ldots, p_{n-1}\}$). In order to obtain the number of simple projective arrangements of five double pseudolines we used four processors of 2GHz for almost 3 weeks: to say it differently each iteration of the **for-to** loop of our enumeration algorithm takes roughly twenty minutes. The working space was bounded by $\max |S_i^\bullet| = 279\,882$ (times the space of the encoding of a single configuration, i.e., about 80 characters). Finding the number of not necessarily simple arrangements of five double pseudolines is doable using a similar working space but much more process time.

Finally, Fig. 15 shows the evolution of the ratio between the sizes of the sets R_i and $S_i^\bullet$ (during the enumeration of simple projective arrangements of five double pseudolines). We have also observed that $\sum |S_i^\bullet| / \sum |R_i| \simeq 5$, which confirms that hardly any configurations of five convex bodies have symmetries.

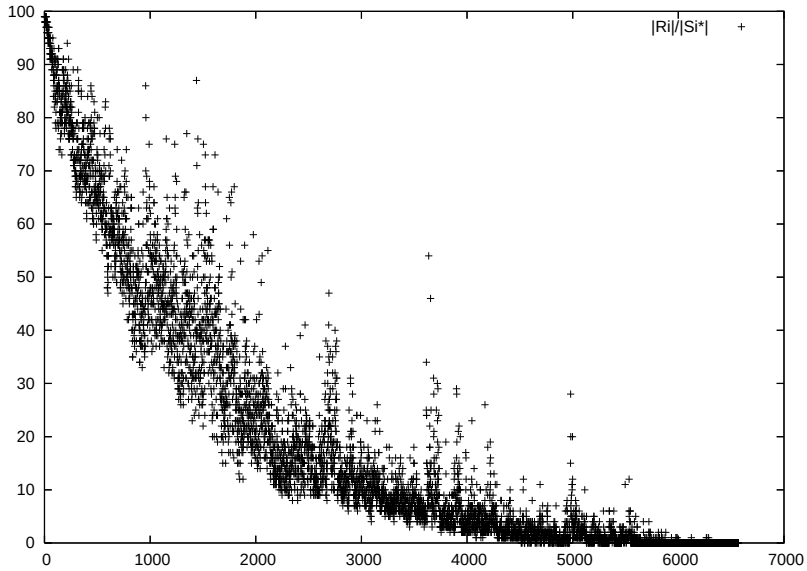

FIGURE 15. Percentage of new configurations.

5.5. **Statistics on mutation graphs.** We provide in Table 5 the distribution of the number x_n of simple projective arrangements of four double pseudolines with n neighboords in their mutation graph (the number of edges of this mutation graph is therefore 54544 and its average degree is 8). Still in Table 5 we provide the distribution of the number y_n of simple projective arrangements of four double pseudolines with n triangular faces. In particular we see that there are two simple projective arrangements of four double pseudolines with the maximum number (16) of triangular faces. They are depicted in Figure 16.

Finally, as a matter of example, we provide in Appendix A the graph of D- and C-mutations on the space of arrangements of three double pseudolines.

n	x_n	y_n
1	6	0
2	28	0
3	69	9
4	149	69
5	317	219
6	655	566
7	951	942
8	1288	1336
9	1228	1306
10	959	1056
11	587	649
12	240	289
13	79	102
14	13	22
15	1	3
16	0	2

TABLE 5. Number x_n of vertices of degree n in the mutation graph of simple arrangements of four double pseudolines and number y_n of simple arrangements of four double pseudolines with n triangular faces.

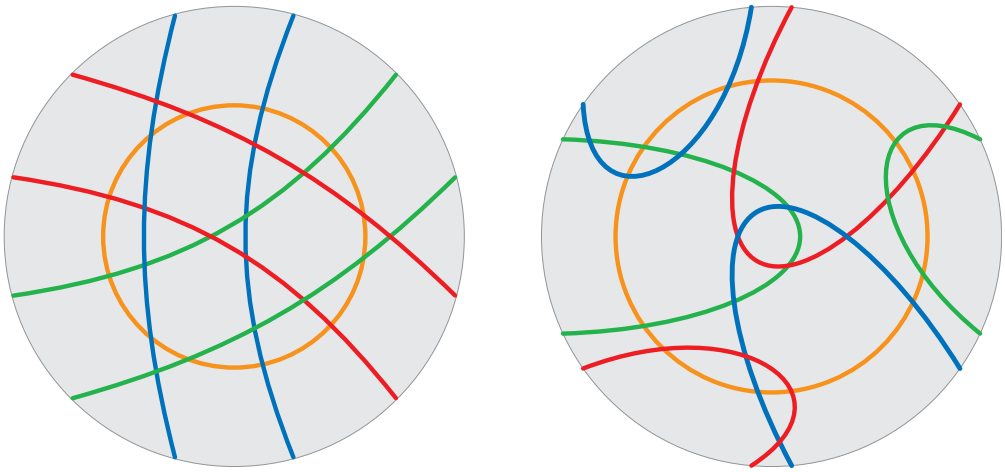

FIGURE 16. The two simple projective arrangements of four double pseudolines with the maximum number (16) of triangular faces.

6. Further developments

The following questions and developments (among others) may be treated in a subsequent paper:

(1) *Developing further implementation*: it will be interesting to add to our implementation the two following functionalities related to the *Geometric Representation Theorem* : computing the dual arrangement of a configuration of convex bodies and conversely, computing a configuration of convex bodies whose dual arrangement is given.

(2) *Drawing an arrangement*: we have seen a method to add a pseudoline in an arrangement. Combined with a planar-graph-drawing algorithm, this provides an algorithm to draw an arrangement in the unit disk. For example, the number of one-marked arrangements composed of m double pseudolines can be interpreted as the number of drawings of the arrangements composed of m double pseudolines with the property that the number of crossings between the arrangement and the boundary of the unit disk is minimum, i.e., $2m - 2$. Similarly, the number of mixed arrangements composed of one pseudoline and m double pseudolines can be interpreted as the number of drawings of the arrangements composed of m double pseudolines with the property that the number of crossings between the arrangement and the boundary of the unit disk is $2m$.

(3) *Generation*: the *Axiomatization Theorem* for double pseudoline arrangements affirms that the huge list of more than one hundred and eighty one millions of arrangements of at most five double pseudolines is an axiomatization of the class of double pseudoline arrangements. Due to its size this list seems hardly computationally workable for the generation of arrangements. However it is possible, as explained in [14], to reduce this axiomatization to a (relatively) short list of axioms with a clear combinatorial or geometrical interpretation of each axiom in a maner very similar to the known simple axiomatizations of pseudoline arrangements [3, 15]. This open the door to the generation of double pseudoline arrangements with prescribed properties using, for example, satisfiability solvers as proposed in [16, 17], see also [5, 7, 2, 8, 6]. In particular it will be interesting to generate the arrangements that maximize the number of connected components of the intersection of the Möbius strips surrounded by the double pseudolines. If we think of the arrangement as the dual of a configuration of (disjoint) convex bodies this means that we seek for the configurations with the maximal number of connected components of transversals.

(4) *Realizability*: it is well-known that certain pseudoline arrangements are not realizable in the standard projective geometry $\mathcal{P}^2(\mathbb{R})$. Inflating pseudolines into thin double pseudolines in such an arrangement give rise to non-realizable double pseudoline arrangements. Are there smaller examples? Are all arrangements of at most five double pseudolines realizable?

Acknowledgments

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Appendix A. A graph of D- and C-mutations

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Node code	order autom.	fvector	nbr simple vertices	Neighbors
#12	24	60	0	#6 #20
#20	6	61	3	#11 #12 #30
#15	6	34	3	#8 #22
#16	2	34	3	#8 #9 #23 #26
#18	2	43	3	#9 #11 #25 #28
#6	6	34	3	#3 #11 #12
#27	2	34	6	#17 #35
#30	4	62	6	#19 #20 #38
#8	2	24	6	#4 #13 #15 #16
#26	2	34	6	#16 #17 #34
#28	1	44	6	#17 #18 #19 #36 #37
#22	2	34	6	#13 #15 #31
#23	1	33	6	#13 #14 #16 #33 #34
#25	2	41	6	#14 #18 #36
#9	1	25	6	#4 #5 #14 #16 #17 #18
#11	2	35	6	#5 #6 #18 #19 #20
#3	4	16	6	#1 #5 #6
#40	4	34	6	#31 #32
#38	6	63	9	#29 #30 #45
#19	2	36	9	#10 #11 #28 #29 #30
#36	1	42	9	#24 #25 #28 #44
#37	2	45	9	#28 #29 #44
#17	1	26	9	#9 #10 #24 #26 #27 #28
#33	2	33	9	#21 #23 #42
#31	2	35	9	#21 #22 #39 #40
#35	1	33	9	#24 #27 #43
#34	1	33	9	#23 #24 #26 #42
#13	1	24	9	#7 #8 #21 #22 #23
#14	1	24	9	#7 #9 #23 #24 #25
#4	2	15	9	#2 #7 #8 #9
#5	2	17	9	#2 #3 #9 #10 #11
#1	6	06	9	#0 #2 #3
#32	2	24	9	#21 #40 #41
#45	24	64	12	#38
#29	6	37	12	#19 #37 #38
#39	12	36	12	#31
#0	24	04	12	#1
#10	4	18	12	#5 #17 #19
#2	6	07	12	#1 #4 #5
#44	2	43	12	#36 #37
#24	1	25	12	#14 #17 #34 #35 #36
#7	2	15	12	#4 #13 #14
#21	2	25	12	#13 #31 #32 #33
#42	2	33	12	#33 #34
#43	2	32	12	#35
#41	4	22	12	#32

TABLE 6. The graph of D- and C-mutations on the space of projective arrangements of three double pseudolines.

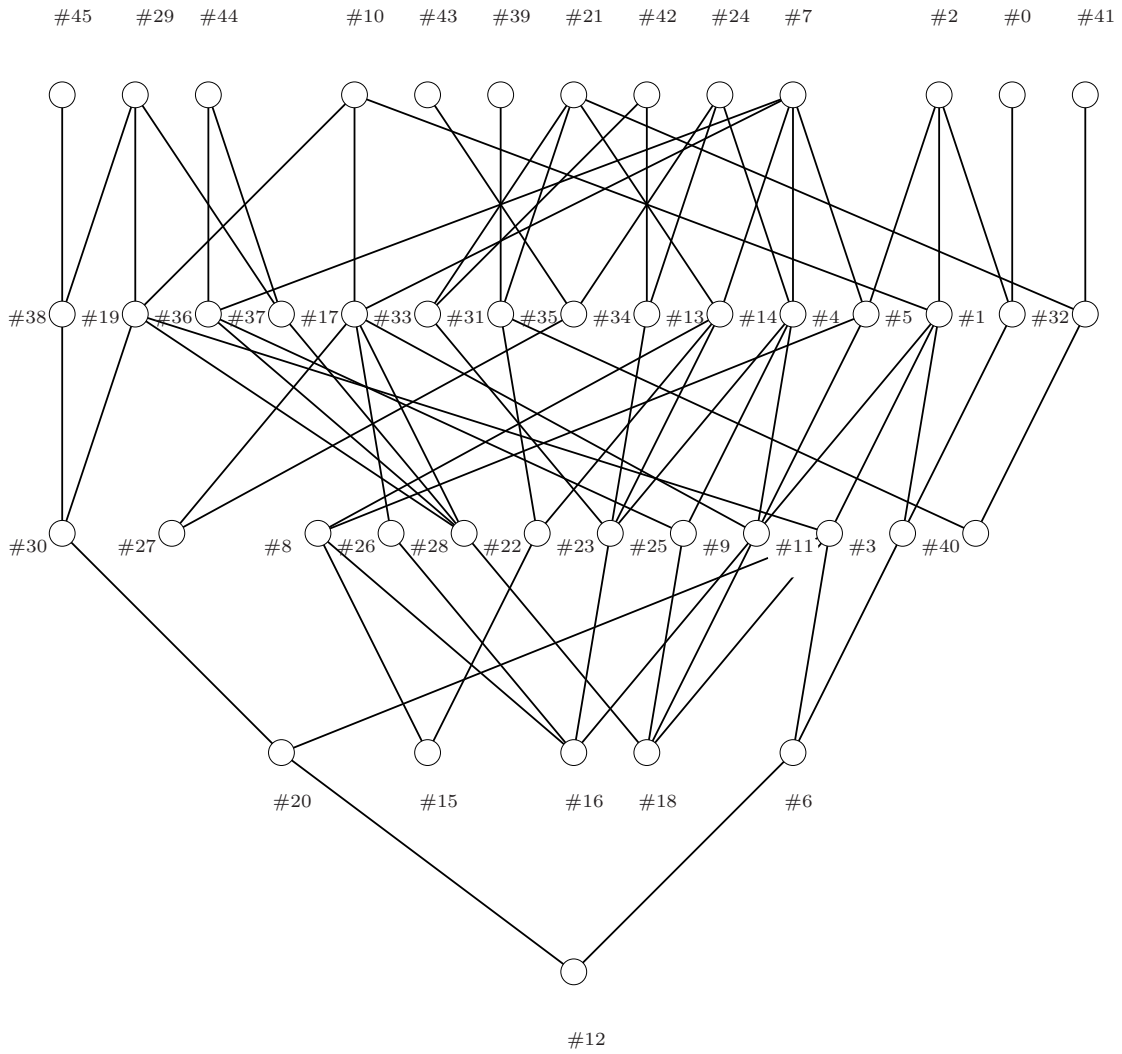

FIGURE 17. The graph of D- and C-mutations on the space of projective arrangements of three double pseudolines.

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