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Helmholtz decomposition of vector fields with mixed boundary conditions and an application to a posteriori finite element error analysis of the Maxwell system

Emmanuel Creusé*, Serge Nicaise†, Zuqi Tang‡

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Abstract

This paper is devoted to the derivation of a Helmholtz decomposition of vector fields in the case of mixed boundary conditions imposed on the boundary of the domain. This particular decomposition allows to obtain a residual a posteriori error estimator for the approximation of magnetostatic problems given in the so-called A -formulation, for which the reliability can be established. Numerical tests confirm the obtained theoretical predictions.

AMS (MOS) subject classification 35Q61; 65N30; 65N15; 65N50.

Key Words Helmholtz decomposition, Finite element methods, A posteriori error estimators.

Introduction

Nowadays, the finite element method is widely used to study electromagnetic systems. In order to carry out an adaptive mesh refinement strategy as well as to control the error obtained between the numerical solution and the exact one, local a posteriori error estimators are needed. Since the pioneering work due to Babuska and Rheinboldt [3, 4], and then several monographies related to this subject [1, 5, 26], an impressive number of papers have been published, considering several kinds of equations, numerical methods or approximation spaces, and providing a large variety of error estimators. In this work, we are particularly interested in explicit residual-based error estimators, devoted to the

*LPP, Université Lille 1, Cité Scientifique, 59655 Villeveuve d'Ascq Cedex & EPI SIMPAF, INRIA Lille Nord Europe, France. creuse@math.univ-lille1.fr.

†LAMAV, Université de Valenciennes et du Hainaut Cambrésis, FR CNRS 2956, Institut des Sciences et Techniques de Valenciennes, F-59313 - Valenciennes Cedex 9 France (corresponding author).

‡L2EP, Université Lille 1, Cité Scientifique, 59655 Villeveuve d'Ascq Cedex, France.

approximation of the solutions of Maxwell equations, discretized by Nédélec $H(\text{curl})$ conforming edge elements.

For the last twenty years, these error estimators have been developed for the computation of quasistatic electromagnetic fields in conductors in the case of regular data and domains [6]. They have been then generalized to the case of anisotropic meshes and non regular data [21], as well as to the one of Lipschitz domains [25]. The robustness of the estimations with the data was also addressed in [11]. More recently, in the same kind of work, namely explicit residual-based a posteriori estimators have also been developed for quasistatic magnetodynamic problems, modeled with harmonic potential formulations [9, 13, 27, 28]. Moreover, some papers are now devoted to the way to design some adaptive mesh loop strategies, in order to ensure the convergence process while the mesh is being refined using the error estimators [10, 17, 18].

In most of the above quoted papers, the reliability of the estimator (namely, the fact that the error is bounded from above by the estimator) is based on a suitable orthogonal Helmholtz decomposition of the error. This one is needed to allow the use of some suitable interpolation operators, which are crucial in the proofs. Nevertheless, to our best knowledge, the case corresponding to mixed boundary conditions imposed on the vector field (see the boundary conditions on A arising in (3) below) have not been reported yet. Consequently, the main objective of this paper is to prove the corresponding Helmholtz decomposition, in order to extend the reliability of residual-type estimator to this case.

The schedule of the paper is the following. In section 1, the context of the problem is recalled. The continuous model we are interested in is given, the existence and uniqueness of the solution are established, and the finite element approximation is briefly described. Section 2 constitutes the main part of this work and is devoted to the Helmholtz decomposition of vector fields. Several lemmas are proved, and used to derive Theorem 2.3. Consequently, in section 3 we are able to derive the a posteriori residual error estimator. With the previous Helmholtz decomposition in hands, the proof can be easily derived from previous papers, but we chose to recall its main steps to make this work self-contained. Finally, section 4 is devoted to numerical tests, to underline the good correspondence between theoretical and numerical results obtained.

1 Setting of the problem

1.1 Modeling and continuous variational formulation

Let us consider an open simply connected domain $\Omega \subset \mathbb{R}^3$ with a Lipschitz boundary $\Gamma = \partial\Omega$. In this paper, we are interested in the magnetic flux density B induced in Ω by

an eddy current J_s , modeled by the following stationary Maxwell equations :

$$\begin{cases} \operatorname{div} B = 0 & \text{in } \Omega & \text{(Gauss equation),} \\ \operatorname{curl} H = J_s & \text{in } \Omega & \text{(Maxwell-Ampère equation),} \end{cases} \quad (1)$$

where H is the magnetic field defined by $B = \mu H$, with $\mu \in L^\infty(\Omega)$ the magnetic permeability of the medium, for which we assume that there exists $\mu_{min} \in \mathbb{R}_+^*$ such that $\mu \geq \mu_{min}$ in Ω . Let us moreover suppose that $\Gamma = \Gamma_N \cup \Gamma_T$, with $\Gamma_N \cap \Gamma_T = \emptyset$ and Γ_N connected. Finally we denote by n the unit outward normal to Ω on Γ . Usual boundary conditions on B associated with problem (1) are given by :

$$\begin{cases} B \cdot n = 0 & \text{on } \Gamma_N, \\ \frac{1}{\mu} B \times n = 0 & \text{on } \Gamma_T. \end{cases} \quad (2)$$

Then, a vector potential A is introduced such that $B = \operatorname{curl} A$, and problem (1)-(2) is reformulated in its potential formulation given by :

$$\begin{cases} \operatorname{curl} \left(\frac{1}{\mu} \operatorname{curl} A \right) = J_s & \text{in } \Omega, \\ A \times n = 0 & \text{on } \Gamma_N, \\ A \cdot n = 0 \text{ and } \frac{1}{\mu} \operatorname{curl} A \times n = 0 & \text{on } \Gamma_T, \\ \operatorname{div} A = 0 & \text{in } \Omega. \end{cases} \quad (3)$$

Let us note that the divergence-free constraint on A corresponds to the gauge condition, needed to ensure the uniqueness of the solution. Here, J_s is supposed to be given such that :

$$\begin{cases} \operatorname{div} J_s = 0 & \text{in } \Omega, \\ J_s \cdot n = 0 & \text{on } \Gamma_T. \end{cases} \quad (4)$$

In the following, for a given domain D , the $L^2(D)$ (or $(L^2(D))^3$) norm will be denoted by $\|\cdot\|_D$. The usual norm and semi-norm of $H^1(D)$ (or $(H^1(D))^3$) will be denoted $\|\cdot\|_{1,D}$ and $|\cdot|_{1,D}$ respectively. We first introduce the following functional spaces:

$$X(\Omega) = \{A \in L^2(\Omega)^3 : \operatorname{curl} A \in L^2(\Omega)^3 \text{ and } A \times n = 0 \text{ on } \Gamma_N\},$$

$$H_{\Gamma_N}^1(\Omega) = \{\varphi \in H^1(\Omega) : \varphi = 0 \text{ on } \Gamma_N\},$$

$$X^0(\Omega) = \left\{ A \in X(\Omega) : \int_{\Omega} A \cdot \nabla \varphi = 0 \forall \varphi \in H_{\Gamma_N}^1(\Omega) \right\},$$

with the natural norm $\|\cdot\|_{X(\Omega)}$ defined on $X(\Omega)$ by :

$$\|A\|_{X(\Omega)}^2 = \|A\|_{\Omega}^2 + \|\operatorname{curl} A\|_{\Omega}^2.$$

Then, we consider the variational formulation associated with (3) by looking for $A \in X^0(\Omega)$ such that, for all $A' \in X^0(\Omega)$, we have :

$$a(A, A') = l(A'), \quad (5)$$

with :

$$a(A, A') = \int_{\Omega} \frac{1}{\mu} \operatorname{curl} A \cdot \operatorname{curl} A' \quad \text{and} \quad l(A') = \int_{\Omega} J_s \cdot A'. \quad (6)$$

In order to address the question of the well-posedness of the problem, let us introduce the space $X_M(\Omega)$ defined by :

$$X_M(\Omega) = \{A \in L^2(\Omega)^3 : \operatorname{curl} A \in L^2(\Omega)^3, \operatorname{div} A \in L^2(\Omega), A \times n = 0 \text{ on } \Gamma_N \text{ and } A \cdot n = 0 \text{ on } \Gamma_T\},$$

with its associated natural norm $\|\cdot\|_{X_M(\Omega)}$ defined by :

$$\|A\|_{X_M(\Omega)}^2 = \|A\|_{\Omega}^2 + \|\operatorname{div} A\|_{\Omega}^2 + \|\operatorname{curl} A\|_{\Omega}^2.$$

Since $X_M(\Omega)$ is compactly embedded in $L^2(\Omega)^3$ [24], and since it can be moreover easily shown that $X^0(\Omega) \subset X_M(\Omega)$, the following Friedrichs-type inequality holds :

There exists $C > 0$ such that for all $A \in X^0(\Omega)$, we have :

$$\|A\|_{\Omega} \leq C \|\operatorname{curl} A\|_{\Omega}.$$

Consequently, we deduce from the Lax-Milgram lemma that problem (5) is well-posed and admits a unique solution $A \in X^0(\Omega)$.

Now, let us derive the following property :

Lemma 1.1 *Let $A \in X^0(\Omega)$ be the unique solution of (5). Then for all $B \in X(\Omega)$, we have*

$$a(A, B) = l(B). \quad (7)$$

Proof. Since $B \in X(\Omega)$, we can decompose it using the following Helmholtz decomposition:

$$B = \Psi + \nabla\tau,$$

with $\Psi \in X^0(\Omega)$ and $\tau \in H_{\Gamma_N}^1(\Omega)$. Indeed, $\tau \in H_{\Gamma_N}^1(\Omega)$ is simply defined by

$$\int_{\Omega} \nabla\tau \cdot \nabla\varphi = \int_{\Omega} B \cdot \nabla\varphi, \quad \forall \varphi \in H_{\Gamma_N}^1(\Omega),$$

and $\Psi = B - \nabla\tau$. So we may write

$$a(A, B) = a(A, \Psi) = l(\Psi).$$

Because of (4) and the fact that $\tau|_{\Gamma_N} = 0$, we clearly have $l(\Psi) = l(B)$, so that (7) holds. ■

1.2 Finite Element approximation

Problem (5) is approximated using a finite element method. The domain Ω is from now supposed to be polyhedral, and is discretized by a conforming mesh \mathcal{T}_h made of tetrahedra denoted by T . The magnetic permeability of the medium μ is supposed to be constant over each tetrahedron. We denote by h_T the diameter of T and by ρ_T the diameter of its largest inscribed ball. For any element T , the ratio h_T/ρ_T is supposed to be bounded from above by a constant $\alpha > 0$ independent of T and of the mesh size $h = \max_{T \in \mathcal{T}_h} h_T$. The faces of \mathcal{T}_h are denoted by F , and we denote by h_F the diameter of the face F . For compatibility reasons, we suppose that only one boundary condition can be prescribed on a given boundary face $F \subset \Gamma$. In other words, if $F \subset \Gamma$, we assume that either $F \subset \Gamma_N$ or $F \subset \Gamma_T$. Finally, the set of the faces of the triangulation is denoted by \mathcal{F} , whereas the set of the internal faces to Ω is denoted by \mathcal{F}_{int} .

The approximation spaces are now defined by :

$$X_h(\Omega) = X(\Omega) \cap \mathcal{ND}_1(\Omega, \mathcal{T}_h) = \left\{ A_h \in X(\Omega) : A_{h|T} \in \mathcal{ND}_1(T), \forall T \in \mathcal{T}_h \right\},$$

$$\mathcal{ND}_1(T) = \left\{ A_h : \begin{array}{l} T \longrightarrow \mathbb{R}^3 \\ x \longrightarrow a + b \times x, a, b \in \mathbb{R}^3 \end{array} \right\},$$

$$\Theta_h^0(\Omega) = \left\{ \xi_h \in H_{\Gamma_N}^1(\Omega) : \xi_{h|T} \in \mathbb{P}_1(T) \forall T \in \mathcal{T}_h \right\},$$

$$X_h^0(\Omega) = \left\{ A_h \in X_h(\Omega) : \int_{\Omega} A_h \cdot \nabla \xi_h = 0 \forall \xi_h \in \Theta_h^0(\Omega) \right\}.$$

The discrete variational formulation now consists in finding $A_h \in X_h^0(\Omega)$ such that, for all $A' \in X_h^0(\Omega)$, we have :

$$a(A_h, A'_h) = l(A'_h). \quad (8)$$

Because of this finite element choice, and from a discrete compactness property of $X_h^0(\Omega)$ [19, 22], we have, as for the continuous case, that the discrete problem (8) is well-posed and admits a unique solution $A_h \in X_h^0(\Omega)$. Let us remark that the approximation is sought on the Nédélec functions that are discrete divergence-free. Nevertheless, in practice, the corresponding gauge condition is not explicitly forced. Indeed, thanks to the particular iterative solver used for the resolution of the linear system, the divergence-free property of the successive vector fields obtained is preserved, so that the gauge condition is implicitly imposed [23].

2 Helmholtz decomposition of vector fields

The aim of this section is to obtain Theorem 2.3, which will be used in section 3 to derive the a posteriori error estimate.

To begin with, for $v \in X_M(\Omega)$, we remark that $\operatorname{curl} v$ belongs to

$$\hat{J}(\Omega) := \{v \in L^2(\Omega)^3 : \operatorname{div} v = 0 \text{ and } v \cdot n = 0 \text{ on } \Gamma_N\}.$$

According to Lemmas 3.1 and 3.2 of [12] we first consider smooth vector potentials for elements of $\hat{J}(\Omega)$. For that purpose we combine some ideas from Theorems 3.1 and 4.1 of [7] (see also [8]). First we state a technical result.

Lemma 2.1 *Let A be a bounded domain of \mathbb{R}^3 with a Lipschitz boundary. Then there exists a linear continuous operator E_A from $H^1(A)$ into $H_{\text{loc}}^1(\mathbb{R}^3)$ such that*

$$E_A 1 = 1, \tag{9}$$

and fulfilling the usual properties of an extension operator:

$$E_A u = u \text{ on } A, \forall u \in H^1(A), \tag{10}$$

and for all open sets B of \mathbb{R}^3 containing \bar{A} , there exists a positive constant C depending only on A and B such that

$$\|E_A u\|_{1,B} \leq C \|u\|_{1,A}, \forall u \in H^1(A).$$

Proof. By Theorem 1.4.3.1 of [16], there exists a linear continuous operator E from $H^1(A)$ into $H^1(\mathbb{R}^3)$ such that

$$E u = u \text{ on } A, \forall u \in H^1(A),$$

and

$$\|E u\|_{1,\mathbb{R}^3} \leq C \|u\|_{1,A}, \forall u \in H^1(A),$$

for some constant $C_1 > 0$. An appropriate choice for E_A is

$$E_A u = E(u - \mathcal{M}_A u) + \mathcal{M}_A u, \forall u \in H^1(A),$$

where $\mathcal{M}_A u$ is the mean of u in A , namely

$$\mathcal{M}_A u = |A|^{-1} \int_A u(x) dx,$$

when $|A|$ is the measure of A . The definition of E_A and the fact that E is linear imply that (9) holds. In the same manner we see that (10) is valid because

$$E(u - \mathcal{M}_A u) = u - \mathcal{M}_A u \text{ on } A.$$

Finally the continuity property of E_A comes from the continuity of E and the fact that

$$\|\mathcal{M}_A u\|_{1,B} = \|\mathcal{M}_A u\|_B = \frac{|B|^{1/2}}{|A|^{1/2}} \|\mathcal{M}_A u\|_A \leq \frac{|B|^{1/2}}{|A|^{1/2}} \|u\|_A.$$

■

Lemma 2.2 For all $f \in \hat{J}(\Omega)$ there exists $w \in H^1(\Omega)^3$ such that

$$w \times n = 0 \text{ on } \Gamma_N, \quad (11)$$

and fulfilling

$$\text{curl } w = f \text{ in } \Omega, \quad (12)$$

and

$$\|w\|_{1,\Omega} \leq C \|f\|_{\Omega}, \quad (13)$$

for some constant $C > 0$ depending only on Ω .

Proof. As said before we combine some arguments from Theorem 3.1 of [7] (case $\Gamma_T = \emptyset$) and from Theorem 4.1 of [7] (case $\Gamma_N = \emptyset$).

Step 1 We consider $\tilde{\Omega}$ a bounded domain of \mathbb{R}^3 with a polyhedral boundary such that

$$\Omega \subset \tilde{\Omega},$$

and such that the common faces between $A := \tilde{\Omega} \setminus \Omega$ and Ω are the faces of Γ_T . As A enters in the framework of Lemma 2.1, we can use the extension operator E_A .

Step 2 Set

$$H_m^1(A) := \{u \in H^1(A) : \int_A u(x) dx = 0\},$$

that can be equipped with the H^1 -semi norm. Let $u \in H_m^1(A)$ be the unique solution of

$$\int_A \nabla u \cdot \nabla w dx = \int_{\Omega} f \cdot \nabla E_A w dx, \forall w \in H_m^1(A). \quad (14)$$

This solution exists and is unique owing to the Lax-Milgram lemma because the right-hand side is a linear and continuous form on $H_m^1(A)$. Furthermore as $E_A 1 = 1$, (14) is valid on the whole of $H^1(A)$, namely we also have

$$\int_A \nabla u \cdot \nabla w dx = \int_{\Omega} f \cdot \nabla E_A w dx, \forall w \in H^1(A). \quad (15)$$

Now we set

$$\tilde{f} := \begin{cases} f & \text{in } \Omega, \\ -\nabla u & \text{in } A \end{cases}$$

Let us check that \tilde{f} belongs to

$$J(\nu, \tilde{\Omega}) := \{v \in L^2(\Omega)^3 : \text{div } v = 0 \text{ and } v \cdot n = 0 \text{ on } \partial\tilde{\Omega}\}.$$

Since it is well known that $\tilde{f} \in J(\nu, \tilde{\Omega})$ if and only if

$$\int_{\tilde{\Omega}} \tilde{f} \cdot \nabla h dx = 0, \forall h \in H^1(\tilde{\Omega}), \quad (16)$$

it remains to check this last property. For an arbitrary $h \in H^1(\tilde{\Omega})$, by the definition of \tilde{f} , we have

$$\int_{\tilde{\Omega}} \tilde{f} \cdot \nabla h \, dx = \int_{\Omega} f \cdot \nabla h \, dx - \int_A \nabla u \cdot \nabla h \, dx.$$

Since u satisfies (15) we get

$$\int_{\tilde{\Omega}} \tilde{f} \cdot \nabla h \, dx = \int_{\Omega} f \cdot \nabla h \, dx - \int_{\Omega} f \cdot \nabla E_A h \, dx = \int_{\Omega} f \cdot \nabla (h - E_A h) \, dx.$$

Hence by Green's formula in Ω we obtain

$$\int_{\tilde{\Omega}} \tilde{f} \cdot \nabla h \, dx = - \int_{\Omega} \operatorname{div} f (h - E_A h) \, dx + \langle f \cdot n; h - E_A h \rangle_{(\tilde{H}^{1/2}(\Gamma_N))' - \tilde{H}^{1/2}(\Gamma_N)},$$

reminding that $h = E_A h$ on Γ_T . Since f belongs to $\hat{J}(\Omega)$, this last right-hand side is zero and we have proven (16).

Step 3 Fix a ball B of \mathbb{R}^3 such that

$$\overline{\tilde{\Omega}} \subset B,$$

and set

$$g := \begin{cases} \tilde{f} & \text{in } \tilde{\Omega}, \\ 0 & \text{in } B \setminus \tilde{\Omega}. \end{cases}$$

As $\tilde{f} \in J(\nu, \tilde{\Omega})$, g belongs to $J(\nu, B)$. Therefore by [7], there exists $w_0 \in H^1(B)^3$ such that

$$\operatorname{curl} w_0 = g \text{ and } \operatorname{div} w_0 = 0 \text{ in } B, \quad (17)$$

with

$$\|w_0\|_{1,B} \leq C_1 \|g\|_B = C_1 \|\tilde{f}\|_{\tilde{\Omega}}, \quad (18)$$

where C_1 is a positive constant that depends only on B . But according to the definition of \tilde{f} ,

$$\|\tilde{f}\|_{\tilde{\Omega}}^2 = \|f\|_{\Omega}^2 + \|\nabla u\|_A^2,$$

and by (14), Cauchy-Schwarz's inequality and Lemma 2.1 we get

$$\|\nabla u\|_A \leq C_2 \|f\|_{\Omega},$$

where C_2 is a positive constant that depends only on A and B . Therefore the estimate (18) reduces to

$$\|w_0\|_{1,B} \leq C_3 \|f\|_{\Omega}. \quad (19)$$

where C_3 is a positive constant that depends only on A and B .

Step 4 As $g = 0$ on $B \setminus \tilde{\Omega}$, we get

$$\operatorname{curl} w_0 = 0 \text{ in } B \setminus \tilde{\Omega}.$$

Since this set is simply connected, there exists $\Phi_A \in H^1(B \setminus \tilde{\Omega})$ such that

$$w_0 = \nabla \Phi_A \text{ in } B \setminus \tilde{\Omega}. \quad (20)$$

This function Φ_A even belongs to $H^2(B \setminus \tilde{\Omega})$ since $w_0 \in H^1(B)^3$.

Denote by Π the extension (linear continuous) operator from $H^2(B \setminus \tilde{\Omega})$ into $H^2(B)$ such that

$$\Pi w = w \text{ in } B \setminus \tilde{\Omega}.$$

Now setting

$$\Phi = \Pi \Phi_A,$$

we notice that

$$w = w_0 - \nabla \Phi$$

yields the conclusion. Indeed w belongs to $H^1(\Omega)^3$ by the regularity of w_0 and Φ . Furthermore

$$\operatorname{curl} w = \operatorname{curl} w_0,$$

and therefore by (17), we obtain (12) (as $g = f$ in Ω). Finally

$$w \times n = w_0 \times n - \nabla \Phi \times n \text{ on } \Gamma_N.$$

and by (20),

$$w \times n = \nabla \Phi_A \times n - \nabla \Pi \Phi_A \times n = 0 \text{ on } \Gamma_N.$$

by the property of the extension operator and proves (11). ■

Theorem 2.3 *If Ω is simply connected and Γ_N is connected, then for any $u \in X_M(\Omega)$ there exist $w \in H^1(\Omega)^3$ satisfying (11) and $\varphi \in H_{\Gamma_N}^1(\Omega) := \{\psi \in H^1(\Omega) : \psi = 0 \text{ on } \Gamma_N\}$ satisfying $\operatorname{div} \nabla \varphi \in L^2(\Omega)$ and such that the following Helmholtz decomposition*

$$u = w + \nabla \varphi \text{ in } \Omega, \quad (21)$$

holds with the estimate

$$\|w\|_{1,\Omega} + \|\varphi\|_{1,\Omega} + \|\operatorname{div} \nabla \varphi\|_{\Omega} \leq C \|u\|_{X_M(\Omega)}, \quad (22)$$

for some constant $C > 0$ depending only on Ω .

Proof. For $u \in X_M(\Omega)$ consider $v = \operatorname{curl} u$ that belongs to $\hat{J}(\Omega)$. Hence by Lemma 2.2, there exists $w \in H^1(\Omega)^3$ satisfying (11) and fulfilling

$$\operatorname{curl} w = \operatorname{curl} u \text{ in } \Omega,$$

and

$$\|w\|_{1,\Omega} \leq C \|\operatorname{curl} u\|_{\Omega}, \quad (23)$$

for some constant $C > 0$ depending only on Ω . This means that $u - w$ is curl free and since Ω is simply connected, there exists $\varphi_0 \in H^1(\Omega)$ such that

$$u - w = \nabla\varphi_0 \text{ in } \Omega. \quad (24)$$

As $(u - w) \times n = 0$ on Γ_N , we deduce that

$$\nabla\varphi_0 \times n = 0 \text{ on } \Gamma_N.$$

Consequently there exists a constant $K \in \mathbb{C}$ such that (reminding that Γ_N is connected)

$$\varphi_0 = K \text{ on } \Gamma_N.$$

The conclusion follows by taking

$$\varphi = \varphi_0 - K,$$

since $\nabla\varphi = \nabla\varphi_0$ and using (23) and (24). ■

Remark 2.4 The previous result still holds if Γ_N is not connected, but in that case φ will be constant on each connected component of Γ_N . Similarly the previous result also holds if Ω is not simply connected, indeed it suffices to apply the previous theorem to each connected component of Ω .

With additional low geometrical assumptions, it is possible to recover homogeneous Neumann boundary conditions for φ involved in the Helmholtz decomposition, this is the subject of Theorem 2.5. Nevertheless, it will not be useful in section 3 to derive the a posteriori error analysis.

Theorem 2.5 *Assume that Ω is simply connected and Γ_N is connected. Assume furthermore that*

- i) for all edges e in common between a face of Γ_T and a face of Γ_N , the interior angle ω_e is different from $\pi/2$,*
- ii) for all corners c , the eigenvalues $\lambda_{c,k}^{\text{mixed}}$ of Δ_c^{mixed} are different from $3/4$, where Δ_c^{mixed} is the non negative Laplace-Beltrami on the unit sphere with mixed boundary conditions (in accordance with the partition of the faces of Γ_N (Dirichlet) and of Γ_T (Neumann) containing c).*

Then for any $u \in X_M(\Omega)$ there exist $w_0 \in H^1(\Omega)^3 \cap X_M(\Omega)$ and $\varphi_0 \in H_{\Gamma_N}^1(\Omega)$ satisfying $\text{div}\nabla\varphi_0 \in L^2(\Omega)$ and

$$\frac{\partial\varphi_0}{\partial n} = 0 \text{ on } \Gamma_T,$$

and such that the following Helmholtz decomposition

$$u = w_0 + \nabla\varphi_0 \text{ in } \Omega, \quad (25)$$

holds with the estimate

$$\|w_0\|_{1,\Omega} + \|\varphi_0\|_{1,\Omega} + \|\text{div}\nabla\varphi_0\|_{\Omega} \leq C\|u\|_{X_M(\Omega)}, \quad (26)$$

for some constant $C > 0$ depending only on Ω .

Proof. In fact the previous Theorem shows that $\varphi \in H_{\Gamma_N}^1(\Omega)$ a weak solution of the boundary value problem

$$\begin{cases} \operatorname{div} \nabla \varphi = \operatorname{div} u - \operatorname{div} w & \text{in } \Omega, \\ \frac{\partial \varphi}{\partial n} = -w \cdot n & \text{on } \Gamma_T. \end{cases}$$

Since the assumptions on u and the regularity of w guarantee that $\operatorname{div} u - \operatorname{div} w$ belongs to $L^2(\Omega)$ and since $w \cdot n$ belongs to $H^{1/2}(\Gamma_T)$, we can apply the results from [14] to obtain the next decomposition of φ :

$$\varphi = \varphi_R + \varphi_0,$$

where $\varphi_R \in H^2(\Omega)$ and $\varphi_0 \in H_{\Gamma_N}^1(\Omega)$ such that

$$\begin{cases} \operatorname{div} \nabla \varphi_0 \in L^2(\Omega), \\ \frac{\partial \varphi_0}{\partial n} = 0 & \text{on } \Gamma_T. \end{cases}$$

with the estimate

$$\|\varphi_R\|_{2,\Omega} + \|\varphi_0\|_{1,\Omega} + \|\operatorname{div} \nabla \varphi_0\|_{\Omega} \leq c(\|\operatorname{div} u - \operatorname{div} w\|_{\Omega} + \|w \cdot n\|_{H^{1/2}(\Gamma_T)}),$$

where c is a positive constant depending only on Ω . We then conclude by taking

$$w_0 = w + \nabla \varphi_R,$$

where $w \in H^1(\Omega)^3$ is the vector potential from Theorem 2.3. ■

3 A posteriori residual error analysis

Let us define π_h as the projection operator from $H(\operatorname{div}, \Omega) = \{u \in L^2(\Omega)^3; \operatorname{div} u \in L^2(\Omega)\}$ to an ad hoc discrete approximation space (which can be chosen for example as the low order Raviart-Thomas finite element space [15] so that it is included in $H(\operatorname{div}, \Omega)$), and let us denote by $[u]_F$ the jump of the quantity u over a face F of the mesh. For each $T \in \mathcal{T}_h$, the local error indicator η_T is defined by :

$$\eta_T^2 = \eta_{T,1}^2 + \eta_{T,2}^2 + \sum_{F \subset \partial T, F \in \mathcal{F}_{int}} \eta_{F,1}^2 + \sum_{F \subset \partial T, F \subset \Gamma_T} \eta_{F,2}^2,$$

with :

$$\begin{aligned} \eta_{T,1} &= h_T \left\| \pi_h J_s - \operatorname{curl} \left(\frac{1}{\mu} \operatorname{curl} A_h \right) \right\|_T, \\ \eta_{T,2} &= h_T \|J_s - \pi_h J_s\|_T, \\ \eta_{F,1} &= h_F^{1/2} \left\| \left[\frac{1}{\mu} \operatorname{curl} A_h \times n \right]_F \right\|_F, \\ \eta_{F,2} &= h_F^{1/2} \left\| \frac{1}{\mu} \operatorname{curl} A_h \times n \right\|_F. \end{aligned}$$

Then, the global error indicator η is defined by :

$$\eta^2 = \sum_{T \in \mathcal{T}_h} \eta_T^2.$$

Theorem 3.1 *Let us suppose that Ω is simply connected and Γ_N is connected, there exists a constant $C_{up} > 0$, which does not depend on A , A_h or h , such that :*

$$\left(\int_{\Omega} \frac{1}{\mu} |\text{curl}(A - A_h)|^2 \right)^{\frac{1}{2}} \leq C_{up} \eta. \quad (27)$$

Proof. The proof can be easily derived from previous more general works (for example [13] which deals with the magnetodynamic case), but in the restricted case $\Gamma_T = \emptyset$ (or, equivalently, $\Gamma_N = \Gamma$). The novelty of this result is the generalization to the case $\Gamma_T \neq \emptyset$, thanks to the new Helmholtz decomposition (21) and to (22). The sketch of the proof is then very similar to the case $\Gamma_T = \emptyset$. Consequently, we only recall here the main steps for the sake of completeness.

1. First, for any $q \in X(\Omega)$, the residual form associated with (8) is defined by :

$$r(q) = l(q) - a(A_h, q). \quad (28)$$

Because of Lemma 1.1, setting $e_A := A - A_h$, we obtain :

$$r(e_A) = \left\| \frac{1}{\sqrt{\mu}} \text{curl} e_A \right\|_{\Omega}^2. \quad (29)$$

Now, we define $\Psi \in H_{\Gamma_N}^1(\Omega)$ by :

$$\int_{\Omega} \nabla \Psi \cdot \nabla \chi = \int_{\Omega} e_A \cdot \nabla \chi, \quad \text{for all } \chi \in H_{\Gamma_N}^1(\Omega).$$

Setting $\tilde{e}_A = e_A - \nabla \Psi$, we have $\tilde{e}_A \in X^0(\Omega) \subset X_M(\Omega)$. Using the Helmholtz decomposition (21), we have :

$$e_A = w + \nabla(\varphi + \Psi),$$

with $w \in H^1(\Omega)^3 \cap X(\Omega)$ and $(\varphi + \Psi) \in H_{\Gamma_N}^1(\Omega)$. From (28) and using (4), similarly to the proof of Lemma 1.1, we get

$$r(e_A) = r(w). \quad (30)$$

The point now is to bound from above the quantity $|r(w)|$ so that (27) occurs from (29) and (30).

2. We introduce an operator \mathcal{P}_{Cl} defined by the identity (32) of [6] where for any edge E belonging to $\overline{\Gamma}_N$, the face F_E has to be chosen in Γ_N :

$$\begin{aligned} \mathcal{P}_{Cl} : H^1(\Omega)^3 \cap X(\Omega) &\rightarrow X_h(\Omega), \\ v &\rightarrow \mathcal{P}_{Cl}v. \end{aligned}$$

From lemma 5 of [6], we get :

$$\left\{ \begin{array}{l} \sum_{T \in \mathcal{T}_h} \frac{1}{h_T^2} \|v - \mathcal{P}_{Cl}v\|_T^2 \leq C_1 \|v\|_{1,\Omega}^2, \\ \sum_{F \in \mathcal{F}_{int} \cup (\mathcal{F} \cap \Gamma_T)} \frac{1}{h_T^2} \|v - \mathcal{P}_{Cl}v\|_F^2 \leq C_2 \|v\|_{1,\Omega}^2, \end{array} \right. \quad (31)$$

where the constants C_1 and C_2 do neither depend on v nor on the mesh size h , but only on the shape regularity of the mesh. We use the fact that $r(w) = r(w - \mathcal{P}_{Cl}w)$, so that, with classical Green formula, we get :

$$\begin{aligned} r(w) &= \sum_{T \in \mathcal{T}_h} \int_T \left(J_s - \operatorname{curl} \left(\frac{1}{\mu} \operatorname{curl} A_h \right) \right) \cdot (w - \mathcal{P}_{Cl}w) \\ &\quad - \sum_{F \in \mathcal{F}_{int}} \int_F \left[\frac{1}{\mu} \operatorname{curl} A_h \times n \right]_F \cdot (w - \mathcal{P}_{Cl}w) \\ &\quad - \sum_{F \in \mathcal{F} \cap \Gamma_T} \int_F \left(\frac{1}{\mu} \operatorname{curl} A_h \times n \right) \cdot (w - \mathcal{P}_{Cl}w). \end{aligned}$$

3. Finally, the use of Cauchy-Schwarz inequalities, discrete Cauchy-Schwarz inequalities, the properties (31) on \mathcal{P}_{Cl} and the stability relation (22) yield (27).

■

For completeness we now state the efficiency of the estimator (namely, the fact that the local estimator is bounded from above by the local error) :

Theorem 3.2 *Let us define the patch of the element T by $\mathcal{P}_T = \bigcup_{T' \cap T \neq \emptyset} T'$. Then, there exists a constant $C_{down} > 0$, which does not depend on A , A_h or h , such that:*

$$\eta_T \leq C_{down} \left\| \frac{1}{\sqrt{\mu}} \operatorname{curl} (A - A_h) \right\|_{\mathcal{P}_T}.$$

Proof. The proof is here exactly the same as the one corresponding to the case $\Gamma_T = \emptyset$, since we do not need any Helmholtz decomposition, but only need to use classical bubble functions and inverse inequalities. It is available in previous papers (see for example [13]).

■

4 Numerical validation

4.1 Analytical solution

Let us consider here the unit cube $\Omega = [0, 1/2]^3$. We want to compute the magnetic flux density $B = \text{curl } A$ given by (3) with the following data (see Figure 1) :

$$\left\{ \begin{array}{l} J_s = (0, 0, 10^7)^T, \\ \Gamma_N = \left\{ (x, y, z) \in \partial\Omega \text{ such that } xyz(z - \frac{1}{2}) = 0 \right\}, \\ \Gamma_T = \partial\Omega \setminus \Gamma_N, \\ \mu = 4\pi \cdot 10^{-7}. \end{array} \right.$$

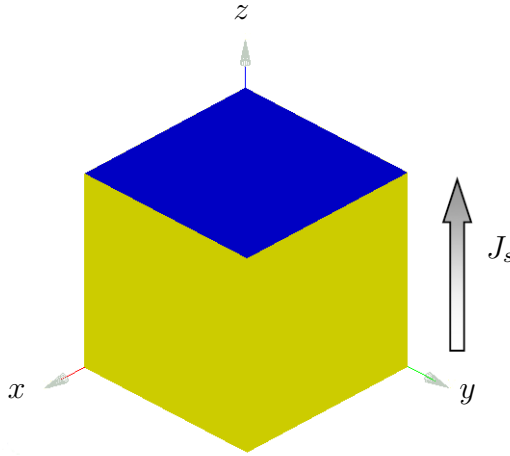


Figure 1: Domain configuration and eddy current J_s .

The solution of this problem is known analytically and is given by (see [2]) :

$$\left\{ \begin{array}{l} B_x(x, y, z) = \frac{16 \cdot 10^7 \mu}{\pi^3} \sum_{n=0}^{+\infty} \sum_{p=0}^{+\infty} \frac{\sin((2n+1)\pi x) \cos((2p+1)\pi y)}{(2n+1)[(2n+1)^2 + (2p+1)^2]}, \\ B_y(x, y, z) = -\frac{16 \cdot 10^7 \mu}{\pi^3} \sum_{n=0}^{+\infty} \sum_{p=0}^{+\infty} \frac{\cos((2n+1)\pi x) \sin((2p+1)\pi y)}{(2p+1)[(2n+1)^2 + (2p+1)^2]}, \\ B_z(x, y, z) = 0. \end{array} \right.$$

Setting $B_h = \text{curl } A_h$, the error $\left\| \frac{1}{\sqrt{\mu}}(B - B_h) \right\|_{\Omega}$ is plotted versus DoF in Figure 2(a) (where DoF is the number of edges in the mesh), to show that the numerical approximated value of the magnetic flux density B_h converges towards the exact one B at order 1 in h (or order $-1/3$ in DoF since $DoF = O(1/h^3)$), as theoretically expected by an ad hoc a priori convergence analysis (see for example [20]). The solution B_h is moreover plotted in

figure 2(b) in the plane $z = 1/2$ to show its qualitatively good behavior. Then, the ratio $\left\| \frac{1}{\sqrt{\mu}}(B - B_h) \right\|_{\Omega} / \eta$ is plotted versus DoF in Figure 3, in order to numerically confirm the reliability of the estimator given by (27).

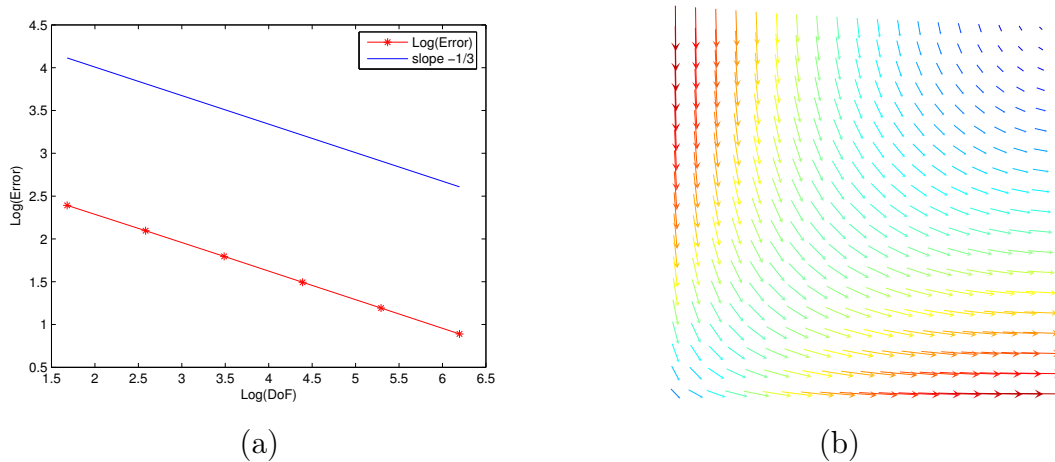


Figure 2: (a) : Convergence rate of the error, (b) : Approximated magnetic flux density field B_h in the plane $z = 1/2$.

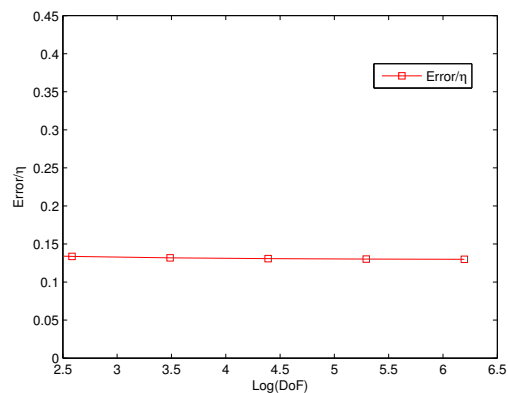


Figure 3: $\left\| \frac{1}{\sqrt{\mu}}(B - B_h) \right\|_{\Omega} / \eta$ versus $\log(DoF)$.

4.2 Singular solution

We consider here a singular solution, in order to show that the estimator detects the elements where the error is the largest. The domain under consideration is defined by $\Omega = L \times (0, 1)$, where L is the two dimensional L-shaped domain $L := (-1, 1)^2 \setminus ([0, 1] \times [-1, 0])$.

The corresponding mesh is composed of 227155 tetrahedra. We want to compute the magnetic flux density $B = \text{curl } A$ given by (3) with the following data (see Figure 4) :

$$\begin{cases} J_s &= (0, 0, 1)^T, \\ \Gamma_N &= \{(x, y, z)^\top \in \partial\Omega : z = 0, \text{ or } z = 1, \text{ or } y = 0\}, \\ \Gamma_T &= \partial\Omega \setminus \Gamma_N, \\ \mu &= 1. \end{cases}$$

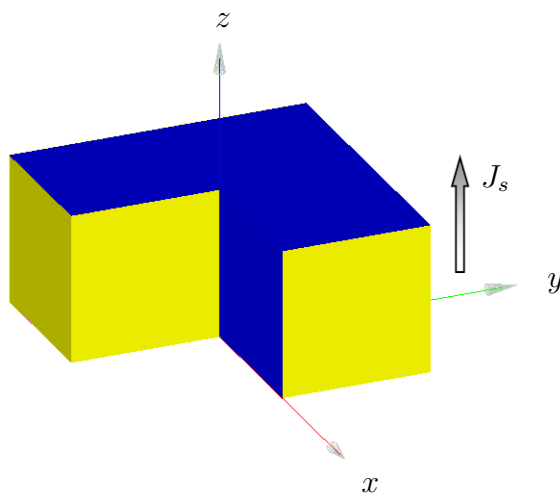


Figure 4: Domain configuration and eddy current J_s .

The exact solution is not known analytically but has an edge singularity along the edge $x = y = 0$. Figure 5 shows that the estimator picks up correctly this singular behavior. Consequently, the efficiency is numerically confirmed (see Theorem 3.2) and a mesh-refinement strategy using this estimator could be successfully used in order to adapt the mesh in the vicinity of the singularity.

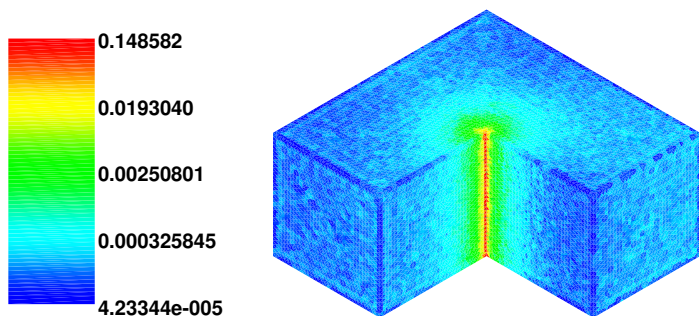


Figure 5: Local estimator map.

Conclusion

In this paper, we were interested in the residual a posteriori error analysis of a magnetostatic problem given in its vectorial potential formulation in the context of a finite element approximation. Here, the model under consideration induces some mixed boundary conditions, what constitutes the originality of the present work. The key point needed in the proof of the reliability of this estimator is a new Helmholtz decomposition of vector fields, which has been established and constitutes the core of this contribution. Consequently, the reliability can be achieved using the same kind of proof as for the case of homogeneous boundary conditions. Numerical tests are then proposed, for a regular as well as for a singular solution, in order to illustrate the obtained theoretical results.

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