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As many antipodes as vertices on convex polyhedra

Joël Rouyer Tewfik Sari

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Abstract

An earlier result states that, on the surface of a convex polyhedron with n vertices endowed with its intrinsic metric, a point cannot have more than n antipodes (farthest points). In this paper we produce examples of polyhedra with n vertices, on which some suitable point admits exactly n antipodes. We also proved that, for any positive number $d \leq 1$, there exist (in the closure of the set of these polyhedra) some convex surfaces on which some point have a set of antipodes of Hausdorff dimension d.

MSC (2000): 52B10, 53C45.

1 Introduction

Geographers define the antipode of some point p belonging to a sphere, or to a slightly flattened ellipsoid of revolution—namely the Earth—as the point diametrically opposite to p. It is also the farthest point from p. Mathematicians will prefer this latter definition which advantageously remains valid in any compact metric spaces, although antipodes are no longer necessarily unique.

Note that the equivalence between the above two definitions is far from obvious in the case of a flattened ellipsoid [13], and false for many surfaces, as a stretched ellipsoid of revolution [14], or the surface of a centrally symmetric polyhedron [11]. Indeed, the equivalence occurs so rarely that H. Steinhaus formulated the conjecture that the sphere is the only convex surface on which the antipodal map is a single-valued involution [7]. The conjecture was not disproved before C. Vîlcu's discovery of some family of counter-examples, including the case of a flattened ellipsoid of revolution [13].

We recall that a convex surface $\mathcal{S}$ is the boundary of some convex body (*i.e.* compact convex subset with nonempty interior) of $\mathbb{R}^3$ endowed with its so-called intrinsic metric: the distance between two points is the length of the shortest curve of $\mathcal{S}$ which joins them.

We said previously that a point may have more than one antipode. It is rather obvious if you consider a long cigar shaped surface. Points near an

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extremity surely have antipodes near the other one, and by the middle, some points must have at least one antipode near each extremity. However the set of points admitting more than one antipode is small [17]. This fact has been investigated beyond the frame of convex surfaces, though the notion of smallness depends on the studied case [8, 9].

A point may also have infinitely many antipodes. As an example, if you identify on a round sphere the points which have the same latitude, and longitudes equal modulo π, on the resulting surface (which is actually isometric to some convex one, by virtue of Alexandrov's gluing theorem [2, p. 315–20]) a point of the equator admits a whole meridian as set of antipodes.

Nevertheless the set of antipodes of some point p of a convex surface cannot be too big. Tudor Zamfirescu proved that its Hausdorff dimension is at most 1, and the quotient of its length (1-dimensional Hausdorff measure) and the distance between p and one of its antipodes is never more than π. Further, the set of antipodes of a point is always homeomorphic to some compact subset of the interval $[0, 1]$. No more could be said—topologically speaking—about it, for each compact subset of $[0, 1]$ can be realized as the set of antipodes of some suitable point of some suitable convex surface [15, 16].

Special attention was paid to the case of polyhedral surfaces. It is proved in [11]—among other facts—that no point on any polyhedron with n vertices can have more than n antipodes (Theorem 7). The aim of this article is to exhibit some families $\mathcal{F}_n$ of polyhedra with n vertices on which there exists a point which admits n antipodes, thereby proving that the upper bound given in [11] is the best one. Apart from an invocation of Theorem 0, due to A. D. Alexandrov, the construction will involve only very elementary mathematics.

In order to construct one of these polyhedra, we follow the modeler's way and begin to sketch its development on a proper cardboard (Section 2), and then, to cut it out, to fold it, and to glue it (Section 3). Section 4 is devoted to the proof that the obtained solid enjoys the desired property. Section 5 briefly discusses a way we can extend the described family. Section 6 illustrates and supplements the results of T. Zamfirescu. We prove that the family $\bigcup_n \mathcal{F}_n$ admits in its boundary some convex surface with quite a long set of antipodes, as well as surfaces on which some point admits a fractal set of antipodes. We also prove that the Hausdorff dimension of a set of antipodes on convex surfaces can take any real value between 0 and 1.

As a matter of notation, the (non-oriented) angle with vertex B and sides through A and C is denoted by $\widehat{ABC}$, as well as its measure. The parenthesis (AB) stands for the line through A and B, the bracket $[AB]$ for the line segment between A and B. The (Euclidean) distance between A and B will be denoted by AB or $\|A - B\|$. We reserve the notation $d(\mathcal{A}, \mathcal{B})$ for the (Euclidean) distance between sets, *i.e.* the $d(\mathcal{A}, \mathcal{B}) = \inf_{A \in \mathcal{A}, B \in \mathcal{B}} AB$. The intrinsic distance between two points a, b on a convex surface is denoted by $\rho(a, b)$. At last, the length of a rectifiable curve γ is denoted by $L(\gamma)$.

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2 Development

Consider some unit circle $\mathcal{C}$ in the plane $\mathbb{R}^2$. Let O be its center. Let A, P, C_N,C_{N-1}, ...,C_0, be $N+3$ points ($N \geq 1$) lying in this order on one open half of $\mathcal{C}$, see Figure 1. The development of our polyhedron will be entirely defined by the positions of these points, that is, by the $N+2$ positives numbers

$$\begin{aligned}\alpha &= \widehat{AOP}\\ \beta &= \widehat{POC_N}\\ \gamma_i &= \widehat{C_iOC_{i-1}} \quad (i = 1, \cdots, N)\end{aligned}$$

such that $\delta \stackrel{def}{=} \pi - (\alpha + \beta + \gamma_1 + ... + \gamma_N) > 0$. Let γ be the sum of γ_i. The reflection with respect to (OC_0) maps C_i on C'_i ($1 \leq i \leq N$) and P on P'. The reflection with respect to (OA) maps P on P''. Let C''_N be the second (*i.e.* distinct from C_N) point of intersection between (C_NP'') and the circle of radius AC_N centered at A. Let B be the intersection point of the mediators of $[P'P'']$ and $[C'_NC''_N]$. We have the following result

Proposition 1. *The polygon*

$$\mathfrak{P}_1 = APC_NC_{N-1}\dots C_1C_0C'_1\dots C'_NBC''_N$$

together with the following set of rules $\mathfrak{R}_1$

- *glue* (C_0, C_1) *on* (C_0, C'_1)
- *glue* (C_i, C_{i+1}) *on* $\left(C'_i, C'_{i+1}\right)$, $1 \leq i \leq N-1$
- *glue* (A, C_N) *on* (A, C''_N)
- *glue* (B, C'_N) *on* (B, C''_N)

is the development of a non-degenerated polyhedron.

The proof of this result is postponed to Section 3, where the notion of development will be defined precisely. The polygon $\mathfrak{P}_1$ is shown in Figure 1. Before folding it into a polyhedron, we prove for further use the following three lemmas.

Lemma 1. *Triangles* $AP''C''_N$ *and* APC_N *are congruent.*

Proof. Obviously, we have

$$AP = AP'', \qquad AC_N = AC''_N.$$

Since the triangle $AC_NC''_N$ is isosceles, we have

$$\widehat{AC''_NP''} = \widehat{AC_NP''} = \frac{\alpha}{2} = \widehat{AC_NP}.$$

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Moreover

$$\widehat{APC_N} = \pi - \frac{\widehat{AOC_N}}{2} = \pi - \frac{\alpha+\beta}{2},$$

and

$$\widehat{AP''C''_N} = \pi - \widehat{AP''C_N} = \pi - \frac{\alpha+\beta}{2} = \widehat{APC_N}.$$

Hence the triangles are congruent. □

Lemma 2. *The point B lies inside the circle.*

Proof. We will prove this by using complex numbers. In order to limit the amount of symbols, we identify the plane with $\mathbb{C}$ and use the same letter to designate both a point and its affix. We assume—without loss of generality—that $\mathcal{C}$ is the unit circle and that $A = 1$. It follows that

$$P'' = \frac{1}{P}, \qquad C''_N = 1 + \frac{1-C_N}{P}, \qquad C'_N = \frac{C_0^2}{C_N}, \qquad P' = \frac{C_0^2}{P}.$$

The point B lies on the mediator of the segment $[P'P'']$, whose equation is

$$P\left(1 - \frac{1}{C_0^2}\right) z + \frac{1}{P}\left(1 - C_0^2\right)\bar{z} = 0.$$

It also lies on the mediator of $[C'_N C''_N]$, an equation of which is :

$$\left(\frac{C_N}{C_0^2} - 1 - P\frac{C_N - 1}{C_N}\right) z + \left(\frac{C_0^2}{C_N} - 1 - \frac{1-C_N}{P}\right)\bar{z} \\ = 1 - \left(1 + \frac{1-C_N}{P}\right)\left(1 + P\frac{C_N-1}{C_N}\right).$$

Solving this linear system we get

$$B = \frac{C_0^2\left(C_N - 1\right)}{P\left(C_N - C_0^2\right)}. \tag{1}$$

It follows that B lies inside $\mathcal{C}$ if and only if $|C_N - A| < \left|C_N - C_0^2\right|$, which is obvious since the mediator of $\left[C_0^2 A\right]$ is actually (OC_0). □

Lemma 3. *The point O lies inside the triangle AC_0B.*

Proof. We prove the lemma by showing that O can lie on none of the lines (AB), (AC_0) or (BC_0). So, the point O has to be either always inside or always outside the triangle. It is easy to see, for some simple particular values (*e.g.* $\alpha = \beta = \gamma = \frac{\pi}{2}$), that O actually lies inside. Since $\alpha + \beta + \gamma < \pi$, obviously $O \notin (AC_0)$. Assume that $O \in (AB)$. Since B belongs to the mediator of $[P'P'']$ which cuts (OA) at O, necessarily $B = 0$, in contradiction with (1). Assume now that $O \in (BC_0)$, *i.e.*

$$\frac{B}{C_0} = \frac{C_0\left(C_N - 1\right)}{P\left(C_N - C_0^2\right)} \in \mathbb{R}.$$

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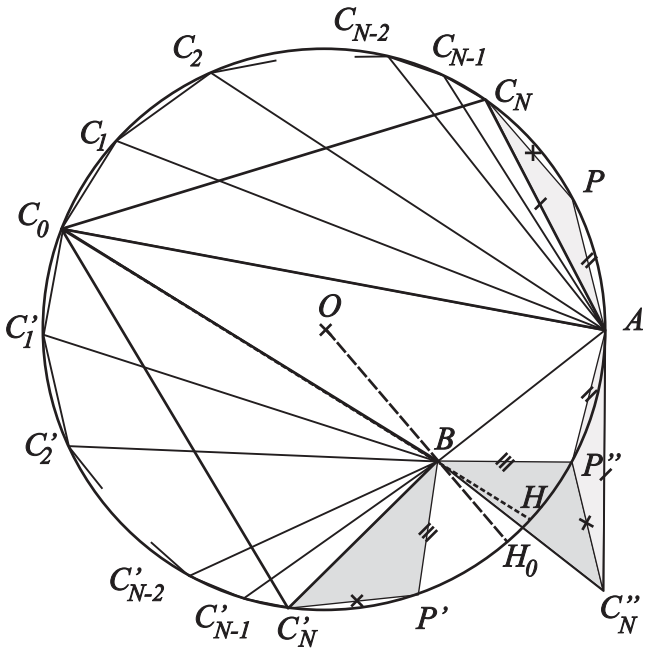

Figure 1: The development of the polyhedron.

A simple calculation shows that

$$\frac{B}{C_0} - \overline{\left(\frac{B}{C_0}\right)} = \frac{C_0\left(C_N - 1\right)\left(1 - P^2\right)}{P\left(C_N - C_0^2\right)} \neq 0,$$

and another contradiction is found. □

3 From development to polyhedron

The fact that the above development will fold into a convex polyhedron follows from the (polyhedral version) of Alexandrov's gluing theorem [3]. In order to state it, we need to precise some points of vocabulary. A development $(\mathfrak{P}, \mathfrak{R})$ will be the data of one or several (planar) disjoint polygons, the union of which is denoted by $\mathfrak{P}$, together with a set $\mathfrak{R}$ of *gluing rules*. A gluing rule says which edge should be glued onto which other, in which direction. Hence a gluing rule is given by a pair of ordered pairs of vertices of the polygons. Of course, gluing rules cannot be chosen freely and have to satisfy some axioms, namely

- Each ordered pair involved in a gluing rule is the pair of endpoints of some edge of $\mathfrak{P}$.

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- Two edges glued together must have the same length.
- Each edge is glued to exactly one other edge.

It is clear that the polygon $\mathfrak{P}_1$ of the preceding section together with the set $\mathfrak{R}_1$ of gluing rules satisfies these axioms, so it is a development.

To any development, we can associate a metric space in the following way. The set of rules $\mathfrak{R}$ induces an equivalence relation, also denoted by $\mathfrak{R}$, in a natural way: $\mathfrak{R}$ is the smallest relation of equivalence such that, for any gluing rule of the form "glue (X, Y) on (X', Y')" and any $\lambda \in [0, 1]$ the points $\lambda X + (1 - \lambda) Y$ and $\lambda X' + (1 - \lambda) Y'$ are equivalent.

Denote by Π the canonical surjection from $\mathfrak{P}$ to $\mathfrak{P}/\mathfrak{R}$. It is clear that a point which is not a vertex is equivalent to exactly one other point if it lies on an edge of the development, and to no other point if it belongs to the interior of $\mathfrak{P}$. A vertex is equivalent to an arbitrary number of points, which are clearly vertices too. A vertex of $\mathfrak{P}/\mathfrak{R}$ is by definition the equivalence class of some vertex of $\mathfrak{P}$. The *curvature* ω_x of some vertex $x = \{X_1, \ldots, X_k\} \in \mathfrak{P}/\mathfrak{R}$ is defined as 2π minus the sum of the angles of $\mathfrak{P}$, measured toward $\mathfrak{P}$, at points X_i. If all but a finite number of points of some curve in the topological quotient $\mathfrak{P}/\mathfrak{R}$ lie in the image under Π of the interior of $\mathfrak{P}$, then its (possibly infinite) length is well defined in a natural way. The distance between two points $x, y \in \mathfrak{P}/\mathfrak{R}$ will be by definition the infimum of the set of the lengths of those curves joining x to y.

A development $(\mathfrak{P}', \mathfrak{R}')$ is said to be obtained from a development $(\mathfrak{P}, \mathfrak{R})$ by *cutting and gluing*, if one can 'cut' $\mathfrak{P}$ into finitely many polygons and move (apply an affine isometry of the plane) the obtained tiles in order to rebuild $\mathfrak{P}'$, such that any two coinciding edges of two moved tiles were glued by some rule in $\mathfrak{R}$. Of course $\mathfrak{R}'$ is derived from $\mathfrak{R}$ and the partitions of $\mathfrak{P}$ and $\mathfrak{P}'$ in a natural way, namely the rules of $\mathfrak{R}'$ are the rules of $\mathfrak{R}$, save those which correspond to some coinciding edges of moved tiles, together with the new rules that identify edges of the tiles which have been separated by cutting and moving.

Remark 1. If $(\mathfrak{P}', \mathfrak{R}')$ is obtained from $(\mathfrak{P}, \mathfrak{R})$ by cutting and gluing, then the quotients $\mathfrak{P}'/\mathfrak{R}'$ and $\mathfrak{P}/\mathfrak{R}$ are isometric [3, p. 51].

Consider the disjoint union of two isometric convex polygons $X_0X_2 \ldots X_k$ and $Y_0Y_2 \ldots Y_k$, labeled in such a way that the isometry maps X_i on Y_i, and the $k + 1$ gluing rules "glue (X_i, X_{i+1}) on (Y_i, Y_{i+1})" (the index i is taken modulo $k + 1$). The resulting metric space is called the *double* of $X_0X_2 \ldots X_k$. It is not isometric to any convex surface. However it is easy to see that it is the limit of a sequence of prisms with fixed base $X_0X_2 \ldots X_k$ and heights tending to zero. Such a metric space is called a *degenerated convex polyhedron*. Now we are in a position to state the polyhedral version of Alexandrov's gluing theorem [3].

Theorem 0. *Let $(\mathfrak{P}, \mathfrak{R})$ be a development such that*

1. *$\mathfrak{P}/\mathfrak{R}$ is homeomorphic to the 2-dimensional sphere.*
2. *Each vertex of $\mathfrak{P}/\mathfrak{R}$ has non-negative curvature.*

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Then, $\mathfrak{P}/\mathfrak{R}$ is isometric to the surface of a (possibly degenerated) convex polyhedron. Moreover the polyhedron is unique up to isometry.

Note that, with the above meaning of vertices, some vertices of $\mathfrak{P}/\mathfrak{R}$ (precisely those having a zero curvature) are not vertices of the resulting polyhedron.

Now, let us return to the proof of Proposition 1. We denote with a lowercase letter the vertex of $\mathfrak{P}_1/\mathfrak{R}_1$ corresponding to the vertices of $\mathfrak{P}_1$ which are denoted by the same uppercase letter(s). So $c_N = \{C_N, C'_N, C''_N\}$, $b = \{B\}$, $c_i = \{C_i, C'_i\}$, $1 \le i < N$, etc. It is quite clear that $\mathfrak{P}_1/\mathfrak{R}_1$ is homeomorphic to the sphere: one can easily compute the Euler characteristic in order to obtain a numerical argument.

It is also clear that the curvatures at points a, b and c_0 are positive, since these equivalence classes are singletons. Let us compute

$$\begin{aligned}
\omega_{c_N} &= 2\pi - \widehat{AC_NC_{N-1}} - \widehat{BC'_NC'_{N-1}} - \widehat{AC''_NB} \\
&= 2\pi - \widehat{AC_NC_{N-1}} - \widehat{BC'_NC'_{N-1}} - \widehat{AC''_NP''} - \widehat{P''C''_NB} \\
&= 2\pi - \widehat{AC_NC_{N-1}} - \widehat{BC'_NC'_{N-1}} - \widehat{AC_NP} - \widehat{BC'_NP'} \\
&= 2\pi - \widehat{PC_NC_{N-1}} - \widehat{P'C'_NC'_{N-1}} \\
&= \beta + \gamma_N > 0
\end{aligned}$$

and, for $i = 1, \dots, N-1$,

$$\omega_{c_i} = 2\pi - 2\widehat{C_{i-1}C_iC_{i+1}} = \widehat{C_{i+1}OC_{i-1}} = \gamma_{i+1} + \gamma_i > 0.$$

Hence Theorem 0 applies, and $\mathfrak{P}_1/\mathfrak{R}_1$ is isometric to the surface of some possibly degenerated convex polyhedron $\mathcal{P}$. Moreover, the curvatures of a, b, c_0, …, c_N are positive, whence $\mathcal{P}$ has exactly $N+3$ vertices.

Assume now that $\mathcal{P}$ is degenerated. So is it of the tetrahedron abc_0c_N, which is obviously included (in the sense of inclusion of subsets of the 3-dimensional Euclidean space) in $\mathcal{P}$. In particular, the solid angle at vertex c_0 should vanish, that is, one of the following statements holds:

$$\widehat{C_NC_0A} + \widehat{BC_0C'_N} = \widehat{AC_0B} \tag{2}$$

$$\widehat{BC_0C'_N} + \widehat{AC_0B} = \widehat{C_NC_0A} \tag{3}$$

$$\widehat{C_NC_0A} + \widehat{AC_0B} = \widehat{BC_0C'_N}. \tag{4}$$

Let H be the other point of intersection of (C_0B) and $\mathcal{C}$, and H_0 the midpoint of the circle-arc $P''P'$. Since O lies inside the triangle ABC_0 (Lemma 3), H lies on the circle arc H_0A, whence

$$\zeta \stackrel{def}{=} \widehat{AOH} < \alpha + \delta.$$

Note that

$$\widehat{C_NC_0A} = \frac{\alpha+\beta}{2}, \qquad \widehat{BC_0C'_N} = \frac{2\pi - 2\gamma - \alpha - \beta - \zeta}{2}, \qquad \widehat{AC_0B} = \frac{\zeta}{2}.$$

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It follows that

$$\widehat{C_N C_0 A} + \widehat{B C_0 C'_N} - \widehat{A C_0 B} = \pi - \gamma - \zeta > \beta > 0,$$

whence (2) never holds. If (3) held, then (AC_0) would be the bisector of $\widehat{C_N C_0 C'_n}$, *i.e.* $(AC_0) = (OC_0)$, which is obviously impossible. In the same way, (4) would infer $O \in (BC_0)$, which is impossible by virtue of Lemma 3. It follows that the polyhedron $\mathcal{P}$ is not degenerated. This ends the proof of Proposition 1.

4 As many antipodes as vertices

The main result is now almost at hand. It remains to prove that $a, p, c_0, \ldots, c_N$ are actually antipodes of o. We need for this purpose the following

Lemma 4. [6, (13.10) and (13.11)] *On a convex surface, if two shortest paths have a common segment, then one of is incuded in the other. It follows that, if σ is a shortest path from x to y, and z is a point interior to σ, then the portion of σ delimitated by x and z is the only shortest path from x to z.*

Remark 2. The above lemma can be stated in a quite more general frame. It is indeed a basic property of Alexandrov spaces with curvature bounded below (see for instance [5] or [12]).

Theorem 1. *The point $o \in \mathcal{P}$ has exactly $N+3$ antipodes, namely $a, c_0, \ldots, c_N$, and p.*

Proof. Our first claim is that the distance between o and any point of $\mathcal{P}$ is never greater than 1. Consider the polygon

$$\mathfrak{P}_2 = C_0 C_1 \ldots C_N P A P'' B P' C'_N C'_{N-1} \ldots C'_2 C'_1$$

and the following set $\mathfrak{R}_2$ of gluing rules

- Glue (C_0, C_1) on (C_0, C'_1)
- Glue (C_i, C_{i+1}) on $\left(C'_i, C'_{i+1}\right)$, $1 \leq i < N$
- Glue (A, P) on (A, P'')
- Glue (B, P'') on (B, P')
- Glue (P', C'_N) on (P, C_N).

Since the triangles APC_N and $AP''C''_N$, and the triangles $BP''C''_N$ and $BP'C'_N$ are congruent, $(\mathfrak{P}_2, \mathfrak{R}_2)$ is obtained from $(\mathfrak{P}_1, \mathfrak{R}_1)$ by cutting (along $[P''A] \cup [P''B] \cup [P''C'']$) and gluing $((C''_N, A)$ on (C_N, A) and (C''_N, B) on $(C'_N, B))$, whence they are developments of the same polyhedron $\mathcal{P}$ (see Remark 1). Since $\mathfrak{P}_2$ is included in the unit disc, the distance from o cannot exceed 1.

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Our second claim is that any shortest path σ between o and any other point of $\mathcal{P}$ corresponds to a line segment of $\mathfrak{P}_2$. In other words, such a path cannot cross

$$\Gamma \stackrel{def}{=} [ap] \cup [bp] \cup [c_n p] \cup \bigcup_{i=0}^{N-1} [c_i c_{i+1}] .$$

Assume on the contrary that σ is crossing Γ and denote by $y \in \mathcal{P} = \mathfrak{P}_2/\mathfrak{R}_2$ the first (counted from o) crossing point. As a minimizing geodesic, σ cannot pass through any vertex (see for instance [6, (12.4)]), so y should lie in the relative interior of one of the segments $[ap]$, $[bp]$, $[c_n p]$, $[c_i c_{i+1}]$. Hence y is a pair $\{Y, Y'\}$ of points of $\mathfrak{P}_2$, and the part of σ between o and y is unfolded into some line segment $[OY]$ or $[OY']$, say $[OY]$. Note that Y' is the point symmetric to Y with respect to (OA) (if $y \in [ap]$), to (OH_0) (if $y \in [bp]$) or to (OC_0) (in the other cases), whence $OY = OY'$. The path of $\mathcal{P}$ corresponding to $[OY']$ is also a shortest path between o and y, in contradiction with Lemma 4. This prove our second claim.

It follows that, for any $x \in \mathcal{P} = \mathfrak{P}_2/\mathfrak{R}_2$ and any $X \in x$, $\rho(o, x)$ equals OX, whence $\rho(o, p) = \rho(o, c_i) = \rho(o, a) = 1$. □

Remark 3. The curve Γ is the cut-locus of o.

In order to conclude the section, we will examine what happens when one or several parameters $\alpha, \beta, \gamma_1, ..., \gamma_N, \delta$ vanish.

First, there is almost nothing to say when one or several (but not all) γ_i vanish: this case reduce to a lower N case. If all γ_i vanish, then $B \in \mathcal{C}$ and the resulting $\mathcal{P}$ is nothing but the double of the acute triangle ABC_0. The point p is the orthocentre of the face abc_0 which does not contain o. The point o has 4 antipodes (a, b, c_0, and p), while $\mathcal{P}$ has only three vertices. However, since $\mathcal{P}$ is degenerated, this is not in contradiction with the main result of [11]. If $\beta = 0$, then $p = c_N$. The $(N+3)$-vertex polyhedron does not degenerate, but o has only $N+2$ antipodes. If $\alpha = 0$ (*i.e.* $A = P = P''$) then O belongs to $(C_0 B)$ and $\mathcal{P}$ is the double of the $(N+3)$-gon $ABC_0 \dots C_N$. At last if $\delta = 0$, then $B = P' = P''$, O lies on (OC_0) and $\mathcal{P}$ is the double of $AC_0 \dots C_N P$.

5 A larger family of tetrahedra

In the above examples, all antipodes of o but one are vertices. Moreover, the only non-vertex antipode of o is joined to o by three distinct shortest paths. Whether these properties are enjoyed by all polyhedra with n vertices admitting a point with n antipodes is a natural question. Some earlier works give a partial answer. The fact that a non-vertex antipode of any point on any polyhedron is always joined by at least three minimizing geodesics is indeed one of the most basic results on this subject [10]. It is also proved in [11] that for such a polyhedron, at least two of the antipodes of the distinguished point should be very acute (*i.e.* with curvature more than π) vertices. This section is devoted to showing that, in the case of the tetrahedron, the two other points can be non-vertex points.

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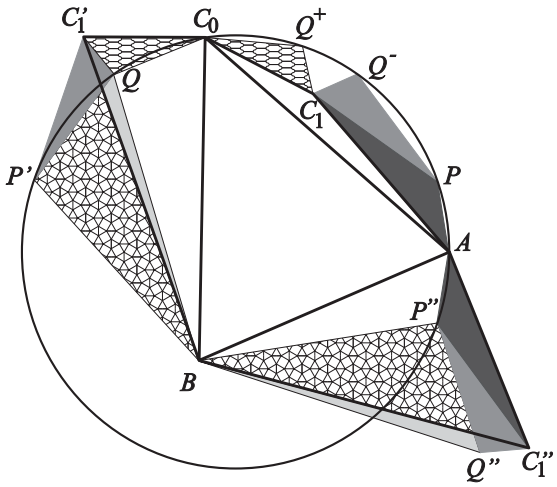

Figure 2: The development of the tetrahedron. Triangles filled with the same gray or pattern are isometric.

In this section, we assume $N = 1$. The main idea is to give a deformation $\mathcal{P}(\varepsilon)$ of the preceding example $\mathcal{P} = \mathcal{P}(0)$ in such a way that c_1 is no more an antipode (*i.e.* C_1 lies inside the circle). As explained before, the point q which will become an antipode of o instead of c_1 must be joined to o by (at least) three shortest paths. As, on $\mathcal{P}(0)$, there were only two shortest paths between o and c_1, one of them has to split. Consider a point $q \in \mathcal{P}(\varepsilon)$ somewhere beyond c_1, looking from o. If q is near enough to c_1 then a shortest path σ between o and q should lie in a neighborhood of the only shortest path between o and c_1. By moving q on a right-left (still from o point of view) degree of liberty, we can ensure that σ lies either on the right side, or on the left side of c_1. Since a shortest path never passes through a vertex [6, (12.4)], there is a right-left position such that two shortest paths, one on each side, exist. Now, if the tetrahedron is not too deformed, by moving q backward, we will obtain a third shortest path, close to the second one between o and c_1 on $\mathcal{P}(0)$. It is not clear, that the deformation could be done in a way that satisfies the equation $\rho(o, q) = 1$. However the space of tetrahedra, up to affine isometries is 6-dimensional, and the fact that A, C_0, P, P' and P'' all lie on a unit circle is described by three equations. Hence it is natural to conjecture that $\mathcal{P}$ can be deformed without breaking the fact that a, p, c_0 are antipodes of o with three degrees of liberty. If we ask moreover that the deformation satisfies $\rho(o, q) = 1$, there still remain two degrees of liberty. One of them corresponds to the variation of γ; it remains one parameter ε. This heuristic description is far beyond a proof, hence we will present a more formal construction.

Choose three positive numbers α, β, γ such that $\alpha + \beta + \gamma < \pi$. For $\varepsilon > 0$

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small enough, consider the points

$$\begin{aligned} A &= (1,0) \\ C_0 &= (\cos(\alpha+\beta+\gamma), \sin(\alpha+\beta+\gamma)) \\ P &= (\cos\alpha, \sin\alpha) \\ Q^+ &= (\cos(\alpha+\beta+\varepsilon), \sin(\alpha+\beta+\varepsilon)) \\ Q^- &= (\cos(\alpha+\beta-\varepsilon), \sin(\alpha+\beta-\varepsilon)), \end{aligned}$$

all lying on the unit circle $\mathcal{C}$. Let Q be the point symmetric to Q^+ with respect to (OC_0). Let P' be the point of $\mathcal{C}$ such that $QP' = Q^-P$ and C_0, Q, P' lie on $\mathcal{C}$ in the direct order. Let R_1 be the rotation which maps the ordered pair (Q^-, P) on (Q, P'). Let R_2 be the rotation of center C_0 which maps Q^+ on Q. The images under R_1 and R_2 of the mediator of $[Q^+Q^-]$ intersect at C_1'; put $C_1 = R_2^{-1}(C_1')$. Let P'' be the point symmetric to P with respect to (OA). The rotation of center A which maps P on P'' maps C_1 on C_1''. The mediators of $[P'P'']$ and $[C_1'C_1'']$ intersect at B.

Figure 2 illustrates the above construction. We let the reader check that the polygon

$$\mathfrak{P}_1 = AC_1''BC_1'C_0C_1$$

together with the following set of rules $\mathfrak{R}_1$

- glue (C_0, C_1) on (C_0, C_1')
- glue (A, C_1) on (A, C_1'')
- glue (B, C_1') on (B, C_1'')

is (for small ε) the development of a tetrahedron with vertices a, b, c_0, c_1. For $\varepsilon = 0$, we get the tetrahedron described in the preceding sections.

Consider the polygon

$$\mathfrak{P}_2 = AC_1''Q''BQC_1'C_0C_1$$

together with the following set of rules $\mathfrak{R}_2$

- glue (C_0, C_1) on (C_0, C_1')
- glue (A, C_1) on (A, C_1'')
- glue (Q, C_1') on (Q'', C_1'')
- glue (B, Q) on (B, Q'').

Since the triangles BQC_1' and $BQ''C_1''$ are congruent, it is clear that $(\mathfrak{P}_2, \mathfrak{R}_2)$ is obtained from $(\mathfrak{P}_1, \mathfrak{R}_1)$ by cutting and gluing and so, is development of the same tetrahedron (see Remark 1).

Consider now the polygon

$$\mathfrak{P}_3 = AP''BP'QC_0Q^+C_1Q^-P$$

together with the following set of rules $\mathfrak{R}_3$

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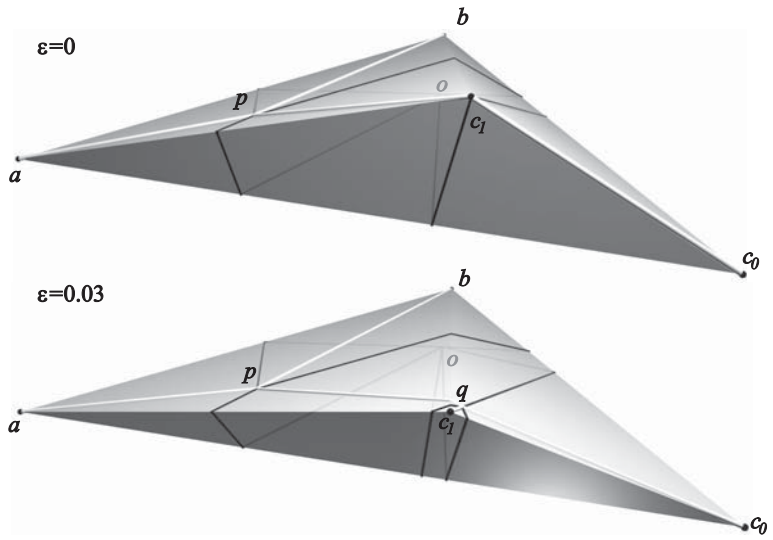

Figure 3: The tetrahedra $\mathcal{P}(0)$ and $\mathcal{P}(0.03)$.

- glue (C_0, Q) on (C_0, Q^+)
- glue (C_1, Q^+) on (C_1, Q^-)
- glue (Q, P') on (Q^-, P)
- glue (A, P) on (A, P'')
- glue (B, P') on (B, P'').

Since the triangles BQP' and $BQ''P''$, the triangles $C_0C_1'Q$ and $C_0C_1Q^+$, the triangles C_1PQ^- and $C_1''P''Q''$ and the triangles C_1AP and $C_1''AP''$ are congruent, it is clear that $(\mathfrak{P}_3, \mathfrak{R}_3)$ is obtained from $(\mathfrak{P}_2, \mathfrak{R}_2)$ by cutting and gluing, and so, is a development of the same tetrahedron (see Remark 1). Since $\mathfrak{P}_3$ is included in the unit disc, the distance from o cannot exceed 1. Thus, the antipodes of o are a, c_0, p and q. The non-vertex antipodes of o are both joined to o by three shortest paths.

Figure 3 represents $\mathcal{P}(0)$ and $\mathcal{P}(0.03)$ for $\alpha = \beta = \gamma = \frac{\pi}{6}$. The white thick line is the set of those points which are joined to o by at least two shortest paths, and the black lines are the shortest paths from o to p and from o to q.

6 Fractal sets of antipodes

The section is devoted to the study of limit cases, when N goes to infinity. The goal of the section is to prove the following

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Theorem 2. *Let $\mathcal{H}$ be an open half-circle. Let K be a compact subset of $\mathcal{H}$ and U a point which does not belong to K. Then there exists a convex surface $\mathcal{S}$ and a point $o \in \mathcal{S}$ such that the set of antipodes of o in $\mathcal{S}$, endowed with the intrinsic distance of $\mathcal{S}$, is locally isometric to $K \cup \{U\}$, endowed with the Euclidean distance of $\mathbb{R}^2$.*

The main idea is easy: given a compact subset K of a half circle, we can choose a sequence of finite sets approaching it. For each finite set F, the preceding section provides a polyhedron $\mathcal{P}_F$ admitting F as a set of antipodes, and $\mathcal{S}$ will be the limit of these polyhedra. However, comparatively to the preceding sections, this proof is somewhat long and technical, and requires not less than ten lemmas.

First, we recall the elementary tools we will have to use. The Hausdorff distance between two compact subsets K_1,$K_2 \subset \mathbb{R}^d$ is the smallest number r such that each point of K_1 is included in a closed ball of radius r centered at some point of K_2, and conversely. The Hausdorff distance is actually a distance on the set of non-empty compact subsets of $\mathbb{R}^d$. We denote it by $h(\cdot,\cdot)$. From now on, convergence of a sequence of compacts is understood with respect to the Hausdorff distance. We will use some well-known facts about this distance, namely that each bounded sequence of compacts in $\mathbb{R}^d$ admits a converging subsequence, that the finite subsets are dense in the set of all non-empty compact subsets, and that the set of convex compact subsets is closed in the space of all nonempty compact subsets. The latter fact admits as a corollary that the limit of a converging sequence of convex surfaces (embedded in $\mathbb{R}^3$) is either a convex surface, or a convex body of dimension less than 3. The following lemma [6, p. 81] states that, if the limit is a convex surface, then the intrinsic distances also converge.

Lemma 5. *Let $\mathcal{S}_n$ be a sequence of convex surfaces converging to a convex surface $\mathcal{S}$. Denote by ρ_n (respectively ρ) the intrinsic distance on $\mathcal{S}_n$ (respectively $\mathcal{S}$). Let x_n, y_n be points of $\mathcal{S}_n$ such that the sequences $(x_n)_n$ and $(y_n)_n$ are converging respectively to x and y in $\mathbb{R}^3$. Then x, $y \in \mathcal{S}$ and $\rho_n(x_n, y_n)$ tends to $\rho(x, y)$.*

Concerning the notions of Hausdorff measure and Hausdorff dimension, we refer (for instance) to [4]. We recall that the d-dimensional Hausdorff measure is preserved by isometries, and that the 1-dimensional measure of a simple curve coincides with its length. We also recall that the Hausdorff dimension is preserved by any bi-Lipschitz map. Since the intrinsic distance of a convex surface and the restriction of the Euclidean distance of $\mathbb{R}^3$ are equivalent (see [6, p. 78]), the Hausdorff dimension of a subset of convex surface does not depend on the considered distance.

Denote by $\mathcal{C}_u^v$ the circle arc $\{(\cos t, \sin t)\,|u < l < v\}$. We fix two positive numbers α, δ such that $\alpha + \delta < \pi$. To any finite subset F of $\mathcal{C}_\alpha^{\pi-\delta}$, we associate the development $(\mathfrak{P}_F, \mathfrak{R}_F)$, corresponding to the development $(\mathfrak{P}_2, \mathfrak{R}_2)$ of Section 4, where $N = card(F)$, $A = (1, 0)$, $P = (\cos\alpha, \sin\alpha)$, $C_0 = (-\cos\delta, \sin\delta)$

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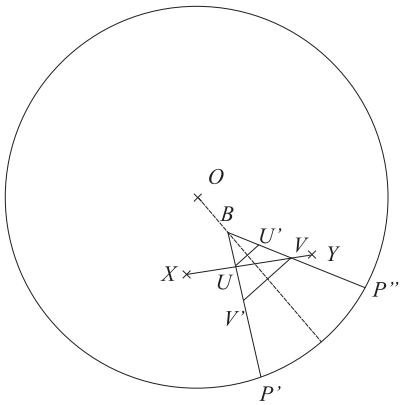

Figure 4: Definition of U, V, U' and V' in Lemma 6.

and $C_1, \ldots, C_N$ are the points of F, numbered from left to right. We put $\beta \overset{def}{=} \widehat{POC_N}$ and $\gamma \overset{def}{=} \widehat{C_N OC_0}$. The development $(\mathfrak{P}_F, \mathfrak{R}_F)$ folds into some polyhedron $\mathcal{P}_F$, uniquely defined up to isometry. If we assume moreover that $o = (0,0,0)$, $a = (1,0,0)$, $b = (x_b, y_b, 0)$ with $y_b > 0$ and that the third coordinate of c_N is positive, then $\mathcal{P}_F$ is uniquely defined as a subset of $\mathbb{R}^3$. This allows us to define the folding map $\Phi_F : \mathfrak{P}_F \to \mathbb{R}^3$. For further use we prove the

Lemma 6. *Let X, Y be two points of $\mathfrak{P}_F$. The distance on $\mathcal{P}_F$ between $\Phi_F(X)$ and $\Phi_F(Y)$ is not greater than XY.*

Proof. If the segment $[XY]$ lies in $\mathfrak{P}_F$, then $\Phi_F([XY])$ is a path between $\Phi_F(X)$ and $\Phi_F(Y)$, and the result holds. Assume now that $[XY]$ is crossing the segments $[BP']$ and $[BP'']$. Denote by U,V, U' and V' the points of intersection and their images under the reflection with respect to (OB), labeled as on Figure 4. We claim that $\Phi_F([XU]) \cup \Phi_F([UV']) \cup \Phi_F([VY])$ is a path shorter than $[XY]$. Indeed, it is sufficient to prove that $UV > UV'$. This is so because in the triangle UVV',

$$\widehat{UV'V} = \widehat{U'VV'} > \widehat{UVV'}.$$

□

Lemma 7. *Let $\gamma : [t_0, t_1] \to \mathbb{R}^2$ be a simple parametrization of a polygonal line inscribed in a half circle. Let $\tau \in [t_0, t_1[$. The function $f : [\tau, t_1] \to \mathbb{R}$ defined by $f(t) = \|\gamma(t) - \gamma(\tau)\|$ is increasing.*

Proof. To prove the result, it is sufficient to notice that, on the one hand, the restriction of f to any subinterval of $[t_0, t_1]$ whose image is a line segment is monotone, and on the other hand, its restriction to the inverse image of the set of vertices is increasing. The details are left to the reader. □

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Lemma 8. *Let* $\mathcal{L} = [PC_N] \cup \bigcup_{k=1}^{N} [C_{k-1}C_k] \subset \mathfrak{P}_F$. *There exists a positive number* r *depending only on* α, δ *such that, for all* $X \in \mathcal{L}$, *the restriction of* Φ_F *to the intersection of* $\mathcal{L}$ *with a ball of radius* r *centered at* X *is an isometry with respect to the intrinsic distance of* $\mathcal{P}_F$.

Proof. We denote by S the symmetry with respect to (OC_0). Let $\mathcal{H}_1$ be the convex hull of $\{O\} \cup \mathcal{C}^0_{-\alpha-\delta}$, $\mathcal{H}_2$ the one of $\mathcal{C}^{\pi-\delta}_{\alpha}$ and $\mathcal{H}_3$ the one of $\mathcal{C}^{2\pi-2\delta-\alpha}_{\pi-\delta}$. Let $r = d(\mathcal{H}_1, \mathcal{H}_2 \cup \mathcal{H}_3)$; r is clearly not greater than the distance between a point of $[AP''] \cup [P''B]$ and a point of $\mathcal{L} \cup S(\mathcal{L})$.

Assume that $XY < r$ and that the abscissa of X is less than the one of Y (otherwise exchange them). Put $X' = S(X)$, $Y' = S(Y)$, $x = \Phi_F(X)$ and $y = \Phi_F(Y)$. Let σ be a minimizing geodesic of $\mathcal{P}_F$ between x and y. If σ does not cross $\mathcal{M} \overset{def}{=} \Phi_F(\mathcal{L}) \cup [bp] \cup [ap]$, then there exist two points $X^* \in \{X, X'\}$ and $Y^* \in \{Y, Y'\}$ such that $\sigma = \Phi_F([X^*Y^*])$. Since $\Phi_F([XY])$ is a path from x to y of length $XY = X'Y' < XY' = X'Y$ we have either $(X^*, Y^*) = (X, Y)$ or $(X^*, Y^*) = (X', Y')$, and so $\rho(x, y) = XY$.

Assume now that σ intersects $\mathcal{M}$ and let u be the first (from x) point of intersection of σ with $\mathcal{M}$. Let σ_u be the part of σ delimitated by x and u. Obviously, σ_u is the image under Φ_F of some segment $[X^*U^*]$, with $X^* \in \{X, X'\}$ and $\Phi_F(U^*) = u$. We discuss three cases.

Case 1: $u \in \Phi_F(\mathcal{L})$. In this case $U^* \in \mathcal{L} \cup S(\mathcal{L})$ and $\Phi_F(S([X^*U^*]))$ is another shortest path from x to u, in contradiction with Lemma 4.

Case 2: $u \in [ap]$. Let $U \in [AP]$ and $U'' \in [AP'']$ be such that $\Phi_F^{-1}(u) = \{U, U''\}$. Assume first that $X^* = X$. By Lemma 7 we have $XY < XU$ and since

$$\begin{aligned}\rho(x, y) &\leq L(\Phi_F([XY])) = XY < XU \\ &= \min(XU, XU'') = L(\sigma_u) < \rho(x, y),\end{aligned}$$

we get a contradiction. Hence $X^* = X'$, and, since $XU < X'U$, $U^* = U''$. Now

$$\rho(x, y) \leq XY < r < X'U'' = L(\sigma_u) < \rho(x, y),$$

and we get another contradiction.

Case 3: $u \in [bp]$. Let $U' \in [BP']$ and $U'' \in [BP'']$ be such that $\Phi_F^{-1}(u) = \{U', U''\}$. Assuming that $U^* = U$ would infer

$$\rho(x, y) \leq XY < r \leq XU^* = L(\sigma_u) < \rho(x, y),$$

whence $U^* = U'$. Let v be the next (along σ, from x to y) point of $\sigma \cap \mathcal{M}$ and denote by σ_v the part of σ delimitated by u and v. Let V'' be the point such that $\sigma_v = \Phi_F([U''V''])$. Since $U''V'' = L(\sigma_V) < \rho(x, y) \leq XY < r$, $V'' \in [AP'']$. Let $V \in [AP]$ be the point symmetric to V'' with respect to (OA). The next (along σ, from x to y) point w of $\sigma \cap \mathcal{M}$ obviously belongs to $\Phi_F(\mathcal{L})$. Let σ_w be the part of σ delimitated by v and w, we have $\sigma_w = \Phi_F([VW])$ for some point $W \in \mathcal{L} \cup S(\mathcal{L})$. Moreover, since $S(\mathcal{L})$ is included in the half plane

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bounded by (OC_0) not containing V, W must lie on $\mathcal{L}$. We claim that $W = Y$. Assume on the contrary that σ is crossing $\Phi_F(\mathcal{L})$ at point w and denote by z the next point of $\mathcal{M} \cap \sigma$. Since σ cannot visit twice the face apc_N of $\mathcal{P}_F$ ([1]), z belongs to $\Phi_F(\mathcal{L})$. Hence the part σ_z of σ delimitated by w and z is the image under Φ_F of some segment $[S(W)Z]$, for some point $Z \in \mathcal{L} \cup S(\mathcal{L})$. In this case $\Phi_F([WS(Z)])$ is also a shortest path from w to z, in contradiction with Lemma 4. This proves the claim. Let R_1 be the rotation of center B which maps P'' on P', R_2 the one of center A which maps P on P'', and put $R = R_1 \circ R_2$. Clearly $L(\sigma) = X'R(Y)$. Since $R(P) = P'$, we have $R(Y)P' = PY = P'Y'$, whence P' belongs to the mediator of $[Y'R(Y)]$. Similarly, R maps C_N on C'_N, whence the mediator of $[R(Y)Y']$ is actually $(P'C'_N)$. It follows that $\rho(x,y) \leq XY = X'Y' \leq X'R(Y) = L(\sigma) = \rho(x,y)$. This completes the third case and the theorem. □

Denote by $\tilde{\mathfrak{P}}$ the compact plane domain (depending on α, δ) delimitated by the line segments $[PA]$, $[AP'']$, $[P''H_0]$, $[H_0P']$, and the circle arc $\mathcal{C}_\alpha^{2\pi-\alpha-2\delta}$. It is clear that $\tilde{\mathfrak{P}}$ contains $\mathfrak{P}_F$. The complement of $\mathfrak{P}_F$ in $\tilde{\mathfrak{P}}$ has $2N+3$ connected components. One of them, denoted by $\mathcal{Q}_F$ is delimitated by the quadrilateral $BP''H_0P'$. All others components are delimitated by a circle arc and the corresponding chord. We denote the union of these components by $\mathcal{T}_F$. We define $\Pi_F : \tilde{\mathfrak{P}} \to \mathfrak{P}_F$ as follows: if $X \in \mathfrak{P}_F$, then $\Pi_F(X) = X$. If $X \in \mathcal{T}_F$ then $\Pi_F(X)$ is the orthogonal projection of X onto the maximal segment included in the boundary of the connected component of $\mathcal{T}_F$ containing X. Finally, if X belongs to $\mathcal{Q}_F$, then $\Pi_F(X)$ is the point of $[P'B] \cup [BP'']$ such that the line $(X\Pi_F(X))$ is parallel to (OH_0).

Lemma 9. *The map Π_F is $\frac{1}{\sin\delta}$-Lipschitz continuous.*

Proof. Given a compact convex set $\mathcal{K} \subset \mathbb{R}^d$, the metrical projection onto $\mathcal{K}$ is the map from $\mathbb{R}^d$ to $\mathcal{K}$ which associates to each point X its closest point in $\mathcal{K}$. Of course this definition needs the unicity of the closest point ([6, (1.7)]). It is well-known that the metrical projection is 1-Lipschitz (follows from [6, (11.2)]). The restriction of Π_F to $\mathfrak{P}_F \cup \mathcal{T}_F$ coincides with the metrical projection onto the convex hull of $\mathfrak{P}_F$ and so is 1-Lipschitz.

Elementary calculus shows that the restriction of Π_F to $\mathcal{Q}_F$ is $\frac{1}{\sin\delta}$-Lipschitz.

Now assume that $X \in \mathcal{Q}_F$ and $Y \in \tilde{\mathfrak{P}} \backslash \mathcal{Q}_F$. Let Z be the intersection of $[XY]$ with $[P'B] \cup [BP'']$. We have

$$\begin{aligned}\|\Pi_F(X) - \Pi_F(Y)\| &\leq \|\Pi_F(X) - \Pi_F(Z)\| + \|\Pi_F(Z) - \Pi_F(Y)\| \\ &\leq \frac{1}{\sin\delta} XZ + ZY \leq \frac{1}{\sin\delta}(XZ + ZY) = \frac{1}{\sin\delta} XY.\end{aligned}$$

□

Assume now that a sequence $(F_k)_{k\geq 0}$ of finite subsets of $\mathcal{C}_\alpha^{\pi-\delta}$ is tending to some compact set $K \subset \overline{\mathcal{C}_\alpha^{\pi-\delta}}$ and put $f_k = \Phi_{F_k} \circ \Pi_{F_k}$.

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Lemma 10. *The restriction to K of Π_{F_k} converges uniformly to the identity of K.*

Proof. Just note that, for any point $X \in K$, we have

$$\|\Pi_{F_k}(X) - X\| \leq d(X, F_k) \leq h(F_k, K) \to 0.$$

□

Lemma 11. *For all k, f_k is $\frac{1}{\sin\delta}$-Lipschitz; therefore the sequence $(f_k)_k$ is equicontinuous.*

Proof. Let X, Y be points of $\tilde{\mathfrak{P}}$ and put $x_k = f_k(X)$, $y_k = f_k(Y)$. The $\mathbb{R}^3$ norm $\|x_k - y_k\|$ is less than or equal to the $\mathcal{P}_{F_k}$-distance between these points, which is by virtue of Lemma 6 less than the $\mathbb{R}^2$ norm $\|\Pi_{F_k}(X) - \Pi_{F_k}(Y)\|$. The conclusion follows from the fact that Π_{F_k} is $\frac{1}{\sin\delta}$-Lipschitz. □

The above lemma, together with Ascoli's well known theorem allows us to extract from (f_n) a subsequence converging uniformly to some $\frac{1}{\sin\delta}$-Lipschitz function $f : \tilde{\mathfrak{P}} \to \mathbb{R}^3$. It follows that $\mathcal{S} \overset{def}{=} f(\mathfrak{P})$ is the limit (with respect to the Hausdorff distance) of $(\mathcal{P}_{F_k})$, and therefore is a convex surface.

Lemma 12. *The image by f of a point of $\overline{\mathcal{C}_\alpha^{\pi-\delta}}$ is an antipode of o if and only if it belongs to $K \cup \{C_0, P\}$.*

Proof. Let $X \in K \cup \{C_0, P\}$. Since F_k is tending to K, we can chose $X_k \in F_k \cup \{C_0, P\}$ such that the sequence (X_k) tends to X. By the definition of F_k, the distance in $\mathcal{P}_k$ between o and $f_k(X_k)$ equals 1. Lemma 5 implies that the distance between o and $f(X)$ is also 1. On the other hand, still by virtue of Lemma 5, the distance on $\mathcal{S}$ from o cannot exceed 1, hence $f(X)$ is an antipode of o.

Assume that $X \in \mathcal{C}_\alpha^{\pi-\delta}$ does not belongs to K, It follows that there exists a positive number ε, such that, for k large enough, no point of F_k lies on $\mathcal{C}_{\xi-\varepsilon}^{\xi+\varepsilon}$. Hence $XX_k \geq 1 - \cos\varepsilon$. Moreover $\widehat{XX_kO}$ is obtuse, whence

$$1 = OX^2 \geq OX_k^2 + XX_k^2 \geq OX_k^2 + 1 + \cos^2\varepsilon - 2\cos\varepsilon.$$

Thus $OX_k \leq \sqrt{\cos\varepsilon(2 - \cos\varepsilon)} < 1$. Since OX_k is also the distance between o and $f_k(X)$ on $\mathcal{P}_k$, by Lemma 5, the distance on $\mathcal{S}$ between o and $f(X)$ is also less than or equal to $\sqrt{\cos\varepsilon(2 - \cos\varepsilon)}$, and $f(X)$ cannot be an antipode of o. □

Lemma 13. *The set of antipodes of $o \in \mathcal{S}$ is $\{c_0, a, p\} \cup f(K)$.*

Proof. Let $x = f(X)$ be a point of $\mathcal{S}$. As seen in the proof of Lemma 12, x is an antipode of o if and only if $\rho(o, x) = 1$. We discuss three (overlapping) cases:

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Case 1: There are infinitely many $k \in \mathbb{N}$ such that $X \in \mathfrak{P}_{F_k}$. By extraction of a suitable subsequence, we can assume that X belongs to $\mathfrak{P}_{F_k}$ for all k. By virtue of Lemmas 5 and 6,

$$\rho\left(f\left(X\right), o\right) = \lim \rho\left(f_k\left(X\right), o\right) = \lim \rho\left(\Phi_{F_k}\left(X\right), o\right) \leq \left\|X\right\|.$$

Hence, if x is an antipode of o, then $X \in \overline{\mathcal{C}_\alpha^{2\pi-2\delta-\alpha}} \cup \{A, P''\}$. By Lemma 5, $a = f\left(A\right)$ and $p = f\left(P\right) = f\left(P'\right) = f\left(P''\right)$ are antipodes of o. If X belongs to $\overline{\mathcal{C}_\alpha^{\pi-\delta}}$, then the conclusion follows from Lemma 12. If X belongs to $\overline{\mathcal{C}_{\pi-\delta}^{2\pi-2\delta-\alpha}}$, then X and its image X' by the symmetry of axis (OC_0) have the same image under f. Hence we can apply Lemma 5 to X' to obtain the conclusion.

Case 2: There are infinitely many $k \in \mathbb{N}$ such that $X \in \mathcal{Q}_{F_k}$. By extraction of a suitable subsequence, we can assume that X belongs to $\mathcal{Q}_{F_k}$ for all k. By virtue of Lemmas 5 and 6,

$$\rho\left(f\left(X\right), o\right) = \lim \rho\left(f_k\left(X\right), o\right) \leq \limsup \left\|\Pi_{F_k}\left(X\right)\right\| \leq \left\|X\right\|.$$

Since $\liminf \beta$ (β actually depends on k) cannot vanish, by (1), $\|B\|$ (which depends on k too) is less than 1, and so $\mathcal{Q}_{F_k}$ does not intersect the unique circle. It follows that $f\left(X\right)$ cannot be an antipode of o in this case.

Case 3: There are infinitely many $k \in \mathbb{N}$ such that $X \in \mathcal{T}_{F_k}$. By extraction of a suitable subsequence, we can assume that X belongs to $\mathcal{T}_{F_k}$ for all k. The ray emanating from $\Pi_{F_k}\left(X\right)$ through X intersects $\mathcal{C}_\alpha^{2\pi-2\delta-\alpha}$ at point X_k. By extraction of a suitable subsequence, we can assume that (X_k) converges to some point $X' \in \overline{\mathcal{C}_\alpha^{2\pi-2\delta-\alpha}}$. Since

$$f\left(X\right) = \lim f_k\left(X\right) = \lim f_k\left(X_k\right) = f\left(X'\right),$$

the conclusion follows from Lemma 12. □

Lemma 14. *The restriction to K of the map $f : \tilde{\mathfrak{P}} \to \mathcal{S}$ is a local isometry, with respect to the norm of $\mathbb{R}^2$ and the intrinsic distance of $\mathcal{S}$.*

Proof. Let r be the number defined in the Lemma 8 and choose $\varepsilon \in \left]0, \frac{r}{2}\right[$. By virtue of Lemma 10, for k large enough, we have

$$\forall X \in K \;\; \left\|\Pi_{F_k}\left(X\right) - X\right\| < \frac{\varepsilon}{2}. \tag{5}$$

Hence, for k large enough, if $X, Y \in K$ are such that $XY \leq \frac{r}{2} < r - \varepsilon$, then, by virtue of Lemma 8,

$$\begin{aligned}\rho\left(f_k\left(X\right), f_k\left(Y\right)\right) &= \rho\left(\Phi_{F_k}\left(\Pi_{F_k}\left(X\right)\right), \Phi_{F_k}\left(\Pi_{F_k}\left(Y\right)\right)\right)\\ &= \left\|\Pi_{F_k}\left(X\right) - \Pi_{F_k}\left(Y\right)\right\|.\end{aligned}$$

On the other hand, by (5), it is clear that

$$XY - \varepsilon \leq \left\|\Pi_{F_k}\left(X\right) - \Pi_{F_k}\left(Y\right)\right\| \leq XY + \varepsilon,$$

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whence, for $XY < \frac{r}{2}$ and k large enough,

$$|\rho(f_k(X), f_k(Y)) - XY| \leq \varepsilon.$$

Passing to the limit we find that $|\rho(f(X), f(Y)) - XY|$ is at most ε for the arbitrary small positive number ε, and so it is zero. □

Now we are in a position to prove Theorem 2.

Proof of Theorem 2. We can assume without loss of generality that $\mathcal{H} = \mathcal{C}_0^\pi$. Let $C_0 \in \mathcal{H}$ be the point of K with the lowest abscissa, and P the one with the greatest abscissa. Let $(F_k)_{k\geq 0}$ be a sequence of finite subsets of $K \setminus \{C_0, P\}$, such that $(F_k \cup \{C_0, P\})$ converges to K with respect to the Hausdorff metric. Let f_k, f and $\mathcal{S}$ be defined as usual. By virtue of Lemma 13, the set of antipodes of o is $\{a\} \cup f(K)$. By virtue of Lemma 14, $f(K)$ is locally isometric to K, thus $K \cup \{U\}$ is locally isometric to $\{a\} \cup f(K)$. □

Corollary 1. *For any $\varepsilon > 0$, there exists a convex surface $\mathcal{S}$ and a point $o \in S$ such that the length (1-dimensional Hausdorff measure) of the set of antipodes of o, divided by the distance between o and one of its antipodes is greater than $\pi - \varepsilon$.*

Proof. Apply Theorem 2 with $K = \mathcal{C}_{\varepsilon/4}^{\pi-\varepsilon/4}$. □

Remark 4. Tudor Zamfirescu exhibited two other examples of surfaces with such large sets of antipodes [15].

Corollary 2. *Any number $d \in [0, 1]$ can be realized as the Hausdorff dimension of the set of antipodes of some point of some convex surface.*

Proof. It is a well-known fact that the Hausdorff dimension of a Cantor set which is the disjoint union of two λ-scaled copies of itself has dimension $-\frac{\log 2}{\log \lambda}$. Consequently, if we apply Theorem 2 with a Cantor set K formed of two $(2^{-1/d})$-scaled copies of itself, we obtain a set of antipodes of dimension d. □

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