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# Existence of global weak solution for compressible fluid models with a capillary tensor for discontinuous interfaces

Boris Haspot \*

## Abstract

This work is devoted to proving the global existence of weak solution for a general isothermal model of capillary fluids derived in his modern form by C. Rohde in [21], which can be used as a phase transition model. First, inspired by the result by P.-L. Lions in [19] on the Navier-Stokes compressible system we show the global stability of weak solutions for our system with isentropic pressure and next with general pressure. Finally, we consider perturbations of a stable equilibrium.

## 1 Introduction

### 1.1 Presentation of the model

The correct mathematical description of liquid-vapor phase interfaces and their dynamical behavior in compressible fluid flow has a long history. We are concerned with compressible fluids endowed with internal capillarity. One of the first model which takes into consideration the variation of density on the interface between two phases, originates from the XIXth century work by Van der Waals and Korteweg [17]. It was actually derived in his modern form in the 1980s using the second gradient theory, see for instance [16, 22]. Korteweg suggests a modification of the Navier-Stokes system to account additionally for phase transition phenomena in introducing a term of capillarity. He assumed that the thickness of the interfaces was not null as in the *sharp interface approach*. This is called the *diffuse interface approach*. Korteweg-type models are based on an extended version of nonequilibrium thermodynamics, which assumes that the energy of the fluid not only depends on standard variables but on the gradient of the density. In terms of the free energy, this principle takes the form of a generalized Gibbs relation, see [22]. In the present paper, we follow a new approach introduced by Coquel *et al* in [5]. They remark that the local diffuse interface approach requires more regular solutions than in the original sharp interface approach. Indeed the interfaces are assumed of nonzero thickness, so that the density varies continuously across the interface, whereas in the sharp interface models, the density may have jumps. Coquel *et al* present an alternative model with a capillarity

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term which does not involve spatial derivatives of the density. The model reads:

$$(NSK) \quad \begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu) \nabla \operatorname{div} u + \nabla(P(\rho)) = \kappa \rho \nabla D[\rho] \\ (\rho_{t=0}, u_{t=0}) = (\rho_0, u_0) \end{cases}$$

where  $\rho \geq 0$  denotes the density of the fluid and  $u \in \mathbb{R}^N$ , the velocity. We assume that the viscosity coefficients  $\mu$  and  $\lambda$  satisfy  $\mu > 0$  and  $\lambda + 2\mu > 0$  and that the capillarity coefficient  $\kappa$  is nonnegative. Finally,  $P$  stands for the pressure function. We supplement the system with the following conditions at infinity:

$$u(t, x) \rightarrow 0, \quad \rho(t, x) \rightarrow \bar{\rho} \quad \text{as } |x| \rightarrow +\infty, \quad (1.1)$$

where  $\bar{\rho}$  is a given nonnegative real number.

As System (NSK) is relevant for the study of phase transitions, we expect the pressure to be of Van der Waals type, namely

$$P : (0, b) \rightarrow (0, +\infty) \\ P(\rho) = \frac{RT_* \rho}{b - \rho} - a\rho^2$$

where  $a, b, R, T_*$  are positive constants,  $R$  being the specific gas constant. However, we shall first concentrate on isentropic pressure laws:  $P(\rho) = a\rho^\gamma$  for some  $a > 0$  and  $\gamma \geq 1$  and on the case of vanishing density at infinity, namely  $\bar{\rho} = 0$ .

In the last section, more general situations will be considered: general monotone pressure laws and Van der Waals type pressure laws and also nonzero conditions at infinity for the density. The term  $\kappa \rho \nabla D[\rho]$  accounts for the capillarity effects close to phase transitions. The classical Korteweg's capillarity term is  $D[\rho] = \Delta \rho$  (see [17]). Based on Korteweg's original ideas Coquel *et al* in [5] and Rohde in [21] choose a nonlocal capillarity term  $D$  which penalizes rapid variations in the density field close from the interfaces. They introduce the following capillarity term:  $D[\rho] = \phi * \rho - \rho$  where  $\phi$  is chosen so that:

$$\phi \in L^\infty(\mathbb{R}^N) \cap C^1(\mathbb{R}^N) \cap W^{1,1}(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} \phi(x) dx = 1, \quad \phi \text{ even, and } \phi \geq 0.$$

This choice of capillarity term allows to get solution with jumps, i.e with sharp interfaces.

## 1.2 Energy spaces

Before tackling the global stability theory for the system (NSK), let us derive formally the uniform bounds available on  $(\rho, u)$ . Let  $\Pi$  (free energy) be defined by:

$$\Pi(s) = s \left( \int_0^s \frac{P(z)}{z^2} dz \right), \quad (1.2)$$

so that  $P(s) = s\Pi'(s) - \Pi(s)$ ,  $\Pi'(\bar{\rho}) > 0$  and if we renormalize the mass equation:

$$\partial_t \Pi(\rho) + \operatorname{div}(u \Pi(\rho)) + P(\rho) \operatorname{div}(u) = 0 \quad \text{in } \mathcal{D}'((0, T) \times \mathbb{R}^N).$$

Notice that  $\Pi$  is convex whenever  $P$  is nondecreasing. Multiplying the equation of momentum conservation by  $u$  and integrating by parts over  $\mathbb{R}^N$ , we obtain the following energy estimate:

$$\begin{aligned} \int_{\mathbb{R}^N} \left( \frac{1}{2} \rho |u|^2 + \Pi(\rho) + E_{global}[\rho(\cdot, t)] \right)(x) dx(t) + \int_0^t \int_{\mathbb{R}^N} (\mu D(u) : D(u) \\ + (\lambda + \mu) |\operatorname{div} u|^2) dx \leq \int_{\mathbb{R}^N} \left( \frac{|m_0|^2}{2\rho} + \Pi(\rho_0) + E_{global}[\rho_0] \right) dx, \end{aligned} \quad (1.3)$$

where we have:

$$E_{global}[\rho(\cdot, t)](x) = \frac{\kappa}{4} \int_{\mathbb{R}^N} \phi(x-y) (\rho(y, t) - \rho(x, t))^2 dy.$$

The only non-standard term is the energy term  $E_{global}$  which comes from the product of  $u$  with the capillarity term  $\kappa \rho \nabla(\phi * \rho - \rho)$ . Indeed we have:

$$\begin{aligned} \kappa \int_{\mathbb{R}^N} u(t, x) \rho(t, x) \cdot \nabla([\phi * \rho(t, \cdot)](x) - \rho(t, x)) dx \\ = -\kappa \int_{\mathbb{R}^N} \operatorname{div}(u(t, x) \rho(t, x)) ([\phi * \rho(t, \cdot)](x) - \rho(t, x)) dx, \\ = \kappa \int_{\mathbb{R}^N} \frac{\partial}{\partial t} \rho(t, x) ([\phi * \rho(t, \cdot)](x) - \rho(t, x)) dx, \\ = -\frac{d}{dt} \int_{\mathbb{R}^N} E_{global}[\rho(t, \cdot)](x) dx. \end{aligned}$$

To derive the last equality we use the relation:

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^N} E_{global}[\rho(t, \cdot)](x) dx &= \frac{\kappa}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \phi(x-y) (\rho(t, y) - \rho(t, x)) \frac{\partial}{\partial t} \rho(t, y) dy dx \\ &\quad + \frac{\kappa}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \phi(y-x) (\rho(t, x) - \rho(t, y)) \frac{\partial}{\partial t} \rho(t, x) dy dx, \\ &= \kappa \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \phi(x-y) (\rho(t, y) - \rho(t, x)) \frac{\partial}{\partial t} \rho(t, y) dy dx, \\ &= -\kappa \int_{\mathbb{R}^N} ([\phi * \rho(t, \cdot)](x) - \rho(t, x)) \frac{\partial}{\partial t} \rho(t, x) dx. \end{aligned}$$

where we just use integration by parts. In the sequel we will note:

$$\mathcal{E}(\rho, \rho u)(t) = \int_{\mathbb{R}^N} \frac{1}{2} \rho |u|^2(t, x) + \Pi(\rho)(t, x) + E_{global}[\rho(\cdot, t)](t, x) dx, \quad (1.4)$$

We are interested to use the above inequality energy to determine the functional space we must work with. So if we expand  $E_{global}[\rho(\cdot, t)](x)$  we get:

$$E_{global}[\rho(t, \cdot)](x) = \frac{\kappa}{4} (\rho^2 + \phi * \rho^2 - 2\rho(\phi * \rho)).$$

Because by the mass equation we obtain that  $\rho$  is bounded in  $L^\infty(0, T; L^1(\mathbb{R}^N))$  if we suppose that  $\rho_0 \in L^1$  and we have supposed that  $\phi \in L^\infty(\mathbb{R}^N)$ , we obtain that  $\rho(\phi * \rho)$

is bounded in  $L^\infty(0, T; L^1(\mathbb{R}^N))$ . So we get that  $\rho^2 + \phi * \rho^2 \in L^\infty(0, T; L^1(\mathbb{R}^N))$  and as  $\phi \geq 0$  and  $\rho \geq 0$  we get a control of  $\rho$  in  $L^\infty(0, T; L^2(\mathbb{R}^N))$  (a property which turns out to be important to taking advantage of the theory of renormalized solutions, indeed  $\rho$  in  $L^\infty(0, T; L^2(\mathbb{R}^N))$  implies that  $\rho \in L^2_{loc}(\mathbb{R}^+ \times \mathbb{R}^N)$  so that one may use the theorem of Di Perna-Lions on renormalized solutions, see [18]). In view of (1.3), we can specify initial conditions on  $\rho|_{t=0} = \rho_0$  and  $\rho u|_{t=0} = m_0$  where we assume that:

- $\rho_0 \geq 0$  a.e in  $\mathbb{R}^N$ ,  $\rho_0 \in L^1(\mathbb{R}^N) \cap L^s(\mathbb{R}^N)$  with  $s = \max(2, \gamma)$ ,
  - $m_0 = 0$  a.e on  $\rho_0 = 0$ ,
  - $\frac{|m_0|^2}{\rho_0}$  (defined to be 0 on  $\rho_0 = 0$ ) is in  $L^1(\mathbb{R}^N)$ .
- (1.5)

We deduce the following a priori bounds which give us the energy space in which we will work:

- $\rho \in L^\infty(0, T; L^1(\mathbb{R}^N) \cap L^s(\mathbb{R}^N))$ ,
- $\rho|u|^2 \in L^\infty(0, T; L^1(\mathbb{R}^N))$ ,
- $\nabla u \in L^2((0, T) \times \mathbb{R}^N)^N$ .

We will use this uniform bound in our result of compactness. Let us emphasize at this point that the above a priori bounds do not provide any control on  $\nabla \rho$  in contrast with the case of  $D[\rho] = \Delta \rho$  studied in [6].

### 1.3 Notion of weak solutions

We now explain what we mean by renormalized weak solutions, weak solutions, and bounded energy weak solution of problem (NSK). Multiplying mass equation by  $b'(\rho)$ , we obtained the so-called renormalized equation (see [18]):

$$\frac{\partial}{\partial t} b(\rho) + \operatorname{div}(b(\rho)u) + (\rho b'(\rho) - b(\rho)) \operatorname{div} u = 0. \quad (1.6)$$

with:

$$b \in C^0([0, +\infty)) \cap C^1((0, +\infty)), \quad |b'(t)| \leq ct^{-\lambda_0}, \quad t \in (0, 1], \quad \lambda_0 < 1 \quad (1.7)$$

with growth conditions at infinity:

$$|b'(t)| \leq ct^{\lambda_1}, \quad t \geq 1, \quad \text{where } c > 0, \quad -1 < \lambda_1 < \frac{s}{2} - 1. \quad (1.8)$$

**Definition 1.1** *A couple  $(\rho, u)$  is called a renormalized weak solution of problem (NSK) whenever:*

- *the equation (1.6) holds in  $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^N)$  for any function  $b$  verifying (1.7) and (1.8).*

**Definition 1.2** *Let the couple  $(\rho_0, u_0)$  satisfy*

- $\rho_0 \in L^1(\mathbb{R}^N)$ ,  $\Pi(\rho_0) \in L^1(\mathbb{R}^N)$ ,  $E_{global}[\rho_0] \in L^1(\mathbb{R}^N)$ ,  $\rho_0 \geq 0$  a.e in  $\mathbb{R}^N$ .

- $\rho_0 u_0 \in (L^1(\mathbb{R}^N))^N$  and  $\rho_0 |u_0|^2 1_{\rho_0 > 0} \in L^1(\mathbb{R}^N)$
- $\rho_0 u_0 = 0$  whenever  $x \in \{\rho_0 = 0\}$ ,

where the quantity  $\Pi$  is defined in (1.2). We have the following definitions:

1. A couple  $(\rho, u)$  is called a weak solution of problem (NSK) on  $\mathbb{R}$  if:

- (a)  $\rho \in L^r(L^r(\mathbb{R}^N))$  for  $s \leq r \leq +\infty$ ,
- (b)  $P(\rho) \in L^\infty(L^1(\mathbb{R}^N))$ ,  $\rho \geq 0$  a.e in  $\mathbb{R} \times \mathbb{R}^N$ ,
- (c)  $\nabla u \in L^2(L^2(\mathbb{R}^N))$ ,  $\rho |u|^2 \in L^\infty(L^1(\mathbb{R}^N))$ .
- (d) Mass equation holds in  $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^N)$ .
- (e) Momentum equation holds in  $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^N)^N$ .
- (f)  $\lim_{t \rightarrow 0^+} \int_{\mathbb{R}^N} \rho(t) \varphi = \int_{\mathbb{R}^N} \rho_0 \varphi$ ,  $\forall \varphi \in \mathcal{D}(\mathbb{R}^N)$ ,
- (g)  $\lim_{t \rightarrow 0^+} \int_{\mathbb{R}^N} \rho u(t) \cdot \phi = \int_{\mathbb{R}^N} \rho_0 u_0 \cdot \phi$ ,  $\forall \phi \in \mathcal{D}(\mathbb{R}^N)^N$ .

2. A couple  $(\rho, u)$  is called a bounded energy weak solution of problem (NSK) if in addition to (1d), (1e), (1f), (1g) we have:

- The quantity  $\mathcal{E}_0$  is finite and inequality (1.3) with  $\mathcal{E}$  defined by (1.4) and with  $\mathcal{E}_0$  in place of  $\mathcal{E}(\rho(0), \rho u(0))$  holds a.e in  $\mathbb{R}$ .

## 1.4 Mathematical results

We wish to prove global stability results for (NSK) with  $D[\rho] = \phi * \rho - \rho$  in functional spaces very close to energy spaces. In the non capillary case and  $P(\rho) = a\rho^\gamma$ , P-L. Lions in [19] proved the global existence of weak solutions  $(\rho, u)$  to (NSK) with  $\kappa = 0$  (that is for the compressible isotherm system of Navier-Stokes) for  $\gamma > \frac{N}{2}$  if  $N \geq 4$ ,  $\gamma \geq \frac{3N}{N+2}$  if  $N = 2, 3$  and initial data  $(\rho_0, m_0)$  such that:

$$\rho_0, \rho_0^\gamma, \frac{|m_0|^2}{\rho_0} \in L^1(\mathbb{R}^N).$$

where we agree that  $m_0 = 0$  on  $\{x \in \mathbb{R}^N / \rho_0(x) = 0\}$ . More precisely, he obtains the existence of global weak solutions  $(\rho, u)$  to (NSK) with  $\kappa = 0$  such that for all  $t \in (0, +\infty)$ :

- $\rho \in L^\infty(0, T; L^\gamma(\mathbb{R}^N))$  and  $\rho \in C([0, T], L^p(\mathbb{R}^N))$  if  $1 \leq p < \gamma$ ,
- $\rho \in L^q((0, T) \times \mathbb{R}^N)$  for  $q = \gamma - 1 + \frac{2\gamma}{N} > \gamma$ .
- $\rho |u|^2 \in L^\infty(0, T; L^1(\mathbb{R}^N))$  and  $Du \in L^2((0, T) \times \mathbb{R}^N)$ .

Notice that the main difficulty for proving Lions' theorem consists in exhibiting strong compactness properties of the density  $\rho$  in  $L^p_{loc}(\mathbb{R}^+ \times \mathbb{R}^N)$  spaces required to pass to the limit in the pressure term  $P(\rho) = a\rho^\gamma$ . Let us mention that Feireisl in [9] generalized the result to any  $\gamma > \frac{N}{2}$  in establishing that we can obtain renormalized solution without imposing that  $\rho \in L^2_{loc}(\mathbb{R}^+ \times \mathbb{R}^N)$  (a property that was needed in Lions' approach in

dimension  $N = 2, 3$  giving the further condition  $\gamma - 1 + \frac{2\gamma}{N} \geq 2$ , for this he introduces the concept of oscillation defect measure evaluating the loss of compactness. We refer to the book of Novotný and Straškraba for more details (see [20]).

Let us mention here that the existence of strong solution with  $D[\rho] = \Delta\rho$  is known since the work by Hattori and Li in [13], [14] in the whole space  $\mathbb{R}^N$ . In [6], Danchin and Desjardins study the well-posedness of the problem for the isothermal case with constant coefficients in critical Besov spaces. We recall too the results by Rohde in [21] who obtains the existence and uniqueness in finite time for two-dimensional initial data in  $H^4(\mathbb{R}^2) \times H^4(\mathbb{R}^2)$  (less regular data in any dimension  $N \geq 2$  have been considered recently in [10]).

In the present paper, we aim at showing the global stability of weak solutions in the energy spaces for the system (NSK). This work is composed of four parts, the first one concerns estimates on the density. We aim at getting more integrability on the density in order to pass to the weak limit in the term of pressure and of capillarity. The second part is the passage to the weak limit in the non-linear terms of the density and the velocity according to Lions' methods. The idea is to use renormalized solution to test the weak limit on convex test functions. For the time being, we focus on the case of pressure laws of type  $P(\rho) = a\rho^\gamma$ . We get the following theorem where  $(\rho_n, u_n)_{n \in \mathbb{N}}$  is a sequence of regular bounded energy weak solutions of (NSK) and, in addition, the sequence  $\rho_n$  is bounded in  $L^r((0, T) \times \mathbb{R}^N) \cap L^1((0, T) \times \mathbb{R}^N)$  for some  $r > \max(\gamma, 2)$ .

**Theorem 1.1** *Let  $N = 2, 3$ . Let  $\gamma > N/2$  if  $N \geq 4$  and  $\gamma \geq 1$  else.*

*Let the couple  $(\rho_0^n, u_0^n)$  satisfy:*

- $\rho_0^n$  is uniformly bounded in  $L^1(\mathbb{R}^N) \cap L^s(\mathbb{R}^N)$  with  $s = \max(\gamma, 2)$  and  $\rho_0^n \geq 0$  a.e in  $\mathbb{R}^N$ ,
- $\rho_0^n |u_0^n|^2$  is uniformly bounded in  $L^1(\mathbb{R}^N)$ ,
- $\rho_0^n u_0^n = 0$  whenever  $x \in \{\rho_0 = 0\}$ .

*In addition we suppose that  $\rho_0^n$  converges in  $L^1(\mathbb{R}^N)$  to  $\rho_0$ . Then, up to a subsequence,  $(\rho_n, u_n)$  converges strongly to a weak solution  $(\rho, u)$  of the system (NSK) satisfying the initial condition  $(\rho_0, u_0)$  as in (1.5). Moreover we have the following convergence:*

- $\rho_n \rightarrow_n \rho$  in  $C([0, T], L^p(\mathbb{R}^N)) \cap L^r((0, T) \times \mathbb{R}^N)$  for all  $1 \leq p < s$ ,  $1 \leq r < q$ , with  $q = s + \frac{N\gamma}{2} - 1$  if  $N = 3$ .
- $\rho_n \rightarrow_n \rho$  in  $C([0, T], L^p(\mathbb{R}^2)) \cap L^r((0, T) \times K)$  for all  $1 \leq p < s$ ,  $1 \leq r < q$ , with  $K$  an arbitrary compact in  $\mathbb{R}^2$  if  $N = 2$ .

*In addition we have:*

- $\rho_n u_n \rightarrow \rho u$  in  $L^p(0, T; L^r(\mathbb{R}^N))$  for all  $1 \leq p < +\infty$  and  $1 \leq r < \frac{2s}{s+1}$ ,
- $\rho_n(u_i)_n(u_j)_n \rightarrow \rho_n u_i u_j$  in  $L^p(0, T; L^1(\mathbb{R}^N))$  for all  $1 \leq p < +\infty$ ,  $1 \leq i, j \leq N$  if  $N = 3$ .
- $\rho_n(u_i)_n(u_j)_n \rightarrow \rho_n u_i u_j$  in  $L^p(0, T; L^1(\Omega))$  for all  $1 \leq p < +\infty$ ,  $1 \leq i, j \leq N$  with  $\Omega$  an arbitrary bounded open set in  $\mathbb{R}^2$  if  $N = 2$ .

**Remark 1** For the cases  $N \geq 4$ , we refer for more details to [12].

Our paper unfolds as follows. In the next part, we prove Theorem 1.1. In the third part we focus on more general pressure laws (including Van der Waal's pressure laws). In the last part of the paper, we consider initial data close to a positive constant  $\bar{\rho}$ .

## 2 Existence of weak solution for a isentropic pressure law

### 2.1 A priori estimates on the density

In this part we are interested by getting a gain of integrability on the density and we consider the case where  $P(\rho) = a\rho^\gamma$ . This will enable us to pass to the weak limit in the pressure and the Korteweg terms. It is expressed by the following theorem:

**Theorem 2.2** *Let  $N = 2, 3$  and  $\gamma \geq 1$ . Let  $(\rho, u)$  be a regular bounded energy weak solution of the system (NSK) with  $\rho \geq 0$  and  $\rho \in L^\infty(L^1 \cap L^{s+\varepsilon})$  where we define  $\varepsilon$  below.*

$$\int_{(0,T) \times \mathbb{R}^N} (\rho^{\gamma+\varepsilon} + \rho^{2+\varepsilon}) dx dt \leq M \text{ for any } 0 < \varepsilon \leq \frac{2}{N}\gamma - 1.$$

with  $M$  depending only on the initial conditions and on the time  $T$ .

**Proof:**

We will begin with the case where  $N = 3$  and we treat after the specific case  $N = 2$ .

**Case  $N = 3$ :**

Applying the operator  $(-\Delta)^{-1}\text{div}$  to the momentum equation yields

$$a\rho^\gamma = \frac{\partial}{\partial t}(-\Delta)^{-1}\text{div}(\rho u) + (-\Delta)^{-1}\partial_{i,j}^2(\rho u_i u_j) + (2\mu + \lambda)\text{div} u - \kappa(-\Delta)^{-1}\text{div}(\rho \nabla(\phi * \rho - \rho)), \quad (2.9)$$

so that multiplying by  $\rho^\varepsilon$  with  $0 < \varepsilon \leq \min(\frac{1}{N}, \frac{2}{N}\gamma - 1)$ , we get

$$a\rho^{\gamma+\varepsilon} + \frac{\kappa}{2}\rho^{2+\varepsilon} = -\kappa\rho^\varepsilon(-\Delta)^{-1}\text{div}(\rho(\nabla\phi * \rho)) + \rho^\varepsilon(-\Delta)^{-1}\partial_{i,j}^2(\rho u_i u_j) + \frac{\partial}{\partial t}(\rho^\varepsilon(-\Delta)^{-1}\text{div}(\rho u)) - [\frac{\partial}{\partial t}\rho^\varepsilon](-\Delta)^{-1}\text{div}(\rho u) + (\mu + \zeta)\text{div} u, \quad (2.10)$$

where we note  $\xi = \lambda + \mu$ . We now rewrite the previous equality as follows:

$$a\rho^{\gamma+\varepsilon} + \frac{\kappa}{2}\rho^{2+\varepsilon} = -\kappa\rho^\varepsilon(-\Delta)^{-1}\text{div}(\rho(\nabla\phi * \rho)) + \rho^\varepsilon(-\Delta)^{-1}\partial_{i,j}^2(\rho(u_i)(u_j)) + \frac{\partial}{\partial t}(\rho^\varepsilon(-\Delta)^{-1}\text{div}(\rho u)) + \text{div}[u\rho^\varepsilon(-\Delta)^{-1}\text{div}(\rho u)] + (\mu + \zeta)\text{div} u - (\rho)^\varepsilon u \cdot \nabla(-\Delta)^{-1}\text{div}(\rho u) + (1 - \varepsilon)(\text{div} u)\rho^\varepsilon(-\Delta)^{-1}\text{div}(\rho u). \quad (2.11)$$



Next we integrate (2.11) in time on  $[0, T]$  and in space. We get:

$$\begin{aligned} \int_{(0,T) \times \mathbb{R}^N} (a\rho^{\gamma+\varepsilon} + \frac{\kappa}{2}\rho^{2+\varepsilon}) dx dt &= \int_{(0,T) \times \mathbb{R}^N} \left( \frac{\partial}{\partial t} [\rho^\varepsilon (-\Delta)^{-1} \operatorname{div}(\rho u)] + \right. \\ &(\mu + \zeta)(\operatorname{div} u) \rho^\varepsilon + (1 - \varepsilon)(\operatorname{div} u) \rho^\varepsilon (-\Delta)^{-1} \operatorname{div}(\rho u) + \rho^\varepsilon [R_i R_j (\rho u_i u_j) \\ &\left. - u_i R_i R_j (\rho u_j)] + \operatorname{div}[u \rho^\varepsilon (-\Delta)^{-1} \operatorname{div}(\rho u)] - \kappa \rho^\varepsilon (-\Delta)^{-1} \operatorname{div}(\rho \nabla(\phi * \rho)) \right) dx dt, \end{aligned} \quad (2.12)$$

where  $R_i$  is the classical Riesz transform. Now we want to control the term  $\int_0^T \int_{\mathbb{R}^N} (\rho^{\gamma+\varepsilon} + \frac{\kappa}{2}\rho^{2+\varepsilon}) dx dt$ . As  $\rho$  is positive, it will enable us to control  $\|\rho\|_{L_{t,x}^{\gamma+\varepsilon}}$  and  $\|\rho\|_{L_{t,x}^{2+\varepsilon}}$ . This may be achieved by bounding each term on the right side of (2.12).

We start by treating the term  $\int_{(0,T) \times \mathbb{R}^N} \frac{\partial}{\partial t} [\rho^\varepsilon (-\Delta)^{-1} \operatorname{div}(\rho u)]$ . So we need to control  $\rho^\varepsilon (-\Delta)^{-1} \operatorname{div}(\rho u)$  in  $L^\infty(0, T; L^1(\mathbb{R}^N))$  and  $\rho_0^\varepsilon (-\Delta)^{-1} \operatorname{div}(\rho_0 u_0)$  because:

$$\begin{aligned} \int_{(0,T) \times \mathbb{R}^N} \frac{\partial}{\partial t} [\rho^\varepsilon (-\Delta)^{-1} \operatorname{div}(\rho u)](t, x) dt dx &= \int_{\mathbb{R}^N} [\rho^\varepsilon (-\Delta)^{-1} \operatorname{div}(\rho u)](t, x) dx \\ &\quad - \int_{\mathbb{R}^N} [\rho_0^\varepsilon (-\Delta)^{-1} \operatorname{div}(\rho_0 u_0)](x) dx. \end{aligned}$$

We recall that  $\rho$ ,  $\rho^2$ ,  $\rho^\gamma$  and  $\rho|u|^2$  are bounded in  $L^\infty(L^1)$  while  $Du$  is bounded in  $L^2((0, T) \times \mathbb{R}^N)$  and  $u$  is bounded in  $L^2(0, T; L^{\frac{2N}{N-2}}(\mathbb{R}^N))$  by Sobolev embedding. In particular by Hölder inequalities we get that  $\rho u$  is bounded in  $L^\infty(0, T, (L^{\frac{2\gamma}{\gamma+1}} \cap L^{\frac{4}{3}})(\mathbb{R}^N))$ . Thus we get by using Hölder inequalities and Sobolev embedding:  $\rho^\varepsilon (-\Delta)^{-1} \operatorname{div}(\rho u) \in L^\infty(0, T, L^1 \cap L^\alpha)$  with:

$$\frac{1}{\alpha} = \frac{\varepsilon}{s} + \min\left(\frac{\gamma+1}{2\gamma}, \frac{3}{4}\right) - \frac{1}{N} < 1.$$

The fact that  $\rho^\varepsilon (-\Delta)^{-1} \operatorname{div}(\rho u) \in L^\infty(0, T, L^1)$  is obtained by interpolation because  $\rho \in L^\infty(L^1)$  and in using less integrability in Sobolev embedding. Next we have the same type of estimates for  $\|\rho_0^\varepsilon (-\Delta)^{-1} \operatorname{div}(\rho_0 u_0)\|_{L^1(\mathbb{R}^N)}$ .

Finally (2.12) is rewritten on the following form in using Green formula:

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^N} (\rho^{\gamma+\varepsilon} + \frac{\kappa}{2}\rho^{2+\varepsilon}) dx dt &\leq C \left( 1 + \int_0^T \int_{\mathbb{R}^N} [|\operatorname{div} u| \rho^\varepsilon (1 + |(-\Delta)^{-1} \operatorname{div}(\rho u)|) \right. \\ &\quad \left. + \rho^\varepsilon |R_i R_j (\rho u_i u_j) - u_i R_i R_j (\rho u_j)| + \kappa \rho^\varepsilon |(-\Delta)^{-1} \operatorname{div}(\rho \nabla(\phi * \rho))|] dt dx \right). \end{aligned}$$

Now we will treat each term of the right hand side. We treat all the terms with the same type of estimates than P.-L. Lions in [19], excepted the capillarity term.

We start with the term  $|\operatorname{div} u| \rho^\varepsilon |(-\Delta)^{-1} \operatorname{div}(\rho u)|$ . If  $\gamma < 6$  then we may write

$$|\operatorname{div} u| \in L^2(L^2), \quad \rho^\varepsilon \in L^\infty(L^{\frac{s}{\varepsilon}}), \quad \rho u \in L^2(0, T, L^r(\mathbb{R}^N))$$

with  $\frac{1}{r} = \frac{1}{s} + \frac{N-2}{2N}$  and by Sobolev embedding  $|(-\Delta)^{-1} \operatorname{div}(\rho u)| \in L^2(L^{s'})$  with  $\frac{1}{s'} = \frac{1}{s} - \frac{1}{N}$  (this is possible only if  $r < N$ ). We are in a critical case for the Sobolev embedding (i.e  $r \geq N$ ) only when  $\gamma \geq 6$ . So by Hölder inequalities we get  $|\operatorname{div} u| \rho^\varepsilon |(-\Delta)^{-1} \operatorname{div}(\rho u)| \in$

$L^1(L^{s_1})$  with:  $\frac{1}{s_1} = \frac{1}{s'} + \frac{\varepsilon}{s} + \frac{1}{2} = 1 - \frac{2}{N} + \frac{1+\varepsilon}{s} \leq 1$  as we have  $s > \frac{N}{2}$ .

Moreover by interpolation  $|\operatorname{div} u| \rho^\varepsilon |(-\Delta)^{-1} \operatorname{div}(\rho u)|$  belongs to  $L^1(0, T; L^1(\mathbb{R}^N))$ . Let us now treat the case  $N = 3$  and  $\gamma \geq 6$  where we choose case  $\varepsilon = \frac{2}{N}\gamma - 1$ . We have:

$$\begin{aligned} \| |\operatorname{div} u| \rho^\varepsilon |(-\Delta)^{-1} \operatorname{div}(\rho u)| \|_{L^1} &\leq \|Du\|_{L^2(L^2)} \|\rho\|_{L^{\gamma+\varepsilon}}^\varepsilon \|\rho u\|_{L^{\frac{2(\gamma+\varepsilon)}{\gamma-\varepsilon}}(L^{\frac{6(\gamma+\varepsilon)}{5\gamma-\varepsilon}})}, \\ &\leq C \|\rho\|_{L^{\gamma+\varepsilon}}^\varepsilon \|\rho u\|_{L^{\frac{10\gamma-6}{\gamma+3}}(L^{\frac{3(10\gamma-6)}{13\gamma+3}})} \leq C \|\rho\|_{L^{\gamma+\varepsilon}}^\varepsilon \|\rho u\|_{L^2(L^{\frac{\gamma+3}{5\gamma-3}}(L^{\frac{2(2\gamma-3)}{5\gamma-3}}))} \|\rho u\|_{L^\infty(L^2)}, \\ &\leq C \|\rho\|_{L^{\gamma+\varepsilon}}^\varepsilon \|\rho u\|_{L^2(L^{\frac{6\gamma}{\gamma+6}})}^{\frac{5\gamma}{5\gamma-3}} \|\rho u\|_{L^\infty(L^{\frac{2\gamma}{\gamma+1}})}^{\frac{2(2\gamma-3)}{5\gamma-3}}, \\ &\leq C \|\rho\|_{L^{\gamma+\varepsilon}}^\varepsilon, \end{aligned}$$

since we have  $\frac{1}{2} + \frac{\varepsilon}{\gamma+\varepsilon} + \frac{\gamma-\varepsilon}{2(\gamma+\varepsilon)} = 1$ ,  $\frac{1}{2} + \frac{\varepsilon}{\gamma+\varepsilon} + \frac{5\gamma-\varepsilon}{6(\gamma+\varepsilon)} - \frac{1}{3} = 1$ , and  $\frac{6(\gamma+\varepsilon)}{5\gamma-\varepsilon} = 3\frac{10\gamma-6}{13\gamma+3} < 3$ .

We now want to treat the term:  $\rho^\varepsilon |(-\Delta)^{-1} \operatorname{div}(\rho \nabla(\phi * \rho))|$ , so we have:  $\rho \nabla(\phi * \rho) = \rho(\nabla \phi * \rho) \in L^\infty(L^1 \cap L^{\frac{s}{2}})$  by Hölder inequalities and the fact that we have  $\rho \in L^\infty(L^1)$  and  $\nabla \phi \in L^1$ .

After we get that  $\rho^\varepsilon(-\Delta)^{-1} \operatorname{div}(\rho \nabla(\phi * \rho)) \in L^\infty(L^{r_1})$  with:  $\frac{1}{r_1} = \frac{\varepsilon}{s} + \frac{2}{s} - \frac{1}{N} = \frac{2+\varepsilon}{s} - \frac{1}{N} < 1$ . We conclude that  $\rho^\varepsilon(-\Delta)^{-1} \operatorname{div}(\rho \nabla(\phi * \rho))$  is  $L^\infty(L^1)$  by using interpolation when  $N = 2, 3$ . Indeed we have  $\rho \nabla(\phi * \rho) \in L^\infty(L^1)$  and in choosing  $\varepsilon = \frac{2}{N}s - 1$  we have:  $1 - \frac{1}{N} + \frac{2}{N}s - 1 \geq 1$  when  $s \geq \frac{N}{2}(1 + \frac{1}{N})$ . This is the case when  $N = 2, 3$ .

We have after the term  $(\operatorname{div}(u))\rho^\varepsilon$ . We recall that  $\rho^\varepsilon$  is in  $L^\infty(L^{\frac{1}{\varepsilon}} \cap L^{\frac{s}{\varepsilon}})$ . If  $\varepsilon \geq \frac{1}{2}$  (i.e  $s \geq \frac{3}{4}N$ ), the bound is obvious because  $\frac{1}{2} + \varepsilon \geq 1$  and  $\frac{1}{2} + \frac{\varepsilon}{s} < 1$ , we can then conclude by interpolation. On the other hand, this rather simple term presents a technical difficulty when  $\varepsilon \leq \frac{1}{2}$  since we do not know in that case if  $\operatorname{div} u \rho^\varepsilon \in L^1(\mathbb{R}^N \times (0, T))$ . One way to get round the difficulty is to multiply (2.9) by  $\rho^\varepsilon 1_{\{\rho \geq 1\}}$ . Then we obtain an estimate on  $\rho^{s+\varepsilon} 1_{\{\rho \geq 1\}}$  in  $L^1((0, T) \times \mathbb{R}^N)$  as  $\rho^\varepsilon 1_{\{\rho \geq 1\}} |\operatorname{div} u| \leq \rho |\operatorname{div} u| \in L^1((0, T) \times \mathbb{R}^N)$  (where  $\varepsilon \leq \frac{1}{2}$ ) and we can conclude since  $0 \leq \rho^{s+\varepsilon} 1_{\{\rho < 1\}} \leq \rho$  on  $(0, T) \times \mathbb{R}^N$  and  $\rho \in L^\infty(L^1)$ .

We end with the following term  $\rho^\varepsilon(R_i R_j(\rho u_i u_j) - u_i R_i R_j(\rho u_j))$ . In the same way than in the previous inequalities we have  $\rho^\varepsilon R_i R_j(\rho u_i u_j)$  is bounded in  $L^1(0, T; L^1(\mathbb{R}^N))$ . Indeed we have by Hölder inequalities and the fact that  $R_i$  is continuous from  $L^p$  in  $L^p$  with  $1 < p < +\infty$ :  $\frac{1}{s} + 2\frac{N-2}{2N} + \frac{\varepsilon}{s} = 1 - \frac{2}{N} + \frac{1+\varepsilon}{s} \leq 1$  (because  $s > \frac{N}{2}$ ). And we conclude by interpolation. We treat the term  $\rho^\varepsilon u_i R_i R_j(\rho u_j)$  similarly.

We have to treat now the case  $N = 2$  where we have to modify the estimates when we are in critical cases for Sobolev embedding.

### Case $N = 2$ :

In the case  $N = 2$  most of the proof given above stay exact except for the slightly more delicate terms  $\rho^\varepsilon |\operatorname{div} u|(-\Delta)^{-1} \operatorname{div}(\rho u)|$  and  $\rho^\varepsilon(R_i R_j(\rho u_i u_j) - u_i R_i R_j(\rho u_j))$ .

We start with the term  $|\rho^\varepsilon \operatorname{div} u(-\Delta)^{-1} \operatorname{div}(\rho u)|$ . In our previous estimate it was possible to use Sobolev embedding on the term  $(-\Delta)^{-1} \operatorname{div}(\rho u)|$  only if  $r \geq N$  (see above the notation), so in the case where  $N = 2$  we are in a critical case for the Sobolev embedding when  $\gamma \geq 2$ .

This may be overcome by using that, by virtue of Sobolev embedding, we have:

$$\| |\operatorname{div} u| \rho^\varepsilon |(-\Delta)^{-1} \operatorname{div}(\rho u)| \|_{L^1} \leq C \|\rho\|_{L^{\gamma+\varepsilon}(L^{\gamma+\varepsilon})}^\varepsilon \|\rho u\|_{L^{2(\gamma+\varepsilon)}(L^{\frac{2(\gamma+\varepsilon)}{\gamma+\varepsilon+1}})}$$

Indeed by Hölder inequality, we have:

$$\frac{1}{2} + \frac{\varepsilon}{\gamma + \varepsilon} + \frac{\gamma + \varepsilon + 1}{2(\gamma + \varepsilon)} - \frac{1}{2} = \frac{1}{2} + \frac{2\varepsilon + 1}{2\varepsilon + 2\gamma} \leq 1 = \frac{1}{2} + \frac{\varepsilon}{\gamma + \varepsilon} + \frac{1}{2(\gamma + \varepsilon)} \leq 1,$$

thus:

$$\frac{1}{2} + \frac{\varepsilon}{\gamma + \varepsilon} + \frac{1}{2(\gamma + \varepsilon)} \leq 1.$$

Moreover we have as  $\rho u = \sqrt{\rho}\sqrt{\rho}u$ , hence:

$$\|\rho u\|_{L^{2(\gamma+\varepsilon)}(L^{\frac{2(\gamma+\varepsilon)}{\gamma+\varepsilon+1}})} \leq C\|\rho\|_{L^{\gamma+\varepsilon}(L^{\gamma+\varepsilon})}^{\frac{1}{2}}$$

and thus:

$$\| |\operatorname{div} u| \rho^\varepsilon |(-\Delta)^{-1} \operatorname{div}(\rho u)| \|_{L^1(L^1)} \leq C\|\rho\|_{L^{\gamma+\varepsilon}(L^{\gamma+\varepsilon})}^{\varepsilon+\frac{1}{2}}.$$

Next we are interested by the term  $\rho^\varepsilon(R_i R_j(\rho u_i u_j) - u_i R_i R_j(\rho u_j))$ . We use the fact that  $u$  is bounded in  $L^2(0, T; \dot{H}^1)$  and thus in  $L^2(0, T; BMO)$ . Then by the Coifman-Rochberg-Weiss commutator theorem in [3], we have for almost all  $t \in [0, T]$ :

$$\|R_i R_j(\rho u_i u_j) - u_i R_i R_j(\rho u_j)\|_{L^{\frac{2(\gamma+\varepsilon)}{\gamma+\varepsilon+1}}(L^{\frac{2(\gamma+\varepsilon)}{\gamma+\varepsilon+1}})} \leq C\|u\|_{L^2(BMO)}\|\rho u\|_{L^{2(\gamma+\varepsilon)}(L^{\frac{2(\gamma+\varepsilon)}{\gamma+\varepsilon+1}})}.$$

So we have:

$$\|\rho^\varepsilon(R_i R_j(\rho u_i u_j) - u_i R_i R_j(\rho u_j))\|_{L^1} \leq C\|\rho\|_{L^{\gamma+\varepsilon}(L^{\gamma+\varepsilon})}^{\varepsilon+\frac{1}{2}}.$$

In view of the previous inequalities we get finally:

$$\|\rho\|_{L^{\gamma+\varepsilon}(L^{\gamma+\varepsilon})}^{\gamma+\varepsilon} \leq C(1 + \|\rho\|_{L^{\gamma+\varepsilon}(L^{\gamma+\varepsilon})}^{\frac{1}{2}+\varepsilon})$$

and the  $L^{\gamma+\varepsilon}(L^{\gamma+\varepsilon})$  bound on  $\rho$  is proven since  $\frac{1}{2} + \varepsilon < \gamma + \varepsilon$ .  $\square$

## 2.2 Compactness results in the case of isentropic pressure

So by following the theorem 2.2, we assume that  $\gamma \geq 1$  such that if  $(\rho, u)$  is a regular approximate solution (it means built by a mollifying processus) then  $\rho \in L^q((0, T) \times \mathbb{R}^N)$  with  $q = \gamma + 1 - \frac{2\gamma}{N}$ . We can observe that in this case  $q > s = \max(\gamma, 2)$ . We will see that it will be very useful in the sequel to justify the passage to the weak limit in some terms to get a gain of integrability on the density. Indeed the key point to proving the existence of weak solutions is the passage to the limit in the term of pressure and in the term of capillarity  $\rho \nabla(\phi * \rho - \rho)$ .

First, we assume that a sequence  $(\rho_n, u_n)_{n \in \mathbb{N}}$  of approximate weak solutions has been constructed by a mollifying process, which have suitable regularity to justify the formal estimates like the energy estimate (1.3) and the theorem 2.2.  $(\rho_n, u_n)_{n \in \mathbb{N}}$  has the initial data of the theorem 1.1 with uniform bounds.

In addition:

- $\rho_n$  is bounded uniformly in  $L^\infty(0, T; L^1 \cap L^s(\mathbb{R}^N)) \cap C([0, T]; L^p(\mathbb{R}^N))$  for  $1 \leq p < \max(2, \gamma)$ ,

- $\rho_n \geq 0$  a.e. and  $\rho_n$  is bounded uniformly in  $L^q(0, T, \mathbb{R}^N)$  for some  $q > s$ ,
- $\nabla u_n$  is bounded in  $L^2(0, T; L^2(\mathbb{R}^N))$ ,  $\rho_n |u_n|^2$  is bounded in  $L^\infty(0, T; L^1(\mathbb{R}^N))$ ,
- $u_n$  is bounded in  $L^2(0, T; L^{\frac{2N}{N-2}}(\mathbb{R}^N))$  for  $N = 3$ .

Passing to the weak limit in the previous bound in extracting subsequence if necessary, one can assume that:

- $\rho_n \rightarrow \rho$  weakly in  $L^s((0, T) \times \mathbb{R}^N)$ ,
- $u_n \rightarrow u$  weakly in  $L^2(0, T, \dot{H}^1(\mathbb{R}^N))$ ,
- $\rho_n^\gamma \rightarrow \overline{\rho^\gamma}$  weakly in  $L^r((0, T) \times \mathbb{R}^N)$  for  $r = \frac{q}{\gamma} > 1$ ,
- $\rho_n^2 \rightarrow \overline{\rho^2}$  weakly in  $L^{r_1}((0, T) \times \mathbb{R}^N)$  for  $r_1 = \frac{q}{2} > 1$ .

**Notation 1** We will always write in the sequel  $\overline{B(\rho)}$  to mean the weak limit of the sequence  $B(\rho_n)$  bounded in appropriate space that we will precise.

We recall that the main difficulty will be to pass to the limit in the pressure term and the capillary term. The idea of the proof will be to test the convergence of the sequence  $(\rho_n)_{n \in \mathbb{N}}$  on convex functions  $B$  in order to use their properties of lower semi-continuity with respect to the weak topology in  $L^1(\mathbb{R}^N)$ . In this goal we will use the theory of renormalized solutions introduced by Di Perna and Lions in [7]. So we will obtain strong convergence of  $\rho_n$  in appropriate spaces.

### 2.3 Outline of the proof

We here give a sketchy proof of the theorem 1.1. First, we can rewrite mass conservation of the regular solution  $(\rho_n, u_n)_{n \in \mathbb{N}}$  on the form:

$$\frac{\partial}{\partial t}(B(\rho_n)) + \operatorname{div}(u_n B(\rho_n)) = (B(\rho_n) - \rho_n B'(\rho_n)) \operatorname{div} u_n.$$

Supposing that  $B(\rho_n)$  is bounded in appropriate space we can pass to the weak limit where we have in the energy space  $\rho_n \rightharpoonup \rho$  and  $u_n \rightharpoonup u$ , so we get:

$$\frac{\partial}{\partial t}(\overline{B(\rho)}) + \operatorname{div}(u \overline{B(\rho)}) = \overline{b(\rho) \operatorname{div} u} \quad \text{with } b(\rho) = B(\rho) - \rho B'(\rho). \quad (2.13)$$

Arguing like P-L. Lions in [19] p 13 we will get:

$$\frac{\partial}{\partial t} \rho + \operatorname{div}(\rho u) = 0. \quad (2.14)$$

After we will just have to verify the passage to the limit for the product  $\rho u$ . Next we will use the theorem on the renormalized solutions of Di Perna-Lions in [18] on (2.14) in recalling that  $\rho \in L^\infty(L^2)$ . So we get:

$$\frac{d}{dt}(B(\rho)) + \operatorname{div}(u B(\rho)) = b(\rho) \operatorname{div}(u). \quad (2.15)$$

Next we subtract (2.13) to (2.15), so we obtain:

$$\frac{d}{dt}(\overline{B(\rho)} - B(\rho)) + \operatorname{div}(u(\overline{B(\rho)} - B(\rho))) = \overline{b(\rho)\operatorname{div}u} - b(\rho)\operatorname{div}u. \quad (2.16)$$

Consequently, in order to estimate the difference  $\overline{B(\rho)} - B(\rho)$  which tests the convergence of  $\rho_n$ , we need to estimate the difference  $\overline{b(\rho)\operatorname{div}(u)} - b(\rho)\operatorname{div}(u)$ . We choose then  $B$  a concave function and we have:

$$\overline{B(\rho)} - B(\rho) \leq 0.$$

The goal will be now to prove the reverse inequality in order to justify that  $B(\rho_n)$  tends to  $B(\rho)$  a.e.

So now we aim at estimating the difference  $\overline{b(\rho)\operatorname{div}u} - b(\rho)\operatorname{div}u$ . This may be achieved by introducing the effective viscous pressure  $P_{eff} = P - (2\mu + \lambda)\operatorname{div}u$  after D. Hoff in [15], which satisfies some important properties of weak convergence.

In fact owing to the capillarity term we adapt Hoff's concept to our equation in setting:

$$\tilde{P}_{eff} = P + \frac{\kappa}{2}\rho^2 - (2\mu + \lambda)\operatorname{div}u.$$

### Proof of theorem 1.1

We begin with the case  $N \geq 3$ , and next we will complete the proof by the case  $N = 2$  in specifying the changes to bring. Before getting into the heart of the proof, we first recall that we obtain easily the convergence in distribution sense of  $\rho_n u_n$  to  $\rho u$  and  $\rho_n(u_n)_i(u_n)_j$  to  $\rho u_i u_j$ . We refer to the classical result by Lions (see [19]) or the book of Novotný and Straškraba [20].

#### Case $N = 3$

As explained in section 2.3 our goal is to compare  $\overline{B(\rho)}$  and  $B(\rho)$  for certain concave functions  $B$ . From the mass equation we have obtained:

$$\partial_t(\overline{B(\rho)} - B(\rho)) + \operatorname{div}(u(\overline{B(\rho)} - B(\rho))) = \overline{b(\rho)\operatorname{div}(u)} - b(\rho)\operatorname{div}(u). \quad (2.17)$$

So before comparing  $\overline{B(\rho)}$  and  $B(\rho)$ , we have to investigate the expression  $\overline{b(\rho)\operatorname{div}(u)} - b(\rho)\operatorname{div}(u)$ . By virtue of theorem 2.2 which gives a gain of integrability we can take the function  $B(x) = x^\varepsilon$ , as we control for  $\varepsilon$  small enough  $\rho^{s+\varepsilon}$ . Our goal now is to exhibit the effective pressure  $\tilde{P}_{eff}$ , and to multiply it by  $\rho^\varepsilon$  to extract  $\overline{\operatorname{div}u b(\rho)}$ . We will see in the sequel how to compare it with  $b(\rho)\operatorname{div}(u)$ .

#### Control of the term $\overline{\operatorname{div}u b(\rho)}$

Taking the div of the momentum equation satisfied by the regular solution yields:

$$\frac{\partial}{\partial t}\operatorname{div}(\rho_n u_n) + \partial_{ij}^2(\rho_n u_n^i u_n^j) - \zeta \Delta \operatorname{div}u_n + \Delta(a\rho_n^\gamma) = \kappa \operatorname{div}(\rho_n(\nabla\phi * \rho_n)) - \frac{\kappa}{2}\Delta(\rho_n^2),$$

with  $\zeta = \lambda + 2\mu$ . Applying the operator  $(-\Delta)^{-1}$  to previous inequality, we obtain:

$$\begin{aligned} \frac{\partial}{\partial t}(-\Delta)^{-1}\operatorname{div}(\rho_n u_n) + (-\Delta)^{-1}\partial_{ij}^2(\rho_n u_n^i u_n^j) + [\zeta \operatorname{div} u_n - a\rho_n^\gamma - \frac{\kappa}{2}\rho_n^2] \\ = \kappa(-\Delta)^{-1}\operatorname{div}(\rho_n(\nabla\phi * \rho_n)) . \end{aligned} \quad (2.18)$$

After we multiply (2.18) by  $\rho_n^\varepsilon$  with  $\varepsilon$  that we choose small enough in  $(0, 1)$ :

$$\begin{aligned} [\zeta \operatorname{div} u_n - a\rho_n^\gamma - \frac{\kappa}{2}\rho_n^2]\rho_n^\varepsilon = \kappa\rho_n^\varepsilon(-\Delta)^{-1}\operatorname{div}(\rho_n(\nabla\phi * \rho_n)) \\ - \rho_n^\varepsilon \frac{\partial}{\partial t}(-\Delta)^{-1}\operatorname{div}(\rho_n u_n) - \rho_n^\varepsilon(-\Delta)^{-1}\partial_{ij}^2(\rho_n u_n^i u_n^j). \end{aligned} \quad (2.19)$$

So if we rewrite (2.19), we have:

$$\begin{aligned} [\zeta \operatorname{div} u_n - a\rho_n^\gamma - \frac{\kappa}{2}\rho_n^2]\rho_n^\varepsilon = \kappa\rho_n^\varepsilon(-\Delta)^{-1}\operatorname{div}(\rho_n(\nabla\phi * \rho_n)) - \rho_n^\varepsilon(-\Delta)^{-1}\partial_{ij}^2(\rho_n u_n^i u_n^j) \\ - \frac{\partial}{\partial t}((\rho_n)^\varepsilon(-\Delta)^{-1}\operatorname{div}(\rho_n u_n)) + [\frac{\partial}{\partial t}(\rho_n)^\varepsilon](-\Delta)^{-1}\operatorname{div}(\rho_n u_n), \end{aligned}$$

Next we have:

$$\begin{aligned} [\zeta \operatorname{div} u_n - a\rho_n^\gamma - \frac{\kappa}{2}\rho_n^2]\rho_n^\varepsilon = \kappa\rho_n^\varepsilon(-\Delta)^{-1}\operatorname{div}(\rho_n(\nabla\phi * \rho_n)) - \rho_n^\varepsilon(-\Delta)^{-1}\partial_{ij}^2(\rho_n u_n^i u_n^j) \\ - \frac{\partial}{\partial t}[(\rho_n)^\varepsilon(-\Delta)^{-1}\operatorname{div}(\rho_n u_n)] - \operatorname{div}[u_n(\rho_n)^\varepsilon(-\Delta)^{-1}\operatorname{div}(\rho_n u_n)] \\ + (\rho_n)^\varepsilon u_n \cdot \nabla(-\Delta)^{-1}\operatorname{div}(\rho_n u_n) + (1 - \varepsilon)(\operatorname{div} u_n)(\rho_n)^\varepsilon(-\Delta)^{-1}\operatorname{div}(\rho_n u_n), \end{aligned} \quad (2.20)$$

or finally:

$$\begin{aligned} [\zeta \operatorname{div} u_n - a\rho_n^\gamma - \frac{\kappa}{2}\rho_n^2]\rho_n^\varepsilon = \kappa\rho_n^\varepsilon(-\Delta)^{-1}\operatorname{div}(\rho_n(\nabla\phi * \rho_n)) \\ - \frac{\partial}{\partial t}[\rho_n^\varepsilon(-\Delta)^{-1}\operatorname{div}(\rho_n u_n)] - \operatorname{div}[u_n(\rho_n)^\varepsilon(-\Delta)^{-1}\operatorname{div}(\rho_n u_n)] \\ + (\rho_n)^\varepsilon[u_n \cdot \nabla(-\Delta)^{-1}\operatorname{div}(\rho_n u_n) - (-\Delta)^{-1}\partial_{ij}^2(\rho u_n^i u_n^j)] \\ + (1 - \varepsilon)(\operatorname{div} u_n)(\rho_n)^\varepsilon(-\Delta)^{-1}\operatorname{div}(\rho_n u_n) . \end{aligned} \quad (2.21)$$

Now like in Lions [19] we want to pass to the limit in the distribution sense in (2.21) in order to estimate  $\operatorname{div} u(\rho)^\varepsilon$ .

### Passing to the weak limit in (2.21)

We shall use the following lemma by P-L Lions in [19]:

**Lemma 1** *Let  $\Omega$  be an open set of  $\mathbb{R}^N$ . Let  $(g_n, h_n)$  converge weakly to  $(g, h)$  in  $L^{p_1}(0, T, L^{p_2}(\Omega)) \times L^{q_1}(0, T, L^{q_2}(\Omega))$  where  $1 \leq p_1, p_2, q_1, q_2 \leq +\infty$  satisfy,  $\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = 1$ . Assume in addition that:*

$$\frac{\partial g^n}{\partial t} \text{ is bounded in } L^1(0, T, W^{-m, 1}(\Omega)) \text{ for some } m \geq 0 \text{ independent of } n. \quad (2.22)$$

and that:

$$\|h^n - h^n(\cdot, \cdot + \xi)\|_{L^{q_1}(0, T, L^{q_2}(\Omega))} \rightarrow 0 \quad \text{as } |\xi| \rightarrow 0, \text{ uniformly in } n. \quad (2.23)$$

Then,  $g^n h^n$  converges to  $gh$  (in the sense of distribution on  $\Omega \times (0, T)$ ).

So we use the above lemma to pass to the weak limit in the following four non-linear terms of (2.21):

$$\begin{aligned} T_n^1 &= u_n \rho_n^\varepsilon (-\Delta)^{-1} \operatorname{div}(\rho_n u_n), & T_n^2 &= \rho_n^\varepsilon (-\Delta)^{-1} \operatorname{div}(\rho_n u_n), \\ T_n^3 &= \rho_n^\varepsilon (-\Delta)^{-1} \operatorname{div}(\rho_n (\nabla \phi * \rho_n)), & T_n^4 &= (\operatorname{div} u_n) (\rho_n)^\varepsilon (-\Delta)^{-1} \operatorname{div}(\rho_n u_n). \end{aligned}$$

So we choose the different  $g_n^i$  and  $h_n^i$  as follows:

$$\begin{aligned} \text{for } T_n^1 & \quad g_n^1 = u_n (\rho_n)^\varepsilon & g^1 &= u \overline{\rho^\varepsilon} & h_n^1 &= (-\Delta)^{-1} \operatorname{div}(\rho_n u_n) \\ \text{for } T_n^2 & \quad g_n^2 = \rho_n^\varepsilon & g^2 &= \overline{\rho^\varepsilon} & h_n^2 &= (-\Delta)^{-1} \operatorname{div}(\rho_n u_n) \\ \text{for } T_n^3 & \quad g_n^3 = \rho_n^\varepsilon & g^3 &= \overline{\rho^\varepsilon} & h_n^3 &= (-\Delta)^{-1} \operatorname{div}(\rho_n (\nabla \phi * \rho_n)) \\ \text{for } T_n^4 & \quad g_n^4 = (\operatorname{div} u_n) (\rho_n)^\varepsilon & g^4 &= \overline{\operatorname{div} u \rho^\varepsilon} & h_n^4 &= (-\Delta)^{-1} \operatorname{div}(\rho_n u_n). \end{aligned}$$

To show that  $u_n (\rho_n)^\varepsilon$  converges in distribution sense to  $u \overline{\rho^\varepsilon}$  we apply lemma 1 with  $h_n = u_n$  and  $g_n = \rho_n^\varepsilon$ . We now want to examine each term and apply the above lemma to pass to the limit in the weak sense. We start with the first term  $T_n^1$ . We have that  $\rho_n^\varepsilon u_n \in L^\infty(L^q) \cap L^2(L^r)$  with  $\frac{1}{q} = \frac{\varepsilon}{2s} + \frac{1}{2}$  and  $\frac{1}{r} = \frac{(N-2)}{2N} + \frac{\varepsilon}{s} = \frac{1}{2} - \frac{1}{N} + \frac{\varepsilon}{s}$ . In addition the hypothesis (2.22) is immediately verified (use the momentum equation).

We now want to verify the hypothesis (2.23), so we have  $h_n^1$  belongs to  $L^\infty(W_{loc}^{1,q'}(\mathbb{R}^N)) \cap L^2(W_{loc}^{1,r'}(\mathbb{R}^N))$  with  $\frac{1}{q'} = \frac{1}{2} + \frac{1}{2s}$  and  $\frac{1}{r'} = \frac{(N-2)}{2N} + \frac{1}{s} = \frac{1}{2} + \frac{1}{s} - \frac{1}{N}$ . This result enables us to verify the hypothesis (2.23) by Sobolev embedding. So we can choose (with the notation of the above lemma)  $q_1 = 2$  and  $q_2 \in (r', \frac{Nr'}{N-r'})$ ,  $p_1 = 2$ , and  $p_2 = 1 - \frac{1}{q_2}$  which is possible by interpolation. Indeed we have:  $\frac{1}{r'} + \frac{1}{r} = 1 - \frac{2}{N} + \frac{1+\varepsilon}{s} \leq 1$ .

We proceed in the same way for  $T_n^2$  and  $T_n^4$ . We can similarly examine  $T_n^3$ , because  $\rho_n^\varepsilon \in L^\infty(L^{\frac{1}{\varepsilon}} \cap L^{\frac{s}{\varepsilon}})$  and  $\rho_n (\nabla \phi * \rho_n) \in L^\infty(L^1 \cap L^{\frac{s}{2}})$ , we can choose  $p_2 = \frac{1}{\varepsilon}$ , we have then  $(\Delta)^{-1} \operatorname{div} \rho_n (\nabla \phi * \rho_n) \in L^\infty(0, T; W^{1, \frac{s}{2}})$  so that we can choose  $q_1 = 2$ ,  $q_2 \in (1, \frac{N \frac{s}{2}}{N - \frac{s}{2}})$ . We can conclude by interpolation.

Finally we have to study the last non linear following term that we treat similarly as P-L.Lions in [19]:

$$A_n = (\rho_n)^\varepsilon [u_n \cdot \nabla (-\Delta)^{-1} \operatorname{div}(\rho_n u_n) - (-\Delta)^{-1} \partial_{ij}^2 (\rho_n (u_i)_n (u_j)_n)].$$

We can express this term  $A_n$  as follows:

$$A_n = (\rho_n)^\varepsilon [u_n^j, R_{ij}] (\rho_n u_n^i).$$

where  $R_{ij} = (-\Delta)^{-1} \partial_{ij}^2$  with  $R_i$  the classical Riesz transform. Next, we use a result by Coifman, Lions, Meyer, Semmes on this type of commutator (see [4]) to take advantage of the regularity of  $[u_n^j, R_{ij}] (\rho_n u_n^i)$ .

**Theorem 2.3** *The following map is continuous for any  $N \geq 2$ :*

$$\begin{aligned} W^{1,r_1}(\mathbb{R}^N)^N \times L^{r_2}(\mathbb{R}^N) &\rightarrow W^{1,r_3}(\mathbb{R}^N)^N \\ (a, b) &\rightarrow [a_j, R_i R_j] b_i \end{aligned} \tag{2.24}$$

with:  $\frac{1}{r_3} = \frac{1}{r_1} + \frac{1}{r_2}$ .

To pass to the weak limit in  $A_n$  we will use the previous lemma. We start with the case with  $s > 3$ . This quantity belongs to the space  $L^1(W^{1,q})$  provided that  $Du_n \in L^2(L^2)$  and  $\rho u^j \in L^2(L^r)$  where  $\frac{1}{r} = \frac{N-2}{2N} + \frac{1}{s} = \frac{1}{2} - \frac{1}{N} + \frac{1}{s}$  in which case  $\frac{1}{q} = \frac{1}{r} + \frac{1}{2} = 1 - \frac{1}{N} + \frac{1}{s} \leq 1$ . After we can use the above lemma applied to  $h_n = [R_{ij}, u_n^j](\rho_n u_n^i)$  and  $g_n = \rho_n^\varepsilon$ . We can show easily in using again lemma 1 that  $h_n$  converges in distribution sense to  $[R_{ij}, u_j](\rho_n u_i)$ . So we can take:  $q_1 = 1$ ,  $p_1 = +\infty$  and  $q_2 \in (q, \frac{qN}{N-q})$ ,  $p_2 = 1 - \frac{1}{q_2}$ , this one because we can use interpolation and we can localize as we want limit in distribution sense.

In the case where  $s \leq 3$ , a simple interpolation argument can be used to accommodate the general case. It suffices to fix  $L^{r_2}(\mathbb{R}^N)$  in the application (2.24) and use a result of Riesz-Thorin. Finally passing to the limit in (2.19), we get:

$$\begin{aligned} [\zeta \overline{\operatorname{div} u \rho^\varepsilon} - \overline{(a\rho^{\gamma+\varepsilon})} - \frac{\kappa}{2} \overline{\rho^{2+\varepsilon}}] &= \overline{\rho^\varepsilon} (-\Delta)^{-1} \operatorname{div}(\overline{\rho(\nabla \phi * \rho)}) \\ - \frac{\partial}{\partial t} [\overline{\rho^\varepsilon} (-\Delta)^{-1} \operatorname{div}(\rho u)] &- \operatorname{div}[\overline{\rho^\varepsilon} u (-\Delta)^{-1} \operatorname{div}(\rho u)] + \overline{\rho^\varepsilon} [u \cdot \nabla (-\Delta)^{-1} \operatorname{div}(\rho u) \\ &- (-\Delta)^{-1} \partial_{ij}(\rho u_i u_j)] + (1 - \varepsilon) \overline{\operatorname{div} u \rho^\varepsilon} (-\Delta)^{-1} \operatorname{div}(\rho u). \end{aligned} \quad (2.25)$$

### Inequality between the terms $\overline{\rho^\varepsilon} \operatorname{div} u$ and $\overline{\operatorname{div} u \rho^\varepsilon}$

Now we are interested in estimating the term  $\overline{\rho^\varepsilon} \operatorname{div} u$  in order to describe the quantity  $\overline{\rho^\varepsilon} \operatorname{div} u - \overline{\operatorname{div} u \rho^\varepsilon}$  before considering the quantity  $\rho^\varepsilon \operatorname{div} u - \overline{\operatorname{div} u \rho^\varepsilon}$ . We pass to the weak limit directly in (2.18) and we get in using again lemma 1:

$$\begin{aligned} \frac{\partial}{\partial t} (-\Delta)^{-1} \operatorname{div}(\rho u) + (-\Delta)^{-1} \partial_{ij}^2(\rho u_i u_j) &+ [(\mu + \xi) \operatorname{div} u - \overline{a\rho^\gamma}] = \\ &- (-\Delta)^{-1} \operatorname{div}(\overline{\rho(\nabla \phi * \rho)}) + \frac{\kappa}{2} \overline{\rho^2}. \end{aligned} \quad (2.26)$$

Now we just multiply (2.26) with  $\overline{\rho^\varepsilon}$  and we can see that each term has a distribution sense. So we get by proceeding in the same way as before:

$$\begin{aligned} [\zeta \operatorname{div} u \overline{\rho^\varepsilon} - \overline{(a\rho^\gamma)} \overline{\rho^\varepsilon} - \frac{\kappa}{2} \overline{\rho^2 \rho^\varepsilon}] &= \overline{\rho^\varepsilon} (-\Delta)^{-1} \operatorname{div}(\overline{\rho(\nabla \phi * \rho)}) \\ - \overline{\rho^\varepsilon} \frac{\partial}{\partial t} [\rho (-\Delta)^{-1} \operatorname{div}(\rho u)] &+ \overline{\rho^\varepsilon} [u \cdot \nabla (-\Delta)^{-1} \operatorname{div}(\rho u) - (-\Delta)^{-1} \partial_{ij}(\rho u_i u_j)] \\ &- \operatorname{div}[\rho^\varepsilon u (-\Delta)^{-1} \operatorname{div}(\rho u)] + (1 - \varepsilon) \overline{\operatorname{div} u (\rho)^\varepsilon} (-\Delta)^{-1} \operatorname{div}(\rho u). \end{aligned} \quad (2.27)$$

Subtracting (2.27) from (2.25), we get:

$$\zeta \overline{\operatorname{div} u (\rho)^\varepsilon} - \overline{a\rho^{\gamma+\varepsilon}} - \frac{\kappa}{2} \overline{\rho^{2+\varepsilon}} = \zeta \operatorname{div} u \overline{\rho^\varepsilon} - \overline{a\rho^\gamma} \overline{\rho^\varepsilon} - \frac{\kappa}{2} \overline{\rho^2 \rho^\varepsilon} \quad \text{a.e.}$$

Next we observe that by convexity:

$$(\overline{\rho^{\gamma+\varepsilon}})^{\frac{\varepsilon}{\gamma+\varepsilon}} \geq (\overline{\rho^\varepsilon}), \quad (\overline{\rho^{\gamma+\varepsilon}})^{\frac{\gamma}{\gamma+\varepsilon}} \geq (\overline{\rho^\gamma}) \quad \text{a.e.}$$

So we get:

$$\overline{\operatorname{div} u (\rho)^\varepsilon} \geq \operatorname{div} u \overline{\rho^\varepsilon}. \quad (2.28)$$



### Comparison between $\rho$ and $\overline{\rho^\varepsilon}^{\frac{1}{\varepsilon}}$

As  $(\rho_n, u_n)$  is a smooth approximate solution, applying equality (2.15) to  $B(x) = x^\varepsilon$  yields:

$$\frac{\partial}{\partial t}(\rho_n)^\varepsilon + \operatorname{div}(u_n(\rho_n)^\varepsilon) = (1 - \varepsilon)\operatorname{div}u_n(\rho_n)^\varepsilon. \quad (2.29)$$

Passing to the weak limit in (2.29), we get:

$$\frac{\partial}{\partial t}\overline{\rho^\varepsilon} + \operatorname{div}(u\overline{\rho^\varepsilon}) = (1 - \varepsilon)\overline{\operatorname{div}u\rho^\varepsilon}. \quad (2.30)$$

Combining with (2.28) we thus get:

$$\frac{\partial}{\partial t}(\overline{\rho})^\varepsilon + \operatorname{div}(u(\overline{\rho})^\varepsilon) \geq (1 - \varepsilon)\operatorname{div}u(\overline{\rho})^\varepsilon. \quad (2.31)$$

Now we wish to conclude about the pointwise convergence of  $\rho_n$  in proving that  $(\overline{\rho^\varepsilon})^{\frac{1}{\varepsilon}} = \rho$  and to finish we will use the following theorem (see [9] p 34) applied to  $B(x) = x^{\frac{1}{\varepsilon}}$  which is convex.

**Theorem 2.4** *Let  $(v_n)_{n \in \mathbb{N}}$  be a sequence of functions bounded in  $L^1(\mathbb{R}^N)$  such that:*

$$v_n \rightharpoonup v \quad \text{weakly in } L^1(\mathbb{R}^N).$$

*Let  $\varphi : \mathbb{R} \rightarrow [-\infty, +\infty)$  be a upper semi-continuous strictly concave function such that  $\varphi(v_n) \in L^1(\mathbb{R}^N)$  for any  $n$ , and:*

$$\varphi(v_n) \rightharpoonup \overline{\varphi(v)} \quad \text{weakly in } L^1(\mathbb{R}^N).$$

*Then:*

$$\varphi(v) \geq \overline{\varphi(v)}.$$

*and if  $\varphi(v) = \overline{\varphi(v)}$  then:*

$$v_n(y) \rightarrow v(y) \quad \text{a.e.}$$

*extracting a subsequence as the case may be.*

Now we want to use a type of Di Perna-Lions theorem on inequality (2.31). Our goal is to renormalize this inequality with the function  $B(x) = x^{\frac{1}{\varepsilon}}$  so that one can compare  $\rho$  and  $\overline{\rho^\varepsilon}^{\frac{1}{\varepsilon}}$ . Although (2.31) doesn't correspond exactly to the mass equation, we can use the same technics to renormalize the solution provided that  $\rho \in L^\infty(L^2)$  which is the case.<sup>1</sup> We recall Di Perna-Lions theorem on renormalized solution for the mass equation.

**Theorem 2.5** *Suppose that  $\rho \in L^\infty(L^2)$ ,  $\beta \in C[0, \infty) \cap C^1(0, \infty); \mathbb{R}$  and the function  $b(z) = z\beta'(z) - \beta(z)$  is bounded on  $[0, \infty)$  with moreover  $\beta(0) = b(0) = 0$ .*

*We have then:*

$$\frac{\partial \beta(\rho)}{\partial t} + \operatorname{div}(\beta(\rho) u) = (\beta(\rho) - \rho\beta'(\rho))\operatorname{div}u$$

*in distribution sense.*

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<sup>1</sup>In our case it is very important that  $\rho \in L^\infty(L^2)$ , indeed it avoids to have supplementary conditions on the index  $\gamma$  like for the compressible Navier-Stokes system in [19].

We now want to adapt this theorem for our equation (2.31) with  $\beta(x) = x^{\frac{1}{\varepsilon}}$ , so we may regularize by  $\omega_\alpha$  (with  $\omega_\alpha = \frac{1}{\alpha^N} \omega(\frac{\cdot}{\alpha})$  where  $\omega \in C_0^\infty(\mathbb{R}^N)$ ,  $\text{supp } \omega \in B_1$  and  $\int \omega dx = 1$ ) and find for all  $\beta \in C_0^\infty([0, +\infty))$ :

$$\frac{\partial}{\partial t}(\overline{\rho^\varepsilon} * \omega_\alpha) + \text{div}[u \overline{\rho^\varepsilon} * \omega_\alpha] \geq (1 - \varepsilon) \text{div} u \overline{\rho^\varepsilon} * \omega_\alpha + R_\alpha$$

where we have:

$$R_\alpha = \text{div}[u \overline{\rho^\varepsilon} * \omega_\alpha] - \text{div}(u \overline{\rho^\varepsilon}) * \omega_\alpha + (1 - \varepsilon) [\text{div} u \overline{\rho^\varepsilon}] * \omega_\alpha - (1 - \varepsilon) \text{div} u \overline{\rho^\varepsilon} * \omega_\alpha$$

We get:

$$\begin{aligned} \frac{\partial}{\partial t}(\beta(\overline{\rho^\varepsilon} * \omega_\alpha)) + \text{div}[u \beta(\overline{\rho^\varepsilon} * \omega_\alpha)] &\geq (1 - \varepsilon) \text{div} u \overline{\rho^\varepsilon} * \omega_\alpha \beta'(\overline{\rho^\varepsilon} * \omega_\alpha) \\ &+ (\text{div} u) [\beta(\overline{\rho^\varepsilon} * \omega_\alpha) - \overline{\rho^\varepsilon} * \omega_\alpha \beta'(\overline{\rho^\varepsilon} * \omega_\alpha)] + R_\alpha \beta'(\overline{\rho^\varepsilon} * \omega_\alpha) \\ &= -\varepsilon (\text{div} u) \overline{(\rho^\varepsilon)}^\varepsilon \beta'(\overline{\rho^\varepsilon}) + (\text{div} u) \beta(\overline{\rho^\varepsilon}) + R_\alpha \beta'(\overline{\rho^\varepsilon} * \omega_\alpha). \end{aligned}$$

After we pass to the limit when  $\alpha \rightarrow 0$  and we see that  $R_\alpha$  tends to 0 in using lemma on regularization in [18] p 43. This looks like a rather harmless manipulation but it's at this point that we require to control  $\rho$  in  $L^2(0, T; \mathbb{R}^N)$ . And in our case we don't need to impose  $\gamma > \frac{N}{2}$  for  $N = 2, 3$ . Hence:

$$\frac{\partial}{\partial t}(\beta(\overline{(\rho^\varepsilon)})) + \text{div}[u \beta(\overline{(\rho^\varepsilon)})] \geq -\varepsilon (\text{div} u) \overline{\rho^\varepsilon} \beta'(\overline{\rho^\varepsilon}) + (\text{div} u) \beta(\overline{\rho^\varepsilon}).$$

We then choose  $\beta = (\Psi_M)^{\frac{1}{\varepsilon}}$  where  $\Psi_M = M \Psi(\frac{\cdot}{M})$ ,  $M \geq 1$ ,  $\Psi \in C_0^\infty([0, +\infty))$ ,  $\Psi(x) = x$  on  $[0, 1]$ ,  $\text{supp } \Psi \subset [0, 2]$ , and we obtain:

$$\begin{aligned} \frac{\partial}{\partial t}(\Psi_M(\overline{\rho^\varepsilon})^{\frac{1}{\varepsilon}}) + \text{div}[u \Psi_M(\overline{\rho^\varepsilon})^{\frac{1}{\varepsilon}}] &\geq (\text{div} u) \Psi_M(\overline{\rho^\varepsilon})^{\frac{1}{\varepsilon}-1} \Psi'_M(\overline{\rho^\varepsilon}) \overline{\rho^\varepsilon} + (\text{div} u) \Psi_M(\overline{\rho^\varepsilon})^{\frac{1}{\varepsilon}} \\ &\geq \text{div} u \Psi_M(\overline{\rho^\varepsilon})^{\frac{1}{\varepsilon}-1} [\Psi_M(\overline{\rho^\varepsilon})^{\frac{1}{\varepsilon}} - \Psi'_M(\overline{\rho^\varepsilon}) \overline{\rho^\varepsilon}] 1_{(\overline{\rho^\varepsilon} > M)} \\ &\geq -C_0 |\text{div} u| M^{\frac{1}{\varepsilon}} 1_{(\overline{\rho^\varepsilon} > M)}. \end{aligned}$$

where  $C_0 = \sup\{|\Psi(x)|^{\frac{1}{\varepsilon}-1} |\Psi(x) - x \Psi'(x)|, x \in [0, +\infty)\}$ . Now we claim that:

$$\frac{\partial}{\partial t}(\overline{(\rho^\varepsilon)}^{\frac{1}{\varepsilon}}) + \text{div}(u \overline{(\rho^\varepsilon)}^{\frac{1}{\varepsilon}}) \geq 0. \quad (2.32)$$

For proving that, we notice that by convexity  $\overline{(\rho^\varepsilon)}^{\frac{1}{\varepsilon}} \leq \rho$ , so we get :

$$\| |\text{div} u| M^{\frac{1}{\varepsilon}} 1_{(\overline{\rho^\varepsilon} > M)} \|_{L_T^1(L^1(\mathbb{R}^N))} \leq \| |\text{div} u| \|_{L_T^2(L^2(\mathbb{R}^N))} \| \rho 1_{\rho > M^{\frac{1}{\varepsilon}}} \|_{L_T^2(L^2(\mathbb{R}^N))} \rightarrow 0 \text{ as } M \rightarrow +\infty.$$

We have concluded by dominated convergence. At this stage we subtract the mass equation to (2.32) and we get in setting  $r = \rho - \overline{(\rho^\varepsilon)}^{\frac{1}{\varepsilon}}$ :

$$\frac{\partial}{\partial t}(r) + \text{div}(ur) \leq 0. \quad (2.33)$$

We now want to integrate and to use the fact that  $r \geq 0$  to get that  $r = 0$  a.a. To justify the integration we test our inequality against a cut-off function  $\varphi_R = \varphi(\frac{\cdot}{R})$  where  $\varphi \in C_0^\infty(\mathbb{R}^N)$ ,  $\varphi = 1$  on  $B(0, 1)$ ,  $\text{Supp}\varphi \subset B(0, 2)$  and  $R > 1$ . We get:

$$\int_{[0,T] \times \mathbb{R}^N} \frac{\partial}{\partial t} [r(t, x)] \varphi_R(x) - u(t, x) r(t, x) \frac{1}{R} \nabla \varphi\left(\frac{x}{R}\right) dt dx \leq 0. \quad (2.34)$$

Next we notice that:

$$\begin{aligned} \left| \int_{[0,T] \times \mathbb{R}^N} u(t, x) r(t, x) \frac{1}{R} \nabla \varphi\left(\frac{x}{R}\right) dt dx \right| &\leq \|u\|_{L^1(0,T; L^{\frac{2N}{N-2}}(\mathbb{R}^N))} \|r\|_{L^1(0,T; L^{\frac{2N}{N+2}}(\mathbb{R}^N))} \\ &\quad \times \frac{1}{R} \|\nabla \varphi\|_{L^\infty(\mathbb{R}^N)}. \end{aligned}$$

It implies that:

$$\int_{[0,T] \times \mathbb{R}^N} u(t, x) r(t, x) \frac{1}{R} \nabla \varphi\left(\frac{x}{R}\right) dt dx \rightarrow 0 \quad \text{as } R \rightarrow +\infty.$$

We have then:

$$\int_{[0,T] \times \mathbb{R}^N} \frac{\partial}{\partial t} r(t, x) \varphi_R(x) dt dx = \int_{\mathbb{R}^N} r(T, x) \varphi_R(T, x) dx - \int_{\mathbb{R}^N} r(0, x) \varphi_R(0, x) dx.$$

In order to conclude, it suffices to verify that  $r(0, \cdot) = 0$ . Indeed we will obtain that:

$$\lim_{R \rightarrow +\infty} \int_{\mathbb{R}^N} r(T, x) \varphi_R(T, x) dx \rightarrow \int_{\mathbb{R}^N} r(T, x) dx \leq 0 \quad \text{and} \quad r \geq 0.$$

then  $r = 0$ .

We know that  $\rho_n$  is uniformly bounded in  $L^\infty(L^1 \cap L^s(\mathbb{R}^N))$ , then  $\rho_n^\varepsilon$  is relatively compact in  $C([0, T]; L^p - w)$  with  $1 < p < s$  (where  $L^p - w$  denote the space  $L^p$  endowed with weak topology). As moreover  $(\rho_0^\varepsilon)_n$  converges to  $\rho_0^\varepsilon$ , we deduce that  $r(0) = 0$  a.a. Now as  $r = 0$  we conclude by using the theorem 2.4 that  $\rho_n$  converges a.a to  $\rho$  and that  $\rho_n$  converges to  $\rho$  in  $L^p([0, T] \times B_R)$  for all  $p \in [1, q)$  and in  $L^{p_1}(0, T, L^{p_2}(B_R))$  for all  $p_1 \in [1, +\infty)$ ,  $p_2 \in [1, s)$  and for all  $R \in (0, +\infty)$ .

## Conclusion

We wish now conclude and get the convergence of our theorem in the total space. We aim at proving here the convergence of  $\rho_n$  in  $C([0, T], L^p(\mathbb{R}^N)) \cap L^{q'}(\mathbb{R}^N \times (0, T))$  for all  $1 \leq p < s$ ,  $1 \leq q' < q$ . We have just to show the convergence of  $\rho_n$  to  $\rho$  in  $C([0, T], L^1(\mathbb{R}^N))$ . To this end, we introduce  $d_n = \sqrt{\rho_n}$  which clearly converges to  $\sqrt{\rho}$  in  $L^{2p_1}(0, T, L^{2p_2}(B_R)) \cap L^{2p}(B_R \times (0, T))$  to  $d = \sqrt{\rho}$  for all  $R \in (0, +\infty)$ . We next remark that  $\rho \in C([0, T], L^1(\mathbb{R}^N))$  and thus  $d \in C([0, T], L^2(\mathbb{R}^N))$ . Indeed, using once more the regularization lemma in [18] we obtain the existence of a bounded  $\rho_\alpha \in C([0, T], L^1(\mathbb{R}^N))$  smooth in  $x$  for all  $t$  satisfying:

$$\frac{\partial \rho_\alpha}{\partial t} + \text{div}(u \rho_\alpha) = r_\alpha \quad \text{in } L^1((0, T) \times \mathbb{R}^N) \quad \text{as } \alpha \rightarrow 0_+,$$

with  $r_\alpha = \operatorname{div}(u\rho_\alpha) - \operatorname{div}(\rho u) * \omega_\alpha$  (where  $\omega$  is defined as in the previous part).  $\rho_\alpha \rightarrow \rho$  in  $L^1(\mathbb{R}^N \times (0, T))$ ,  $\rho_\alpha/t=0 \rightarrow \rho/t=0$  in  $L^1(\mathbb{R}^N)$  as  $\alpha \rightarrow 0_+$ . From these facts, it is straightforward to deduce that:

$$\frac{\partial}{\partial t}|\rho_\alpha - \rho_\eta| + \operatorname{div}(u|\rho_\alpha - \rho_\eta|) = |r_\alpha - r_\eta|$$

and thus:

$$\sup_{[0, T]} \int_{\mathbb{R}^N} |\rho_\alpha - \rho_\eta| dx = \int_0^T \int_{\mathbb{R}^N} |r_\alpha - r_\eta| dx.$$

Since  $\rho \in C([0, T], L^p(B_R) - w)$  (for all  $R \in (0, +\infty)$ ,  $1 < p < s$ ), we may then deduce that  $\rho_\alpha$  converges to  $\rho$  in  $C([0, T], L^1(\mathbb{R}^N))$ . Next, we observe that we can justify as we did above that  $d_n$  and  $d$  satisfy:

$$\begin{aligned} \frac{\partial d_n}{\partial t} + \operatorname{div}(u_n d_n) &= \frac{1}{2} d_n \operatorname{div}(u_n), \\ \frac{\partial d}{\partial t} + \operatorname{div}(ud) &= \frac{1}{2} d \operatorname{div}(u). \end{aligned}$$

Therefore once more,  $d_n$  converges to  $d$  in  $C([0, T], L^2(\mathbb{R}^N) - w)$ . Thus in order to conclude, we just have to show that whenever  $t_n \in [0, T]$ ,  $t_n \rightarrow t$ , then  $d_n(t_n) \rightarrow d(t)$  in  $L^2(\mathbb{R}^N)$  or equivalently that:

$$\int_{\mathbb{R}^N} d_n(t_n)^2 dx = \int_{\mathbb{R}^N} \rho_n(t_n) dx \rightarrow_n \int_{\mathbb{R}^N} d(t)^2 dx = \int_{\mathbb{R}^N} \rho(t) dx.$$

This is the case since we deduce from the mass equation, integrating this equation over  $\mathbb{R}^N$  and justifying the integration exactly like previously that:

$$\int_{\mathbb{R}^N} \rho_n(t_n) dx = \int_{\mathbb{R}^N} (\rho_0)_n dx \rightarrow_n \int_{\mathbb{R}^N} \rho_0 dx = \int_{\mathbb{R}^N} \rho(t) dx.$$

We then conclude by uniform continuity that  $\|\rho_n(t_n) - \rho_n(t)\|_{L^1}$  tends to 0.

### Case $N = 2$

First of all, the main difficulty is the fact that we no longer have global  $L^p$  bounds on  $u_n$ . That's why most of the proof is in fact local and we know that  $u_n$  is bounded in  $L^2(0, T; L^p(B_R))$  for all  $p \in [1, +\infty)$ ,  $R \in (0, +\infty)$ . As we need to localize the argument, we get the following limit:

$$(\zeta \operatorname{div} u_n - a \rho_n^\gamma - \frac{1}{2} \rho_n^2) \rho_n^\varepsilon \rightharpoonup_n (\zeta \operatorname{div} u - a \overline{\rho^\gamma} - \frac{1}{2} \overline{\rho^2}) \overline{\rho^\varepsilon} \text{ in } \mathcal{D}'(\mathbb{R}^N \times [0, T]).$$

Let  $\varphi \in C_0^\infty(\mathbb{R}^N)$ ,  $0 \leq \varphi \leq 1$ ,  $\operatorname{supp} \varphi \subset K$  for an arbitrary compact set  $K \in \mathbb{R}^N$ . We apply the operator  $(\Delta)^{-1} \operatorname{div}$  to the momentum equation that we have localized and we pass directly to the weak limit:

$$\begin{aligned} \frac{\partial}{\partial t} (-\Delta)^{-1} \operatorname{div}(\varphi \rho u) + R_{ij}(\varphi \rho u_i u_j) + [\zeta \operatorname{div} u \varphi - \overline{a \rho^\gamma} \varphi] \\ = \kappa (-\Delta)^{-1} \operatorname{div}(\varphi \overline{\rho(\nabla \phi * \rho)} - \frac{\kappa}{2} \varphi \overline{\rho^2}) + (-\Delta)^{-1} \overline{R}. \end{aligned} \tag{2.35}$$

with:

$$\begin{aligned}\bar{R} &= \partial_i \varphi \partial_j (\rho u_i u_j) + (\partial_{ij} \varphi) \rho u_i u_j - \zeta \Delta \varphi \operatorname{div} u - \zeta \nabla \varphi \cdot \nabla \operatorname{div} u + \mu \Delta u \cdot \nabla \varphi \\ &+ \Delta \varphi a \bar{\rho}^\gamma + a \nabla \varphi \cdot \nabla \bar{\rho}^\gamma + \frac{\kappa}{2} \Delta \varphi \bar{\rho}^2 + \frac{\kappa}{2} \nabla \varphi \cdot \nabla \bar{\rho}^2.\end{aligned}$$

Now we multiply (2.35) with  $\bar{\rho}^\varepsilon$  and we verify that each term has a sense. So we get in proceeding in the same way as before, we can verify that  $(\rho_n)^\varepsilon (-\Delta)^{-1} R_n$  converges in distribution sense to  $\bar{\rho}^\varepsilon (-\Delta)^{-1} \bar{R}$  for small enough  $\varepsilon$ . We get as in the previous case for  $N \geq 3$ :

$$\varphi [\zeta \operatorname{div} u (\bar{\rho}^\varepsilon)^\varepsilon - a \bar{\rho}^{\gamma+\varepsilon} - \frac{\kappa}{2} \bar{\rho}^{2+\varepsilon}] = \varphi [\zeta (\operatorname{div} u) \varphi \bar{\rho}^\varepsilon - a \bar{\varphi} \bar{\rho}^\gamma \bar{\rho}^\varepsilon - \frac{\kappa}{2} \bar{\varphi} \bar{\rho}^2 \bar{\rho}^\varepsilon] \quad \text{a.e.}$$

We then deduce the following inequalities as in the previous proof:

$$\frac{d}{dt} (\bar{\rho}^\varepsilon)^{\frac{1}{\varepsilon}} + \operatorname{div} (u (\bar{\rho}^\varepsilon)^{\frac{1}{\varepsilon}}) \geq 0 \quad \text{in } \mathcal{D}'((0, T) \times \mathbb{R}^N).$$

We see that the only point left to check is the justification of the integration over  $\mathbb{R}^2$  of terms like  $\operatorname{div}(\bar{\rho}^{\frac{1}{\varepsilon}} u)$  or  $\operatorname{div}(\rho u)$  and more precisely that the integral vanishes. This is in fact straightforward provided we use the bounds on  $\rho \in L^\infty(L^1(\mathbb{R}^N))$  and  $\rho |u|^2 \in L^\infty(L^1(\mathbb{R}^N))$  and so  $\rho u \in L^\infty(L^1(\mathbb{R}^N))$ . Then, letting  $\varphi \in C_0^\infty(\mathbb{R}^2)$ ,  $0 \leq \varphi \leq 1$ ,  $\varphi = 1$  on  $B(0, 1)$  and  $\varphi = 0$  on  ${}^c B(0, 2)$ . We set  $\varphi_R(\cdot) = \varphi(\frac{\cdot}{R})$  for  $R \geq 1$ , we have similarly as in the previous case:

$$|\int_0^T dt \int_{\mathbb{R}^2} \rho u \cdot \nabla \varphi_R(x) dx| \leq \|\nabla \varphi\|_{L^\infty(\mathbb{R}^2)} \frac{1}{R} \|\rho u\|_{L^1(0, T) \times \mathbb{R}^2} \rightarrow 0 \quad \text{as } R \rightarrow +\infty.$$

We can then conclude as in the previous proof in using the fact that that  $0 \leq \bar{\rho}^{\frac{1}{\varepsilon}} \leq \rho$ .  
□

### Proof of the convergence assertion on $\rho_n u_n$

We now want to show the convergence of  $\rho_n u_n$  to have informations on strong convergence of  $u_n$  modulo the vacuum. We recall in this part some classical inequalities to get the convergence of  $\rho_n u_n$ , for more details see Lions in [19]. We use once more a mollifier  $k_\alpha = \frac{1}{\alpha^N} k(\frac{\cdot}{\alpha})$  where  $k \in C_0^\infty(\mathbb{R}^N)$  and we let  $g_\alpha = g * k_\alpha$  for an arbitrary function  $g$ . We first observe that we have for all  $\frac{N}{2} < p < s$ :

$$\begin{aligned}|((\rho_n u_n)_\alpha - \rho_n u_n)(x)| &= |\int_{\mathbb{R}^d} [\rho_n(t, y) - \rho_n(t, x)] u_n(t, y) k_\alpha(x - y) dy \\ &\quad + \rho_n(t, x) ((u_n)_\alpha - u_n)(t, x)|\end{aligned}$$

We have in using Hölder inequalities with the measure  $k_\alpha(x - y) dy$ :

$$\begin{aligned}|((\rho_n u_n)_\alpha - \rho_n u_n)(x)| &\leq [\int_{\mathbb{R}^d} |\rho_n(t, y) - \rho_n(t, x)|^p k_\alpha(x - y) dy]^{\frac{1}{p}} (|u_n|^{\frac{p}{p-1}})_\alpha^{\frac{p-1}{p}} \\ &\quad + \rho_n |(u_n)_\alpha - u_n|(t, x).\end{aligned}$$

Hence for all  $t \geq 0$

$$\begin{aligned} \int_{\mathbb{R}^d} |((\rho_n u_n)_\alpha - \rho_n u_n)(x)| dx &\leq \left[ \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} |\rho_n(t, y) - \rho_n(t, x)|^p k_\alpha(x - y) dy \right]^{\frac{1}{p}} \\ &\quad \times \left\| \left( |u_n|^{\frac{p}{p-1}} \right)_\alpha \right\|_{L^1}^{\frac{p-1}{p}} + \|\rho_n\|_{L^p} \|(u_n)_\alpha - u_n\|_{L^{\frac{p-1}{p}}}, \\ &\leq \left[ \sup_{|z| \leq \alpha} \|\rho_n(\cdot + z) - \rho_n\|_{L^p} \right] \|u_n\|_{L^{\frac{p-1}{p}}} + \|\rho_n\|_{L^p} \|(u_n)_\alpha - u_n\|_{L^{\frac{p-1}{p}}}. \end{aligned}$$

Next if we choose  $p > \frac{2N}{N+2}$ , so that  $\frac{p}{p-1} < \frac{2N}{N-2}$  then  $\|(u_n)_\alpha - u_n\|_{L^2(0, T; L^{\frac{p-1}{p}})}$  converges to 0 as  $\alpha$  goes to  $0_+$  uniformly in  $n$ . In addition, the convergence on  $\rho_n$  assure that  $\sup_{|z| \leq \alpha} \|\rho_n(\cdot + z) - \rho_n\|_{L^p}$  converge to 0 as  $\alpha$  goes to  $0_+$  uniformly in  $n$ . Therefore in conclusion,  $(\rho_n u_n)_\alpha - \rho_n u_n$  converge to 0 in  $L^2(0, T; L^1)$  as  $\alpha$  goes to  $0_+$  uniformly in  $n$ . Next  $(\rho_n u_n)_\alpha$  is smooth in  $x$ , uniformly in  $n$  and in  $t \in [0, T]$ . Therefore, remarking that  $\frac{\partial}{\partial t}(\rho_n u_n)_\alpha$  is bounded in a  $L^2(0, T; H^m)$  for any  $m \geq 0$ , we deduce that  $(\rho_n u_n)_\alpha$  converge to  $(\rho u)_\alpha$  as  $n$  goes to  $+\infty$  in  $L^1((0, T) \times \mathbb{R}^N)$  for each  $\alpha$ . Then using the bound on  $\rho_n u_n$  in  $L^\infty(L^{\frac{2s}{s+1}})$ , we deduce that  $\rho_n u_n$  converges to  $\rho u$  in  $L^1((0, T) \times \mathbb{R}^N)$  and we can conclude by interpolation. The last convergence result is a consequence of the strong convergence of  $\rho_n$  and  $\rho_n u_n$ .  $\square$

### 3 The case of a general pressure law

In the sequel we focus on the cases  $N = 2, 3$ . We now want to extend our previous result to more general and physical pressure laws. In particular we are now interested by two cases, the first one concerns monotonous pressure law (close in a certain sense that we will precise to  $\rho^\gamma$  pressure), the second one is the case of a slightly modified Van der Waals pressure. The technics of proof will be very similar to the previous proof, only technical points change.

#### 3.1 Monotonous pressure

In this section, we shall investigate an extension of the preceding results to the case of a general monotonous pressure  $P$ , i.e  $P$  is assumed to be a  $C^1$  non-decreasing function on  $[0, +\infty)$  vanishing at 0.

We want here to mention in the general situation our new energy inequality, we recall the inequality (1.3):

$$\begin{aligned} \int_{\mathbb{R}^N} \left( \frac{1}{2} \rho |u|^2 + \Pi(\rho) + E_{global}[\rho(\cdot, t)] \right)(x) dx(t) + \int_0^t \int_{\mathbb{R}^N} (\mu D(u) : D(u) \\ + (\lambda + \mu) |\operatorname{div} u|^2) dx \leq \int_{\mathbb{R}^N} \left( \frac{|m_0|^2}{2\rho} + \Pi(\rho_0) + E_{global}[\rho_0] \right) dx. \end{aligned}$$

where we define  $\Pi$  by:  $\frac{\partial}{\partial t} \left( \frac{\Pi(t)}{t} \right) = \frac{P(t)}{t^2}$  for all  $t > 0$ . There are two cases worth considering: first of all if  $P(t)$  is such that  $\int_0^1 \frac{P(s)}{s^2} ds < +\infty$  then we can choose  $\Pi(\rho) = \rho \int_0^\rho \frac{P(s)}{s^2} ds$ . In the other case, i.e  $\lim_{t \rightarrow 0} \frac{P(t)}{t} = c > 0$ , we can choose  $\Pi(\rho) = \rho \int_\rho^1 \frac{P(s)}{s^2} ds$  and  $\Pi$  behaves like  $\rho \log \rho$  as  $\rho$  goes to 0.

We now consider a sequence of solutions  $(\rho_n, u_n)$  and we make the same assumptions on this sequence as in the previous section except that we need to modify the assumptions on  $\rho_n$ . We assume always that  $\rho_n$  is bounded in  $C([0, T], L^1(\mathbb{R}^N))$ ,  $P(\rho_n)$  is bounded in  $L^\infty(0, T, L^1(\mathbb{R}^N))$ ,

$$\begin{aligned} (\rho_n)_{n \geq 1} & \text{ is bounded in } L^\infty(0, T, L^2), \\ (\rho_n^\varepsilon P(\rho_n))_{n \geq 1} & \text{ is bounded in } L^1(K \times (0, T)), \end{aligned}$$

for some  $\varepsilon > 0$ , where  $K$  is an arbitrary compact set included in  $\mathbb{R}^N$ .

**Theorem 3.6** *Let the assumptions of theorem 1.1 be satisfied with in addition  $P$  a monotone pressure. Then there exists a renormalized finite energy weak solution to problem (NSK) in the sense of definitions 1.1 and 1.2. Moreover  $P(\rho_n)$  converges to  $P(\rho)$  in  $L^1(K \times (0, T))$  for any compact set  $K$ .*

**Proof:**

The proof of theorem 3.6 is based on the same compactness argument as in the theorem 1.1. In particular, there is essentially one observation which allows us to adapt the proof of theorem 1.1. Namely we still obtain the following identity for the effective viscous flux:

$$\beta(\rho_n)(\zeta \operatorname{div} u_n - P(\rho_n) - \frac{\kappa}{2} \rho_n^2) \rightharpoonup_n \overline{\beta(\rho)}(\zeta \operatorname{div} u - \overline{P(\rho)} - \frac{\kappa}{2} \overline{\rho^2}),$$

with  $\beta(\rho) = \rho^\varepsilon$  for  $\varepsilon$  small enough. We have then:

$$\zeta \overline{\rho^\varepsilon \operatorname{div} u} - \overline{P(\rho) \rho^\varepsilon} - \frac{\kappa}{2} \overline{\rho^{2+\varepsilon}} = \zeta \overline{\operatorname{div} u \rho^\varepsilon} - \overline{P(\rho) \rho^\varepsilon} - \frac{\kappa}{2} \overline{\rho^2 \rho^\varepsilon}, \quad (3.36)$$

Now we can recall a lemma coming from P.-L. Lions in [19]:

**Lemma 2** *Let  $p_1, p_2 \in C([0, \infty))$  be non-decreasing functions. We assume that  $p_1(\rho_n)$ ,  $p_2(\rho_n)$  and  $p_1(\rho_n)p_2(\rho_n)$  are relatively weakly compact in  $L^1(K \times (0, T))$  for any compact set  $K \subset \mathbb{R}^N$ . Then, we have:*

$$\overline{p_1(\rho)p_2(\rho)} \geq \overline{p_1(\rho)} \overline{p_2(\rho)} \quad \text{a.e.}$$

We get finally as in the proof of theorem 1.1 by using lemma 2:  $\overline{\operatorname{div} u \rho^\varepsilon} \geq \operatorname{div} u \overline{\rho^\varepsilon}$ . All remaining argumentation of the proof of theorem 1.1 can be performed to conclude.  $\square$

### 3.2 Pressure of Van der Waals type

In this section we are interested by pressure of Van der Waals type which consequently are not necessarily non-decreasing. That's why in the following proof we will proceed slightly differently. So we consider the pressure law:

$$P(\rho) = \frac{RT_* \rho}{b - \rho} - a \rho^2 \quad \text{for } \rho \leq b - \theta \quad \text{for some small } \theta > 0$$

and we extend the function  $P$  to be strictly increasing on  $\rho \geq b - \theta$ . We have then:

1.  $P'$  is bounded from below, that is:

$$P'(\rho) \geq -\bar{\rho} \quad \text{for all } \rho > 0.$$

2.  $P$  is a strictly increasing function for  $\rho$  large enough.

Under the above conditions, it is easy to see that the pressure can be written as:  $P(\rho) = P_1(\rho) - P_2(\rho)$ . with  $P_1$  a non-decreasing function of  $\rho$ , and

$$P_2 \in C^2[0, +\infty), \quad P_2 \geq 0, \quad P_2 = 0 \quad \text{for } \rho \geq \bar{\rho}.$$

**Remark 2** *The a priori energy estimate give us the bound of  $\rho$  in  $L^\infty(L^2)$ . We have thus:*

$$|\{(t, x) \in (0, T) \times \mathbb{R}^N / |\rho(t, x)| > b\}| \leq \frac{T \|\rho\|_{L^\infty(L^2(\mathbb{R}^N))}^2}{b^2}.$$

Hence the set where  $P$  is different from the Van der Waals law is of finite measure.

**Theorem 3.7** *If in addition to the above assumptions, we assume that  $\rho_n^0$  converges in  $L^1(\mathbb{R}^N)$  to  $\rho_0$  then  $(\rho, u)$  is a weak solution of the system (NSK) satisfying the initial condition and we have:*

$$\rho_n \rightarrow \rho \quad \text{in } C([0, T], L^p(\mathbb{R}^N) \cap L^r((0, T) \times \mathbb{R}^N)) \quad \text{for all } 1 \leq p < 2, 1 \leq r < 1 + \frac{4}{N}.$$

**Proof:**

Most of the proof of theorem 3.7 is similar as theorem 1.1. We will use a approximated sequel  $T_k$  (introduced by Feireisl in [9]) of  $\rho$  by some concave bounded function.

**Definition 3.3** *We define the function  $T \in C^\infty(\mathbb{R}^N)$  as follows:*

$$\begin{aligned} T(z) &= z \quad \text{for } z \in [0, 1], \\ T(z) &\text{ concave on } [0, +\infty), \\ T(z) &= 2 \quad \text{for } z \geq 3, \\ T(z) &= -T(-z) \quad \text{for } z \in (-\infty, 0], \end{aligned}$$

And  $T_k$  is the cut-off function:

$$T_k(z) = kT\left(\frac{z}{k}\right).$$

By following the proof of theorem 1.1, we get:

$$\begin{aligned} \frac{\partial}{\partial t}(\overline{L_k(\rho)} - L_k(\rho)) + \operatorname{div}((\overline{T_k(\rho)} - T_k(\rho))u) + \overline{T_k(\rho)} \operatorname{div} u - \overline{T_k(\rho)} \operatorname{div} u \\ = (T_k(\rho) - \overline{T_k(\rho)}) \operatorname{div} u \quad \text{in } \mathcal{D}'((0, T) \times \mathbb{R}^N), \end{aligned}$$



where  $L_k(\rho) = \rho \log(\rho)$  for  $0 \leq \rho \leq k$ , and  $0 \leq L_k(\rho) \leq \rho \log(\rho)$  otherwise. So we get in integrating in time on  $[t_1, t_2]$ :

$$\begin{aligned} & \int_{\mathbb{R}^N} (\overline{L_k(\rho)} - L_k(\rho))(t_2) dx - \int_{\mathbb{R}^N} (\overline{L_k(\rho)} - L_k(\rho))(t_1) dx \\ & + \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \overline{T_k(\rho) \operatorname{div} u} - \overline{T_k(\rho)} \operatorname{div} u \, dx dt = \int_{t_1}^{t_2} \int_{\mathbb{R}^N} (T_k(\rho) - \overline{T_k(\rho)}) \operatorname{div} u \, dx dt. \end{aligned}$$

We can show that:

$$\|T_k(\rho) - \overline{T_k(\rho)}\|_{L^2((0,T) \times \mathbb{R}^N)} \rightarrow_{k \rightarrow +\infty} 0$$

For proving the previous inequality, we see that  $\|T_k(\rho) - \overline{T_k(\rho)}\|_{L^1((0,T) \times \mathbb{R}^N)} \rightarrow 0$  for  $k \rightarrow +\infty$ . We then conclude by interpolation with  $T_k(\rho) - \overline{T_k(\rho)} \in L^q(0, T) \times \mathbb{R}^N$  with  $q > 2$ . By Hölder inequality we obtain that:

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^N} (T_k(\rho) - \overline{T_k(\rho)}) \operatorname{div} u \, dx dt \rightarrow 0 \quad \text{for } k \rightarrow +\infty.$$

We have then:

$$\begin{aligned} & \lim_{k \rightarrow +\infty} \int_{\mathbb{R}^N} (\overline{L_k(\rho)} - L_k(\rho))(t_2) dx - \int_{\mathbb{R}^N} (\overline{L_k(\rho)} - L_k(\rho))(t_1) dx \\ & = - \lim_{k \rightarrow +\infty} \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \overline{T_k(\rho) \operatorname{div} u} - \overline{T_k(\rho)} \operatorname{div} u \, dx dt. \end{aligned} \tag{3.37}$$

We set:

$$\begin{aligned} dft[\rho_n \rightarrow \rho](t) &= \lim_{k \rightarrow +\infty} \int_{\mathbb{R}^N} (\overline{L_k(\rho)} - L_k(\rho))(t) dx \\ A(k, \rho) &= \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \overline{T_k(\rho) \operatorname{div} u} - \overline{T_k(\rho)} \operatorname{div} u \, dx dt. \end{aligned}$$

We can show as in the previous proof of theorem 1.1 that:

$$\begin{aligned} & \int_{t_1}^{t_2} \int_K \overline{T_k(\rho) \operatorname{div} u} - \overline{T_k(\rho)} \operatorname{div} u \, dx dt \\ & = \lim_{n \rightarrow +\infty} \int_{t_1}^{t_2} \int_K ((P(\rho_n) + \frac{\kappa}{2} \rho_n^2) T_k(\rho_n) - \overline{(P(\rho) + \frac{\kappa}{2} \rho^2) T_k(\rho)}) \, dx dt. \end{aligned}$$

for any compact  $K \subset \mathbb{R}^N$ . Using the lemma 2 we deduce that:

$$\lim_{n \rightarrow +\infty} \int_{t_1}^{t_2} \int_{\mathbb{R}^N} (P_1(\rho_n) + \frac{\kappa}{2} \rho_n^2) T_k(\rho_n) - \overline{(P_1(\rho) + \frac{\kappa}{2} \rho^2) T_k(\rho)} \, dx dt \leq 0.$$

We have then:

$$\begin{aligned} dft[\rho_n \rightarrow \rho](t_2) - dft[\rho_n \rightarrow \rho](t_1) &\leq \lim_{k \rightarrow +\infty} \left( \lim_{n \rightarrow +\infty} \int_{t_1}^{t_2} \int_{\mathbb{R}^N} P_2(\rho_n) T_k(\rho_n) \right. \\ &\quad \left. - \overline{P_2(\rho) T_k(\rho)} \, dx dt \right). \end{aligned}$$

As the sequence  $(\rho_n)_{n \in \mathbb{N}}$  is bounded in  $L^\infty(0, T, L^2(\mathbb{R}^N))$ , and  $P_2$  is a bounded function, we have:

$$\begin{aligned} \lim_{k \rightarrow +\infty} \left( \lim_{n \rightarrow +\infty} \int_{t_1}^{t_2} \int_{\mathbb{R}^N} P_2(\rho_n) T_k(\rho_n) - \overline{P_2(\rho) T_k(\rho)} dx dt \right) \\ = \lim_{n \rightarrow +\infty} \int_{t_1}^{t_2} \int_{\mathbb{R}^N} P_2(\rho_n) \rho_n - \overline{P_2(\rho)} \bar{\rho} dx dt. \end{aligned}$$

Since the function  $P_2$  is twice continuously differentiable and compactly supported in  $[0, +\infty)$ , there exists  $\Lambda > 0$  big enough such that both  $\rho \rightarrow \Lambda \rho \log \rho - \rho P_2(\rho)$  and  $\rho \rightarrow \Lambda \rho \log \rho + \rho P_2(\rho)$  are convex functions of  $\rho$ , indeed the second derivative are positive. As a consequence of weak lower semi-continuity of convex functionals, we obtain:

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_{t_1}^{t_2} \int_{\mathbb{R}^N} P_2(\rho_n) \rho_n - \overline{P_2(\rho)} \bar{\rho} dx dt \\ \leq \Lambda \int_{t_1}^{t_2} \int_{\mathbb{R}^N} (\overline{\rho \log \rho} - \rho \log \rho) dx dt + \int_{t_1}^{t_2} \int_{\mathbb{R}^N} (P_2(\rho) - \overline{P_2(\rho)}) \rho dx dt. \end{aligned}$$

Furthermore we have:

$$\begin{aligned} \int_{t_1}^{t_2} \int_{\mathbb{R}^N} (P_2(\rho) - \overline{P_2(\rho)}) \rho dx dt &\leq \int_{t_1}^{t_2} \int_{\rho \leq \rho_r} (P_2(\rho) - \overline{P_2(\rho)}) \rho \\ &\leq \Lambda \int_{t_1}^{t_2} \int_{\rho \leq \rho_r} (\overline{\rho \log \rho} - \rho \log \rho) dx dt \\ &\leq \Lambda \rho_r \int_{t_1}^{t_2} \int_{\mathbb{R}^N} (\overline{\rho \log \rho} - \rho \log \rho) dx dt \end{aligned}$$

The previous relation gives:

$$dft[\rho_n \rightarrow \rho](t_2) \leq dft[\rho_n \rightarrow \rho](t_1) + \omega \int_{t_1}^{t_2} dft[\rho_n \rightarrow \rho](t) dt.$$

Applying Grönwall's lemma we infer:

$$dft[\rho_n \rightarrow \rho](t_2) \leq dft[\rho_n \rightarrow \rho](t_1) \exp(\omega(t_2 - t_1)).$$

We conclude that  $dft[\rho_n \rightarrow \rho](t) = 0 \ \forall t$ , because  $\rho_n^0$  converges strongly in  $L^1$  to  $\rho_0$ .

## 4 Weak solutions for data close to a stable equilibrium

We consider in this section one situation which is rather different from the three cases considered in the preceding sections. This situation is relevant for practical applications and realistic flow and they involve conditions at infinity different from those studied. We wish to investigate the system  $(NSK)$  with hypothesis close from these of the theorem for strong solutions. We want then to study the system with a density close from a stable equilibrium in the goal to can choose initial data avoiding the vacuum. We look now for a solution  $(\rho, u)$  defined on  $\mathbb{R} \times \mathbb{R}^N$  of the system  $(NSK)$  (where  $P(\rho) = a\rho^\gamma$ ) with  $\rho \geq 0$

on  $\mathbb{R} \times \mathbb{R}^N$ .

In addition we require  $(\rho, u)$  to satisfy the following limit conditions:

$$(\rho, u)(x, t) \rightarrow (\bar{\rho}, 0) \text{ as } |x| \rightarrow +\infty, \text{ for all } t > 0$$

where  $\bar{\rho} > 0$ . Such an analysis requires the use of the Orlicz spaces and for the definition and properties on the Orlicz space we refer to [19]. Our goal is now to get energy estimate. We have to face a new difficulty. Indeed  $\rho, \rho|u|^2, \rho^\gamma$  need not belong to  $L^1$ . We first want to explain how it is possible to obtain natural a priori bounds which correspond to energy-like identities. Next we write the following formal identities:

$$\begin{aligned} & \frac{1}{\gamma-1} \frac{d}{dt} (\rho^\gamma - \bar{\rho}^\gamma - \gamma \bar{\rho}^{\gamma-1} (\rho - \bar{\rho}) + \operatorname{div} [u \frac{\gamma}{\gamma-1} (\rho^\gamma - \bar{\rho}^{\gamma-1} \rho)]) = u \cdot \nabla (\rho^\gamma) \\ & \rho \frac{d}{dt} \frac{|u|^2}{2} + \rho u \cdot \nabla \frac{|u|^2}{2} - \mu \Delta u \cdot u - \xi \nabla \operatorname{div} u \cdot u + a u \cdot \nabla \rho^\gamma = \kappa \rho u \nabla (\phi * \rho - \rho). \end{aligned} \quad (4.38)$$

We may then integrate in space the equality (4.38) and we get:

$$\begin{aligned} & \left( \int_{\mathbb{R}^N} \rho \frac{|u|^2}{2} + \frac{a}{\gamma-1} (\rho^\gamma + (\gamma-1) \bar{\rho}^\gamma - \gamma \bar{\rho}^{\gamma-1} \rho) + E_{\text{global}}[\rho - \bar{\rho}] dx \right) (t) \\ & + \int_0^t ds \int_{\mathbb{R}^N} 2\mu |Du|^2 + 2\xi |\operatorname{div} u|^2 dx \leq \int_{\mathbb{R}^N} \rho_0 \frac{|u_0|^2}{2} + \frac{a}{\gamma-1} (\rho_0^\gamma + (\gamma-1) \bar{\rho}^\gamma - \gamma \bar{\rho}^{\gamma-1} \rho_0) \\ & + E_{\text{global}}[\rho_0 - \bar{\rho}] dx. \end{aligned}$$

In the sequel we will note:  $j_\gamma(\rho) = \rho^\gamma + (\gamma-1) \bar{\rho}^\gamma - \gamma \bar{\rho}^{\gamma-1} \rho$ . Now we can recall a theorem (see [19]) on the Orlicz space concerning this quantity:

**Theorem 4.8**  $j_\gamma(\rho) \in L^1(\mathbb{R}^N)$  if and only if  $\rho - \bar{\rho} \in L_2^\gamma$ .

By following this theorem and our energy estimate we get that  $\rho - \bar{\rho} \in L^\infty(0, T; L_2^\gamma(\mathbb{R}^N))$  for all  $T \in \mathbb{R}$ . Moreover we have:

$$E_{\text{global}}[\rho(t, \cdot) - \bar{\rho}](x) = \frac{\kappa}{4} (\rho - \bar{\rho})^2 + \phi * (\rho - \bar{\rho})^2 - 2(\rho - \bar{\rho}) (\phi * (\rho - \bar{\rho})).$$

Then in using the fact that  $\rho - \bar{\rho} \in L^\infty(0, T; L_2^\gamma(\mathbb{R}^N)) \hookrightarrow L^\infty(L^2(\mathbb{R}^N) + L^\gamma(\mathbb{R}^N))$  and interpolation on  $\nabla \phi$ , we get that  $\rho - \bar{\rho} \in L^\infty(0, T; L^2(\mathbb{R}^N))$ . We may now turn to our compactness result. First of all, we consider sequences of approximate smooth solutions  $(\rho_n, u_n)$  of the system corresponding to some initial conditions  $(\rho_n^0, u_n^0)$ . By using the above energy inequalities, we assume that  $j_\gamma(\rho_n^0)$ ,  $E_{\text{global}}[\rho_n^0 - \bar{\rho}]$  and  $\rho_n^0 |u_n^0|^2$  are bounded in  $L^\infty(L^1(\mathbb{R}^N))$  and that  $\rho_n^0 - \bar{\rho}$  converges weakly in  $L_2^\gamma(\mathbb{R}^N)$  to some  $\rho_0 - \bar{\rho}$ .

We now assume that:

$$j_\gamma(\rho_n), E_{\text{global}}[\rho_n - \bar{\rho}], \rho_n |u_n|^2 \text{ are bounded in } L^\infty(0, T, L^1(\mathbb{R}^N)),$$

Moreover we have for all  $T \in (0, +\infty)$  and for all compact sets  $K \subset \mathbb{R}^N$ :

$$\rho_n - \bar{\rho} \in L^\infty(L^2(\mathbb{R}^N)) \text{ and } \rho_n \text{ is bounded in } L^q((0, T) \times K), \text{ for some } q > s.$$

$Du_n$  is bounded in  $L^2(\mathbb{R}^N \times (0, T))$ ,  $u_n$  is bounded in  $L^2(0, T, H^1(B_R))$  for all  $R, T \in (0, +\infty)$ .

Extracting subsequences if necessary, we may assume that  $\rho_n, u_n$  converge weakly respectively in  $L^2((0, T) \times B_R)$ ,  $L^2(0, T; H^1(B_R))$  to  $\rho, u$  for all  $R, T \in (0, +\infty)$ . We also extract subsequences for which  $\rho_n u_n, \rho_n u_n \otimes u_n$  converge weakly as previously.

**Remark 3** We notice that the conditions at infinity are implicitly contained in the fact that  $(\rho_n - \bar{\rho})^2$  and  $\rho_n |u_n|^2 \in L^1(\mathbb{R}^N)$ .

**Theorem 4.9** Let  $\gamma \geq 1$ . We assume that  $\rho_n^0$  converges in  $L^1(B_R)$  (for all  $R \in (0, +\infty)$ ) to  $\rho_0$ . Then  $(\rho_n, u_n)_{n \in \mathbb{N}}$  converges in distribution sense to  $(\rho, u)$  a solution of (NSK). Moreover we have for all  $R, T \in (0, +\infty)$ :

$\rho_n \rightarrow_n \rho$  in  $C([0, T], L^p(B_R \times (0, T))) \cap L^{s_1}(B_R \times (0, T))$  for all  $1 \leq p < s, 1 \leq s_1 < q$ .

with  $q = s + \frac{4}{N} - 1$ .

**Proof:**

As in the theorem 1.1, we want test the strong convergence of  $\rho_n$  on concave function  $B$ . Since the proof is purely local, we have again for small enough  $\varepsilon > 0$  in  $\mathcal{D}'((0, \infty) \times \mathbb{R}^N)$ :

$$(\rho_n)^\varepsilon ((\mu + \xi) \operatorname{div} u_n - a(\rho_n)^\gamma - \frac{\kappa}{2} \rho_n^2) \rightarrow_n \overline{(\rho)^\varepsilon} ((\mu + \xi) \operatorname{div} u - a \bar{\rho}^\gamma - \frac{\kappa}{2} \bar{\rho}^2).$$

so we obtain:

$$\frac{d}{dt}(\bar{\rho}^\varepsilon) + \operatorname{div}(u \bar{\rho}^\varepsilon) \geq (1 - \varepsilon)(\operatorname{div} u) \bar{\rho}^\varepsilon \quad \text{in } \mathcal{D}'((0, \infty) \times \mathbb{R}^N). \quad (4.39)$$

Next since  $(\bar{\rho}^\varepsilon)^{\frac{1}{\varepsilon}} \in L^2(B_R \times (0, T))$  for all  $R, T \in (0, +\infty)$ , as in the theorem 1.1 in using a result of type Diperna-Lions on renormalized solutions, we get:

$$\frac{d}{dt}(\overline{(\rho)^\varepsilon}^{\frac{1}{\varepsilon}}) + \operatorname{div}(u \overline{(\rho)^\varepsilon}^{\frac{1}{\varepsilon}}) \geq 0 \quad \text{in } \mathcal{D}'((0, \infty) \times \mathbb{R}^N). \quad (4.40)$$

while we have  $\overline{(\rho)^\varepsilon}^{\frac{1}{\varepsilon}} \leq \rho$  a.e in  $\mathbb{R}^N \times (0, +\infty)$  and  $\overline{(\rho)^\varepsilon}^{\frac{1}{\varepsilon}}_{/t=0} = \rho_{/t=0}$  in  $\mathbb{R}^N$ . Now by subtracting the second equality of (4.40) from the mass equation and setting  $f = \rho - \overline{(\rho)^\varepsilon}^{\frac{1}{\varepsilon}}$ , we have:

$$\frac{d}{dt}(f) + \operatorname{div}(uf) \leq 0, \quad f \geq 0 \quad \text{a.e, } f_{/t=0} = 0 \quad \text{in } \mathbb{R}^N. \quad (4.41)$$

Next we want again to show from (4.41) that  $f = 0$ , in integrating (4.41) and in using the fact that  $f \leq 0$  to conclude that  $f = 0$ . The difference with the proof of theorem 1.1 is to justify the integration by parts as we work in different energy space. We need a cut-off function. We introduce  $\varphi \in C_0^\infty(\mathbb{R}^N)$ ,  $0 \leq \varphi \leq 1$ ,  $\operatorname{supp} \varphi \subset B_2$ ,  $\varphi = 1$  on  $B_1$  and we set  $\varphi_R = \varphi(\frac{x}{R})$  for  $R \geq 1$ . Multiplying (4.41) by  $\varphi_R(x)$ , we obtain:

$$\frac{d}{dt} \int_{\mathbb{R}^N} f \varphi_R(x) dx = \int_{\mathbb{R}^N} \frac{1}{R} f u \cdot \nabla \varphi\left(\frac{x}{R}\right). \quad (4.42)$$

Next, if  $T > 0$  is fixed, we see that  $\operatorname{supp} \nabla \varphi(\frac{\cdot}{R}) \subset \{R \leq |x| \leq 2R\}$ , therefore, for  $R$  large enough we have:

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^N} f \varphi_R(x) dx &= \int_{\mathbb{R}^N} \frac{1}{R} f u \cdot \nabla \varphi\left(\frac{x}{R}\right), \\ &\leq \frac{C}{R} \int_{\mathbb{R}^N} f |u| 1_{(R \leq |x| \leq 2R)} dx, \quad \text{for } t \in (0, T), \end{aligned} \quad (4.43)$$

To conclude that  $f = 0$ , we only have to prove that:

$$\frac{1}{R} \int_{\mathbb{R}^N} f|u|1_{(R \leq |x| \leq 2R)} dx \rightarrow 0 \text{ as } R \rightarrow +\infty. \quad (4.44)$$

We now use the fact that  $f \in L^\infty(0, T, L^2(\mathbb{R}^N))$  and  $f|u|^2 \in L^\infty(0, T, L^1(\mathbb{R}^N))$  for all  $T \in (0, +\infty)$  to control (4.44). The second fact is obvious since  $0 \leq \rho$  and  $\rho|u|^2 \in L^\infty(0, T; L^1(\mathbb{R}^N))$ . In order to prove the first claim, we only have to show that  $\overline{(\rho)}^{\frac{1}{\varepsilon}} - \bar{\rho} \in L^\infty(0, T, L^2(\mathbb{R}^N))$ . We rewrite  $(\rho_n)^\varepsilon - (\bar{\rho})^\varepsilon = (\bar{\rho} + (\rho_n - \bar{\rho}))^\varepsilon - (\bar{\rho})^\varepsilon$  which is bounded in  $L^\infty(0, T, L^2(\mathbb{R}^N))$ . So we have  $\sqrt{f} \in L^\infty(L^4(\mathbb{R}^N))$  and we get:

$$\frac{1}{R} \int_0^T dt \int_{\mathbb{R}^N} f|u|1_{(R \leq |x| \leq 2R)} dx \leq \frac{C_0}{R} \text{meas}(C(0, R, 2R))^{\frac{1}{4}}.$$

We recall that:

$$\text{meas}(C(0, R, 2R)) \sim_{R \rightarrow +\infty} C(n)R^N$$

Then we get:

$$\frac{d}{dt} \int_{\mathbb{R}^N} f\varphi_R(x) dx \rightarrow_{R \rightarrow +\infty} 0$$

and we conclude as in the proof of theorem 1.1. At this stage, it only remains to show that, for instance,  $\rho_n$  converges to  $\rho$  in  $C([0, T], L^1(B_R))$  for all  $R, T \in (0, +\infty)$ . In order to do so, we just have to localize the corresponding argument in the proof of theorem 1.1. Therefore we choose for  $R, T \in (0, +\infty)$  fixed,  $\varphi \in C_0^\infty(\mathbb{R}^N)$  such that  $\varphi = 1$  on  $B_R$ ,  $0 \leq \varphi$  on  $\mathbb{R}^N$ . Then, we observe that we have:

$$\begin{aligned} \frac{\partial}{\partial t}(\varphi^2 \rho_n) + \text{div}(u_n(\varphi^2 \rho_n)) &= \rho_n u_n \cdot \nabla \varphi^2, \quad \frac{\partial}{\partial t}(\varphi^2 \rho) + \text{div}(u(\varphi^2 \rho)) = \rho u \cdot \nabla \varphi^2 \\ \frac{\partial}{\partial t}(\varphi \sqrt{\rho_n}) + \text{div}(u_n(\varphi \rho_n)) &= \frac{1}{2}(\text{div} u_n) \varphi \sqrt{\rho_n} + \sqrt{\rho_n} u_n \cdot \nabla \varphi, \\ \frac{\partial}{\partial t}(\varphi \sqrt{\rho}) + \text{div}(u(\varphi \rho)) &= \frac{1}{2}(\text{div} u) \varphi \sqrt{\rho} + \sqrt{\rho} u \cdot \nabla \varphi. \end{aligned}$$

From these equations, we deduce as in the proof of the previous theorem, that  $\varphi^2 \rho \in C([0, +\infty), L^1(\mathbb{R}^N))$ ,  $\varphi \sqrt{\rho} \in C([0, +\infty), L^2(\mathbb{R}^N))$  and that  $\varphi \sqrt{\rho_n}$  converges weakly in  $L^2(\mathbb{R}^N)$ , uniformly in  $t \in [0, T]$ . Therefore, in order to conclude, we just have to show that we have:

$$\int_{\mathbb{R}^N} \varphi^2 \rho_n(t_n) dx \rightarrow \int_{\mathbb{R}^N} \varphi^2 \rho(\bar{t}) dx$$

whenever  $t_n \in [0, T]$ ,  $t_n \rightarrow_n \bar{t}$ , and this is straightforward since we have, in view of the above equation:

$$\begin{aligned} \int_{\mathbb{R}^N} \varphi^2 \rho_n(t_n) dx &= \int_{\mathbb{R}^N} \varphi^2(\rho_0)_n dx + \int_0^{t_n} ds \int_{\mathbb{R}^N} \rho_n u_n \cdot \nabla \varphi^2 dx \\ &\rightarrow_n \int_{\mathbb{R}^N} \varphi^2 \rho_0 dx + \int_0^T ds \int_{\mathbb{R}^N} \rho u \cdot \nabla \varphi^2 dx = \int_{\mathbb{R}^N} \varphi \rho(\bar{t}) dx. \end{aligned}$$

□

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