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# Existence of solutions for compressible fluid models of Korteweg type

Boris Haspot \*

## Abstract

This work is devoted to the study of the initial boundary value problem for a general non isothermal model of capillary fluids derived by J.E Dunn and J.Serrin (1985) in [15, 25], which can be used as a phase transition model.

We distinguish two cases, when the physical coefficients depend only on the density, and the general case. In the first case we can work in critical scaling spaces, and we prove global existence of solution and uniqueness for data close to a stable equilibrium. For general data, existence and uniqueness is stated on a short time interval. In the general case with physical coefficients depending on density and on temperature, additional regularity is required to control the temperature in  $L^\infty$  norm. We prove global existence of solution close to a stable equilibrium and local in time existence of solution with more general data. Uniqueness is also obtained.

## 1 Introduction

### 1.1 Derivation of the Korteweg model

We are concerned with compressible fluids endowed with internal capillarity. The model we consider originates from the XIXth century work by van der Waals and Korteweg [21] and was actually derived in its modern form in the 1980s using the second gradient theory, see for instance [20, 26].

Korteweg-type models are based on an extended version of nonequilibrium thermodynamics, which assumes that the energy of the fluid not only depends on standard variables but on the gradient of the density. Let us consider a fluid of density  $\rho \geq 0$ , velocity field  $u \in \mathbb{R}^N$  ( $N \geq 2$ ), entropy density  $e$ , and temperature  $\theta = (\frac{\partial e}{\partial s})_\rho$ . We note  $w = \nabla \rho$ , and we suppose that the intern specific energy,  $e$  depends on the density  $\rho$ , on the entropy specific  $s$ , and on  $w$ . In terms of the free energy, this principle takes the form of a generalized Gibbs relation :

$$de = \tilde{T}ds + \frac{p}{\rho^2}d\rho + \frac{1}{\rho}\phi^* \cdot dw$$

where  $\tilde{T}$  is the temperature,  $p$  the pressure,  $\phi$  a vector column of  $\mathbb{R}^N$  and  $\phi^*$  the adjoint vector.

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\*Université Paris-Est Laboratoire d'Analyse et de Mathématiques Appliquées, UMR 8050 61 avenue du Général de Gaulle 94 010 CRETEIL Cedex FRANCE, haspot@univ-paris12.fr

In the same way we can write a differential equation for the intern energy per unit volume,  $E = \rho e$ ,

$$dE = \tilde{T}dS + gd\rho + \phi^* \cdot dw$$

where  $S = \rho s$  is the entropy per unit volume and  $g = e - s\tilde{T} + \frac{p}{\rho}$  is the chemical potential. In terms of the free energy, the Gibbs principle gives us:

$$dF = -Sd\tilde{T} + gd\rho + \phi^* \cdot dw.$$

In the present chapter, we shall make the hypothesis that:

$$\phi = \kappa w.$$

The nonnegative coefficient  $\kappa$  is called the capillarity and may depend on both  $\rho$  and  $\tilde{T}$ . All the thermodynamic quantities are sum of their classic version (it means independent of  $w$ ) and of one term in  $|w|^2$ .

In this case the free energy  $F$  decomposes into a standard part  $F_0$  and an additional term due to gradients of density:

$$F = F_0 + \frac{1}{2}\kappa|w|^2.$$

We denote  $v = \frac{1}{\rho}$  the specific volume and  $k = v\kappa$ . Similar decompositions hold for  $S$ ,  $p$  and  $g$ :

$$\begin{aligned} p &= p_0 - \frac{1}{2}K_p|w|^2 & \text{where: } K_p &= k'_v \text{ and } p_0 = -(f_0)'_v \\ g &= g_0 + \frac{1}{2}K_g|w|^2 & \text{where: } K_e &= k - \tilde{T}k'_T \text{ and } e_0 = f_0 - \tilde{T}(f_0)'_{\tilde{T}}. \end{aligned}$$

The model deriving from a Cahn-Hilliard like free energy (see the pioneering work by J.E.Dunn and J.Serrin in [15] and also in [1, 8, 17]), the conservation of mass, momentum and energy read:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u + pI) = \operatorname{div}(K + D) + \rho f, \\ \partial_t(\rho(e + \frac{1}{2}u^2)) + \operatorname{div}(u(\rho e + \frac{1}{2}\rho|u|^2 + p)) = \operatorname{div}((D + K) \cdot u - Q + W) + \rho f \cdot u, \end{cases}$$

with:

$$D = (\lambda \operatorname{div} u)I + \mu(du + \nabla u), \text{ is the diffusion tensor}$$

$$K = (\rho \operatorname{div} \phi)I - \phi w^*, \text{ is the Korteweg tensor}$$

$$Q = -\eta \nabla \tilde{T}, \text{ is the heat flux.}$$

The term

$$W = (\partial_t \rho + u^* \cdot \nabla \rho)\phi = -(\rho \operatorname{div} u)\phi$$

is the interstitial work which is needed in order to ensure the entropy balance and was first introduced by Dunn and Serrin in [15].

The coefficients  $(\lambda, \mu)$  represent the viscosity of the fluid and may depend on both the density  $\rho$  and the temperature  $\tilde{T}$ . The thermal coefficient  $\eta$  is a given non negative function of the temperature  $\tilde{T}$  and of the density  $\rho$ .

Differentiating formally the equation of conservation of the mass, we obtain a law of conservation for  $w$ :

$$\partial_t w + \operatorname{div}(uw^* + \rho du) = 0 .$$

One may obtain an equation for  $e$  by using the mass and momentum conservation laws and the relations:

$$\operatorname{div}((-pI + K + D)u) = (\operatorname{div}(-pI + K + D)) \cdot u - p \operatorname{div}(u) + (K + D) : \nabla u .$$

Multiplying the momentum equation by  $u$  yields:

$$(\operatorname{div}(-pI + K + D)) \cdot u = (\partial_t(\rho u) + \operatorname{div}(\rho u u^*)) \cdot u = \partial_t\left(\frac{\rho|u|^2}{2}\right) + \operatorname{div}\left(\frac{\rho|u|^2}{2}u\right) .$$

We obtain then:

$$\rho(\partial_t e + u^* \cdot \nabla e) + p \operatorname{div}u = (K + D) : \nabla u + \operatorname{div}(W - Q) .$$

In substituting  $K$ , we have (with the summation convention over repeated indices):

$$K : \nabla u = \rho \operatorname{div}\phi \operatorname{div}u - \phi_i w_j \partial_i u_j ,$$

while:

$$-\operatorname{div}W = \operatorname{div}((\rho \operatorname{div}u)\phi) = \rho(\operatorname{div}\phi)(\operatorname{div}u) + (w^* \cdot \phi)\operatorname{div}u + \phi_i \rho \partial_{j,i}^2 u_j .$$

In using  $w_j = \partial_j \rho$ , we obtain:

$$\begin{aligned} K : \nabla u - \operatorname{div}W &= -(w^* \cdot \phi)\operatorname{div}u - \phi_i \partial_j (\rho \partial_i u_j) \\ &= -(w^* \cdot \phi)\operatorname{div}u - (\operatorname{div}(\rho du)) \cdot \phi . \end{aligned}$$

Finally, the equation for  $e$  rewrites:

$$\rho(\partial_t e + u^* \cdot \nabla e) + (p + w^* \cdot \phi)\operatorname{div}u = D : \nabla u - (\operatorname{div}(\rho du)) \cdot \phi - \operatorname{div}Q .$$

From now on, we shall denote:  $d_t = \partial_t + u^* \cdot \nabla$ .

## 1.2 The case of a generalized Van der Waals law

From now on, we assume that there exist two functions  $\Pi_0$  and  $\Pi_1$  such that:

$$\begin{aligned} p_0 &= \tilde{T}\Pi_1'(v) + \Pi_0'(v), \\ e_0 &= -\Pi_0(v) + \varphi(\tilde{T}) - \tilde{T}\varphi'(\tilde{T}). \end{aligned}$$

We now suppose that the coefficients  $\lambda$ ,  $\mu$  depend on the density and on the temperature, and in all the sequel the capillarity  $\kappa$  doesn't depend on the temperature.

Moreover we suppose that the intern specific energy is an increasing function of  $\tilde{T}$ :

$$(A) \quad \Psi'(\tilde{T}) > 0 \text{ with } \Psi(\tilde{T}) = \varphi(\tilde{T}) - \tilde{T}\varphi'(\tilde{T}).$$

We then set  $\theta = \Psi(\tilde{T})$  and we search to obtain an equation on  $\theta$ . In what follows, we assume that  $\kappa$  depends only on the specific volume.

**Obtaining an equation for  $\theta$  :**

As:

$$e = -\Pi_0(v) + \theta + \frac{1}{2}\kappa|w|^2,$$

we thus have:

$$d_t e = -\Pi'_0(v)d_t v + d_t \theta + \frac{1}{2}\kappa'_v|w|^2 d_t v + \kappa w^* \cdot d_t w .$$

By a direct calculus we find:

$$d_t v = v \operatorname{div} u \quad \text{and} \quad w^* \cdot d_t w = -|w|^2 \operatorname{div} u - \operatorname{div}(\rho du) \cdot w .$$

Then we have:

$$d_t \theta = d_t e + v(p - \tilde{T}\Pi'_1(v)) \operatorname{div} u + \kappa|w|^2 \operatorname{div} u + \kappa \operatorname{div}(\rho du) \cdot w .$$

And in using the third equation of the system, we get an equation on  $\theta$ :

$$d_t \theta + v \operatorname{div} Q + v \tilde{T}\Pi'_1(v) \operatorname{div} u = v D : \nabla u + \operatorname{div}(\rho du) \cdot (\kappa w - v \phi) + (\kappa|w|^2 - v w^* \cdot \phi) \operatorname{div} u .$$

But as we have  $\phi = \kappa w$  and  $k = v\kappa$  we conclude that:

$$d_t \theta - v \operatorname{div}(\chi \nabla \theta) + v \Psi^{-1}(\theta) \Pi'_1(v) \operatorname{div} u = v D : \nabla u$$

with:  $\chi(\rho, \theta) = \eta(\rho, \tilde{T})(\Psi^{-1})'(\theta)$  .

**Obtaining a system for  $\rho, u, \theta$ :**

We obtain then for the momentum equation:

$$d_t u - \frac{\operatorname{div} D}{\rho} + \frac{\nabla p_0}{\rho} = \frac{\operatorname{div} K}{\rho} + \frac{1}{2} \frac{\nabla(K_p |w|^2)}{\rho}$$

where  $K_p = \kappa - \rho \kappa'_\rho$ .

And by a calculus we check that:

$$\operatorname{div} K + \frac{1}{2} \nabla(K_p |w|^2) = \rho \nabla(\kappa \Delta \rho) + \frac{\rho}{2} \nabla(\kappa'_\rho |\nabla \rho|^2) .$$

Indeed we have:

$$\begin{aligned} I &= \nabla(\rho \operatorname{div}(\kappa \nabla \rho)) - \operatorname{div}(\kappa w w^*) + \frac{1}{2} \nabla(K_p |w|^2) \\ &= [\rho \nabla(\kappa \Delta \rho) + \kappa \nabla \rho \Delta \rho + \rho \nabla(\kappa'_\rho |\rho|^2) + \kappa'_\rho |\nabla \rho|^2 \nabla \rho] - [\kappa \operatorname{div}(w w^*) + \kappa'_\rho w \cdot \nabla \rho w], \\ &\quad + [\frac{\kappa}{2} \nabla |w|^2 + \frac{\kappa'_\rho}{2} \nabla |w|^2 \nabla \rho - \frac{1}{2} \nabla(\rho \kappa'_\rho |w|^2)], \\ &= [\rho \nabla(\kappa \Delta \rho) + \kappa \nabla \rho \Delta \rho + \frac{\rho}{2} \nabla(\kappa'_\rho |\rho|^2)] - [\kappa \operatorname{div}(w w^*)] + [\frac{\kappa}{2} \nabla |w|^2], \\ &= [\rho \nabla(\kappa \Delta \rho) + \kappa \nabla \rho \Delta \rho + \frac{\rho}{2} \nabla(\kappa'_\rho |\rho|^2)] - [\kappa w \operatorname{div} w + \frac{\kappa}{2} \nabla |w|^2] + [\frac{\kappa}{2} \nabla |w|^2], \\ &= \rho \nabla(\kappa \Delta \rho) + \frac{\rho}{2} \nabla(\kappa'_\rho |\nabla \rho|^2) . \end{aligned}$$

Finally we have obtained the following system:

$$(NHV) \quad \begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t u + u \cdot \nabla u - \frac{\operatorname{div} D}{\rho} - \nabla(\kappa \Delta \rho) + \frac{\nabla(P_0(\rho) + \Psi^{-1}(\theta)P_1(\rho))}{\rho} \\ \hspace{15em} = \nabla\left(\frac{\kappa'}{2}|\nabla \rho|^2\right), \\ \partial_t \theta + u \cdot \nabla \theta - \frac{\operatorname{div}(\chi \nabla \theta)}{\rho} + \Psi^{-1}(\theta) \frac{P_1(\rho)}{\rho} \operatorname{div}(u) = \frac{D : \nabla u}{\rho}, \end{cases}$$

where:  $P_0 = \Pi'_0$ ,  $P_1 = \Pi'_1$  and  $\tilde{T} = \Psi^{-1}(\theta)$ .

We supplement (NHV) with initial conditions:

$$\rho_{t=0} = \rho_0 \geq 0 \quad u_{t=0} = u_0, \quad \text{and} \quad \theta_{t=0} = \theta_0.$$

### 1.3 Classical a priori-estimates

Before getting into the heart of mathematical results, let us derive the physical energy bounds of the (NHV) system when  $\kappa$  is a constant and where the pressure just depends on the density to simplify. Let  $\bar{\rho} > 0$  be a constant reference density, and let  $\Pi$  be defined by:

$$\Pi(s) = s \left( \int_{\bar{\rho}}^s \frac{P_0(z)}{z^2} dz - \frac{P_0(\bar{\rho})}{\bar{\rho}} \right)$$

so that  $P_0(s) = s\Pi'(s) - \Pi(s)$ ,  $\Pi'(\bar{\rho}) = 0$  and:

$$\partial_t \Pi(\rho) + \operatorname{div}(u\Pi(\rho)) + P_0(\rho)\operatorname{div}(u) = 0 \quad \text{in} \quad \mathcal{D}'((0, T) \times \mathbb{R}^N).$$

Notice that  $\Pi$  is convex as far as  $P$  is non decreasing (since  $P'_0(s) = s\Pi''(s)$ ), which is the case for  $\gamma$ -type pressure laws or for Van der Waals law above the critical temperature. Multiplying the equation of momentum conservation in the system (NHV) by  $\rho u$  and integrating by parts over  $\mathbb{R}^N$ , we obtain the following estimate:

$$\begin{aligned} \int_{\mathbb{R}^N} \left( \frac{1}{2}\rho|u|^2 + \rho\theta + (\Pi(\rho) - \Pi(\bar{\rho})) + \frac{\kappa}{2}|\nabla \rho|^2 \right)(t) dx + 2 \int_0^t \int_{\mathbb{R}^N} (2\mu D(u) : D(u) \\ + (\lambda + \mu)|\operatorname{div} u|^2) dx \leq \int_{\mathbb{R}^N} \left( \frac{|m_0|^2}{2\rho} + \rho_0\theta_0 + (\Pi(\rho_0) - \Pi(\bar{\rho})) + \frac{\kappa}{2}|\nabla \rho_0|^2 \right) dx. \end{aligned}$$

It follows that assuming that the initial total energy is finite:

$$\epsilon_0 = \int_{\mathbb{R}^N} \left( \frac{|m_0|^2}{2\rho} + \rho_0\theta_0 + (\Pi(\rho_0) - \Pi(\bar{\rho})) + \frac{\kappa}{2}|\nabla \rho_0|^2 \right) dx < +\infty,$$

then we have the a priori bounds:

$$\begin{aligned} \Pi(\rho) - \Pi(\bar{\rho}), \quad \rho|u|^2, \quad \text{and} \quad \rho\theta \in L^\infty(0, \infty, L^1(\mathbb{R}^N)) \\ \nabla \rho \in L^\infty(0, \infty, L^2(\mathbb{R}^N))^N, \quad \text{and} \quad \nabla u \in L^2(0, \infty, \mathbb{R}^N)^{N^2}. \end{aligned}$$

## 2 Mathematical results

We wish to prove existence and uniqueness results for  $(NHV)$  in functions spaces very close to energy spaces. In the non isothermal non capillary case and  $P(\rho) = a\rho^\gamma$ , with  $a > 0$  and  $\gamma > 1$ , P-L. Lions in [22] proved the global existence of variational solutions  $(\rho, u, \theta)$  to  $(NHV)$  with  $\kappa = 0$  for  $\gamma > \frac{N}{2}$  if  $N \geq 4$ ,  $\gamma \geq \frac{3N}{N+2}$  if  $N = 2, 3$  and initial data  $(\rho_0, m_0)$  such that:

$$\Pi(\rho_0) - \Pi(\bar{\rho}), \frac{|m_0|^2}{\rho_0} \in L^1(\mathbb{R}^N), \text{ and } \rho_0 \theta_0 \in L^1(\mathbb{R}^N).$$

These solutions are weak solutions in the classical sense for the equation of mass conservation and for the equation of the momentum.

On the other hand, the weak solution satisfies only an inequality for the thermal energy equation.

Notice that the main difficulty for proving Lions' theorem consists in exhibiting strong compactness properties of the density  $\rho$  in  $L^p_{loc}$  spaces required to pass to the limit in the pressure term  $P(\rho) = a\rho^\gamma$ .

Let us mention that Feireisl in [16] generalized the result to  $\gamma > \frac{N}{2}$  in establishing that we can obtain renormalized solution without imposing that  $\rho \in L^2_{loc}$ , for this he introduces the concept of oscillation defect measure evaluating the lost of compactness.

We can finally cite a very interesting result from Bresch-Desjardins in [5],[6] where they show the existence of global weak solution for  $(NHV)$  with  $\kappa = 0$  in choosing specific type of viscosity where  $\mu$  and  $\lambda$  are linked. It allows them to get good estimate on the density in using energy inequality and to can treat by compactness all the delicate terms. This result is very new because the energy equation is verified really in distribution sense. In [23], Mellet and Vasseur improve the results of Bresch,Desjardins in generalize to some coefficient  $\mu$  and  $\lambda$  admitting the vacuum in the case of Navier-Stokes isothermal, they use essentially a gain of integrability on the velocity.

In the case  $\kappa > 0$ , we remark then that the density belongs to  $L^\infty(0, \infty, \dot{H}^1(\mathbb{R}^N))$ . Hence, in contrast to the non capillary case one can easily pass to the limit in the pressure term. However let us emphasize at this point that the above a priori bounds do not provide any  $L^\infty$  control on the density from below or from above. Indeed, even in dimension  $N = 2$ ,  $H^1$  functions are not necessarily locally bounded. Thus, vacuum patches are likely to form in the fluid in spite of the presence of capillary forces, which are expected to smooth out the density. Danchin and Desjardins show in [14] that the isothermal model has weak solutions if there exists  $c_1$  and  $M_1$  such that:

$$c_1 \leq |\rho| \leq M_1 \text{ and } |\rho - 1| \ll 1.$$

The vacuum is one of the main difficulties to get weak solutions, and the problem remains open.

In the isothermal capillary case with specific type of viscosity and capillarity  $\mu(\rho) = \mu\rho$  and  $\lambda(\rho) = 0$ , Bresch, Desjardins and Lin in [7] obtain the global existence of weak solutions without smallness assumption on the data. We can precise the space of test functions used depends on the solution itself which are on the form  $\rho\phi$  with  $\phi \in C_0^\infty(\mathbb{R}^N)$ .

The specificity of the viscosity allows to get a gain of one derivative on the density:  $\rho \in L^2(H^2)$ .

Existence of strong solution with  $\kappa$ ,  $\mu$  and  $\lambda$  constant is known since the work by Hattori and Li in [18], [19] in the whole space  $\mathbb{R}^N$ . In [14], Danchin and Desjardins study the well-posedness of the problem for the isothermal case with constant coefficients in critical Besov spaces.

Here we want to investigate the well-posedness of the full non isothermal problem in critical spaces, that is, in spaces which are invariant by the scaling of Korteweg's system. Recall that such an approach is now classical for incompressible Navier-Stokes equation and yields local well-posedness (or global well-posedness for small data) in spaces with minimal regularity.

Let us explain precisely the scaling of Korteweg's system. We can easily check that, if  $(\rho, u, \theta)$  solves  $(NHV)$ , so does  $(\rho_\lambda, u_\lambda, \theta_\lambda)$ , where:

$$\rho_\lambda(t, x) = \rho(\lambda^2 t, \lambda x), \quad u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x) \quad \text{and} \quad \theta_\lambda(t, x) = \lambda^2 \theta(\lambda^2 t, \lambda x)$$

provided the pressure laws  $P_0, P_1$  have been changed into  $\lambda^2 P_0, \lambda^2 P_1$ .

**Definition 2.1** *We say that a functional space is critical with respect to the scaling of the equation if the associated norm is invariant under the transformation:*

$$(\rho, u, \theta) \longrightarrow (\rho_\lambda, u_\lambda, \theta_\lambda)$$

(up to a constant independent of  $\lambda$ ).

This suggests us to choose initial data  $(\rho_0, u_0, \theta_0)$  in spaces whose norm is invariant for all  $\lambda > 0$  by  $(\rho_0, u_0, \theta_0) \longrightarrow (\rho_0(\lambda \cdot), \lambda u_0(\lambda \cdot), \lambda^2 \theta_0(\lambda \cdot))$ .

A natural candidate is the homogeneous Sobolev space  $\dot{H}^{N/2} \times (\dot{H}^{N/2-1})^N \times \dot{H}^{N/2-2}$ , but since  $\dot{H}^{N/2}$  is not included in  $L^\infty$ , we cannot expect to get  $L^\infty$  control on the density when  $\rho_0 \in \dot{H}^{N/2}$ . The same problem occurs in the equation for the temperature when dealing with the non linear term involving  $\Psi^{-1}(\theta)$ .

This is the reason why, instead of the classical homogeneous Sobolev space  $\dot{H}^s(\mathbb{R}^d)$ , we will consider homogeneous Besov spaces with the same derivative index  $B^s = \dot{B}_{2,1}^s(\mathbb{R}^N)$  (for the corresponding definition we refer to section 4).

One of the nice property of  $B^s$  spaces for critical exponent  $s$  is that  $B^{N/2}$  is an algebra embedded in  $L^\infty$ . This allows to control the density from below and from above, without requiring more regularity on derivatives of  $\rho$ . For similar reasons, we shall take  $\theta_0$  in  $B^{\frac{N}{2}}$  in the general case where appear non-linear terms in function of the temperature.

Since a global in time approach does not seem to be accessible for general data, we will mainly consider the global well-posedness problem for initial data close enough to stable equilibria (Section 5). This motivates the following definition:

**Definition 2.2** *Let  $\bar{\rho} > 0, \bar{\theta} > 0$ . We will note in the sequel:*

$$q = \frac{\rho - \bar{\rho}}{\bar{\rho}} \quad \text{and} \quad \mathcal{T} = \theta - \bar{\theta}.$$



One can now state the main results of the paper.

The first three theorems concern the global existence and uniqueness of solution to the Korteweg's system with *small* initial data. In particular the first two results concern Korteweg's system with coefficients depending only on the density and where the intern specific energy is a linear function of the temperature.

**Theorem 2.1** *Let  $N \geq 3$ . Assume that the function  $\Psi$  defined in (A) satisfies  $\Psi(\tilde{T}) = A\tilde{T}$  with  $A > 0$  and that all the physical coefficients are smooth functions depending only on the density. Let  $\bar{\rho} > 0$  be such that:*

$$\kappa(\bar{\rho}) > 0, \quad \mu(\bar{\rho}) > 0, \quad \lambda(\bar{\rho}) + 2\mu(\bar{\rho}) > 0, \quad \eta(\bar{\rho}) > 0 \quad \text{and} \quad \partial_{\rho} P_0(\bar{\rho}) > 0.$$

Moreover suppose that:

$$q_0 \in \tilde{B}^{\frac{N}{2}-1, \frac{N}{2}}, \quad u_0 \in B^{\frac{N}{2}-1}, \quad \mathcal{T}_0 \in \tilde{B}^{\frac{N}{2}-1, \frac{N}{2}-2}.$$

There exists an  $\varepsilon_0$  depending only on the physical coefficients (that we will precise later) such that if:

$$\|q_0\|_{\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}}} + \|u_0\|_{\tilde{B}^{\frac{N}{2}-1}} + \|\mathcal{T}_0\|_{\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}-2}} \leq \varepsilon_0$$

then (NHV) has a unique global solution  $(\rho, u, \mathcal{T})$  in  $E^{N/2}$  where  $E^s$  is defined by:

$$\begin{aligned} E^s = & [C_b(\mathbb{R}_+, \tilde{B}^{s-1, s}) \cap L^1(\mathbb{R}_+, \tilde{B}^{s+1, s+2})] \times [C_b(\mathbb{R}_+, B^{s-1})^N \cap L^1(\mathbb{R}_+, B^{s+1})^N] \\ & \times [C_b(\mathbb{R}_+, \tilde{B}^{s-1, s-2}) \cap L^1(\mathbb{R}_+, \tilde{B}^{s+1, s})]. \end{aligned}$$

**Remark 1** *Above,  $\tilde{B}^{s,t}$  stands for a Besov space with regularity  $B^s$  in low frequencies and  $B^t$  in high frequencies (see definition 3.3).*

The case  $N = 2$  requires more regular initial data because of technical problems involving some nonlinear terms in the temperature equation.

**Theorem 2.2** *Let  $N = 2$ . Under the assumption of the theorem 2.1 for  $\Psi$  and the physical coefficients, let  $\varepsilon' > 0$  and suppose that:*

$$q_0 \in \tilde{B}^{0, 1+\varepsilon'}, \quad u_0 \in \tilde{B}^{0, \varepsilon'}, \quad \mathcal{T}_0 \in \tilde{B}^{0, -1+\varepsilon'}.$$

There exists an  $\varepsilon_0$  depending only on the physical coefficients such that if:

$$\|q_0\|_{\tilde{B}^{0, 1+\varepsilon'}} + \|u_0\|_{\tilde{B}^{0, \varepsilon'}} + \|\mathcal{T}_0\|_{\tilde{B}^{0, -1+\varepsilon'}} \leq \varepsilon_0$$

then (NHV) has a unique global solution  $(\rho, u, \mathcal{T})$  in the space:

$$\begin{aligned} E' = & [C_b(\mathbb{R}_+, \tilde{B}^{0, 1+\varepsilon'}) \cap L^1(\mathbb{R}_+, \tilde{B}^{2, 3+\varepsilon'})] \times [C_b(\mathbb{R}_+, \tilde{B}^{0, \varepsilon'})^2 \cap L^1(\mathbb{R}_+, \tilde{B}^{2, 2+\varepsilon'})^2] \\ & \times [C_b(\mathbb{R}_+, \tilde{B}^{0, -1+\varepsilon'}) \cap L^1(\mathbb{R}_+, \tilde{B}^{2, 1+\varepsilon'})]. \end{aligned}$$

In the following theorem we are interested by showing the global existence of solution for Korteweg's system with general conditions and small initial data. In order to control the non linear terms in temperature more regularity is required. That's why we want control the temperature in norm  $L^\infty$ .

**Theorem 2.3** *Let  $N \geq 2$ . Assume that  $\Psi$  be a regular function depending on  $\theta$ . Assume that all the coefficients are smooth functions of  $\rho$  and  $\theta$  except  $\kappa$  which depends only on the density. Take  $(\bar{\rho}, \bar{T})$  such that:*

$$\kappa(\bar{\rho}) > 0, \quad \mu(\bar{\rho}, \bar{T}) > 0, \quad \lambda(\bar{\rho}, \bar{T}) + 2\mu(\bar{\rho}, \bar{T}) > 0, \quad \eta(\bar{\rho}, \bar{T}) > 0 \quad \text{and} \quad \partial_\rho P_0(\bar{\rho}, \bar{T}) > 0.$$

Moreover suppose that:

$$q_0 \in \tilde{B}^{\frac{N}{2}-1, \frac{N}{2}+1}, \quad u_0 \in \tilde{B}^{\frac{N}{2}-1, \frac{N}{2}}, \quad \mathcal{T}_0 \in \tilde{B}^{\frac{N}{2}-1, \frac{N}{2}}.$$

There exists an  $\varepsilon_1$  depending only on the physical coefficients such that if:

$$\|q_0\|_{\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}+1}} + \|u_0\|_{\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}}} + \|\mathcal{T}_0\|_{\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}}} \leq \varepsilon$$

then (NHV) has a unique global solution  $(\rho, u, \mathcal{T})$  in:

$$\begin{aligned} F^{\frac{N}{2}} = & [C_b(\mathbb{R}_+, \tilde{B}^{\frac{N}{2}-1, \frac{N}{2}+1}) \cap L^1(\mathbb{R}_+, \tilde{B}^{\frac{N}{2}+1, \frac{N}{2}+3})] \times [C_b(\mathbb{R}_+, \tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})^N \\ & \cap L^1(\mathbb{R}_+, \tilde{B}^{\frac{N}{2}+1, \frac{N}{2}+2})^N] [C_b(\mathbb{R}_+, \tilde{B}^{\frac{N}{2}-1, \frac{N}{2}}) \cap L^1(\mathbb{R}_+, \tilde{B}^{\frac{N}{2}+1, \frac{N}{2}+2})]. \end{aligned}$$

In the previous theorem we can observe for the case  $N = 2$  that the initial data are very close from the energy space of Bresch, Desjardins and Lin in [7].

In the following three theorems we are interested by the existence and uniqueness of solution in finite time for *large* data. We distinguish always the different cases  $N \geq 3$  and  $N = 2$  if the coefficients depend only on  $\rho$ , and the case where the coefficients depend also on  $\tilde{T}$ .

**Theorem 2.4** *Let  $N \geq 3$ , and  $\Psi$  and the physical coefficients be as in theorem 2.1. We suppose that  $(q_0, u_0, \mathcal{T}_0) \in B^{\frac{N}{2}} \times (B^{\frac{N}{2}-1})^N \times B^{\frac{N}{2}-2}$  and that  $\rho_0 \geq c$  for some  $c > 0$ .*

*Then there exists a time  $T$  such that system (NHV) has a unique solution in  $F_T$*

$$\begin{aligned} F_T = & [\tilde{C}_T(B^{\frac{N}{2}}) \cap L_T^1(B^{\frac{N}{2}+2})] \times [\tilde{C}_T(B^{\frac{N}{2}-1})^N \cap L_T^1(B^{\frac{N}{2}+1})^N] \\ & \times [\tilde{C}_T(B^{\frac{N}{2}-2}) \cap L_T^1(B^{\frac{N}{2}})]. \end{aligned}$$

For the same reasons as previously in the case  $N = 2$  we can not reach the critical level of regularity.

**Theorem 2.5** *Let  $N = 2$  and  $\varepsilon' > 0$ . Under the assumptions of theorem 2.1 for  $\Psi$  and the physical coefficients we suppose that  $(q_0, u_0, \mathcal{T}_0) \in \tilde{B}^{1, 1+\varepsilon'} \times (\tilde{B}^{0, \varepsilon'})^2 \times \tilde{B}^{-1, -1+\varepsilon'}$  and  $\rho_0 \geq c$  for some  $c > 0$ .*

*Then there exists a time  $T$  such that the system has a unique solution in  $F_T(2)$  with:*

$$\begin{aligned} F_T(2) = & [\tilde{C}_T(\tilde{B}^{1, 1+\varepsilon'}) \cap L_T^1(\tilde{B}^{3, 3+\varepsilon'})] \times [\tilde{C}_T(\tilde{B}^{0, \varepsilon'})^2 \cap L_T^1(\tilde{B}^{2, 2+\varepsilon'})^2] \\ & \times [\tilde{C}_T(\tilde{B}^{-1, -1+\varepsilon'}) \cap L_T^1(\tilde{B}^{1, 1+\varepsilon'})]. \end{aligned}$$

In the last theorem we see the general system without conditions, and like previously we need more regular initial data.

**Theorem 2.6** *Under the hypotheses of theorem 2.3 we suppose that:*

$$(q_0, u_0, \mathcal{T}_0) \in \tilde{B}^{\frac{N}{2}, \frac{N}{2}+1} \times (B^{\frac{N}{2}})^N \times B^{\frac{N}{2}} \quad \text{and} \quad \rho_0 \geq c \quad \text{for some } c > 0.$$

*Then there exists a time  $T$  such that the system has a unique solution in :*

$$F'_T = [\tilde{C}_T(\tilde{B}^{\frac{N}{2}, \frac{N}{2}+1}) \cap L_T^1(\tilde{B}^{\frac{N}{2}+2, \frac{N}{2}+3})] \times [\tilde{C}_T(B^{\frac{N}{2}})^N \cap L_T^1(B^{\frac{N}{2}+2})^N] \\ \times [\tilde{C}_T(B^{\frac{N}{2}}) \cap L_T^1(B^{\frac{N}{2}+2})].$$

This chapter is structured in the following way, first of all we recall in the section 3 some definitions on Besov spaces and some useful theorem concerning Besov spaces. Next we will concentrate in the section 4 on the global existence and uniqueness of solution for our system ( $NHV$ ) with small initial data. In subsection 4.1 we will give some necessary conditions to get the stability of the linear part associated to the system ( $NHV$ ). In subsection 4.2 we will study the case where the specific intern energy is linear and where the physical coefficients are independent of the temperature. In our proof we will distinguish the case  $N \geq 3$  and the case  $N = 2$  for some technical reasons. In the section 5 we will examine the local existence and uniqueness of solution with general initial data. For the same reasons as section 4 we will distinguish the cases in function of the behavior of the coefficients and of the intern specific energy.

## 3 Littlewood-Paley theory and Besov spaces

### 3.1 Littlewood-Paley decomposition

Littlewood-Paley decomposition corresponds to a dyadic decomposition of the space in Fourier variables.

We can use for instance any  $\varphi \in C^\infty(\mathbb{R}^N)$ , supported in  $\mathcal{C} = \{\xi \in \mathbb{R}^N / \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$  such that:

$$\sum_{l \in \mathbb{Z}} \varphi(2^{-l}\xi) = 1 \quad \text{if } \xi \neq 0.$$

Denoting  $h = \mathcal{F}^{-1}\varphi$ , we then define the dyadic blocks by:

$$\Delta_l u = 2^{lN} \int_{\mathbb{R}^N} h(2^l y) u(x - y) dy \quad \text{and} \quad S_l u = \sum_{k \leq l-1} \Delta_k u.$$

Formally, one can write that:

$$u = \sum_{k \in \mathbb{Z}} \Delta_k u.$$

This decomposition is called homogeneous Littlewood-Paley decomposition. Let us observe that the above formal equality does not hold in  $\mathcal{S}'(\mathbb{R}^N)$  for two reasons:

1. The right hand-side does not necessarily converge in  $\mathcal{S}'(\mathbb{R}^N)$ .
2. Even if it does, the equality is not always true in  $\mathcal{S}'(\mathbb{R}^N)$  (consider the case of the polynomials).

However, this equality holds true modulo polynomials hence homogeneous Besov spaces will be defined modulo the polynomials, according to [4].

### 3.2 Homogeneous Besov spaces and first properties

**Definition 3.3** For  $s \in \mathbb{R}$ , and  $u \in \mathcal{S}'(\mathbb{R}^N)$  we set:

$$\|u\|_{B^s} = \left( \sum_{l \in \mathbb{Z}} (2^{ls} \|\Delta_l u\|_{L^2}) \right).$$

A difficulty due to the choice of homogeneous spaces arises at this point. Indeed,  $\|\cdot\|_{B^s}$  cannot be a norm on  $\{u \in \mathcal{S}'(\mathbb{R}^N), \|u\|_{B^s} < +\infty\}$  because  $\|u\|_{B^s} = 0$  means that  $u$  is a polynomial. This enforces us to adopt the following definition for homogeneous Besov spaces, see [4].

**Definition 3.4** Let  $s \in \mathbb{R}$ .

Denote  $m = [s - \frac{N}{2}]$  if  $s - \frac{N}{2} \notin \mathbb{Z}$  and  $m = s - \frac{N}{2} - 1$  otherwise.

- If  $m < 0$ , then we define  $B^s$  as:

$$B^s = \left\{ u \in \mathcal{S}'(\mathbb{R}^N) / \|u\|_{B^s} < \infty \text{ and } u = \sum_{l \in \mathbb{Z}} \Delta_l u \text{ in } \mathcal{S}'(\mathbb{R}^N) \right\}.$$

- If  $m \geq 0$ , we denote by  $\mathcal{P}_m[\mathbb{R}^N]$  the set of polynomials of degree less than or equal to  $m$  and we set:

$$B^s = \left\{ u \in \mathcal{S}'(\mathbb{R}^N) / \mathcal{P}_m[\mathbb{R}^N] / \|u\|_{B^s} < \infty \text{ and } u = \sum_{l \in \mathbb{Z}} \Delta_l u \text{ in } \mathcal{S}'(\mathbb{R}^N) / \mathcal{P}_m[\mathbb{R}^N] \right\}.$$

**Proposition 3.1** The following properties hold:

1. *Density:* If  $|s| \leq \frac{N}{2}$ , then  $C_0^\infty$  is dense in  $B^s$ .
2. *Derivatives:* There exists a universal constant  $C$  such that:
$$C^{-1} \|u\|_{B^s} \leq \|\nabla u\|_{B^{s-1}} \leq C \|u\|_{B^s}.$$
3. *Algebraic properties:* For  $s > 0$ ,  $B^s \cap L^\infty$  is an algebra.
4. *Interpolation:*  $(B^{s_1}, B^{s_2})_{\theta, 1} = B^{\theta s_2 + (1-\theta)s_1}$ .

### 3.3 Hybrid Besov spaces and Chemin-Lerner spaces

Hybrid Besov spaces are functional spaces where regularity assumptions are different in low frequency and high frequency, see [12].

They may be defined as follows:

**Definition 3.5** Let  $s, t \in \mathbb{R}$ . We set:

$$\|u\|_{\tilde{B}^{s,t}} = \sum_{q \leq 0} 2^{qs} \|\Delta_q u\|_{L^2} + \sum_{q > 0} 2^{qt} \|\Delta_q u\|_{L^2}.$$

Let  $m = -[\frac{N}{2} + 1 - s]$ , we then define:

- $\tilde{B}^{s,t} = \{u \in \mathcal{S}'(\mathbb{R}^N) / \|u\|_{\tilde{B}^{s,t}} < +\infty\}$ , if  $m < 0$
- $\tilde{B}^{s,t} = \{u \in \mathcal{S}'(\mathbb{R}^N) / \mathcal{P}_m[\mathbb{R}^N] / \|u\|_{\tilde{B}^{s,t}} < +\infty\}$  if  $m \geq 0$ .

Let us now give some properties of these hybrid spaces and some results on how they behave with respect to the product. The following results come directly from the paradiifferential calculus.

**Proposition 3.2** *We recall some inclusion:*

- We have  $\tilde{B}^{s,s} = B^s$ .
- If  $s \leq t$  then  $\tilde{B}^{s,t} = B^s \cap B^t$ , if  $s > t$  then  $\tilde{B}^{s,t} = B^s + B^t$ .
- If  $s_1 \leq s_2$  and  $t_1 \geq t_2$  then  $\tilde{B}^{s_1,t_1} \hookrightarrow \tilde{B}^{s_2,t_2}$ .

**Proposition 3.3** *For all  $s, t > 0$ , we have:*

$$\|uv\|_{\tilde{B}^{s,t}} \leq C(\|u\|_{L^\infty} \|v\|_{\tilde{B}^{s,t}} + \|v\|_{L^\infty} \|u\|_{\tilde{B}^{s,t}}).$$

For all  $s_1, s_2, t_1, t_2 \leq \frac{N}{2}$  such that  $\min(s_1 + s_2, t_1 + t_2) > 0$  we have:

$$\|uv\|_{\tilde{B}^{s_1+t_1-\frac{N}{2}, s_2+t_2-\frac{N}{2}}} \leq C\|u\|_{\tilde{B}^{s_1,t_1}} \|v\|_{\tilde{B}^{s_2,t_2}}.$$

For a proof of this proposition see [12]. We are now going to define the spaces of Chemin-Lerner in which we will work, which are a refinement of the spaces:

$$L_T^\rho(B^s) := L^\rho(0, T, B^s).$$

**Definition 3.6** *Let  $\rho \in [1, +\infty[$ ,  $T \in [1, +\infty]$  and  $s \in \mathbb{R}$ . We then denote:*

$$\|u\|_{\tilde{L}_T^\rho(\tilde{B}^{s_1,s_2})} = \sum_{l \leq 0} 2^{ls_1} \left( \int_0^T \|\Delta_l u(t)\|_{L^2}^\rho dt \right)^{1/\rho} + \sum_{l > 0} 2^{ls_2} \left( \int_0^T \|\Delta_l u(t)\|_{L^2}^\rho dt \right)^{1/\rho}.$$

And we have in the case  $\rho = \infty$ :

$$\|u\|_{\tilde{L}_T^\infty(\tilde{B}^{s_1,s_2})} = \sum_{l \leq 0} 2^{ls_1} \|\Delta_l u\|_{L^\infty(L^2)} + \sum_{l > 0} 2^{ls_2} \|\Delta_l u\|_{L^\infty(L^2)}.$$

We note that thanks to Minkowsky inequality we have:

$$\|u\|_{L_T^\rho(\tilde{B}^{s_1,s_2})} \leq \|u\|_{\tilde{L}_T^\rho(\tilde{B}^{s_1,s_2})} \quad \text{and} \quad \|u\|_{L_T^1(\tilde{B}^{s_1,s_2})} = \|u\|_{\tilde{L}_T^1(\tilde{B}^{s_1,s_2})}.$$

From now on, we will denote:

$$\begin{aligned} \|u\|_{\tilde{L}_T^-(B^{s_1})} &= \sum_{l \leq 0} 2^{ls_1} \left( \int_0^T \|\Delta_l u(t)\|_{L^p}^\rho dt \right)^{1/\rho} \\ \|u\|_{\tilde{L}_T^+(B^{s_2})} &= \sum_{l > 0} 2^{ls_2} \left( \int_0^T \|\Delta_l u(t)\|_{L^p}^\rho dt \right)^{1/\rho}. \end{aligned}$$

Hence:

$$\|u\|_{\tilde{L}_T^\rho(B^{s_1})} = \|u\|_{\tilde{L}_T^\rho(B^{s_1})}^- + \|u\|_{\tilde{L}_T^\rho(\tilde{B}^{s_1, s_2})}^+$$

We then define the space:

$$\tilde{L}_T^\rho(\tilde{B}^{s_1, s_2}) = \{u \in L_T^\rho(\tilde{B}^{s_1, s_2}) / \|u\|_{\tilde{L}_T^\rho(\tilde{B}^{s_1, s_2})} < \infty\}.$$

We denote moreover by  $\tilde{C}_T(\tilde{B}^{s_1, s_2})$  the set of those functions of  $\tilde{L}_T^\infty(\tilde{B}^{s_1, s_2})$  which are continuous from  $[0, T]$  to  $\tilde{B}^{s_1, s_2}$ . In the sequel we are going to give some properties of this spaces concerning the interpolation and their relationship with the heat equation.

**Proposition 3.4** *Let  $s, t, s_1, s_2 \in \mathbb{R}$ ,  $\rho, \rho_1, \rho_2 \in [1, +\infty]$ . We have:*

1. *Interpolation:*

$$\|u\|_{\tilde{L}_T^\rho(\tilde{B}^{s, t})} \leq \|u\|_{\tilde{L}_T^{\rho_1}(\tilde{B}^{s_1, t_1})}^\theta \|u\|_{\tilde{L}_T^{\rho_2}(\tilde{B}^{s_2, t_2})}^{1-\theta} \quad \text{with } \theta \in [0, 1] \text{ and } \frac{1}{\rho} = \frac{\theta}{\rho_1} + \frac{1-\theta}{\rho_2},$$

$$s = \theta s_1 + (1-\theta)s_2, \quad t = \theta t_1 + (1-\theta)t_2.$$

2. *Embedding:*

$$\tilde{L}_T^\rho(\tilde{B}^{s, t}) \hookrightarrow L_T^\rho(C_0) \quad \text{and} \quad \tilde{C}_T(B^{\frac{N}{2}}) \hookrightarrow C([0, T] \times \mathbb{R}^N).$$

The  $\tilde{L}_T^\rho(B_p^s)$  spaces suit particularly well to the study of smoothing properties of the heat equation. In [9], J-Y. Chemin proved the following proposition:

**Proposition 3.5** *Let  $p \in [1, +\infty]$  and  $1 \leq \rho_2 \leq \rho_1 \leq +\infty$ . Let  $u$  be a solution of:*

$$\begin{cases} \partial_t u - \mu \Delta u = f \\ u_{t=0} = u_0. \end{cases}$$

*Then there exists  $C > 0$  depending only on  $N, \mu, \rho_1$  and  $\rho_2$  such that:*

$$\|u\|_{\tilde{L}_T^{\rho_1}(B^{s+2/\rho_1})} \leq C \|u_0\|_{B^s} + C \|f\|_{\tilde{L}_T^{\rho_2}(B^{s-2+2/\rho_2})}.$$

To finish with, we explain how the product of functions behaves in the spaces of Chemin-Lerner. We have the following properties:

**Proposition 3.6**

*Let  $s > 0$ ,  $t > 0$ ,  $1/\rho_2 + 1/\rho_3 = 1/\rho_1 + 1/\rho_4 = 1/\rho \leq 1$ ,  $u \in \tilde{L}_T^{\rho_3}(\tilde{B}^{s, t}) \cap \tilde{L}_T^{\rho_1}(L^\infty)$  and  $v \in \tilde{L}_T^{\rho_4}(\tilde{B}^{s, t}) \cap \tilde{L}_T^{\rho_2}(L^\infty)$ .*

*Then  $uv \in \tilde{L}_T^\rho(\tilde{B}^{s, t})$  and we have:*

$$\|uv\|_{\tilde{L}_T^\rho(\tilde{B}^{s, t})} \leq C \|u\|_{\tilde{L}_T^{\rho_1}(L^\infty)} \|v\|_{\tilde{L}_T^{\rho_4}(\tilde{B}^{s, t})} + \|v\|_{\tilde{L}_T^{\rho_2}(L^\infty)} \|u\|_{\tilde{L}_T^{\rho_3}(\tilde{B}^{s, t})}.$$

*If  $s_1, s_2, t_1, t_2 \leq \frac{N}{2}$ ,  $s_1 + s_2 > 0$ ,  $t_1 + t_2 > 0$ ,  $\frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{1}{\rho} \leq 1$ ,  $u \in \tilde{L}_T^{\rho_1}(\tilde{B}^{s_1, t_1})$  and  $v \in \tilde{L}_T^{\rho_2}(\tilde{B}^{s_2, t_2})$  then  $uv \in \tilde{L}_T^\rho(\tilde{B}^{s_1+s_2-\frac{N}{2}, t_1+t_2-\frac{N}{2}})$  and:*

$$\|uv\|_{\tilde{L}_T^\rho(\tilde{B}_2^{s_1+s_2-\frac{N}{2}, t_1+t_2-\frac{N}{2}})} \leq C \|u\|_{\tilde{L}_T^{\rho_1}(\tilde{B}^{s_1, t_1})} \|v\|_{\tilde{L}_T^{\rho_2}(\tilde{B}^{s_2, t_2})}.$$

For a proof of this proposition see [12]. Finally we need an estimate on the composition of functions in the spaces  $\tilde{L}_T^\rho(\tilde{B}_p^s)$  (see the proof in the appendix).

**Proposition 3.7** *Let  $s > 0$ ,  $p \in [1, +\infty]$  and  $u_1, u_2, \dots, u_d \in \tilde{L}_T^\rho(B_p^s) \cap L_T^\infty(L^\infty)$ .*

(i) *Let  $F \in W_{loc}^{[s]+2, \infty}(\mathbb{R}^N)$  such that  $F(0) = 0$ . Then  $F(u_1, u_2, \dots, u_d) \in \tilde{L}_T^\rho(B_p^s)$ . More precisely, there exists a constant  $C$  depending only on  $s, p, N$  and  $F$  such that:*

$$\begin{aligned} \|F(u_1, u_2, \dots, u_d)\|_{\tilde{L}_T^\rho(B_p^s)} &\leq C(\|u_1\|_{L_T^\infty(L^\infty)}, \|u_2\|_{L_T^\infty(L^\infty)}, \dots, \|u_d\|_{L_T^\infty(L^\infty)}) \\ &\quad (\|u_1\|_{\tilde{L}_T^\rho(B_p^s)} + \dots + \|u_d\|_{\tilde{L}_T^\rho(B_p^s)}). \end{aligned}$$

(ii) *Let  $u \in \tilde{L}_T^\rho(\tilde{B}^{s_1, s_2})$ ,  $s_1, s_2 > 0$  then we have  $F(u) \in \tilde{L}_T^\rho(\tilde{B}^{s_1, s_2})$  and*

$$\|F(u)\|_{\tilde{L}_T^\rho(\tilde{B}^{s_1, s_2})} \leq C(\|u\|_{L_T^\infty(L^\infty)})\|u\|_{\tilde{L}_T^\rho(\tilde{B}^{s_1, s_2})}.$$

(iii) *If  $v, u \in \tilde{L}_T^\rho(B_p^s) \cap L_T^\infty(L^\infty)$  and  $G \in W_{loc}^{[s]+3, \infty}(\mathbb{R}^N)$  then  $G(u) - G(v)$  belongs to  $\tilde{L}_T^\rho(B_p^s)$  and there exists a constant  $C$  depending only of  $s, p, N$  and  $G$  such that:*

$$\begin{aligned} \|G(u) - G(v)\|_{\tilde{L}_T^\rho(B_p^s)} &\leq C(\|u\|_{L_T^\infty(L^\infty)}, \|v\|_{L_T^\infty(L^\infty)}) (\|v - u\|_{\tilde{L}_T^\rho(B_p^s)} (1 + \|u\|_{L_T^\infty(L^\infty)} \\ &\quad + \|v\|_{L_T^\infty(L^\infty)}) + \|v - u\|_{L_T^\infty(L^\infty)} (\|u\|_{\tilde{L}_T^\rho(B_p^s)} + \|v\|_{\tilde{L}_T^\rho(B_p^s)})). \end{aligned}$$

(iv) *If  $v, u \in \tilde{L}_T^\rho(\tilde{B}_p^{s_1, s_2}) \cap L_T^\infty(L^\infty)$  and  $G \in W_{loc}^{[s]+3, \infty}(\mathbb{R}^N)$  then  $G(u) - G(v)$  belongs to  $\tilde{L}_T^\rho(\tilde{B}_p^{s_1, s_2})$  and it exists a constant  $C$  depending only of  $s, p, N$  and  $G$  such that:*

$$\begin{aligned} \|G(u) - G(v)\|_{\tilde{L}_T^\rho(\tilde{B}_p^{s_1, s_2})} &\leq C(\|u\|_{L_T^\infty(L^\infty)}, \|v\|_{L_T^\infty(L^\infty)}) (\|v - u\|_{\tilde{L}_T^\rho(\tilde{B}_p^{s_1, s_2})} (1 + \|u\|_{L_T^\infty(L^\infty)} \\ &\quad + \|v\|_{L_T^\infty(L^\infty)}) + \|v - u\|_{L_T^\infty(L^\infty)} (\|u\|_{\tilde{L}_T^\rho(\tilde{B}_p^{s_1, s_2})} + \|v\|_{\tilde{L}_T^\rho(\tilde{B}_p^{s_1, s_2})}). \end{aligned}$$

The proof is an adaptation of a theorem by J.Y. Chemin and H. Bahouri in [2], see the proof in the Appendix.

## 4 Existence of solutions for small initial data

### 4.1 Study of the linear part

This section is devoted to the study of the linearization of system  $(NHV)$  in order to get conditions for the existence of solution. We recall the system  $(NHV)$  in the case where





where we set:  $\Lambda^s h = \mathcal{F}^{-1}(|\xi|^s \hat{h})$  (the curl is defined in the appendix).

We finally obtain the following system in projecting on divergence free vector fields and on potential vector fields:

$$(M'_1) \quad \begin{cases} \partial_t q + \Lambda d = F, \\ \partial_t d - \nu \Delta d - \varepsilon \Lambda^3 q - \beta \Lambda q - \gamma \Lambda \mathcal{T} = \Lambda^{-1} \operatorname{div} G, \\ \partial_t \mathcal{T} - \alpha \Delta \mathcal{T} + \delta \Lambda d = H, \\ \partial_t \Omega - \tilde{\mu} \Delta \Omega = \Lambda^{-1} \operatorname{curl} G, \\ u = -\Lambda^{-1} \nabla d - \Lambda^{-1} \operatorname{div} \Omega. \end{cases}$$

The last equation is just a heat equation. Hence we are going to focus on the first three equations. However the last equation gives us an idea of which spaces we can work with. The first three equation can be read as follows:

$$(M'_2) \quad \partial_t \begin{pmatrix} \hat{q}(t, \xi) \\ \hat{d}(t, \xi) \\ \hat{\mathcal{T}}(t, \xi) \end{pmatrix} + A(\xi) \begin{pmatrix} \hat{q}(t, \xi) \\ \hat{d}(t, \xi) \\ \hat{\mathcal{T}}(t, \xi) \end{pmatrix} = \begin{pmatrix} \hat{F}(t, \xi) \\ \Lambda^{-1} \operatorname{div} \hat{G}(t, \xi) \\ \hat{H}(t, \xi) \end{pmatrix}$$

where we have:

$$A(\xi) = \begin{pmatrix} 0 & |\xi| & 0 \\ -\varepsilon |\xi|^3 - \beta |\xi| & \nu |\xi|^2 & -\gamma |\xi| \\ 0 & \delta |\xi| & \alpha |\xi|^2 \end{pmatrix}.$$

The eigenvalues of the matrix  $-A(\xi)$  are of the form  $|\xi|^2 \lambda_\xi$  with  $\lambda_\xi$  being the roots of the following polynomial:

$$P_\xi(X) = X^3 + (\nu + \alpha)X^2 + \left( \varepsilon + \nu\alpha + \frac{\gamma\delta + \beta}{|\xi|^2} \right) X + \left( \alpha\varepsilon + \frac{\alpha\beta}{|\xi|^2} \right).$$

For very large  $\xi$ , the roots tend to those of the following polynomial (by virtue of continuity of the roots in function of the coefficients):

$$X^3 + (\nu + \alpha)X^2 + (\varepsilon + \nu\alpha)X + \alpha\varepsilon.$$

The roots are  $-\alpha$  and  $-\frac{\nu}{2}(1 \pm \sqrt{1 - \frac{4\varepsilon}{\nu^2}})$ .

The system  $(M'_1)$  is well-posed if and only if for  $|\xi|$  tending to  $+\infty$  the real part of the eigenvalues associated to  $A(\xi)$  stay non positive. Hence, we must have:

$$\varepsilon, \nu, \alpha \geq 0.$$

Let us now state a necessary and sufficient condition for the global stability of  $(M')$ .

**Proposition 4.8** *The linear system  $(M')$  is globally stable if and only if the following conditions are verified:*

$$(*) \quad \nu, \varepsilon, \alpha \geq 0, \quad \alpha\beta \geq 0, \quad \gamma\delta(\nu + \alpha) + \nu\beta \geq 0, \quad \gamma\delta + \beta \geq 0.$$

If all the inequalities are strict, the solutions tend to 0 in the sense of distributions and the three eigenvalues  $\lambda_1(\xi)$ ,  $\lambda_+(\xi)$ ,  $\lambda_-(\xi)$  have the following asymptotic behavior when  $\xi$  tends to 0:

$$\lambda_1(\xi) \sim -\left(\frac{\alpha\beta}{\beta + \gamma\delta}\right)|\xi|^2, \quad \lambda_{\pm}(\xi) \sim -\left(\frac{\gamma\delta(\nu + \alpha) + \nu\beta}{2(\gamma\delta + \beta)}\right)|\xi|^2 \pm i|\xi|\sqrt{\gamma\delta + \beta}.$$

**Proof :**

We already know that the system is well-posed if and only if  $\nu, \alpha \geq 0$ . We want that all the eigenvalues have a negative real part for all  $\xi$ .

We have to distinguish two cases: either all the eigenvalues are real or there are two complex conjugated eigenvalues.

**First case:**

The eigenvalues are real. A necessary condition for negativity of the eigenvalues is that  $P(X) \geq 0$  for  $X \geq 0$ . We must have in particular:

$$P_{\xi}(0) = \alpha\varepsilon + \frac{\alpha\beta}{|\xi|^2} \geq 0 \quad \forall \xi \neq 0.$$

This imply that  $\alpha\beta \geq 0$  and  $\alpha\varepsilon \geq 0$ . Hence, given that  $\alpha \geq 0$ , we must have  $\beta \geq 0$  and  $\varepsilon \geq 0$ . For  $\xi$  tending to 0, we have:

$$P_{\xi}(\lambda) \sim \frac{\lambda(\gamma\delta + \beta) + \alpha\beta}{|\xi|^2}.$$

Making  $\lambda$  tend to infinity, we must have  $P_{\xi}(\lambda) \geq 0$  and so  $\gamma\delta + \beta \geq 0$ . The converse is trivial.

**Second case:**

$P_{\xi}$  has two complex roots  $z_{\pm} = a \pm ib$  and one real root  $\lambda$ , we have:

$$P_{\xi}(X) = (X - \lambda)(X^2 - 2aX + |z_{\pm}|^2).$$

A necessary condition to have the real parts negative is in the same way that  $P_{\xi}(X) \geq 0$  for all  $X \geq 0$ .

If  $\gamma\delta + \beta > 0$ , we are in the case where  $\xi$  tends to 0 (and we see that  $P_{\xi}$  is increasing). We can observe the terms of degree 2 and we get:  $\lambda + 2a = -\alpha - \nu$  then  $\lambda$  and  $\alpha$  are non positive if and only if  $P_{\xi}(-\alpha - \nu) \leq 0$  (for this it suffices to rewrite  $P_{\xi}$  like  $P_{\xi}(X) = (X - \lambda)(X^2 - 2aX + |z_{\pm}|^2)$ ). Calculate:

$$P_{\xi}(-\alpha - \nu) = -\nu\varepsilon - \nu^2\alpha - \frac{\nu\beta + \nu\gamma\delta + \alpha\gamma\delta}{|\xi|^2}.$$

With the hypothesis that we have made, we deduce that  $P_{\xi}(-\alpha - \nu) \leq 0$  for  $\xi$  tending to 0 if and only if  $\nu\beta + \nu\gamma\delta + \alpha\gamma\delta \geq 0$ .

**Behavior of the eigenvalues in low frequencies:**

Let us now study the asymptotic behavior of the eigenvalues when  $\xi$  tends to 0 and all the inequalities in (A) are strict.

We remark straight away that the condition  $\gamma\delta + \beta > 0$  ensures the strict monotonicity of the function:  $\lambda \rightarrow P_\xi(\lambda)$  for  $\xi$  small. Then there's only one real eigenvalue  $\lambda_1(\xi)$  and two complex eigenvalues  $\lambda_\pm(\xi) = a(\xi) \pm ib(\xi)$ .

Let  $\varepsilon^- < -\frac{\alpha\beta}{\gamma\delta+\beta} < \varepsilon^+ < 0$ . When  $\xi$  tends to 0, we have:

$$P_\xi(\lambda) \sim |\xi|^{-2}(\lambda(\gamma\delta + \beta) + \alpha\beta).$$

Then  $P_\xi(\varepsilon^-) < 0$  and  $P_\xi(\varepsilon^+) > 0$  and  $P_\xi$  has a unique real root included between  $\varepsilon^-$  and  $\varepsilon^+$ . These considerations give the asymptotic value of  $\lambda_1(\xi)$ .

Finally, we have:

$$\lambda_1(\xi) + 2a(\xi) = -\alpha - \nu \text{ and } -(a(\xi)^2 + b(\xi)^2)\lambda(\xi) = \alpha\xi + \frac{\alpha\beta}{|\xi|^2} \sim \frac{\alpha\beta}{|\xi|^2},$$

whence the result. □

We summarize this results in the following remark.

**Remark 2** *According to the analysis made in proposition 4.8, we expect the system (M) to be locally well-posed close to the equilibrium  $(\bar{\rho}, 0, \bar{T})$  if and only if we have:*

$$(C) \quad \mu(\bar{\rho}, \bar{\theta}) \geq 0, \lambda(\bar{\rho}, \bar{\theta}) + 2\mu(\bar{\rho}, \bar{\theta}) \geq 0, \kappa(\bar{\rho}) \geq 0, \text{ and } \chi(\bar{\rho}, \bar{T}) \geq 0.$$

By the calculus we have:

$$\beta = \partial_\rho p_0(\bar{\rho}, \bar{T}), \gamma = \frac{\partial_T p_0(\bar{\rho}, \bar{T})}{\bar{\rho} \partial_T e_0(\bar{\rho}, \bar{T})}, \delta = \frac{\bar{T} \partial_T p_0(\bar{\rho}, \bar{T})}{\bar{\rho}}.$$

We remark that  $\gamma\delta \geq 0$  if  $\partial_T e_0(\bar{\rho}, \bar{T}) \geq 0$ . In the case where  $\eta$  verifies  $\eta(\bar{\rho}, \bar{T}) > 0$ , the supplementary condition giving the global stability reduces to:

$$(D) \quad \partial_\rho p_0(\bar{\rho}, \bar{T}) \geq 0.$$

Now that we know the stability conditions on the coefficients of the system  $(M')$ , we aim at proving estimates in the space  $E^{\frac{N}{2}}$ .

We add a condition in this following proposition compared with the proposition 4.8 which is:  $\gamma\delta > 0$ , but it's not so important because in the system  $(NHV)$  we are interested in, we have effectively  $\gamma\delta = \frac{1}{T\Psi'(T)} > 0$ .

**Proposition 4.9 :** *Under the conditions of proposition 4.8 with strict inequalities and with the condition  $\gamma\delta > 0$ , let  $(q, d, \mathcal{T})$  be a solution of the system  $(M')$  on  $[0, T)$  with initial data  $(q_0, u_0, \mathcal{T}_0)$  such that:*

$$q_0 \in \tilde{B}^{s-1, s}, d_0 \in B^{s-1}, \mathcal{T}_0 \in \tilde{B}^{s-1, s-2} \text{ for some } s \in \mathbb{R}.$$

Moreover we suppose that for some  $1 \leq r_1 \leq +\infty$ , we have:

$$F \in \tilde{L}_T^{r_1}(\tilde{B}^{s-3+\frac{2}{r_1}, s-2+\frac{2}{r_1}}), \quad G \in \tilde{L}_T^{r_1}(B^{s-3+\frac{2}{r_1}}), \quad H \in \tilde{L}_T^{r_1}(\tilde{B}^{s-3+\frac{2}{r_1}, s-4+\frac{2}{r_1}}).$$

We then have the following estimate for all  $r_1 \leq r \leq +\infty$ :

$$\begin{aligned} & \|q\|_{\tilde{L}_T^r(\tilde{B}^{s-1+\frac{2}{r}, s+\frac{2}{r}})} + \|\mathcal{T}\|_{\tilde{L}_T^r(\tilde{B}^{s-1+\frac{2}{r}, s-2+\frac{2}{r}})} + \|u\|_{\tilde{L}_T^r(B^{s-1+\frac{2}{r}})} \lesssim \|q_0\|_{\tilde{B}^{s-1, s}} + \|u_0\|_{B^{s-1}} \\ & + \|\mathcal{T}_0\|_{\tilde{B}^{s-1, s-2}} + \|F\|_{L_T^{r_1}(\tilde{B}^{s-3+\frac{2}{r_1}, s-2+\frac{2}{r_1}})} + \|G\|_{L_T^{r_1}(B^{s-3+\frac{2}{r_1}})} + \|H\|_{L_T^{r_1}(\tilde{B}^{s-3+\frac{2}{r_1}, s-4+\frac{2}{r_1})}. \end{aligned}$$

**Proof:**

We are going to separate the case of the low, medium and high frequencies, particularly the low and high frequencies which have a different behavior, and depend on the indice of Besov space.

### 1) Case of low frequencies:

Let us focus on just the first three equation because the last one is a heat equation that we can treat independently. Applying operator  $\Delta_l$  to the system  $(M_1')$ , we obtain then in setting:

$$q_l = \Delta_l q, \quad d_l = \Delta_l d, \quad \mathcal{T}_l = \Delta_l \mathcal{T}$$

the following system:

$$\partial_t q_l + \Lambda d_l = F_l, \tag{4.1}$$

$$\partial_t d_l - \nu \Delta d_l - \varepsilon \Lambda^3 q_l - \beta \Lambda q_l - \gamma \Lambda \mathcal{T}_l = \Lambda^{-1} \operatorname{div} G_l, \tag{4.2}$$

$$\partial_t \mathcal{T}_l - \alpha \Delta \mathcal{T}_l + \delta \Lambda d_l = H_l. \tag{4.3}$$

Throughout the proof, we assume that  $\delta \neq 0$ : if not we have just a heat equation on (4.3) and we can use the proposition 3.5 to have the estimate on  $\mathcal{T}$  and we have just to deal with the first two equations. Denoting by  $W(t)$  the semi-group associated to (4.1 – 4.3) we have:

$$\begin{pmatrix} q(t) \\ u(t) \\ \theta(t) \end{pmatrix} = W(t) \begin{pmatrix} q_0 \\ u_0 \\ \theta_0 \end{pmatrix} + \int_0^t W(t-s) \begin{pmatrix} F(s) \\ G(s) \\ H(s) \end{pmatrix} ds.$$

We set:

$$f_l^2 = \beta \|q_l\|_{L^2}^2 + \|d_l\|_{L^2}^2 + \frac{\gamma}{\delta} \|\mathcal{T}_l\|_{L^2}^2 - 2K \langle \Lambda q_l, d_l \rangle$$

for some  $K \geq 0$  to be fixed hereafter and  $\langle \cdot, \cdot \rangle$  noting the  $L^2$  inner product.

To begin with, we consider the case where  $F = G = H = 0$ .

Then we take the inner product of (4.2) with  $d_l$ , of (4.1) with  $\beta q_l$  and of (4.3) with  $\gamma \mathcal{T}_l$ .

We get:

$$\frac{1}{2} \frac{d}{dt} (\|d_l\|_{L^2}^2 + \beta \|q_l\|_{L^2}^2 + \frac{\gamma}{\delta} \|\mathcal{T}_l\|_{L^2}^2) + \nu \|\nabla d_l\|_{L^2}^2 - \varepsilon \langle \Lambda^3 q_l, d_l \rangle + \frac{\gamma \alpha}{\delta} \|\nabla \mathcal{T}_l\|_{L^2}^2 = 0. \tag{4.4}$$

Next, we apply the operator  $\Lambda$  to (4.2) and take the inner product with  $q_l$ , and we take the scalar product of (4.1) with  $\Lambda d_l$  to control the term  $\frac{d}{dt}\langle \Lambda q_l, d_l \rangle$ . Summing the two resulting equalities, we get:

$$\frac{d}{dt}\langle \Lambda q_l, d_l \rangle + \|\Lambda d_l\|_{L^2}^2 - \nu\langle \Delta d_l, \Lambda q_l \rangle - \varepsilon\|\Lambda^2 q_l\|_{L^2}^2 - \beta\|\Lambda q_l\|_{L^2}^2 - \gamma\langle \Lambda \mathcal{T}_l, \Lambda q_l \rangle = 0. \quad (4.5)$$

We obtain then in summing (4.4) and (4.5):

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}f_l^2 + (\nu\|\nabla d_l\|_{L^2}^2 - K\|\Lambda d_l\|_{L^2}^2) + (K\beta\|\Lambda q_l\|_{L^2}^2 + K\varepsilon\|\Lambda^2 q_l\|_{L^2}^2) + \frac{\gamma\alpha}{\delta}\|\nabla \mathcal{T}_l\|_{L^2}^2 \\ + K\nu\langle \Delta d_l, \Lambda q_l \rangle + K\gamma\langle \Lambda \mathcal{T}_l, \Lambda q_l \rangle - \varepsilon\langle \Lambda^3 q_l, d_l \rangle = 0. \end{aligned} \quad (4.6)$$

Like indicated, we are going to focus on low frequencies so assume that  $l \leq l_0$  for some  $l_0$  to be fixed hereafter. We have then  $\forall c, b, d > 0$  :

$$\begin{aligned} |\langle \Delta d_l, \Lambda q_l \rangle| &\leq \frac{b}{2}\|\Lambda q_l\|_{L^2}^2 + \frac{1}{2b}\|\Delta d_l\|_{L^2}^2 \\ &\leq \frac{b}{2}\|\Lambda q_l\|_{L^2}^2 + \frac{C2^{2l_0}}{2b}\|\Lambda d_l\|_{L^2}^2, \\ |\langle \Lambda^3 q_l, d_l \rangle| = |\langle \Lambda^2 q_l, \Lambda d_l \rangle| &\leq \frac{C2^{2l_0}}{2c}\|\Lambda q_l\|_{L^2}^2 + \frac{c}{2}\|\Lambda d_l\|_{L^2}^2. \end{aligned} \quad (4.7)$$

Moreover we have:  $\|\nabla d_l\|_{L^2}^2 = \|\Lambda d_l\|_{L^2}^2$ . Finally we obtain:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}f_l^2 + \left[\nu - \left(K + \frac{C2^{2l_0}}{2b}K\nu + \frac{c\varepsilon}{2}\right)\right]\|\Lambda d_l\|_{L^2}^2 + \left[\frac{\gamma\alpha}{\delta} - \frac{K\gamma}{2d}\right]\|\Lambda \mathcal{T}_l\|_{L^2}^2 \\ + K\left[\beta + \varepsilon C2^{2l_0} - \nu\frac{b}{2} - \varepsilon\frac{C2^{2l_0}}{2c} - \gamma\frac{d}{2}\right]\|\Lambda q_l\|_{L^2}^2 \leq 0. \end{aligned}$$

Then we choose  $(b, c, d)$  such that:

$$b = \frac{\beta}{2\nu}, \quad c = \frac{\nu}{\varepsilon}, \quad d = \frac{\beta}{2\gamma},$$

which is possible if  $\gamma > 0$  as  $\nu > 0$ ,  $\varepsilon > 0$ . In the case where  $\gamma \leq 0$ , we recall that  $\gamma$  and  $\delta$  have the same sign, we have then no problem because with our choice the first and third following inequalities will be satisfied and if  $\gamma \leq 0$  in the second equation the term  $\gamma\frac{d}{2}$  is positive in taking  $d > 0$ . So we assume from now on that  $\gamma > 0$  and so with this choice, we want that:

$$\begin{aligned} \frac{\nu}{2} - K\left(1 + C2^{2l_0}\frac{\nu^2}{\beta}\right) &> 0, \\ \frac{\beta}{2} + \varepsilon C2^{2l_0} - C2^{2l_0}\frac{\varepsilon^2}{2\nu} &> 0, \\ \frac{\gamma\alpha}{\delta} - K\frac{\gamma}{2} &> 0. \end{aligned}$$

We recall that in your case  $\nu > 0$ ,  $\beta > 0$ ,  $\alpha > 0$  and  $\gamma > 0$ ,  $\delta > 0$ . So it suffices to choose  $K$  and  $l_0$  such that:

$$K < \min\left(\frac{\nu}{2(1 + C2^{2l_0}\frac{\nu^2}{2\beta})}, \frac{2\alpha}{\delta}\right) \quad \text{and} \quad 2^{2l_0} < \min\left(\frac{\beta\nu}{6C\varepsilon^2}, \frac{1}{6\varepsilon C}\right).$$

Finally we conclude in using Proposition 3.1 part (ii) with a  $c'$  small enough. We get:

$$\frac{1}{2} \frac{d}{dt} f_l^2 + c' 2^{2l} f_l^2 \leq 0 \quad \text{for } l \leq l_0. \quad (4.8)$$

## 2) Case of high frequencies:

We are going to work with  $l \geq l_1$  where we will determine  $l_1$  hereafter. We set then:

$$f_l^2 = \varepsilon B \|\Lambda q_l\|_{L^2}^2 + B \|d_l\|_{L^2}^2 + \|\Lambda^{-1} \mathcal{T}_l\|_{L^2}^2 - 2K \langle \Lambda q_l, d_l \rangle,$$

and we choose  $B$  and  $K$  later on.

Then we take the inner product of (4.2) with  $d_l$ :

$$\frac{1}{2} \frac{d}{dt} \|d_l\|_{L^2}^2 + \nu \|\nabla d_l\|_{L^2}^2 - \varepsilon \langle \Lambda^3 q_l, d_l \rangle - \beta \langle \Lambda q_l, d_l \rangle - \gamma \langle \Lambda \mathcal{T}_l, d_l \rangle = 0. \quad (4.9)$$

Moreover we have in taking the scalar product of (4.1) with  $\Lambda^2 q_l$ :

$$\frac{1}{2} \frac{d}{dt} \|\Lambda q_l\|_{L^2}^2 + \langle \Lambda^2 d_l, \Lambda q_l \rangle = 0. \quad (4.10)$$

And in the same way with (4.3), we have:

$$\frac{1}{2} \frac{d}{dt} \|\Lambda^{-1} \mathcal{T}_l\|_{L^2}^2 + \alpha \|\mathcal{T}_l\|_{L^2}^2 + \delta \langle d_l, \Lambda^{-1} \mathcal{T}_l \rangle = 0. \quad (4.11)$$

After we sum (4.9), (4.10) and (4.11) to get:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (B \|d_l\|_{L^2}^2 + \varepsilon B \|\Lambda q_l\|_{L^2}^2 + \|\Lambda^{-1} \mathcal{T}_l\|_{L^2}^2) + B \nu \|\nabla d_l\|_{L^2}^2 + \alpha \|\mathcal{T}_l\|_{L^2}^2 \\ - B \beta \langle \Lambda q_l, d_l \rangle - B \gamma \langle \Lambda \mathcal{T}_l, d_l \rangle + \delta \langle d_l, \Lambda^{-1} \mathcal{T}_l \rangle = 0. \end{aligned} \quad (4.12)$$

Then like previously we can play with  $\langle \Lambda q_l, d_l \rangle$  to obtain a term in  $\|\Lambda q_l\|_{L^2}^2$ . We have then again the following equation:

$$\frac{d}{dt} \langle \Lambda q_l, d_l \rangle + \|\Lambda d_l\|_{L^2}^2 - \nu \langle \Delta d_l, \Lambda q_l \rangle - \varepsilon \|\Lambda^2 q_l\|_{L^2}^2 - \beta \|\Lambda q_l\|_{L^2}^2 - \gamma \langle \Lambda \mathcal{T}_l, \Lambda q_l \rangle = 0. \quad (4.13)$$

We sum all these expressions and get:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} f_l^2 + [B \nu \|\nabla d_l\|_{L^2}^2 - K \|\Lambda d_l\|_{L^2}^2] + \alpha \|\mathcal{T}_l\|_{L^2}^2 + K [\beta \|\Lambda q_l\|_{L^2}^2 + \varepsilon \|\Lambda^2 q_l\|_{L^2}^2] \\ - B \beta \langle \Lambda q_l, d_l \rangle - B \gamma \langle \Lambda \mathcal{T}_l, d_l \rangle + \delta \langle d_l, \Lambda^{-1} \mathcal{T}_l \rangle + K \nu \langle \Delta d_l, \Lambda q_l \rangle + \gamma K \langle \Lambda \mathcal{T}_l, \Lambda q_l \rangle = 0. \end{aligned} \quad (4.14)$$

The main term in high frequencies will be:  $\|\Lambda^2 q_l\|_{L^2}^2$ . The other terms may be treated by mean of Young's inequality:

$$\begin{aligned} |\langle \Lambda q_l, d_l \rangle| &\leq \frac{1}{2a} \|\Lambda q_l\|_{L^2}^2 + \frac{a}{2} \|\Lambda d_l\|_{L^2}^2, \\ &\leq \frac{1}{2a c 2^{2l_1}} \|\Lambda^2 q_l\|_{L^2}^2 + \frac{a}{2} \|\Lambda d_l\|_{L^2}^2. \end{aligned}$$

We do as before with the others terms in the second line of (4.14) and we obtain:

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} f_l^2 + (B\nu - K) \|\Lambda d_l\|_{L^2}^2 + \alpha \|\mathcal{T}_l\|_{L^2}^2 + K \left( \frac{\beta}{c 2^{2l_1}} + \varepsilon \right) \|\Lambda^2 q_l\|_{L^2}^2 \leq \\ &B\gamma \left[ \frac{1}{2a} \|\mathcal{T}_l\|_{L^2}^2 + \frac{a}{2} \|\Lambda d_l\|_{L^2}^2 \right] + K \left[ \frac{\nu b}{2} \|\Lambda^2 q_l\|_{L^2}^2 + \frac{2\nu}{b} \|\Lambda d_l\|_{L^2}^2 + \frac{\gamma}{2c'} \|\mathcal{T}_l\|_{L^2}^2 + \frac{\gamma c'}{2} \|\Lambda^2 q_l\|_{L^2}^2 \right] \\ &+ B\beta \left[ \frac{1}{2d} \frac{1}{c 2^{2l_1}} \|\Lambda^2 q_l\|_{L^2}^2 + \frac{d}{2} \frac{1}{c 2^{2l_1}} \|\Lambda d_l\|_{L^2}^2 \right] + \delta \left[ \frac{1}{2e} \frac{1}{c 2^{2l_1}} \|\mathcal{T}_l\|_{L^2}^2 + \frac{e}{2} \frac{1}{c 2^{2l_1}} \|\Lambda d_l\|_{L^2}^2 \right]. \end{aligned}$$

We obtain then for some  $a, b, c', d, e$  to be chosen:

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} f_l^2 + \left[ B\nu - \left( K + B\gamma \frac{a}{2} + K\nu \frac{2}{b} + B\beta \frac{d}{2c 2^{2l_1}} + \delta \frac{e}{2} \frac{1}{c 2^{2l_1}} \right) \right] \|\Lambda d_l\|_{L^2}^2 \\ &+ \left[ \alpha - \left( B\gamma \frac{1}{2a} + \gamma K \frac{1}{2c'} + \delta \frac{1}{2e} \frac{1}{c 2^{2l_1}} \right) \right] \|\mathcal{T}_l\|_{L^2}^2 \\ &+ \left[ \frac{\beta K}{c 2^{2l_1}} + \varepsilon K - K\nu \frac{b}{2} - \gamma K \frac{c'}{2} - B\beta \frac{1}{2d} \frac{1}{c 2^{2l_1}} \right] \|\Lambda^2 q_l\|_{L^2}^2 \leq 0. \end{aligned} \quad (4.15)$$

We claim that  $a, b, c', d, e, l_1, K$  may be chosen so that:

$$B\nu - \left( K + B\gamma \frac{a}{2} + K\nu \frac{2}{b} + B\beta \frac{d}{2} + \delta \frac{e}{2} \frac{1}{2^{2l_1}} \right) > 0, \quad (4.16)$$

$$\alpha - \left( B\gamma \frac{1}{2a} + \gamma K \frac{1}{2c'} + \delta \frac{1}{2e} \frac{1}{c 2^{2l_1}} \right) > 0, \quad (4.17)$$

$$\frac{\beta K}{c 2^{2l_1}} + \varepsilon K - K\nu \frac{b}{2} - \gamma K \frac{c'}{2} - B\beta \frac{1}{2d} \frac{1}{c 2^{2l_1}} > 0. \quad (4.18)$$

We want at once that for (4.16) and (4.18):

$$\nu - \gamma \frac{a}{2} - \beta \frac{d}{2} > 0, \quad (4.19)$$

$$\varepsilon - \nu \frac{b}{2} - \gamma \frac{c'}{2} > 0. \quad (4.20)$$

So we take:

$$\begin{aligned} e = 1, \quad a = 2h\delta, \quad d = 2h, \quad h = \frac{\nu}{2(\gamma\delta + \beta)}, \quad b = 2\beta h', \\ c' = 2\delta(\nu + \alpha)h' \quad \text{and} \quad h' = \frac{\varepsilon}{2(\gamma\delta(\nu + \alpha) + \nu\beta)}. \end{aligned}$$

With this choice, we get (4.19) and (4.20). In what follows it suffices to choice  $B, K$  small enough and  $l_1$  large enough. We have then:

$$f_l \simeq \text{Max}(1, 2^l) \|q_l\|_{L^2} + \|d_l\|_{L^2} + \text{Min}(1, 2^l) \|\mathcal{T}_l\|_{L^2}$$

We have so obtain for  $l \leq l_0, l \geq l_1$  and for a  $c'$  small enough:

$$\frac{1}{2} \frac{d}{dt} f_l^2 + c' 2^{2l} f_l^2 \leq 0.$$

### 3) Case of Medium frequencies:

For  $l_0 \leq l \leq l_1$ , there is only a finite number of terms to treat. So it suffices to find a  $C$  such that for all these terms:

$$(B) \quad \|q_l\|_{L_T^r(L^2)} \leq C, \|d_l\|_{L_T^r(L^2)} \leq C, \|\mathcal{T}_l\|_{L_T^r(L^2)} \leq C \quad \text{for all } T \in [0, +\infty] \\ \text{and } r \in [1, +\infty]$$

with  $C$  large enough independent of  $T$ .

And this is true because the system is globally stable: indeed according to proposition 4.8, we have:

$$\left\| W(t) \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\|_{L^2} \lesssim e^{-c_1(\xi)t} \left\| \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\|_{L^2} \quad \forall a, b, c \in L^2$$

with  $c_1(\xi) = \min_{2^{l_0} \leq |\xi| \leq 2^{l_1}} (\operatorname{Re}(\lambda_1(\xi)), \operatorname{Re}(\lambda_2(\xi)), \operatorname{Re}(\lambda_3(\xi)))$  where the  $\lambda_i(\xi)$  correspond to the eigenvalues of the system. We have then in using the estimate in low and high frequencies in part 4.1 and the continuity of  $c_1(\xi)$  the fact that there exists  $c_1$  such that:

$$c_1(\xi) \geq c_1 > 0.$$

So that we have:

$$\left( \int_0^T \left( \begin{array}{l} \|q_l(t)\|_{L^2}^r \\ \|u_l(t)\|_{L^2}^r \\ \|\mathcal{T}_l(t)\|_{L^2}^r \end{array} \right) dt \right)^{\frac{1}{r}} \lesssim \left( \int_0^T e^{-c_1 r s} ds \right)^{\frac{1}{r}} \left( \begin{array}{l} \|(q_0)_l\|_{L^2} \\ \|(u_0)_l\|_{L^2} \\ \|(\mathcal{T}_0)_l\|_{L^2} \end{array} \right) \quad \text{for } l_0 \leq l \leq l_1.$$

And so we have the result (B).

### 4) Conclusion:

In using Duhamel formula for  $W$  and in taking  $C$  large enough we have for all  $l$ :

$$\max(1, 2^l) \|q_l(t)\|_{L^2} + \|d_l(t)\|_{L^2} + \min(1, 2^{-l}) \|\mathcal{T}_l(t)\|_{L^2} \leq C e^{-c 2^{2l} t} (\max(1, 2^l) \|(q_0)_l\|_{L^2} \\ + \|(d_0)_l\|_{L^2} + \min(1, 2^{-l}) \|(\mathcal{T}_0)_l\|_{L^2}) + C \int_0^t e^{-c 2^{2l}(t-s)} (\max(1, 2^{2l}) \|F_l\|_{L^2} + \|G_l\|_{L^2} \\ + \min(1, 2^{-l}) \|H_l\|_{L^2}) ds.$$

Now we take the  $L^r$  norm in time and we sum in multiplying by  $2^{l(s-1+\frac{2}{r})}$  for the low frequencies and we sum in multiplying by  $2^{l(s+\frac{2}{r})}$  for the high frequencies.



This yields:

$$\begin{aligned}
& \|q\|_{\tilde{L}_T^r(\tilde{B}^{s-1+\frac{2}{r}, s+\frac{2}{r}})} + \|\mathcal{T}\|_{\tilde{L}_T^r(\tilde{B}^{s-1+\frac{2}{r}, s-2+\frac{2}{r}})} + \|d\|_{\tilde{L}_T^r(B^{s-1+\frac{2}{r}})} \leq \|q_0\|_{\tilde{B}^{s-1, s}} + \|\mathcal{T}_0\|_{\tilde{B}^{s-1, s-2}} \\
& + \|d_0\|_{B^{s-1}} + \sum_{l \leq 0} 2^{l(s-1+\frac{2}{r})} \int_0^T \left( \int_0^T e^{c(t-\tau)} (\|F_l(\tau)\|_{L^2} + \|G_l(\tau)\|_{L^2} + \|H_l(\tau)\|_{L^2}) d\tau \right)^r dt \Big)^{\frac{1}{r}} \\
& + \sum_{l \geq 0} 2^{l(s+\frac{2}{r})} \left( \int_0^T \left( \int_0^T e^{c(t-\tau)} (\|\nabla F_l(\tau)\|_{L^2} + \|G_l(\tau)\|_{L^2} + \|\Lambda^{-1}H_l(\tau)\|_{L^2}) d\tau \right)^r dt \right)^{\frac{1}{r}}.
\end{aligned}$$

Bounding the right hand-side may be done by taking advantage of convolution inequalities. To complete the proof of proposition 4.9, it suffices to use that  $u = -\Lambda^{-1}\nabla d - \Lambda^{-1}\text{div}\Omega$  and to apply proposition 3.5.  $\square$

## 4.2 Global existence for temperature independent coefficients

This section is devoted to the proof of theorem 2.1 and 2.3. Let us first recall the spaces in which we work with for the theorem 2.1:

$$\begin{aligned}
E^s &= [C_b(\mathbb{R}_+, \tilde{B}^{s-1, s}) \cap L^1(\mathbb{R}_+, \tilde{B}^{s+1, s+2})] \times [C_b(\mathbb{R}_+, B^{s-1})^N \cap L^1(\mathbb{R}_+, B^{s+1})^N] \\
&\quad \times [C_b(\mathbb{R}_+, \tilde{B}^{s-1, s-2}) \cap L^1(\mathbb{R}_+, \tilde{B}^{s+1, s})].
\end{aligned}$$

In what follows, we assume that  $N \geq 3$ .

### Proof of theorem 2.1:

We shall use a contracting mapping argument for the function  $\psi$  defined as follows:

$$\psi(q, u, \mathcal{T}) = W(t, \cdot) * \begin{pmatrix} q_0 \\ u_0 \\ \mathcal{T}_0 \end{pmatrix} + \int_0^t W(t-s) \begin{pmatrix} F(q, u, \mathcal{T}) \\ G(q, u, \mathcal{T}) \\ H(q, u, \mathcal{T}) \end{pmatrix} ds. \quad (4.21)$$

In what follows we set:

$$\rho = \bar{\rho}(1+q), \quad \theta = \bar{\theta} + \mathcal{T}, \quad \tilde{T} = \Psi^{-1}(\theta).$$

The non linear terms  $F, G, H$  are defined as follows:

$$\begin{aligned}
F &= -\text{div}(qu), \\
G &= -u \cdot \nabla u + \nabla \left( \frac{K'}{2} |\nabla \rho|^2 \right) + \left[ \frac{\mu(\rho)}{\rho} - \frac{\mu(\bar{\rho})}{\bar{\rho}} \right] \Delta u + \left[ \frac{\zeta(\rho)}{\rho} - \frac{\zeta(\bar{\rho})}{\bar{\rho}} \right] \nabla \text{div} u \\
&\quad + (\nabla((K(\rho) - K(\bar{\rho}))\Delta\rho) + \left[ \frac{P'_0(\rho) + \tilde{T}P'_1(\rho)}{\rho} - \frac{P'_0(\bar{\rho}) + \tilde{T}P'_1(\bar{\rho})}{\bar{\rho}} \right] \nabla \rho \\
&\quad + \left[ \frac{P_1(\rho)}{\rho\Psi'(\mathcal{T})} - \frac{P_1(\bar{\rho})}{\bar{\rho}\Psi'(\tilde{T})} \right] \nabla \theta + \frac{\lambda'(\rho)\nabla\rho\text{div}u}{\rho} + \frac{(du + \nabla u)\mu'(\rho)\nabla\rho}{\rho},
\end{aligned} \quad (4.22)$$

where we note:  $\zeta = \lambda + \mu$ , and:

$$H = \left( \frac{\operatorname{div}(\chi(\rho)\nabla\theta)}{\rho} - \frac{\bar{\chi}}{\bar{\rho}}\Delta\theta \right) + \left[ \frac{\bar{T}P_1(\bar{\rho})}{\bar{\rho}} - \frac{\tilde{T}P_1(\rho)}{\rho} \right] \operatorname{div}u - u^* \cdot \nabla\theta + \frac{D : \nabla u}{\rho}. \quad (4.23)$$

### 1) First step, uniform bounds:

Let:

$$\eta = \|q_0\|_{\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}}} + \|u_0\|_{B^{\frac{N}{2}-1}} + \|\mathcal{T}_0\|_{\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}}}.$$

We are going to show that  $\psi$  maps the ball  $B(0, R)$  into itself if  $R$  is small enough. According to proposition 4.9, we have:

$$\|W(t, \cdot) * \begin{pmatrix} q_0 \\ u_0 \\ \mathcal{T}_0 \end{pmatrix}\|_{E^{\frac{N}{2}}} \leq C\eta. \quad (4.24)$$

We have then according (4.21), proposition 4.9 and 4.24:

$$\begin{aligned} \|\psi(q, u, \mathcal{T})\|_{E^{\frac{N}{2}}} &\leq C\eta + \|F(q, u, \mathcal{T})\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} + \|G(q, u, \mathcal{T})\|_{L^1(B^{\frac{N}{2}-1})} \\ &\quad + \|H(q, u, \mathcal{T})\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}-2})}. \end{aligned} \quad (4.25)$$

Moreover we suppose for the moment that:

$$(\mathcal{H}) \quad \|q\|_{L^\infty(\mathbb{R} \times \mathbb{R}^N)} \leq 1/2.$$

We will use the different theorems on the paradifferential calculus to obtain estimates on

$$\|F(q, u, \mathcal{T})\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})}, \|G(q, u, \mathcal{T})\|_{L^1(B^{\frac{N}{2}-1})} \text{ and } \|H(q, u, \mathcal{T})\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}-2})}.$$

1) Let us first estimate  $\|F(q, u, \mathcal{T})\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})}$ . According to proposition 3.6, we have:

$$\|\operatorname{div}(qu)\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} \leq \|qu\|_{L^1(B^{\frac{N}{2}})} + \|qu\|_{L^1(B^{\frac{N}{2}+1})}$$

and:

$$\|qu\|_{L^1(B^{\frac{N}{2}})} \leq \|q\|_{L^2(B^{\frac{N}{2}})} \|u\|_{L^2(B^{\frac{N}{2}})}$$

$$\|qu\|_{L^1(B^{\frac{N}{2}+1})} \leq \|q\|_{L^\infty(B^{\frac{N}{2}})} \|u\|_{L^1(B^{\frac{N}{2}+1})} + \|q\|_{L^2(B^{\frac{N}{2}+1})} \|u\|_{L^2(B^{\frac{N}{2}})}.$$

Because  $\tilde{B}^{\frac{N}{2}, \frac{N}{2}+1} \hookrightarrow B^{\frac{N}{2}}$  and  $\tilde{B}^{\frac{N}{2}, \frac{N}{2}+1} \hookrightarrow B^{\frac{N}{2}+1}$  (from proposition 3.2), we get:

$$\|\operatorname{div}(qu)\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} \leq \|q\|_{L^\infty(\tilde{B}^{\frac{N}{2}, \frac{N}{2}+1})} \|u\|_{L^1(B^{\frac{N}{2}+1})} + \|q\|_{L^2(\tilde{B}^{\frac{N}{2}, \frac{N}{2}+1})} \|u\|_{L^2(B^{\frac{N}{2}})}.$$

2) We have to estimate  $\|G(q, u, \mathcal{T})\|_{L^1(B^{\frac{N}{2}-1})}$ . We see straight away that:

$$\left[ \frac{\mu(\rho)}{\rho} - \frac{\mu(\bar{\rho})}{\bar{\rho}} \right] \Delta u = K(q) \Delta u$$

for some smooth function  $K$  such that  $K(0) = 0$ . Hence by propositions 3.7, 3.6 and 3.2 yield:

$$\begin{aligned} \left\| \left[ \frac{\mu(\rho)}{\rho} - \frac{\mu(\bar{\rho})}{\bar{\rho}} \right] \Delta u \right\|_{L^1(B^{\frac{N}{2}-1})} &\lesssim \|K(q)\|_{L^\infty(B^{\frac{N}{2}})} \|u\|_{L^1(B^{\frac{N}{2}+1})}, \\ &\lesssim \|q\|_{L^\infty(B^{\frac{N}{2}})} \|u\|_{L^1(B^{\frac{N}{2}+1})}, \\ &\lesssim \|q\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} \|u\|_{L^1(B^{\frac{N}{2}+1})}. \end{aligned}$$

In the same way we have:

$$\begin{aligned} \left\| \left[ \frac{\zeta(\rho)}{\rho} - \frac{\zeta(\bar{\rho})}{\bar{\rho}} \right] \nabla \operatorname{div} u \right\|_{L^1(B^{\frac{N}{2}-1})} &\lesssim \|q\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} \|u\|_{L^1(B^{\frac{N}{2}+1})}, \\ \left\| \nabla (K(\rho) - K(\bar{\rho})) \Delta q \right\|_{L^1(B^{\frac{N}{2}-1})} &\lesssim \|q\|_{L^\infty(B^{\frac{N}{2}})} \|q\|_{L^1(B^{\frac{N}{2}+2})}, \\ \left\| \left[ \frac{P'_0(\rho)}{\rho} - \frac{P'_0(\bar{\rho})}{\bar{\rho}} \right] \nabla \rho \right\|_{L^1(B^{\frac{N}{2}-1})} &\lesssim \|q\|_{L^\infty(B^{\frac{N}{2}-1})} \|q\|_{L^1(B^{\frac{N}{2}+1})}, \\ &\lesssim \|q\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} \|q\|_{L^1(\tilde{B}^{\frac{N}{2}+1, \frac{N}{2}+2})}. \end{aligned}$$

After it remains two terms to treat:

$$\begin{aligned} \left\| \left[ \frac{\tilde{T}P'_1(\rho)}{\rho} - \frac{\bar{T}P'_1(\bar{\rho})}{\bar{\rho}} \right] \nabla \rho \right\|_{L^1(B^{\frac{N}{2}-1})} &\lesssim \left\| \left[ \left( \frac{P'_1(\rho)}{\rho} - \frac{P'_1(\bar{\rho})}{\bar{\rho}} \right) \bar{\theta} \right] \nabla q \right\|_{L^1(B^{\frac{N}{2}-1})} \\ &\quad + \left\| \frac{P'_1(\bar{\rho})}{\bar{\rho}} \mathcal{T} \nabla q \right\|_{L^1(B^{\frac{N}{2}-1})} + \left\| \mathcal{T} \left( \frac{P'_1(\rho)}{\rho} - \frac{P'_1(\bar{\rho})}{\bar{\rho}} \right) \nabla q \right\|_{L^1(B^{\frac{N}{2}-1})}, \\ \left\| \left[ \frac{\tilde{T}P'_1(\rho)}{\rho} - \frac{\bar{T}P'_1(\bar{\rho})}{\bar{\rho}} \right] \nabla \rho \right\|_{L^1(B^{\frac{N}{2}-1})} &\lesssim \|q\|_{L^\infty(B^{\frac{N}{2}-1})} \|q\|_{L^1(B^{\frac{N}{2}+1})} + \|\mathcal{T} \nabla q\|_{L^1(B^{\frac{N}{2}-1})} \\ &\quad + \|K_1(q) \mathcal{T} \nabla q\|_{L^1(B^{\frac{N}{2}-1})}, \end{aligned}$$

According to proposition 3.7, we have:

$$\begin{aligned} \|\mathcal{T} \nabla q\|_{L^1(B^{\frac{N}{2}-1})} &\leq \|\mathcal{T}\|_{L^2(\tilde{B}^{\frac{N}{2}, \frac{N}{2}-1})} \|q\|_{L^2(\tilde{B}^{\frac{N}{2}, \frac{N}{2}+1})}, \\ \|K_1(q) \mathcal{T} \nabla q\|_{L^1(B^{\frac{N}{2}-1})} &\leq C \|q\|_{L^\infty(B^{\frac{N}{2}})} \|\mathcal{T} \nabla q\|_{L^1(B^{\frac{N}{2}-1})}. \end{aligned}$$

Therefore:

$$\begin{aligned} \left\| \left[ \frac{\tilde{T}P'_1(\rho)}{\rho} - \frac{\bar{T}P'_1(\bar{\rho})}{\bar{\rho}} \right] \nabla \rho \right\|_{L^1(B^{\frac{N}{2}-1})} &\lesssim \|q\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} \|q\|_{L^1(B^{\frac{N}{2}+1})} \\ &\quad + (1 + \|q\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})}) \|\mathcal{T}\|_{L^2(\tilde{B}^{\frac{N}{2}, \frac{N}{2}-1})} \|q\|_{L^2(\tilde{B}^{\frac{N}{2}, \frac{N}{2}+1})}. \end{aligned}$$

In the same spirit:

$$\left\| \left( \frac{P_1(\rho)}{\rho} - \frac{P_1(\bar{\rho})}{\bar{\rho}} \right) \nabla \theta \right\|_{L^1(B^{\frac{N}{2}-1})} \lesssim \|q\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} \|\mathcal{T}\|_{L^1(\tilde{B}^{\frac{N}{2}+1, \frac{N}{2}})},$$

$$\begin{aligned} \left\| \left( \frac{P_1(\rho)}{A\rho} - \frac{P_1(\bar{\rho})}{A\bar{\rho}} \right) \nabla \theta \right\|_{L^1(B^{\frac{N}{2}-1})} &\leq \|q\|_{L^2(\tilde{B}^{\frac{N}{2}, \frac{N}{2}+1})} \|\mathcal{T}_{BF}\|_{L^2(B^{\frac{N}{2}})} \\ &\quad + \|q\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} \|\mathcal{T}_{HF}\|_{L^1(B^{\frac{N}{2}})}, \end{aligned}$$

where we have:

$$\mathcal{T}_{BF} = \sum_{l \leq 0} \Delta_l \mathcal{T} \quad \text{and} \quad \mathcal{T}_{HF} = \sum_{l > 0} \Delta_l \mathcal{T}.$$

Next we have the following term:

$$\|u^* \cdot \nabla u\|_{L^1(B^{\frac{N}{2}-1})} \lesssim \|u\|_{L^\infty(B^{\frac{N}{2}-1})} \|u\|_{L^1(B^{\frac{N}{2}+1})}.$$

And finally we have the terms coming from  $\operatorname{div}(D)$  which are of the form:

$$\left\| \frac{\lambda'(\rho) \nabla \rho \operatorname{div} u}{\rho} \right\|_{L^1(B^{\frac{N}{2}-1})} \leq \|L(q) \nabla \rho \operatorname{div} u\|_{L^1(B^{\frac{N}{2}-1})} + \left\| \frac{\lambda'(\bar{\rho})}{\bar{\rho}} \nabla \rho \operatorname{div} u \right\|_{L^1(B^{\frac{N}{2}-1})}$$

where we have set:

$$L(x_1) = \frac{\lambda'(\bar{\rho}(1+x_1))}{\bar{\rho}(1+x_1)} - \frac{\lambda'(\bar{\rho})}{\bar{\rho}}.$$

Afterwards we can apply proposition 3.7 to get:

$$\begin{aligned} \|\nabla \rho \operatorname{div} u\|_{L^1(B^{\frac{N}{2}-1})} &\lesssim \|u\|_{L^1(B^{\frac{N}{2}+1})} \|q\|_{L^\infty(B^{\frac{N}{2}})}, \\ \|L(q) \nabla \rho \operatorname{div} u\|_{L^1(B^{\frac{N}{2}-1})} &\leq \|L(q)\|_{L^\infty(B^{\frac{N}{2}})} \|\nabla \rho \operatorname{div} u\|_{L^1(B^{\frac{N}{2}-1})}. \end{aligned}$$

As we assumed that  $(\mathcal{H})$  is satisfied, we have in using proposition 3.7:

$$\|L(q)\|_{L^\infty(B^{\frac{N}{2}})} \leq C \|q\|_{L^\infty(B^{\frac{N}{2}})}.$$

So we have:

$$\left\| \frac{\lambda'(\rho) \nabla \rho \operatorname{div} u}{\rho} \right\|_{L^1(B^{\frac{N}{2}-1})} \lesssim \|u\|_{L^1(B^{\frac{N}{2}+1})} \|q\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} (1 + \|q\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})}).$$

In the same way we have in using 3.6, 3.7 and 3.2:

$$\begin{aligned} \left\| \frac{(du + \nabla u) \nabla \rho \mu'(\rho)}{\rho} \right\|_{L^1(B^{\frac{N}{2}-1})} &\leq C \|u\|_{L^1(B^{\frac{N}{2}+1})} \|q\|_{L^\infty(B^{\frac{N}{2}})} (1 + \|q\|_{L^\infty(B^{\frac{N}{2}})}). \\ \|\nabla \left( \frac{K'_\rho}{2} |\nabla \rho|^2 \right)\|_{L^1(B^{\frac{N}{2}-1})} &\lesssim \left\| \left( \frac{K'_\rho}{2} - \frac{K'_{\bar{\rho}}}{2} \right) |\nabla \rho|^2 \right\|_{L^1(B^{\frac{N}{2}})} + \left\| \frac{K'_{\bar{\rho}}}{2} |\nabla \rho|^2 \right\|_{L^1(B^{\frac{N}{2}})}, \\ &\lesssim \|L(q)\|_{L^\infty(B^{\frac{N}{2}})} \|\nabla \rho\|_{L^1(B^{\frac{N}{2}})}^2 + \|\nabla \rho\|_{L^1(B^{\frac{N}{2}})}^2, \\ &\lesssim \|q\|_{L^\infty(B^{\frac{N}{2}})} \|\nabla \rho\|_{L^2(B^{\frac{N}{2}})}^2 + \|\nabla \rho\|_{L^2(B^{\frac{N}{2}})}^2, \\ &\lesssim \|q\|_{L^\infty(B^{\frac{N}{2}})} \|q\|_{L^2(B^{\frac{N}{2}+1})}^2 + \|q\|_{L^2(B^{\frac{N}{2}+1})}^2. \end{aligned}$$

where  $L(q) = \frac{K'_\rho}{2} - \frac{K'_\bar{\rho}}{2}$ .

3) Let us finally estimate  $\|H(q, u, \mathcal{T})\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}-2})}$ :

$$\begin{aligned} \left\| \frac{\operatorname{div}(\chi(\rho)\nabla\theta)}{\rho} - \frac{\bar{\chi}}{\bar{\rho}}\Delta\theta \right\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}-2})} &\leq \|K(q)\operatorname{div}(K_1(q)\nabla\theta)\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}-2})} \\ &+ \|\operatorname{div}(K_1(q)\nabla\theta)\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}-2})} + \|K(q)\Delta\theta\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}-2})}, \end{aligned}$$

and we have:

$$\|\operatorname{div}(K_1(q)\nabla\theta)\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}-2})} \leq C\|q\|_{L^\infty(B^{\frac{N}{2}})}\|\mathcal{T}\|_{L^1(\tilde{B}^{\frac{N}{2}+1, \frac{N}{2}})}.$$

So finally:

$$\begin{aligned} \left\| \frac{\operatorname{div}(\chi(\rho)\nabla\theta)}{\rho} - \frac{\bar{\chi}}{\bar{\rho}}\Delta\theta \right\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}-2})} &\lesssim \|q\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})}\|\mathcal{T}\|_{L^1(\tilde{B}^{\frac{N}{2}+1, \frac{N}{2}})} \\ &\times (2 + \|q\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})}). \end{aligned}$$

Next we have:

$$\begin{aligned} \left\| \left( \frac{\theta P_1(\rho)}{A\rho} - \frac{\bar{\theta} P_1(\bar{\rho})}{A\bar{\rho}} \right) \operatorname{div} u \right\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}-2})} &\lesssim \|\mathcal{T} \operatorname{div} u\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}-2})} \\ &+ \|\mathcal{T} L_1(q) \operatorname{div} u\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}-2})} + \|L_1(q) \operatorname{div} u\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}-2})}, \end{aligned}$$

where we denote:

$$L_1(x) = \frac{P_1(\bar{\rho}(1+x))}{\bar{\rho}(1+x)} - \frac{P_1(\bar{\rho})}{\bar{\rho}}.$$

On one hand,

$$\begin{aligned} \|\mathcal{T} \operatorname{div} u\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}-2})} &\lesssim \|\mathcal{T}\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}-2})} \|u\|_{L^1(B^{\frac{N}{2}+1})}, \\ \|L_1(q) \operatorname{div} u\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}-2})} &\lesssim \|L_1(q)\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}-2})} \|u\|_{L^1(B^{\frac{N}{2}+1})}, \end{aligned}$$

whence the desired result:

$$\begin{aligned} \left\| \left( \frac{\theta P_1(\rho)}{A\rho} - \frac{\bar{\theta} P_1(\bar{\rho})}{A\bar{\rho}} \right) \operatorname{div} u \right\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}-2})} &\lesssim \|q\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}-2})} \|u\|_{L^1(B^{\frac{N}{2}+1})} \\ &+ \|\mathcal{T}\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}-2})} \|u\|_{L^1(B^{\frac{N}{2}+1})} (1 + \|q\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}-2})}). \end{aligned}$$

We proceed in the same way for the others terms which are similar, and we finish with the last two following terms:

$$\begin{aligned} \|u^* \cdot \nabla \theta\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}-2})} &\lesssim \|\mathcal{T}\|_{L^1(\tilde{B}^{\frac{N}{2}+1, \frac{N}{2}})} \|u\|_{L^\infty(B^{\frac{N}{2}-1})}, \\ \left\| \frac{D : \nabla u}{\rho} \right\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}-2})} &\lesssim \|K(q)\nabla u : \nabla u\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}-2})} + \|\nabla u : \nabla u\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}-2})}. \end{aligned}$$

and:

$$\|K(q)\nabla u : \nabla u\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}-2})} \lesssim \|q\|_{L^\infty(B^{\frac{N}{2}})} \|u\|_{L^2(B^{\frac{N}{2}})}.$$

so the result:

$$\left\| \frac{D : \nabla u}{\rho} \right\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}-2})} \lesssim (1 + \|q\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})}) \|u\|_{L^2(B^{\frac{N}{2}})}.$$

Finally in using (4.24), (4.25) and all the previous bound, we get:

$$\|\psi(q, u, \mathcal{T})\|_{E^{\frac{N}{2}}} \leq C((C+1)\eta + R)^2. \quad (4.26)$$

Let  $c$  be such that  $\|\cdot\|_{B^{\frac{N}{2}}} \leq c$  implies that:  $\|\cdot\|_{L^\infty} \leq 1/3$ . Then we choose  $R$  and  $\eta$  such that:

$$R \leq \inf((3C)^{-1}, c, 1), \text{ and } \eta \leq \frac{\inf(R, c)}{C+1}.$$

So  $(\mathcal{H})$  is verified and we have:

$$\psi(B(0, R)) \subset B(0, R).$$

## 2) Second step: Property of contraction

We consider  $(q'_1, u'_1, \mathcal{T}'_1), (q'_2, u'_2, \mathcal{T}'_2)$  in  $B(0, R)$  where we note:

$$\theta_i = \mathcal{T}_i + \bar{\theta}, \quad \tilde{T}_i = \Psi^{-1}(\theta_i)$$

and we set:

$$(\delta q = q'_2 - q'_1, \delta u = u'_2 - u'_1, \delta \mathcal{T} = \mathcal{T}'_2 - \mathcal{T}'_1).$$

We have according to proposition 4.9 and (4.21):

$$\begin{aligned} & \|\psi_{(q_L, u_L, \mathcal{T}_L)}(q'_2, u'_2, \mathcal{T}'_2) - \psi_{(q_L, u_L, \mathcal{T}_L)}(q'_1, u'_1, \mathcal{T}'_1)\|_{E^{\frac{N}{2}}} \lesssim \\ & \|F(q_2, u_2, \mathcal{T}_2) - F(q_1, u_1, \mathcal{T}_1)\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} \\ & + \|G(q_2, u_2, \mathcal{T}_2) - G(q_1, u_1, \mathcal{T}_1)\|_{L^1(B^{\frac{N}{2}-1})} \\ & + \|H(q_2, u_2, \mathcal{T}_2) - H(q_1, u_1, \mathcal{T}_1)\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}-2})}. \end{aligned} \quad (4.27)$$

where we have:

$$F(q_2, u_2, \mathcal{T}_2) - F(q_1, u_1, \mathcal{T}_1) = -\operatorname{div}(q_2 u_2) + \operatorname{div}(q_1 u_1)$$

$$G(q_2, u_2, \mathcal{T}_2) - G(q_1, u_1, \mathcal{T}_1) = \delta u^* \cdot \nabla u_2 + u_1^* \cdot \nabla \delta u + \nabla \left( \frac{1}{2} (K'_{\rho_2} - K'_{\rho_1}) |\nabla \rho_2|^2 \right)$$

$$+ \nabla \left( \frac{1}{2} K'_{\rho_1} (|\nabla \rho_2|^2 - |\nabla \rho_1|^2) - \mu(\bar{\rho}) \Delta \delta u + \frac{\mu(\rho_1)}{\rho_1} \Delta \delta u - \zeta(\bar{\rho}) \nabla \operatorname{div}(\delta u) \right)$$

$$+ \nabla (K(\rho_1) \Delta(\bar{\rho} \delta q)) + \nabla ((K(\rho_2) - K(\rho_1)) \Delta \rho_2) + \left[ \frac{P'_0(\rho_2)}{\rho_2} - \frac{P'_0(\rho_1)}{\rho_1} \right] \bar{\rho} \nabla \delta q$$

$$\begin{aligned}
& + \left( \frac{P'_0(\rho_1)}{\rho_1} - \frac{P'_0(\bar{\rho})}{\bar{\rho}} \right) \nabla(\delta q) + \frac{P'_0(\bar{\rho})}{\bar{\rho}} \nabla(\delta q) + \bar{T} P'_1(\bar{\rho}) \nabla \delta q + \bar{\rho} A \delta \mathcal{T} P'_1(\rho_2) \\
& + \bar{\rho} A \theta_1 (P'_1(\rho_2) - P'_1(\rho_1)) \nabla q_2 + \frac{1}{A} [P_1(\rho_2) - P_1(\rho_1)] \nabla \theta_2 + \frac{1}{A} [P_1(\rho_1) - P_1(\bar{\rho})] \nabla \delta \mathcal{T} \\
& + \bar{\rho} A \theta_1 P'_1(\rho_1) \nabla \delta q + \left( \frac{\lambda'(\rho_2)}{\rho_2} - \frac{\lambda'(\rho_1)}{\rho_1} \right) \nabla \rho_2 \operatorname{div} u_2 + \frac{\lambda'(\rho_1)}{\rho_1} (\bar{\rho} \nabla \delta q \operatorname{div} u_2 \\
& + \nabla \rho_1 \operatorname{div} \delta u) + \left( \frac{\mu'(\rho_2)}{\rho_2} - \frac{\mu'(\rho_1)}{\rho_1} \right) (du_2 + \nabla u_2) \nabla \rho_2 + \frac{\mu'(\rho_1)}{\rho_1} (d(\delta u) + \nabla \delta u \nabla \rho_1 \\
& + \bar{\rho} \nabla(\delta q)(du_2 + \nabla u_2)).
\end{aligned}$$

And we have for the part pertaining to  $H$ :

$$\begin{aligned}
H(q_2, u_2, \mathcal{T}_2) - H(q_1, u_1, \mathcal{T}_1) & = \left( \frac{1}{\rho_2} - \frac{1}{\rho_1} \right) \operatorname{div}(\chi(\rho_2) \nabla \theta_2) + \frac{1}{\rho_1} \operatorname{div}((\chi(\rho_2) - \chi(\rho_1)) \nabla \theta_2) \\
& + \frac{1}{\rho_1} \operatorname{div}(\chi(\rho_1) \nabla \delta \mathcal{T}) + \left[ \frac{P_1(\rho_1)}{\rho_1} - \frac{P_1(\rho_2)}{\rho_2} \right] \frac{\theta_1}{A} \operatorname{div} u_1 + \frac{P_1(\rho_2)}{\rho_2} \frac{\delta \mathcal{T}}{A} \operatorname{div} u_1 - \delta u^* \cdot \nabla \theta_2 \\
& + \frac{P_1(\rho_2)}{\rho_2} \frac{\theta_2}{A} \operatorname{div} \delta u - u_1^* \nabla \delta \mathcal{T} + \left( \frac{1}{\rho_2} - \frac{1}{\rho_1} \right) D_2 : \nabla u_2 - \frac{1}{\rho_1} D_1 : \nabla \delta u - \frac{1}{\rho_1} (D_2 - D_1) : \nabla u_2
\end{aligned}$$

Let us first estimate  $\|F(q_2, u_2, \mathcal{T}_2) - F(q_1, u_1, \mathcal{T}_1)\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})}$ . We have:

$$\begin{aligned}
\|F(q_2, u_2, \mathcal{T}_2) - F(q_1, u_1, \mathcal{T}_1)\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} & \leq \|\operatorname{div}((q_2 - q_1)u_2)\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} \\
& \quad + \|\operatorname{div}(q_1(u_2 - u_1))\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})}, \\
& \lesssim \|\delta q\|_{L^2(B^{\frac{N}{2}})} \|u_2\|_{L^2(B^{\frac{N}{2}})} + \|\delta q\|_{L^\infty(B^{\frac{N}{2}})} \|u_2\|_{L^1(B^{\frac{N}{2}+1})} + \|\delta q\|_{L^2(B^{\frac{N}{2}+1})} \|u_2\|_{L^2(B^{\frac{N}{2}})} \\
& \quad + \|q_1\|_{L^2(B^{\frac{N}{2}})} \|\delta u\|_{L^2(B^{\frac{N}{2}})} + \|q_1\|_{L^\infty(B^{\frac{N}{2}})} \|\delta u\|_{L^1(B^{\frac{N}{2}+1})} + \|q_1\|_{L^2(B^{\frac{N}{2}+1})} \|\delta u\|_{L^2(B^{\frac{N}{2}})}.
\end{aligned}$$

Next, we have to bound  $\|G(q_2, u_2, \mathcal{T}_2) - G(q_1, u_1, \mathcal{T}_1)\|_{L^1(B^{\frac{N}{2}-1})}$ . We treat only one typical term, the others are of the same form.

We use essentially the proposition 3.7 to treat the product and the composition, so we get :

$$\left\| \frac{\mu(\rho_1)}{\rho_1} \Delta(u_2 - u_1) \right\|_{L^1(B^{\frac{N}{2}-1})} \lesssim (1 + \|q_1\|_{L^\infty(B^{\frac{N}{2}})}) \|\delta u\|_{L^1(B^{\frac{N}{2}+1})}.$$

Bounding  $\|H(q_2, u_2, \mathcal{T}_2) - H(q_1, u_1, \mathcal{T}_1)\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}-2})}$  is left to the reader. So we get in using the proposition 4.9 :

$$\begin{aligned}
\|\Psi(q'_2, u'_2, \mathcal{T}'_2) - \Psi(q'_1, u'_1, \mathcal{T}'_1)\|_{E^{\frac{N}{2}}} & \leq C \|(\delta q, \delta u, \delta \mathcal{T})\|_{E^{\frac{N}{2}}} \left( \|(q'_1, u'_1, \mathcal{T}'_1)\|_{E^{\frac{N}{2}}} \right. \\
& \quad \left. + \|(q'_2, u'_2, \mathcal{T}'_2)\|_{E^{\frac{N}{2}}} + 2\|(q_L, u_L, \mathcal{T}_L)\|_{E^{\frac{N}{2}}} \right).
\end{aligned}$$

If one chooses  $R$  small enough, we end up with in using (4.27) and the previous estimates:

$$\|\Psi(q'_2, u'_2, \mathcal{T}'_2) - \Psi(q'_1, u'_1, \mathcal{T}'_1)\|_{E^{\frac{N}{2}}} \leq \frac{3}{4} \|(\delta q, \delta u, \delta \mathcal{T})\|_{E^{\frac{N}{2}}}.$$

We thus have the property of contraction and so by the fixed point theorem, we have existence of a solution to  $(NHV)$ . Indeed we can see easily that  $E^{\frac{N}{2}}$  is a Banach space.

### 3) Uniqueness of the solution:

The proof is similar to the proof of contraction, hence we will have the same type of estimates. So consider two solutions in  $\tilde{E}^{\frac{N}{2}}$ :  $(q_1, u_1, \mathcal{T}_1)$  and  $(q_2, u_2, \mathcal{T}_2)$  of the system  $(NHV)$  with the same initial data. With no loss of generality, one can assume that  $(q_1, u_1, \mathcal{T}_1)$  is the solution found in the previous section.

We thus have:

$$(\mathcal{H}) \quad \|q_1\|_{L^\infty([0, T] \times \mathbb{R}^N)} \leq \frac{1}{2}.$$

Let  $\bar{T}$  be the largest time such that  $q_2$  verifies  $(\mathcal{H})$ . By continuity, we have  $0 < \bar{T} \leq T$ . Next we see that:

$$\delta q = q_2 - q_1, \quad \delta u = u_2 - u_1, \quad \delta \mathcal{T} = \mathcal{T}_2 - \mathcal{T}_1$$

verifies the system:

$$\left\{ \begin{array}{l} \partial_t \delta q + \operatorname{div} \delta u = F(q_2, u_2, \mathcal{T}_2) - F(q_1, u_1, \mathcal{T}_1), \\ \partial_t \delta u - \frac{\bar{\mu}}{\bar{\rho}} \Delta \delta u - \frac{\bar{\zeta}}{\bar{\rho}} \nabla \operatorname{div} \delta u - \bar{\rho} \bar{K} \nabla \Delta \delta q + (P'_0(\bar{\rho}) + \bar{T} P'_1(\bar{\rho})) \nabla \delta q + \frac{P_1(\bar{\rho})}{\bar{\rho} \psi'(\bar{T})} \nabla \delta \mathcal{T} \\ \hspace{15em} = G(q_2, u_2, \mathcal{T}_2) - G(q_1, u_1, \mathcal{T}_1), \\ \partial_t \delta \mathcal{T} - \frac{\bar{\chi}}{\bar{\rho}} \Delta \delta \mathcal{T} + \frac{\bar{T} P_1(\bar{\rho})}{\bar{\rho}} \operatorname{div} \delta u = H(q_2, u_2, \mathcal{T}_2) - H(q_1, u_1, \mathcal{T}_1). \end{array} \right.$$

We apply the proposition 4.9 on  $[0, T_1]$  with  $0 < T_1 \leq \bar{T}$  and we have:

$$\|(\delta q, \delta u, \delta \mathcal{T})\|_{\tilde{E}^{\frac{N}{2}}} \leq A(T_1) \|(\delta q, \delta u, \delta \mathcal{T})\|_{\tilde{E}^{\frac{N}{2}}}$$

where we have for  $T_1$  enough small  $A(T_1) \leq \frac{1}{2}$ .

And we thus have:  $\delta q = 0$ ,  $\delta u = 0$ ,  $\delta \mathcal{T} = 0$  on  $[0, T_1]$  for  $T_1$  small enough and we conclude after by connectivity.  $\square$

We treat now the specific case of  $N = 2$ , where we need more regularity for the initial data because we cannot use the proposition 3.6 in the case  $N = 2$  with the previous initial data. Indeed we cannot treat some non-linear terms such as  $\|\mathcal{T} \operatorname{div} u\|_{L^1(\tilde{B}^{0, -1})}$  or  $\|u^* \cdot \nabla \theta\|_{L^1(\tilde{B}^{0, -1})}$  because if we want to use proposition 3.6, we are in the case  $s_1 + s_2 = 0$ . This is the reason why more regularity is required.



We recall the space in which we are working:

$$E' = [C_b(\mathbb{R}_+, \tilde{B}^{0,1+\varepsilon'}) \cap L^1(\mathbb{R}_+, \tilde{B}^{2,3+\varepsilon'})] \times [C_b(\mathbb{R}_+, \tilde{B}^{0,\varepsilon'})^N \cap L^1(\mathbb{R}_+, \tilde{B}^{2,2+\varepsilon'})^N] \\ \times [C_b(\mathbb{R}_+, \tilde{B}^{0,-1+\varepsilon'}) \cap L^1(\mathbb{R}_+, \tilde{B}^{2,1+\varepsilon'})]$$

with  $\varepsilon' > 0$ ,  $E'$  being the space in which we have a solution. And  $\tilde{E}'$  corresponds to the space where we show the uniqueness of solution.

$$\tilde{E}' = [C_b(\mathbb{R}_+, \tilde{B}^{0,1+\varepsilon'}) \cap L^2(\mathbb{R}_+, \tilde{B}^{1,2+\varepsilon'})] \times [C_b(\mathbb{R}_+, \tilde{B}^{0,\varepsilon'})^N \cap L^2(\mathbb{R}_+, \tilde{B}^{1,1+\varepsilon'})^N] \\ \times [C_b(\mathbb{R}_+, \tilde{B}^{0,-1+\varepsilon'}) \cap L^2(\mathbb{R}_+, \tilde{B}^{1,\varepsilon'})].$$

### Proof of theorem 2.2

The proof is similar to the previous one except that we have changed the functional space, in which the fixed point theorem is applied. So we want verify that the function  $\psi$  is contracting to apply the fixed point. We denote by  $(q_L, u_L, \mathcal{T}_L)$  the solution of the linear system  $(M')$  with  $F = G = H = 0$  and with initial data  $(q_0, u_0, \mathcal{T}_0)$

Arguing as before, we get:

$$\|\psi(q, u, \mathcal{T})\|_{E'} \leq C\eta + \|F(q, u, \mathcal{T})\|_{L^1(\tilde{B}^{0,1+\varepsilon'})} \\ + \|G(q, u, \mathcal{T})\|_{L^1(\tilde{B}^{0,\varepsilon'})} + \|H(q, u, \mathcal{T})\|_{L^1(\tilde{B}^{0,-1+\varepsilon'})}. \quad (4.28)$$

if:

$$\|q_0\|_{\tilde{B}^{0,1+\varepsilon'}} + \|u_0\|_{\tilde{B}^{0,\varepsilon'}} + \|\mathcal{T}_0\|_{\tilde{B}^{0,-1+\varepsilon'}} \leq \eta.$$

Let us estimate  $\|F(q, u, \mathcal{T})\|_{L^1(\tilde{B}^{0,1+\varepsilon'})}$ ,  $\|G(q, u, \mathcal{T})\|_{L^1(\tilde{B}^{0,\varepsilon'})}$  and  $\|H(q, u, \mathcal{T})\|_{L^1(\tilde{B}^{0,-1+\varepsilon'})}$ , we just give two examples of estimates in the space  $E'$ , the other estimates are left to the reader.

$$\|\operatorname{div}(qu)\|_{L^1(\tilde{B}^{0,1+\varepsilon'})} \leq \|qu\|_{L^1(B^1)} + \|qu\|_{L^1(B^{2+\varepsilon'})},$$

and:

$$\|qu\|_{L^1(B^1)} \lesssim \|q\|_{L^2(B^1)} \|u\|_{L^2(B^1)},$$

$$\|qu\|_{L^1(B^{2+\varepsilon'})} \lesssim \|q\|_{L^\infty(B^1)} \|u\|_{L^1(B^{2+\varepsilon'})} + \|q\|_{L^2(B^{2+\varepsilon'})} \|u\|_{L^2(B^1)}.$$

We do similarly for  $\|G(q, u, \mathcal{T})\|_{L^1(\tilde{B}^{0,\varepsilon'})}$ . The new difficulty appears on the last term  $\|H(q, u, \mathcal{T})\|_{L^1(\tilde{B}^{0,-1+\varepsilon'})}$ . In fact it's only for this term that that additional regularity is needed. Proposition 3.6 enables us to write:

$$\|\mathcal{T} \operatorname{div} u\|_{L^1(\tilde{B}^{0,-1+\varepsilon'})} \lesssim \|\mathcal{T}\|_{L^\infty(\tilde{B}^{0,-1+\varepsilon'})} \|u\|_{L^1(B^2)}.$$

$$\|u^* \cdot \nabla \theta\|_{L^1(\tilde{B}^{0,-1+\varepsilon'})} \lesssim \|\mathcal{T}\|_{L^1(\tilde{B}^{1,1+\varepsilon'})} \|u\|_{L^\infty(B^0)}.$$

To conclude we follow the previous proof. Uniqueness in  $\tilde{E}'$  goes along the lines of the proof of uniqueness in dimension  $N \geq 3$ .  $\square$

### 4.3 Existence of a solution in the general case with small initial data

In this section we are interested by the general case where all the coefficients depend on the density and the temperature except  $\kappa$ . In this case to control the non-linear terms we need that  $\theta$  be bounded, that's why we need to take more regular initial data to preserve the  $L^\infty$  bound.

As the initial data are more regular, we need to obtain new estimates in Besov spaces on the linear system  $(M')$ .

**Proposition 4.10** *Under conditions of proposition 4.8 with strict inequality, let  $(q, u, \mathcal{T})$  be a solution of the system  $(M')$  on  $[0, T)$  with initial conditions  $(q_0, u_0, \mathcal{T}_0)$  such that:*

$$q_0 \in \tilde{B}^{s-1, s+1}, u_0 \in \tilde{B}^{s-1, s}, \mathcal{T}_0 \in \tilde{B}^{s-1, s}.$$

Moreover we suppose  $1 \leq r_1 \leq +\infty$  and:

$$F \in \tilde{L}_T^{r_1}(\tilde{B}^{s-3+\frac{2}{r_1}, s-1+\frac{2}{r_1}}), \quad G \in \tilde{L}_T^{r_1}(\tilde{B}^{s-3+\frac{2}{r_1}, s-2+\frac{2}{r_1}}), \quad H \in \tilde{L}_T^{r_1}(\tilde{B}^{s-3+\frac{2}{r_1}, s-2+\frac{2}{r_1}}).$$

We then have the following estimate for all  $r \in [r_1, +\infty]$ :

$$\begin{aligned} & \|q\|_{\tilde{L}_T^r(\tilde{B}^{s-1+\frac{2}{r}, s+1+\frac{2}{r}})} + \|u\|_{\tilde{L}_T^r(\tilde{B}^{s-1+\frac{2}{r}, s+\frac{2}{r}})} + \|\mathcal{T}\|_{\tilde{L}_T^r(\tilde{B}^{s-1+\frac{2}{r}, s+\frac{2}{r}})} \lesssim \|q_0\|_{\tilde{B}^{s-1, s+1}} + \|u_0\|_{\tilde{B}^{s-1, s}} \\ & + \|\mathcal{T}_0\|_{\tilde{B}^{s-1, s}} + \|F\|_{L_T^{r_1}(\tilde{B}^{s-3+\frac{2}{r_1}, s-1+\frac{2}{r_1}})} + \|G\|_{L_T^{r_1}(\tilde{B}^{s-3+\frac{2}{r_1}, s-2+\frac{2}{r_1}})} + \|H\|_{L_T^{r_1}(\tilde{B}^{s-3+\frac{2}{r_1}, s-2+\frac{2}{r_1}})}. \end{aligned}$$

**Proof:**

The proof is similar to that of proposition 4.9. Low frequencies are treated as in proposition 4.9 because we don't change the regularity index for the low frequencies. On the other hand in the case of high frequencies the regularity index has changed so that we have to see what is new. For the medium frequencies we can proceed as in proposition 4.9.

**Case of high frequencies:**

We are going to work with  $l \geq l_1$  where we will determine  $l_1$  hereafter. We set:

$$f_l^2 = \varepsilon B \|\Lambda q_l\|_{L^2}^2 + B \|d_l\|_{L^2}^2 + \|\mathcal{T}_l\|_{L^2}^2 - 2K \langle \Lambda q_l, d_l \rangle$$

where  $B$  and  $K$  will be chosen later on.

Then we take the scalar product of (4.3) with  $\mathcal{T}_l$ , we get:

$$\frac{1}{2} \frac{d}{dt} \|\mathcal{T}_l\|_{L^2}^2 + \alpha \|\nabla \mathcal{T}_l\|_{L^2}^2 + \delta \langle \Lambda d_l, \mathcal{T}_l \rangle = 0. \quad (4.29)$$

After we sum (4.9), (4.10) and (4.29) to get:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (B \|d_l\|_{L^2}^2 + \varepsilon B \|\Lambda q_l\|_{L^2}^2 + \|\mathcal{T}_l\|_{L^2}^2) + B \nu \|\nabla d_l\|_{L^2}^2 + \alpha \|\nabla \mathcal{T}_l\|_{L^2}^2 \\ & - B \beta \langle \Lambda q_l, d_l \rangle - B \gamma \langle \Lambda \mathcal{T}_l, d_l \rangle + \delta \langle \Lambda d_l, \mathcal{T}_l \rangle = 0. \end{aligned} \quad (4.30)$$

We sum (4.30) and (4.13) and we get:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} f_l^2 + [B\nu \|\nabla d_l\|_{L^2}^2 - K \|\Lambda d_l\|_{L^2}^2] + \alpha \|\nabla \mathcal{T}_l\|_{L^2}^2 + [\beta K \|\Lambda q_l\|_{L^2}^2 + \varepsilon K \|\Lambda^2 q_l\|_{L^2}^2] \\ & - B\beta \langle \Lambda q_l, d_l \rangle - B\gamma \langle \Lambda \mathcal{T}_l, d_l \rangle + \delta \langle \Lambda d_l, \mathcal{T}_l \rangle + K\nu \langle \Delta d_l, \Lambda q_l \rangle + \gamma K \langle \Lambda \mathcal{T}_l, \Lambda q_l \rangle = 0. \end{aligned} \quad (4.31)$$

We interest us after only to the terms of high frequencies, so arguing as in proposition 4.9 we get:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} f_l^2 + [B\nu - (K + B\gamma \frac{a}{2c2^{2l_1}} + K\nu \frac{2}{b} + B\beta \frac{d}{2c2^{2l_1}} + \delta \frac{e}{2})] \|\Lambda d_l\|_{L^2}^2 \\ & + [\alpha - (B\gamma \frac{1}{2a} + \gamma K \frac{1}{2c'} + \delta \frac{1}{2e} \frac{1}{c2^{2l_1}})] \|\Lambda \mathcal{T}_l\|_{L^2}^2 \\ & + (\frac{\beta K}{c2^{2l_1}} + \varepsilon K - K\nu \frac{b}{2} - \gamma K \frac{c'}{2c2^{2l_1}} - B\beta \frac{1}{2d} \frac{1}{2^{2l_1}}) \|\Lambda^2 q_l\|_{L^2}^2 \leq 0. \end{aligned} \quad (4.32)$$

Let us assume that:

$$\begin{aligned} & B\nu - (K + B\gamma \frac{a}{2c2^{2l_1}} + K\nu \frac{2}{b} + B\beta \frac{d}{2c2^{2l_1}} + \delta \frac{e}{2}) > 0, \\ (1) \quad & \alpha - (B\gamma \frac{1}{2a} + \gamma K \frac{1}{2c'} + \delta \frac{1}{2e} \frac{1}{2^{2l_1}}) > 0, \\ & \frac{\beta K}{c2^{2l_1}} + \varepsilon K - K\nu \frac{b}{2} - \gamma K \frac{c'}{2c2^{2l_1}} - B\beta \frac{1}{2d} \frac{1}{2^{2l_1}} > 0. \end{aligned}$$

We recall that  $\nu > 0$ , and  $\alpha > 0$ . Next we want to have:

$$\varepsilon - \nu \frac{b}{2} > 0.$$

So we take:  $b = \frac{\nu}{\varepsilon}$  (we recall that  $\varepsilon > 0$ ). So with this choice we get (1) in taking  $B$ ,  $K$  small enough and  $l_1$  big enough in following the same type of estimate as in the proof of the proposition 4.9. We have then for  $l \leq l_0$ ,  $l \geq l_1$  and  $c'$  small enough:

$$\frac{1}{2} \frac{d}{dt} f_l^2 + c' 2^{2l} f_l^2 \leq 0.$$

and:

$$f_l \simeq \max(1, 2^l) \|q_l\|_{L^2} + \|d_l\|_{L^2} + \|\mathcal{T}_l\|_{L^2}$$

Next we conclude in a similar way as in proposition 4.9.  $\square$

In the general case the coefficients depend on the temperature and we have to control the norm  $L^\infty$  in order to apply the theorems of composition. This motivates us to work in the following spaces:

$$\begin{aligned} F^{\frac{N}{2}} = & [C_b(\mathbb{R}_+, \tilde{B}^{\frac{N}{2}-1, \frac{N}{2}+1}) \cap L^1(\mathbb{R}_+ \tilde{B}^{\frac{N}{2}+1, \frac{N}{2}+3})] \times [C_b(\mathbb{R}_+, \tilde{B}^{\frac{N}{2}-1, \frac{N}{2}}) \cap \\ & L^1(\mathbb{R}_+, \tilde{B}^{\frac{N}{2}+1, \frac{N}{2}+2})]^N \times [C_b(\mathbb{R}_+, \tilde{B}^{\frac{N}{2}-1, \frac{N}{2}}) \cap L^1(\mathbb{R}_+, \tilde{B}^{\frac{N}{2}+1, \frac{N}{2}+2})]. \end{aligned}$$

**Proof of theorem 2.3:**

The principle of the proof is similar to the previous one and we use the same notation. We define the map  $\psi$  as before with the same  $F$ ,  $G$  and  $H$  except that our coefficients depends on the density and the temperature. We will verify only that  $\psi$  maps a ball  $B(0, R)$  into itself, the end is left to the reader.

**1) First step, uniform Bounds:**

We set:

$$\alpha_0 = \|q_0\|_{\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}+1}} + \|u_0\|_{\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}}} + \|\mathcal{T}_0\|_{\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}}}.$$

We denote  $(q_L, u_L, \mathcal{T}_L)$  the solution of  $(M')$  with initial data  $(q_0, u_0, \mathcal{T}_0)$ . We have so in accordance with proposition 4.10 the following estimates:

$$\|(q_L, u_L, \mathcal{T}_L)\|_{F^{\frac{N}{2}}} \leq C\alpha_0, \quad (4.33)$$

$$\begin{aligned} \|\psi(q, u, \mathcal{T})\|_{F^{\frac{N}{2}}} &\leq C\alpha_0 + \|F(q, u, \mathcal{T})\|_{L_T^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}+1})} \\ &\quad + \|G(q, u, \mathcal{T})\|_{L_T^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} + \|H(q, u, \mathcal{T})\|_{L_T^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})}. \end{aligned} \quad (4.34)$$

Moreover we suppose for the moment that:

$$(\mathcal{H}) \quad \|q\|_{L^\infty(\mathbb{R} \times \mathbb{R}^N)} \leq 1/2 \quad \text{and} \quad \|\mathcal{T}\|_{L^\infty(\mathbb{R} \times \mathbb{R}^N)} \leq 1/2.$$

We will now treat each term:  $\|F(q, u, \mathcal{T})\|_{L_T^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}+1})}$ ,  $\|G(q, u, \mathcal{T})\|_{L_T^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})}$  and  $\|H(q, u, \mathcal{T})\|_{L_T^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})}$ .

1) We notice that:

$$\begin{aligned} \|\operatorname{div}(qu)\|_{L^1(B^{\frac{N}{2}-1})} &\leq \|q\|_{L^2(B^{\frac{N}{2}})} \|u\|_{L^2(B^{\frac{N}{2}})}, \\ &\leq \|q\|_{L^2(B^{\frac{N}{2}+2})} \|u\|_{L^2(B^{\frac{N}{2}})} + \|u\|_{L^1(B^{\frac{N}{2}+2})} \|q\|_{L^\infty(B^{\frac{N}{2}})}. \end{aligned}$$

2) After we focus on  $\|G(q, u, \mathcal{T})\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})}$ . We have according to proposition 3.7:

$$\begin{aligned} \left\| \left[ \frac{\mu(\rho, \theta)}{\rho} - \frac{\mu(\bar{\rho}, \bar{\theta})}{\bar{\rho}} \right] \Delta u \right\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} &\leq \|K(q, \mathcal{T})\|_{L^\infty(B^{\frac{N}{2}})} \|u\|_{L^1(\tilde{B}^{\frac{N}{2}+1, \frac{N}{2}+2})} \\ &\lesssim (\|q\|_{L^\infty(B^{\frac{N}{2}})} + \|\mathcal{T}\|_{L^\infty(B^{\frac{N}{2}})}) \|u\|_{L^1(\tilde{B}^{\frac{N}{2}+1, \frac{N}{2}+2})}. \end{aligned}$$

We proceed in a similar way for the term:

$$(\zeta(\rho) - \zeta(\bar{\rho})) \nabla \operatorname{div} u.$$

Next we have in using propositions 3.6 and 3.2:

$$\begin{aligned} \|\rho \nabla(\kappa(\rho) - \kappa(\bar{\rho})) \Delta q\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} &\lesssim \left( \|q\|_{L^1(\tilde{B}^{\frac{N}{2}+2, \frac{N}{2}+3})} \|q\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}+1})} \right. \\ &\quad \left. + \|q\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}+1})} \|q\|_{L^1(B^{\frac{N}{2}+2})} \right) (1 + \|q\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}+1})}) \\ \left\| \left[ \frac{\bar{\rho} P'_0(\rho)}{\rho} - P'_0(\bar{\rho}) \right] \nabla q \right\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} &\leq \|q\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} \|q\|_{L^1(B^{\frac{N}{2}+1})}. \end{aligned}$$

Next, we have to treat the following terms:

$$\begin{aligned} \left\| \left[ \frac{\tilde{T} \bar{\rho} P'_1(\rho)}{\rho} - \bar{T} P'_1(\bar{\rho}) \right] \nabla q \right\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} &\lesssim \|L_1(q) L_2(\mathcal{T}) \nabla q\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} \\ &\quad + \|L_1(q) \nabla q\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} + \|L_2(\mathcal{T}) \nabla q\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})}, \end{aligned}$$

where  $L_1$  and  $L_2$  are regular function in the sense of proposition 3.7. And we have:

$$\begin{aligned} \|L_1(q) \nabla q\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} &\lesssim \|q\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} \|q\|_{L^1(B^{\frac{N}{2}+1})}, \\ \|L_2(\mathcal{T}) \nabla q\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} &\lesssim \|\mathcal{T}\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} \|q\|_{L^1(B^{\frac{N}{2}+1})}. \end{aligned}$$

Finally:

$$\begin{aligned} \left\| \left[ \frac{\tilde{T} \bar{\rho} P'_1(\rho)}{\rho} - \bar{T} P'_1(\bar{\rho}) \right] \nabla q \right\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} &\lesssim \|q\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}+1})} \|\mathcal{T}\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} \|q\|_{L^1(B^{\frac{N}{2}+1})} \\ &\quad + (\|q\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}+1})} + \|\mathcal{T}\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})}) \|q\|_{L^1(\tilde{B}^{\frac{N}{2}+1, \frac{N}{2}+3})} \end{aligned}$$

and:

$$\begin{aligned} \left\| \left( \frac{P_1(\rho)}{\rho \Psi'(\tilde{T})} - \frac{P_1(\bar{\rho})}{\bar{\rho} \Psi'(\bar{T})} \right) \nabla \theta \right\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} &\lesssim \|L_1(q) \nabla \mathcal{T}\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} + \|L_2(\mathcal{T}) \nabla \theta\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} \\ &\quad + \|L_1(q) L_2(\mathcal{T}) \nabla \theta\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})}, \\ \|L_2(\mathcal{T}) \nabla \theta\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} &\lesssim \|\mathcal{T}\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} \|\mathcal{T}\|_{L^1(B^{\frac{N}{2}+1})}, \\ \|L_1(q) \nabla \theta\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} &\lesssim \|q\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}+1})} \|\mathcal{T}\|_{L^1(\tilde{B}^{\frac{N}{2}+1, \frac{N}{2}+2})}. \end{aligned}$$

Finally:

$$\begin{aligned} \left\| \left( \frac{P_1(\rho)}{\rho \Psi'(\tilde{T})} - \frac{P_1(\bar{\rho})}{\bar{\rho} \Psi'(\bar{T})} \right) \nabla \theta \right\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} &\lesssim (\|q\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}+1})})^2 \|\mathcal{T}\|_{L^1(\tilde{B}^{\frac{N}{2}+1, \frac{N}{2}+2})} \\ &\quad + (\|\mathcal{T}\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} + \|q\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}+1})}) \|\mathcal{T}\|_{L^1(B^{\frac{N}{2}+1, \frac{N}{2}+2})}. \end{aligned}$$

After we have the following terms:

$$\|u^* \cdot \nabla u\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} \leq \|u\|_{L^2(\tilde{B}^{\frac{N}{2}, \frac{N}{2}+1})}^2.$$

And we have the terms coming from  $\operatorname{div}(D)$ . We will treat this one:

$$\begin{aligned} \left\| \frac{\lambda'_1(\rho, \theta) \nabla \rho \operatorname{div} u}{\rho} \right\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} &\lesssim \|L(q, \mathcal{T}) \nabla \rho \operatorname{div} u\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} + \|\nabla \rho \operatorname{div} u\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} \\ &\lesssim (1 + \|q\|_{L^\infty(B^{\frac{N}{2}-1, \frac{N}{2}+1})} + \|\mathcal{T}\|_{L^\infty(B^{\frac{N}{2}-1, \frac{N}{2}})}) \|q\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}+1})} \|u\|_{L^1(\tilde{B}^{\frac{N}{2}+1, \frac{N}{2}+2})}. \end{aligned}$$

Afterwards in the same way we can treat the terms of the type:

$$\frac{(du + \nabla u) \nabla \rho \mu'_1(\rho, \theta)}{\rho}, \quad \frac{(du + \nabla u) \nabla \theta \mu'_2(\rho, \theta)}{\rho} \quad \text{and} \quad \frac{\lambda'_2(\rho, \theta) \nabla \theta \operatorname{div} u}{\rho}.$$

Finally, we have:

$$\begin{aligned} \|\nabla(K'_\rho |\nabla \rho|^2)\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} &\lesssim \|(K'_\rho - K'_\bar{\rho}) |\nabla \rho|^2\|_{L^1(\tilde{B}^{\frac{N}{2}, \frac{N}{2}+1})} + \| |\nabla \rho|^2 \|_{L^1(\tilde{B}^{\frac{N}{2}, \frac{N}{2}+1})} \\ &\lesssim (1 + \|q\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}+1})}) \|q\|_{L^2(\tilde{B}^{\frac{N}{2}+1, \frac{N}{2}+2})}^2. \end{aligned}$$

3) Let us finally estimate  $\|H(q, u, \mathcal{T})\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})}$ :

$$\begin{aligned} \left\| \frac{\operatorname{div}(\chi(\rho, \theta) \nabla \theta)}{\rho} - \frac{\chi(\bar{\rho}, \bar{\theta})}{\bar{\rho}} \Delta \theta \right\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} &\leq \|K(q) \operatorname{div}(K_1(q, \mathcal{T}) \nabla \theta)\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} \\ &\quad + \|\operatorname{div}(K_1(q, \mathcal{T}) \nabla \theta)\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} + \|K(q) \Delta \theta\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} \end{aligned}$$

and in using the propositions 3.7 and 3.6 we get:

$$\begin{aligned} \|\operatorname{div}(K_1(q, \mathcal{T}) \nabla \theta)\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} &\lesssim (\|q\|_{L^\infty(B^{\frac{N}{2}})} + \|\mathcal{T}\|_{L^\infty(B^{\frac{N}{2}})}) \|\mathcal{T}\|_{L^1(\tilde{B}^{\frac{N}{2}+1, \frac{N}{2}+2})} \\ &\quad + (\|q\|_{L^2(\tilde{B}^{\frac{N}{2}, \frac{N}{2}+1})} + \|\mathcal{T}\|_{L^2(\tilde{B}^{\frac{N}{2}, \frac{N}{2}+1})}) \|\mathcal{T}\|_{L^2(B^{\frac{N}{2}+1})}. \end{aligned}$$

Next we have:

$$\begin{aligned} \left\| \left( \frac{TP_1(\rho)}{\rho} - \frac{\bar{TP}_1(\bar{\rho})}{\bar{\rho}} \right) \operatorname{div} u \right\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} &\lesssim \|L_1(q) \operatorname{div} u\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} \\ &\quad + \|L_1(q) L_2(\mathcal{T}) \operatorname{div} u\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} + \|L_2(\mathcal{T}) \operatorname{div} u\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})}, \end{aligned}$$

where:

$$\begin{aligned} \|L_1(q) \operatorname{div} u\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} &\lesssim \|q\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} \|u\|_{L^1(B^{\frac{N}{2}+1})} \\ \|L_2(\mathcal{T}) \operatorname{div} u\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} &\lesssim \|\mathcal{T}\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} \|u\|_{L^1(B^{\frac{N}{2}+1})} \end{aligned}$$

so we get:

$$\begin{aligned} \left\| \left( \frac{TP_1(\rho)}{\rho} - \frac{\bar{T}P_1(\bar{\rho})}{\bar{\rho}} \right) \operatorname{div} u \right\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} &\lesssim \|\mathcal{T}\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} \|u\|_{L^1(B^{\frac{N}{2}+1})} \|q\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} \\ &+ (\|q\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} + \|\mathcal{T}\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})}) \|u\|_{L^1(B^{\frac{N}{2}+1})}. \end{aligned}$$

To end with, we have the last two terms:

$$\begin{aligned} \|u^* \cdot \nabla \theta\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} &\leq \|\mathcal{T}\|_{L^1(\tilde{B}^{\frac{N}{2}+1, \frac{N}{2}+2})} \|u\|_{L^\infty(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})}, \\ \left\| \frac{D : \nabla u}{\rho} \right\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} &\leq \|K(q) \nabla u : \nabla u\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} + \|\nabla u : \nabla u\|_{L^1(\tilde{B}^{\frac{N}{2}-1, \frac{N}{2}})} \\ &\lesssim (1 + \|q\|_{L^\infty(\tilde{B}^{\frac{N}{2}, \frac{N}{2}+1})}) \|u\|_{L^2(\tilde{B}^{\frac{N}{2}, \frac{N}{2}+1})} \|u\|_{L^2(B^{\frac{N}{2}+1})}. \end{aligned}$$

Finally we have in using (4.33), (4.34) and the previous bounds:

$$\|\psi(q, u', \mathcal{T}')\|_{E^{N/2}} \leq C((C+1)\eta + R)^2 \quad (4.35)$$

Let  $c$  such that  $\|\cdot\|_{B^{N/2}} \leq c$  implies that:  $\|\cdot\|_{L^\infty} \leq 1/3$  then we choose  $R$  and  $\alpha_0$  such that:

$$R \leq \inf((3C)^{-1}, c, 1), \quad \alpha_0 \leq \inf \frac{(R, c)}{C+1}.$$

So  $(\mathcal{H})$  is verified and we have then:

$$\psi(B(0, R)) \subset B(0, R).$$

Next one can proceed as in the proof of the theorem 2.1, we have to show the contraction of the application  $\psi$  to use the theorem of the fixed point.

The uniqueness of the solution in the space  $F^{\frac{N}{2}}$  follows the same lines as in theorem 2.1. The details are left to the reader.

## 5 Local theory for large data

In this part we are interested in results of existence in finite time for general initial data with density bounded away from zero. We focus on the case where the coefficients depend only on the density with linear specific energy, and next we will treat the general case. As a first step, we shall study the linear part of the system  $(NHV)$  about non constant reference density and temperature, that is:

$$(N) \quad \begin{cases} \partial_t q + \operatorname{div} u = F, \\ \partial_t u - \operatorname{div}(a \nabla u) - \nabla(b \operatorname{div} u) - \nabla(c \Delta q) = G, \\ \partial_t \mathcal{T} - \operatorname{div}(d \nabla \mathcal{T}) = H, \end{cases}$$

## 5.1 Study of the linearized equation

We want to prove a priori estimates for system (N) with the following hypotheses on  $a, b, c, d$ :

$$\begin{aligned} 0 < c_1 \leq a < M_1 < \infty, \quad 0 < c_2 \leq a + b < M_2 < \infty, \quad 0 < c_3 \leq c < M_3 < \infty, \\ 0 < c_4 \leq d < M_4 < \infty. \end{aligned}$$

We remark that the last equation is just a heat equation with variable coefficients so that one can apply the following proposition proved in [13].

**Proposition 5.11** *Let  $\mathcal{T}$  solution of the heat equation:*

$$\partial_t \mathcal{T} - \operatorname{div}(d \nabla \mathcal{T}) = H,$$

we have so for all index  $\tau$  such that  $-\frac{N}{2} - 1 < \tau \leq \frac{N}{2} - 1$  the following estimate for all  $\alpha \in [1, +\infty]$ :

$$\|\mathcal{T}\|_{\tilde{L}_T^\alpha(B^{\tau+\frac{2}{\alpha}})} \leq \|\mathcal{T}_0\|_{B^\tau} + \|H\|_{\tilde{L}_T^1(B^\tau)} + \|\nabla d\|_{\tilde{L}_T^\infty(B^{\frac{N}{2}-1})} \|\nabla \mathcal{T}\|_{\tilde{L}_T^1(B^{\tau+1})}.$$

We are now interested by the first two equations of the system (N).

$$(N') \quad \begin{cases} \partial_t q + \operatorname{div} u = F \\ \partial_t u - \operatorname{div}(a \nabla u) - \nabla(b \operatorname{div} u) - \nabla(c \Delta q) = G \end{cases}$$

where we keep the same hypothesis on  $a, b$  and  $c$ . We have then the following estimate of the solution in the spaces of Chemin-Lerner:

**Proposition 5.12** *Let  $1 \leq r_1 \leq r \leq +\infty$ ,  $0 \leq s \leq 1$ ,  $(q_0, u_0) \in B^{\frac{N}{2}+s} \times (B^{\frac{N}{2}-1+s})^N$ , and  $(F, G) \in \tilde{L}_T^{r_1}(B^{\frac{N}{2}-2+s+2/r_1}) \times (\tilde{L}_T^{r_1}(B^{\frac{N}{2}-3+s+2/r_1}))^N$ .*

*Suppose that  $\nabla a$ ,  $\nabla b$ ,  $\nabla c$  belong to  $\tilde{L}_T^2(B^{\frac{N}{2}})$  and that  $\partial_t c \in L_T^1(L^\infty)$ .*

*Let  $(q, u) \in (\tilde{L}_T^r(B^{\frac{N}{2}+s+2/r}) \cap \tilde{L}_T^2(B^{\frac{N}{2}+s+1})) \times ((\tilde{L}_T^r(B^{\frac{N}{2}+s-1+2/r}))^N \cap (\tilde{L}_T^2(B^{\frac{N}{2}+s})^N))$  be a solution of the system (N').*

*Then there exists a constant  $C$  depending only on  $r, r_1, \bar{\lambda}, \bar{\mu}, \bar{\kappa}, c_1, c_2, M_1$  and  $M_2$  such that:*

$$\begin{aligned} \|(\nabla q, u)\|_{\tilde{L}_T^r(B^{\frac{N}{2}-1+s+2/r})} (1 - C \|\nabla c\|_{L_T^2(L^\infty)}) &\leq \|(\nabla q_0, u_0)\|_{B^{\frac{N}{2}}} + \|(\nabla F, G)\|_{\tilde{L}_T^{r_1}(B^{\frac{N}{2}-3+s+2/r_1})} \\ &+ \|\nabla q\|_{\tilde{L}_T^\infty(B^{\frac{N}{2}-1+s})} \|\partial_t c\|_{L_T^1(L^\infty)} + \|(\nabla q, u)\|_{\tilde{L}_T^2(B^{\frac{N}{2}+s})} (\|\nabla a\|_{\tilde{L}_T^2(B^{\frac{N}{2}})} + \|\nabla b\|_{\tilde{L}_T^2(B^{\frac{N}{2}})} \\ &+ \|\nabla c\|_{\tilde{L}_T^2(B^{\frac{N}{2}})}). \end{aligned}$$



**Proof:**

Like previously we are going to show estimates on  $q_l$  and  $u_l$ . So we apply to the system the operator  $\Delta_l$ , and we have then:

$$\partial_t q_l + \operatorname{div} u_l = F_l \quad (5.36)$$

$$\partial_t u_l - \operatorname{div}(a \nabla u_l) - \nabla(b \operatorname{div} u_l) - \nabla(c \Delta q_l) = G_l + R_l \quad (5.37)$$

where we denote:

$$R_l = \operatorname{div}([a, \Delta_l] \nabla u) - \nabla([b, \Delta_l] \operatorname{div} u_l) - \nabla([c, \Delta_l] \Delta q).$$

Performing integrations by parts and usinf (5.36) we have:

$$\begin{aligned} - \int_{\mathbb{R}^N} u_l \nabla(c \Delta q_l) dx &= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} c |\nabla q_l|^2 dx - \int_{\mathbb{R}^N} (\operatorname{div} u_l (\nabla q_l \cdot \nabla c) + \frac{|\nabla q_l|^2}{2} \partial_t c \\ &\quad + c \cdot \nabla q_l \cdot \nabla F_l) dx. \end{aligned}$$

Next, we take the inner product of (5.37) with  $u_l$  and we use the previous equality, we have then:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|u_l\|_{L^2}^2 + \int_{\mathbb{R}^N} c |\nabla q_l|^2 dx) + \int_{\mathbb{R}^N} (a |\nabla u_l|^2 + b |\operatorname{div} u_l|^2) dx &= \int_{\mathbb{R}^N} ((G_l + R_l) \cdot u_l) dx \\ &\quad + \int_{\mathbb{R}^N} ((\operatorname{div} u_l (\nabla c \cdot \nabla q_l) + \frac{|\nabla q_l|^2}{2} \partial_t c + c \nabla q_l \cdot \nabla F_l)) dx. \end{aligned}$$

In order to recover some terms in  $\Delta q_l$  we take the inner product of the gradient of (5.36) with  $u_l$ , the inner product scalar of (5.37) with  $\nabla q_l$  and we sum, we obtain then:

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^N} \nabla q_l \cdot u_l dx + \int_{\mathbb{R}^N} c (\Delta q_l)^2 dx &= \int_{\mathbb{R}^N} ((G_l + R_l) \cdot \nabla q_l + |\operatorname{div} u_l|^2 + u_l \cdot \nabla F_l \\ &\quad - a \nabla u_l : \nabla^2 q_l - b \Delta q_l \operatorname{div} u_l) dx. \end{aligned} \quad (5.38)$$

Let  $\alpha > 0$  small enough. We define:

$$k_l^2 = \|u_l\|_{L^2}^2 + \int_{\mathbb{R}^N} (\bar{\kappa} c |\nabla q_l|^2 + 2\alpha \nabla q_l \cdot u_l) dx. \quad (5.39)$$

In using the previous inequality and the fact that  $a_1 b_1 \leq \frac{1}{2} a_1^2 + \frac{1}{2} b_1^2$ , we have in summing:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} k_l^2 + \frac{1}{2} \int_{\mathbb{R}^N} (a |\nabla u_l|^2 + \alpha b |\Delta q_l|^2) dx &\lesssim (\|G_l\|_{L^2} + \|R_l\|_{L^2}) \\ \times (\alpha \|\nabla q_l\|_{L^2} + \|u_l\|_{L^2}) + \|\nabla F_l\|_{L^2} (\alpha \|u_l\|_{L^2} + \|c \nabla q_l\|_{L^2}) &+ \frac{1}{2} \|\partial_t c\|_{L^\infty} \|\nabla q_l\|_{L^2}^2 \\ &+ \|\nabla c\|_{L^\infty} \|\nabla q_l\|_{L^2} \|\nabla u_l\|_{L^2}. \end{aligned} \quad (5.40)$$

For small enough  $\alpha$ , we have according (5.39):

$$\frac{1}{2}k_l^2 \leq \|u_l\|^2 + \int_{\mathbb{R}^N} \bar{\kappa}c|\nabla q_l|^2 dx \leq \frac{3}{2}k_l^2. \quad (5.41)$$

Hence according to (5.40) and (5.41):

$$\frac{1}{2} \frac{d}{dt} k_l^2 + K2^{2l}k_l^2 \leq k_l (\|G_l\|_{L^2} + \|R_l\|_{L^2} + \|\nabla F_l\|_{L^2}) \|\partial_t c\|_{L^\infty} \|\nabla q_l\|_{L^2} + 2^{2l}k_l^2 \|\nabla c\|_{L^2}.$$

By integrating with respect to the time, we obtain:

$$k_l(t) \leq e^{-K2^{2l}t} k_l(0) + C \int_0^t e^{-K2^{2l}(t-\tau)} (\|\partial_t c\|_{L^\infty} \|\nabla q_l(\tau)\|_{L^2} + \|\nabla F_l(\tau)\|_{L^2} + \|G_l(\tau)\|_{L^2} + \|R_l(\tau)\|_{L^2} + 2^l k_l(\tau) \|\nabla c(\tau)\|_{L^2}) d\tau.$$

After convolution inequalities imply that:

$$\begin{aligned} \|k_l\|_{L^r([0,T])} &\leq (2^{-\frac{2l}{r}} k_l(0) + (2^{-2l(1+1/r-1/r_1)}) \|(\nabla F_l, G_l)\|_{L_T^{r_1}(L^2)} + 2^{-\frac{2l}{r}} \|R_l\|_{L_T^1(L^2)} \\ &\quad + 2^{-\frac{2l}{r}} \|\nabla q_l\|_{L_T^\infty(L^2)} \|\partial_t c\|_{L_T^1(L^\infty)} + \|\nabla c\|_{L_T^2(L^\infty)}) \|k_l\|_{L^r([0,T])}. \end{aligned} \quad (5.42)$$

Moreover we have:

$$C^{-1} k_l \leq \|\nabla q_l\|_{L^2} + \|u_l\|_{L^2} \leq C k_l.$$

Finally multiplying by  $2^{(\frac{N}{2}-1+s+\frac{2}{r})l}$  and using (5.41), we end up with:

$$\begin{aligned} \|(\nabla q, u)\|_{L_T^r(B^{\frac{N}{2}-1+s+2/r})} (1 - C\|\nabla c\|_{L^2(L^\infty)}) &\leq \|(\nabla F, G)\|_{\tilde{L}_T^{r_1}(B^{\frac{N}{2}-3+s+2/r_1})} \\ &\quad \|(\nabla q_0, u_0)\|_{B^{\frac{N}{2}-1+s}} + \|\nabla q\|_{\tilde{L}_T^\infty(B^{\frac{N}{2}-1+s})} \|\partial_t c\|_{L_T^1(L^\infty)} + \sum_{l \in \mathbb{Z}} 2^{l(\frac{N}{2}+s-1)} \|R_l\|_{L_T^1(L^2)}. \end{aligned}$$

Finally, applying lemma 1 on the appendix to bound the remainder term completes the proof

$$\begin{aligned} \sum_{l \in \mathbb{Z}} 2^{l(\frac{N}{2}+s-1)} \|R_l\|_{L_T^1(L^2)} &\leq C \|a\|_{\tilde{L}_T^2(B^{\frac{N}{2}+1})} \|u\|_{\tilde{L}_T^2(B^{\frac{N}{2}})} + C \|b\|_{\tilde{L}_T^2(B^{\frac{N}{2}+1})} \|u\|_{\tilde{L}_T^2(B^{\frac{N}{2}})} \\ &\quad + C \|c\|_{\tilde{L}_T^2(B^{\frac{N}{2}})} \|q\|_{\tilde{L}_T^2(B^{\frac{N}{2}+1})}. \end{aligned}$$

□

## 5.2 Local existence Theorem for temperature independent coefficients

We recall the space we will work with:

$$F_T = [\tilde{C}_T(B^{\frac{N}{2}}) \cap L_T^1(B^{\frac{N}{2}+2})] \times [\tilde{C}_T(B^{\frac{N}{2}-1})^N \cap L_T^1(B^{\frac{N}{2}+1})^N] \times [\tilde{C}_T(B^{\frac{N}{2}-2}) \cap L_T^1(B^{\frac{N}{2}})]$$

endowed with the following norm:

$$\begin{aligned} \|(q, u, \mathcal{T})\|_{F_T} = & \|q\|_{L_T^1(B^{\frac{N}{2}+2})} + \|q\|_{\tilde{L}_T^\infty(B^{\frac{N}{2}})} + \|u\|_{L_T^1(B^{\frac{N}{2}+1})} + \|u\|_{\tilde{L}_T^\infty(B^{\frac{N}{2}-1})} \\ & + \|\mathcal{T}\|_{L_T^1(B^{\frac{N}{2}})} + \|\mathcal{T}\|_{\tilde{L}_T^\infty(B^{\frac{N}{2}-2})}. \end{aligned}$$

We will now prove the local existence of a solution for general initial data with a linear specific intern energy and coefficients independent of the temperature. The functional space we shall work with is larger than previously, the reason why is that the low frequencies don't play an important role as far as one is interested in *local* results.

In what follows,  $N \geq 3$  is assumed.

### Proof of the theorem 2.4:

Let:

$$q^n = q^0 + \bar{q}^n, \quad \rho^n = \bar{\rho}(1 + q^n), \quad u^n = u^0 + \bar{u}^n, \quad \mathcal{T}^n = \mathcal{T}^0 + \bar{\mathcal{T}}^n \quad \text{and} \quad \theta^n = \bar{\theta} + \mathcal{T}^n$$

where  $(q^0, u^0, \mathcal{T}^0)$  stands for the solution of:

$$\begin{cases} \partial_t q^0 - \Delta q^0 = 0, \\ \partial_t u^0 - \Delta u^0 = 0, \\ \partial_t \mathcal{T}^0 - \Delta \mathcal{T}^0 = 0, \end{cases}$$

supplemented with initial data:

$$q^0(0) = q_0, \quad u^0(0) = u_0, \quad \mathcal{T}^0(0) = \mathcal{T}_0.$$

Let  $(\bar{q}_n, \bar{u}_n, \bar{\mathcal{T}}_n)$  be the solution of the following system:

$$(N_1) \quad \begin{cases} \partial_t \bar{q}_{n+1} + \operatorname{div}(\bar{u}_{n+1}) = F_n, \\ \partial_t \bar{u}_{n+1} - \operatorname{div}\left(\frac{\mu(\rho^n)}{\rho^n} \nabla \bar{u}^{n+1}\right) - \nabla\left(\frac{\zeta(\rho^n)}{\rho^n} \operatorname{div}(\bar{u}^{n+1})\right) - \nabla(K(\rho^n) \Delta \bar{q}^{n+1}) = G_n, \\ \partial_t \bar{\mathcal{T}}_{n+1} - \operatorname{div}\left(\frac{\chi(\rho^n)}{1 + q^n} \bar{\mathcal{T}}_{n+1}\right) = H_n, \\ (\bar{q}_{n+1}, \bar{u}_{n+1}, \bar{\mathcal{T}}_{n+1})_{t=0} = (0, 0, 0), \end{cases}$$

where:

$$\begin{aligned} F_n &= -\operatorname{div}(q^n u^n) - \Delta q^0 - \operatorname{div}(u^0), \\ G_n &= -(u^n)^* \cdot \nabla u^n + \nabla\left(\frac{K'(\rho^n)}{2} |\nabla \rho^n|^2\right) - \nabla\left(\frac{\mu(\rho^n)}{\rho^n}\right) \operatorname{div} u^n + \nabla\left(\frac{\zeta(\rho^n)}{\rho^n}\right) \operatorname{div} u^n \\ &\quad + \frac{\lambda'(\rho^n) \nabla \rho^n \operatorname{div} u^n}{1 + q^n} + \frac{(du^n + \nabla u^n) \mu'(\rho^n) \nabla \rho^n}{1 + q^n} + \left[ \frac{P_1(\rho^n)}{\rho^n \psi'(\bar{\mathcal{T}}^n)} \right] \nabla \theta^n \end{aligned}$$

$$\begin{aligned}
& + \frac{[P'_0(\rho^n) + \tilde{T}^n P'_1(\rho^n)] \nabla q^n}{1 + q^n} - \Delta u^0 + \operatorname{div} \left( \frac{\mu(\rho^n)}{1 + q^n} \nabla u^0 \right) \\
& + \nabla \left( \frac{\mu(\rho^n) + \lambda(\rho^n)}{1 + q^n} \operatorname{div}(u^0) \right) + \nabla (K(\rho^n) \Delta q^0), \\
H_n & = \nabla \left( \frac{1}{1 + q^n} \right) \cdot \nabla \theta^n \chi(\rho^n) - \frac{\tilde{T}^n P_1(\rho^n)}{\rho^n} \operatorname{div} u^n - (u^n)^* \cdot \nabla \theta^n + \frac{D_n : \nabla u^n}{\rho^n} \\
& - \Delta \theta_0 + \operatorname{div} \left( \frac{\chi(\rho^n)}{1 + q^n} \nabla \theta^0 \right).
\end{aligned}$$

### 1) First Step , Uniform Bound

Let  $\varepsilon$  be a small parameter and choose  $T$  small enough so that in using the estimate of the heat equation stated in proposition 3.5 we have:

$$\begin{aligned}
(\mathcal{H}_\varepsilon) \quad & \|\mathcal{T}^0\|_{L_T^1(B^{\frac{N}{2}})} + \|u^0\|_{L_T^1(B^{\frac{N}{2}+1})} + \|q^0\|_{L_T^1(B^{\frac{N}{2}+2})} \leq \varepsilon, \\
& \|\mathcal{T}^0\|_{\tilde{L}_T^\infty(B^{\frac{N}{2}-2})} + \|u^0\|_{\tilde{L}_T^\infty(B^{\frac{N}{2}-1})} + \|q^0\|_{\tilde{L}_T^\infty(B^{\frac{N}{2}})} \leq A_0.
\end{aligned}$$

We are going to show by induction that:

$$(\mathcal{P}_n) \quad \|(\bar{q}^n, \bar{u}^n, \bar{\mathcal{T}}^n)\|_{F_T} \leq \varepsilon.$$

As  $(\bar{q}_0, \bar{u}_0, \bar{\mathcal{T}}_0) = (0, 0, 0)$  the result is true for  $n = 0$ . We suppose now  $(\mathcal{P}_n)$  true and we are going to show  $(\mathcal{P}_{n+1})$ .

To begin with we are going to show that  $1 + q^n$  is positive. Using the fact that  $B^{\frac{N}{2}} \hookrightarrow L^\infty$  and that we take  $\varepsilon$  small enough, we have for  $t \in [0, T]$ :

$$\begin{aligned}
\|q^n - q_0\|_{L^\infty((0, T) \times \mathbb{R}^N)} & \lesssim \|\operatorname{div} \bar{u}^n\|_{L_T^1(B^{\frac{N}{2}})} + \|\operatorname{div}(q^{n-1} u^{n-1})\|_{L_T^1(B^{\frac{N}{2}})} + \|\operatorname{div} u^0\|_{L_T^1(B^{\frac{N}{2}})}, \\
& \lesssim 2\varepsilon + \|q^{n-1} u^{n-1}\|_{L_T^1(B^{\frac{N}{2}+1})},
\end{aligned}$$

and:

$$\|q^{n-1} u^{n-1}\|_{L_T^1(B^{\frac{N}{2}+1})} \leq \|q^{n-1}\|_{L_T^\infty(B^{\frac{N}{2}})} \|u^{n-1}\|_{L_T^1(B^{\frac{N}{2}+1})} + \|q^{n-1}\|_{L_T^2(B^{\frac{N}{2}+1})} \|u^{n-1}\|_{L_T^2(B^{\frac{N}{2}})}.$$

Hence:

$$\|q^n - q_0\|_{L^\infty((0, T) \times \mathbb{R}^N)} \leq C_1(2\varepsilon + (A_0 + \varepsilon)\varepsilon).$$

Finally we thus have:

$$\begin{aligned}
\|1 + q_0\|_{L^\infty((0, T) \times \mathbb{R}^N)} - \|q^n - q_0\|_{L^\infty((0, T) \times \mathbb{R}^N)} & \leq 1 + q^n \leq \|1 + q_0\|_{L^\infty((0, T) \times \mathbb{R}^N)} \\
& + \|q^n - q_0\|_{L^\infty((0, T) \times \mathbb{R}^N)},
\end{aligned}$$

whence if  $\varepsilon$  is small enough:

$$\frac{c}{2\bar{\rho}} \leq 1 + q^n \leq 1 + \frac{\|\rho_0\|_{L^\infty}}{\bar{\rho}}.$$

In order to bound  $(\bar{q}^n, \bar{u}^n, \bar{\mathcal{T}}^n)$  in  $F_T$ , we shall use proposition 5.12. For that we must check that the different hypotheses of this proposition adapted to our system  $(N_1)$  are satisfied, so we study the following terms:

$$a^n = \frac{\mu(\rho^n)}{1+q^n}, \quad b^n = \frac{\zeta(\rho^n)}{1+q^n}, \quad c^n = K(\rho^n), \quad d^n = \frac{\chi(\rho^n)}{1+q^n}.$$

In using  $(\mathcal{P}_n)$  and by continuity of  $\mu$  and the fact that  $\mu$  is positive on  $[\bar{\rho}(1+\min(q_0)) - \alpha, \bar{\rho}(1+\max(q_0)) + \alpha]$ , we have:

$$0 < c_1 \leq a^n = \frac{\mu(\rho^n)}{1+q^n} \leq M_1.$$

We proceed similarly for the others terms.

Next, notice that:

$$\begin{aligned} \|\nabla a^n\|_{\tilde{L}_T^2(B^{\frac{N}{2}})} &\leq \left\| \frac{\mu(\rho^n)}{1+q^n} - \mu(\bar{\rho}) \right\|_{\tilde{L}_T^2(B^{\frac{N}{2}+1})} \leq C \|q^n\|_{\tilde{L}_T^2(B^{\frac{N}{2}+1})}. \\ \|\nabla b^n\|_{\tilde{L}_T^2(B^{\frac{N}{2}})} &\leq \left\| \frac{\zeta(\rho^n)}{1+q^n} - \zeta(\bar{\rho}) \right\|_{\tilde{L}_T^2(B^{\frac{N}{2}+1})} \leq C \|q^n\|_{\tilde{L}_T^2(B^{\frac{N}{2}+1})} \\ \|\nabla c^n\|_{\tilde{L}_T^2(B^{\frac{N}{2}})} &\leq C \|q^n\|_{\tilde{L}_T^2(B^{\frac{N}{2}+1})}. \end{aligned}$$

To end on our hypotheses we have to control  $\partial_t c^n$  in norm  $\|\cdot\|_{L_T^1(L^\infty)}$ . As  $B^{\frac{N}{2}} \hookrightarrow L^\infty$ , it actually suffices to bound  $\|\partial_t c^n\|_{L_T^1(B^{\frac{N}{2}})}$ . We have:

$$\partial_t c^n = K'(\rho^n) \partial_t q^n = K'(\rho^n) (\operatorname{div}(q^{n-1} u^{n-1}) - \operatorname{div}(u^n)).$$

And we have in using the propositions 3.6 and 3.7:

$$\begin{aligned} \|K'(\rho^n) (\operatorname{div}(q^{n-1} u^{n-1}) - \operatorname{div}(u^n))\|_{L_T^1(B^{\frac{N}{2}})} &\leq \|K'(\rho^n) \operatorname{div}(q^{n-1} u^{n-1})\|_{L_T^1(B^{\frac{N}{2}})} \\ &\quad + \|K'(\rho^n) \operatorname{div}(u^n)\|_{L_T^1(B^{\frac{N}{2}})}, \\ &\lesssim (1 + \|q^n\|_{L_T^\infty(B^{\frac{N}{2}})}) (\|u^n\|_{L_T^1(B^{\frac{N}{2}+1})} + \|q^{n-1} u^{n-1}\|_{L_T^1(B^{\frac{N}{2}+1})}) \\ &\lesssim (1 + \|q^n\|_{L_T^\infty(B^{\frac{N}{2}})}) (\|u^n\|_{L_T^1(B^{\frac{N}{2}+1})} + \|q^{n-1}\|_{L_T^\infty(B^{\frac{N}{2}})} \|u^{n-1}\|_{L_T^1(B^{\frac{N}{2}+1})} \\ &\quad + \|q^{n-1}\|_{L_T^2(B^{\frac{N}{2}+1})} \|u^{n-1}\|_{L_T^2(B^{\frac{N}{2}})}). \end{aligned}$$

We now use proposition 5.11 to get the bound on  $\bar{\mathcal{T}}^n$ , so we obtain in taking  $\tau = \frac{N}{2} - 2$ :

$$\begin{aligned} \|\mathcal{T}^n\|_{L_T^1(B^{\frac{N}{2}}) \cap L_T^\infty(B^{\frac{N}{2}-2})} &\lesssim (\|H_n\|_{L_T^1(B^{\frac{N}{2}-2})} + \|\nabla(\frac{\chi(\rho^n)}{\rho^n})\|_{L_T^\infty(B^{\frac{N}{2}-1})}) \\ &\quad \times \|\mathcal{T}^n\|_{L_T^1(B^{\frac{N}{2}})}. \end{aligned} \tag{5.43}$$

So we need to bound  $d^n$  in  $L_T^\infty(B^{\frac{N}{2}})$ :

$$\|\nabla d^n\|_{L_T^\infty(B^{\frac{N}{2}-1})} \leq C \|q^n\|_{L_T^\infty(B^{\frac{N}{2}})}.$$

Now we show by induction ( $\mathcal{P}_{n+1}$ ). Finally, applying the estimates of propositions 5.12 and 5.11, we conclude that:

$$\begin{aligned} & \|(\bar{q}^{n+1}, \bar{u}^{n+1}, \bar{\mathcal{T}}^{n+1})\|_{F_T} (1 - C(\|a^n\|_{L_T^2(B^{\frac{N}{2}+1})} + \|b^n\|_{L_T^2(B^{\frac{N}{2}+1})} + \|c^n\|_{L_T^2(B^{\frac{N}{2}+1})} \\ & + \|d^n\|_{L_T^\infty(B^{\frac{N}{2}})} + \|\partial_t c^n\|_{L_T^1(B^{\frac{N}{2}})}) \leq \|(\nabla F_n, G_n)\|_{L_T^1(B^{\frac{N}{2}-1})} + \|H_n\|_{L_T^1(B^{\frac{N}{2}-2})}. \end{aligned} \quad (5.44)$$

Bounding the right-hand side may be done by applying propositions 3.6 and 3.7. For instance, we have:

$$\|F_n\|_{L_T^1(B^{N/2})} \leq \|\operatorname{div}(q^n u^n)\|_{L_T^1(B^{N/2})} + \|\operatorname{div} u^0\|_{L_T^1(B^{N/2})} + \|\Delta q^0\|_{L_T^1(B^{N/2})}.$$

Since:

$$\|u^n q^n\|_{L_T^1(B^{N/2+1})} \lesssim \|q^n\|_{L_T^\infty(B^{N/2})} \|u^n\|_{L_T^1(B^{N/2+1})} + \|q^n\|_{L_T^2(B^{N/2+1})} \|u^n\|_{L_T^2(L^\infty)},$$

we can conclude that:

$$\|F_n\|_{L_T^1(B^{N/2})} \leq C(A_0 + \varepsilon + \sqrt{\varepsilon})^2.$$

Next we want to control the different terms of  $G_n$ . According to propositions 3.6 and 5.12, we have:

$$\begin{aligned} & \|(u^n)^* \cdot \nabla u^n\|_{L_T^1(B^{\frac{N}{2}-1})} \lesssim \|u^n\|_{L_T^\infty(B^{\frac{N}{2}-1})} \|u^n\|_{L_T^1(B^{\frac{N}{2}+1})} \\ & \left\| \nabla \left( \frac{K'_{\rho^n}}{2} |\nabla \rho_n|^2 \right) \right\|_{L_T^1(B^{\frac{N}{2}-1})} \lesssim \|L(q^n) |\nabla \rho_n|^2\|_{L_T^1(B^{\frac{N}{2}})} + \| |\nabla \rho_n|^2 \|_{L_T^1(B^{\frac{N}{2}})} \\ & \lesssim (1 + \|q^n\|_{L_T^\infty(B^{\frac{N}{2}})}^2) \|q^n\|_{L_T^2(B^{\frac{N}{2}+1})}^2. \end{aligned}$$

After we have:

$$\begin{aligned} & \left\| \nabla \left( \frac{\mu(\rho^n)}{\rho^n} \operatorname{div} u_n \right) \right\|_{L_T^1(B^{\frac{N}{2}-1})} \lesssim \|\operatorname{div} u_n\|_{L_T^1(B^{\frac{N}{2}})} \left\| \nabla \left( \frac{\mu(\rho^n)}{\rho^n} - \frac{\mu(\bar{\rho})}{\bar{\rho}} \right) \right\|_{L_T^\infty(B^{\frac{N}{2}-1})} \\ & \leq C \|u^n\|_{L_T^1(B^{\frac{N}{2}+1})} \|q^n\|_{L_T^\infty(B^{\frac{N}{2}})}. \end{aligned}$$

$$\left\| \nabla \left( \frac{\zeta(\rho^n)}{\rho^n} \operatorname{div} u_n \right) \right\|_{L_T^1(B^{\frac{N}{2}-1})} \leq C \|u^n\|_{L_T^1(B^{\frac{N}{2}+1})} \|q^n\|_{L_T^\infty(B^{\frac{N}{2}})}.$$

After we study the term coming from  $\operatorname{div}(D)$ :

$$\begin{aligned} & \left\| \frac{\lambda'(\rho^n) \nabla \rho^n \operatorname{div}(u^n)}{1 + q^n} \right\|_{L_T^1(B^{\frac{N}{2}-1})} \lesssim \|L(q^n) \nabla \rho^n \operatorname{div}(u^n)\|_{L_T^1(B^{\frac{N}{2}-1})} + \|\nabla \rho^n \operatorname{div}(u^n)\|_{L_T^1(B^{\frac{N}{2}-1})} \\ & \lesssim (1 + \|q^n\|_{L_T^\infty(B^{\frac{N}{2}})}) \|u^n\|_{L_T^1(B^{\frac{N}{2}+1})} \|q^n\|_{L_T^\infty(B^{\frac{N}{2}})}. \end{aligned}$$

We proceed similarly for the following term:

$$\frac{(du^n + \nabla u^n)\mu'(\rho^n)\nabla\rho^n}{1+q^n}.$$

Next we study the last terms:

$$\begin{aligned} \left\| \frac{[P'_0(\rho^n) + \mathcal{T}^n P'_1(\rho^n)]\nabla q^n}{1+q^n} \right\|_{L^1_T(B^{\frac{N}{2}-1})} &\lesssim \|K(q^n)\nabla q^n\|_{L^1_T(B^{\frac{N}{2}-1})} + \|K(q^n)\mathcal{T}^n\nabla q^n\|_{L^1_T(B^{\frac{N}{2}-1})} \\ &\quad + \|\mathcal{T}^n\nabla q^n\|_{L^1_T(B^{\frac{N}{2}-1})}, \\ &\lesssim T\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})}^2 + \|\mathcal{T}^n\|_{L^2_T(B^{\frac{N}{2}-1})}\|q^n\|_{L^2_T(B^{\frac{N}{2}+1})}(1 + \|q^n\|_{L^\infty_T(B^{\frac{N}{2}})}). \end{aligned}$$

$$\left\| \left[ \frac{P_1(\rho^n)}{\rho^n} \right] \nabla \theta^n \right\|_{L^1_T(B^{\frac{N}{2}-1})} \leq C(\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})} + 1)\|\mathcal{T}^n\|_{L^1_T(B^{\frac{N}{2}})},$$

$$\|\operatorname{div}\left(\frac{\mu(\rho^n)}{1+q^n}\nabla u^0\right)\|_{L^1_T(B^{\frac{N}{2}-1})} \lesssim (1 + \|q^n\|_{L^\infty_T(B^{\frac{N}{2}})})\|u^0\|_{L^1_T(B^{\frac{N}{2}+1})}.$$

We proceed similarly with the other terms:

$$-\Delta u^0, \quad \nabla\left(\frac{\zeta(\rho^n)}{1+q^n}\operatorname{div}(u^0)\right), \quad \nabla(K(\rho^n)\Delta q^0).$$

Let us estimate now  $\|H_n\|_{L^1_T(B^{\frac{N}{2}-2})}$ . We obtain:

$$\begin{aligned} \left\| \nabla\left(\frac{1}{1+q^n}\right) \cdot \nabla \theta^n \chi(\rho^n) \right\|_{L^1_T(B^{\frac{N}{2}-2})} &\lesssim \|K(q^n)\nabla\left(\frac{q^n}{1+q^n}\right) \cdot \nabla \theta^n\|_{L^1_T(B^{N/2-2})} \\ &\quad + C\left\| \nabla\left(\frac{q^n}{1+q^n}\right) \cdot \nabla \theta^n \right\|_{L^1_T(B^{\frac{N}{2}-2})}, \\ &\lesssim (1 + \|q^n\|_{L^2_T(B^{\frac{N}{2}})})\|q^n\|_{L^2_T(B^{\frac{N}{2}+1})}\|\mathcal{T}^n\|_{L^2_T(B^{\frac{N}{2}-1})}. \end{aligned}$$

We have after these last two terms:

$$\begin{aligned} \left\| \frac{\mathcal{T}^n P_1(\rho^n)}{\rho^n} \operatorname{div} u^n \right\|_{L^1_T(B^{\frac{N}{2}-2})} &\lesssim \|K(q^n)\operatorname{div} u^n\|_{L^1_T(B^{\frac{N}{2}-2})} + \|K(q^n)K_1(\mathcal{T}^n)\operatorname{div} u^n\|_{L^1_T(B^{\frac{N}{2}-2})} \\ &\quad + \|K_1(\mathcal{T}^n)\operatorname{div} u^n\|_{L^1_T(B^{\frac{N}{2}-2})}, \end{aligned}$$

with  $K$  and  $K_1$  regular in sense of the proposition 3.7 and:

$$\begin{aligned} \|K(q^n)\operatorname{div} u^n\|_{L^1_T(B^{\frac{N}{2}-2})} &\lesssim T\|q^n\|_{L^\infty_T(B^{\frac{N}{2}})}\|u^n\|_{L^\infty_T(B^{\frac{N}{2}-1})}, \\ \|K_1(\mathcal{T}^n)\operatorname{div} u^n\|_{L^1_T(B^{N/2-2})} &\lesssim \|\mathcal{T}^n\|_{L^\infty_T(B^{\frac{N}{2}-2})}\|u^n\|_{L^1_T(B^{\frac{N}{2}+1})}, \end{aligned}$$

so finally:

$$\begin{aligned} \left\| \frac{\mathcal{T}^n P_1(\rho^n)}{\rho^n} \operatorname{div} u^n \right\|_{L_T^1(B^{\frac{N}{2}-2})} &\lesssim (T \|q^n\|_{L_T^\infty(B^{\frac{N}{2}})} + \|\mathcal{T}^n\|_{L_T^\infty(B^{\frac{N}{2}-2})}) \|u^n\|_{L_T^\infty(B^{\frac{N}{2}-1})} \\ &\quad + \|q^n\|_{L_T^\infty(B^{\frac{N}{2}})} \|\mathcal{T}^n\|_{L_T^\infty(B^{\frac{N}{2}-2})} \|u^n\|_{L_T^1(B^{\frac{N}{2}+1})}. \end{aligned}$$

and, since  $N \geq 3$ :

$$\begin{aligned} \|u^n \cdot \nabla \theta^n\|_{L_T^1(B^{\frac{N}{2}-2})} &\leq \|u^n\|_{L_T^\infty(B^{\frac{N}{2}-1})} \|\theta^n\|_{L_T^1(B^{\frac{N}{2}})}, \\ \|\nabla u^n : \nabla u^n\|_{L_T^1(B^{\frac{N}{2}-2})} &\leq \|u^n\|_{L_T^2(B^{\frac{N}{2}})}^2. \end{aligned}$$

We obtain in using (5.44) and the different previous inequalities:

$$\|(\bar{q}_{n+1}, \bar{u}_{n+1}, \bar{\mathcal{T}}_{n+1})\|_{F_T} (1 - C2\sqrt{\varepsilon}(A_0 + \sqrt{\varepsilon})) \leq C_1(\varepsilon(A_0 + \sqrt{\varepsilon})^2 + T(A_0 + \sqrt{\varepsilon})).$$

In taking  $T$  and  $\varepsilon$  small enough we have  $(\mathcal{P}_{n+1})$ , so we have shown by induction that  $(q^n, u^n, \mathcal{T}^n)$  is bounded in  $F_T$ .

### Second Step: Convergence of the sequence

We will show that  $(q^n, u^n, \mathcal{T}^n)$  is a Cauchy sequence in the Banach space  $F_T$ , hence converges to some  $(q, u, \mathcal{T}) \in F_T$ .

Let:

$$\delta q^n = q^{n+1} - q^n, \quad \delta u^n = u^{n+1} - u^n, \quad \delta \mathcal{T}^n = \mathcal{T}^{n+1} - \mathcal{T}^n.$$

The system verified by  $(\delta q^n, \delta u^n, \delta \mathcal{T}^n)$  reads:

$$\left\{ \begin{array}{l} \partial_t \delta q^n + \operatorname{div} \delta u^n = F_n - F_{n-1}, \\ \partial_t \delta u^n - \operatorname{div} \left( \frac{\mu(\rho^n)}{\rho^n} \nabla \delta u^n \right) - \nabla \left( \frac{\zeta(\rho^n)}{\rho^n} \operatorname{div}(\delta u^n) \right) - \nabla (K(\rho^n) \Delta \delta q^n) = \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad G_n - G_{n-1} + G'_n - G'_{n-1}, \\ \partial_t \delta \mathcal{T}^n - \operatorname{div} \left( \frac{\chi(\rho^n)}{1 + q^n} \nabla \delta \mathcal{T}^n \right) = H_n - H_{n-1} + H'_n - H'_{n-1}, \\ \delta q^n(0) = 0, \quad \delta u^n(0) = 0, \quad \delta \mathcal{T}^n(0) = 0, \end{array} \right.$$

where we define:

$$\begin{aligned} G'_n &= -\operatorname{div} \left( \left( \frac{\mu(\rho^{n+1})}{\rho^{n+1}} - \frac{\mu(\rho^n)}{\rho^n} \right) \nabla u^{n+1} \right) - \nabla \left( (K(\rho^{n+1}) - K(\rho^n)) \Delta q^{n+1} \right) \\ &\quad - \nabla \left( \left( \frac{\zeta(\rho^{n+1})}{\rho^{n+1}} - \frac{\zeta(\rho^n)}{\rho^n} \right) \operatorname{div}(u^{n+1}) \right). \end{aligned}$$

In the same way we have:

$$H'_n = \operatorname{div} \left( \left( \frac{\chi(\rho^{n+1})}{1 + q^{n+1}} - \frac{\chi(\rho^n)}{1 + q^n} \right) \nabla \theta^{n+1} \right).$$



Applying propositions 5.11, 5.12, and using  $(\mathcal{P}_n)$ , we get:

$$\begin{aligned} \|(\delta q^n, \delta u^n, \delta \mathcal{T}^n)\|_{F_T} &\leq C(\|F_n - F_{n-1}\|_{L_T^1(B^{N/2})} + \|G_n - G_{n-1} + G'_n - G'_{n-1}\|_{L_T^1(B^{N/2-1})}) \\ &\quad + \|H_n - H_{n-1} + H'_n - H'_{n-1}\|_{L_T^1(B^{N/2-2})}, \end{aligned}$$

And by the same type of estimates as before, we get:

$$\|(\delta q^n, \delta u^n, \delta \mathcal{T}^n)\|_{F_T} \leq C\sqrt{\varepsilon}(1 + A_0)^3 \|(\delta q^{n-1}, \delta u^{n-1}, \delta \mathcal{T}^{n-1})\|_{F_T}.$$

So in taking  $\varepsilon$  enough small we have that  $(q^n, u^n, \mathcal{T}^n)$  is Cauchy sequence, so the limit  $(q, u, \mathcal{T})$  is in  $F_T$  and we verify easily that this is a solution of the system.

### Third step: Uniqueness

Suppose that  $(q_1, u_1, \mathcal{T}_1)$  and  $(q_2, u_2, \mathcal{T}_2)$  are solutions with the same initial conditions, and  $(q_1, u_1, \mathcal{T}_1)$  corresponds to the previous solution.

Assume moreover that we have:

$$\|q_1(t)\|_{L^\infty} \leq \alpha, \quad \forall t \in [0, T].$$

We set then:

$$\delta q = q_2 - q_1, \quad \delta u = u_2 - u_1, \quad \delta \mathcal{T} = \mathcal{T}_2 - \mathcal{T}_1.$$

The triplet  $(\delta q, \delta u, \delta \mathcal{T})$  satisfies the following system:

$$\begin{cases} \partial_t \delta q + \operatorname{div} \delta u = F_2 - F_1, \\ \partial_t \delta u - \operatorname{div} \left( \frac{\mu(\rho_2)}{\rho_2} \nabla \delta u \right) - \nabla \left( \frac{\zeta(\rho_2)}{\rho_2} \operatorname{div}(\delta u) \right) - \nabla (K(\rho_2) \Delta \delta q) = G_2 - G_1 + G' \\ \partial_t \delta \mathcal{T} - \operatorname{div} \left( \frac{\chi(\rho_2)}{1 + q_2} \nabla \delta \mathcal{T} \right) = H_2 - H_1 + H', \\ \delta q(0) = 0, \quad \delta u(0) = 0, \quad \delta \mathcal{T}(0) = 0 \end{cases}$$

with:

$$\begin{aligned} G' &= -\operatorname{div} \left( \left( \frac{\mu(\rho_2)}{\rho_2} - \frac{\mu(\rho_1)}{\rho_1} \right) \nabla u_2 \right) - \nabla \left( (K(\rho_2) - K(\rho_1)) \Delta q_2 \right) \\ &\quad - \nabla \left( \left( \frac{\zeta(\rho_2)}{\rho_2} - \frac{\zeta(\rho_1)}{\rho_1} \right) \operatorname{div}(u_1) \right), \\ H' &= \operatorname{div} \left( \left( \frac{\chi(\rho_2)}{1 + q_2} - \frac{\chi(\rho_1)}{1 + q_1} \right) \nabla \theta_2 \right). \end{aligned}$$

Let  $\bar{T}$  the largest time such that:  $\|q_2\|_{L^\infty((0, \bar{T}) \times \mathbb{R}^N)} \leq \alpha$ . As  $q_2 \in C([0, T]; B^{N/2})$ , we have by continuity  $0 < \bar{T} \leq T$ .

We are going to work on the interval  $[0, T_1]$  with  $0 < T_1 \leq \bar{T}$  and we use the proposition 5.12, so we obtain in using the same type of estimates than in the part on the contraction:

$$\|(\delta q, \delta u, \delta \mathcal{T})\|_{\tilde{F}_T^{\frac{N}{2}}} \leq Z(T) \|(\delta q, \delta u, \delta \mathcal{T})\|_{\tilde{F}_T^{\frac{N}{2}}}$$

with  $Z(T) \rightarrow_{T \rightarrow 0} 0$ .

We have then for  $T_1$  small enough:  $(\delta q, \delta u, \delta \mathcal{T}) = (0, 0, 0)$  on  $[0, T_1]$  and by connectivity we finally conclude that:

$$q_1 = q_2, u_1 = u_2, \mathcal{T}_1 = \mathcal{T}_2 \text{ on } [0, T].$$

□

### Proof of the theorem 2.5

In the special case  $N = 2$ , we need to take more regular initial data for the same reasons as in theorem 2.2. Indeed some terms like  $\Psi(\theta)\text{div}u$  or  $u^*.\nabla\theta$  can't be controlled without more regularity.

The proof is similar to the previous proof of theorem 2.4 except that we have changed the functional space  $F_T(2)$ , in which the fixed point theorem is going to be applied. As we explain above we can use the paraproduct because we have more regularity, so we just see the term  $u^*.\nabla\theta$ . The other terms and the details are left to the reader.

We then have:

$$\|u^*.\nabla\theta\|_{L_T^1(\tilde{B}^{-1, -1+\varepsilon'})} \lesssim \|\mathcal{T}\|_{L_T^1(\tilde{B}^{0, 1+\varepsilon'})} \|u\|_{L_T^\infty(B^0)}.$$

□

### 5.3 Local existence theorem in the general case

Now we suppose that all the coefficients depend on the temperature and on the density, and that conditions (C) and (D) are satisfied with strict inequalities.

One of the problem in the general case is the control of the  $L^\infty$  norm of the temperature  $\theta$  in order to have control on the non linear terms where the physical coefficients appear. Indeed in the theorem of composition we need to control the norm  $L^\infty$ .

So we must impose that  $\theta_0$  is in  $B^{\frac{N}{2}}$  to hope a  $L^\infty$  control. And in consequence the others initial data have to be also more regular.

#### Proof of theorem 2.6:

We proceed exactly like in theorem 2.4 except that we ask more regularity for the initial data. We define then:

$$q^n = q^0 + \bar{q}^n, u^n = u^0 + \bar{u}^n, \mathcal{T}^n = \mathcal{T}^0 + \bar{\mathcal{T}}^n \text{ and } \theta^n = \mathcal{T}^n + \bar{\mathcal{T}}$$

where  $(q^0, u^0, \mathcal{T}^0)$  stands for the solution of:

$$\begin{cases} \partial_t q^0 - \Delta q^0 = 0, \\ \partial_t u^0 - \Delta u^0 = 0, \\ \partial_t \mathcal{T}^0 - \Delta \mathcal{T}^0 = 0, \end{cases}$$

supplemented with initial data:

$$q^0(0) = q_0, u^0(0) = u_0, \theta^0(0) = \theta_0.$$

Let  $(\bar{q}_n, \bar{u}_n, \bar{\theta}_n)$  be the solution of the following system:

$$\begin{cases} \partial_t \bar{q}^{n+1} + \operatorname{div}(\bar{u}^{n+1}) = F_n, \\ \partial_t \bar{u}^{n+1} - \operatorname{div}\left(\frac{\mu(\rho^n, \theta^n)}{\rho^n} \nabla \bar{u}^{n+1}\right) - \nabla\left(\frac{\zeta(\rho^n, \theta^n)}{\rho^n} \operatorname{div}(\bar{u}^{n+1})\right) - \nabla(\kappa(\rho^n) \Delta \bar{q}^{n+1}) = G_n, \\ \partial_t \bar{T}^{n+1} - \operatorname{div}\left(\frac{\chi(\rho^n, \theta^n)}{1+q^n} \bar{T}^{n+1}\right) = H_n, \\ (\bar{q}^{n+1}, \bar{u}^{n+1}, \bar{T}^{n+1})_{t=0} = (0, 0, 0). \end{cases}$$

where:

$$F_n = -\operatorname{div}(q^n u^n) - \Delta q^0 - \operatorname{div}(u^0),$$

$$\begin{aligned} G_n &= -(u^n)^* \cdot \nabla u^n + \nabla\left(\frac{K' \rho^n}{2} |\nabla \rho^n|^2\right) - \nabla\left(\frac{\mu(\rho^n, \theta^n)}{\rho^n}\right) \operatorname{div} u^n + \nabla\left(\frac{\zeta(\rho^n, \theta^n)}{\rho^n}\right) \operatorname{div} u^n \\ &\quad + \frac{\lambda'_1(\rho^n, \theta^n) \nabla \rho^n \operatorname{div} u^n}{1+q^n} + \frac{(du^n + \nabla u^n) \mu'_1(\rho^n, \theta^n) \nabla \rho^n}{1+q^n} \\ &\quad + \frac{\lambda'_2(\rho^n, \theta^n) \nabla \theta^n \operatorname{div} u^n}{1+q^n} + \frac{(du^n + \nabla u^n) \mu'_2(\rho^n, \theta^n) \nabla \theta^n}{1+q^n} \\ &\quad + \frac{[P'_0(\rho^n) + \tilde{T}^n P'_1(\rho^n)] \nabla q^n}{1+q^n} + \left[\frac{P_1(\rho^n)}{\rho^n \psi'(T^n)}\right] \nabla \theta^n \\ &\quad - \Delta u^0 + \operatorname{div}\left(\frac{\mu(\rho^n, \theta^n)}{1+q^n} \nabla u^0\right) + \nabla\left(\frac{\mu(\rho^n) + \lambda(\rho^n)}{1+q^n} \operatorname{div}(u^0)\right) + \nabla(\kappa(\rho^n) \Delta q^0), \\ H_n &= \nabla\left(\frac{1}{1+q^n}\right) \cdot \nabla \theta^n \chi(\rho^n, \theta^n) - \frac{\tilde{T}^n P_1(\rho^n)}{\rho^n} \operatorname{div} u^n - (u^n)^* \cdot \nabla \theta^n + \frac{D^n : \nabla u^n}{\rho^n} \\ &\quad - \Delta \theta_0 + \operatorname{div}\left(\frac{\chi(\rho^n, \theta^n)}{1+q^n} \nabla \theta^0\right). \end{aligned}$$

### 1) First Step , Uniform Bound

Let  $\varepsilon$  be a small positive parameter and choose  $T$  small enough so that in using the estimate of the proposition 3.5 we have:

$$\begin{aligned} (\mathcal{H}_\varepsilon) \quad & \|\mathcal{T}^0\|_{L^1_T(B^{\frac{N}{2}+2})} + \|u^0\|_{L^1_T(B^{\frac{N}{2}+2})} + \|q^0\|_{L^1_T(\tilde{B}^{\frac{N}{2}+2, \frac{N}{2}+3})} \leq \varepsilon, \\ & \|\mathcal{T}^0\|_{\tilde{L}^\infty_T(B^{\frac{N}{2}})} + \|u^0\|_{\tilde{L}^\infty_T(B^{\frac{N}{2}})} + \|q^0\|_{\tilde{L}^\infty_T(\tilde{B}^{\frac{N}{2}, \frac{N}{2}+1})} \leq A_0. \end{aligned}$$

After we are going to show by induction that:

$$(\mathcal{P}_n) \quad \|(\bar{q}_n, \bar{u}_n, \bar{T}_n)\|_{F_T} \leq \varepsilon.$$

As  $(\bar{q}_0, \bar{u}_0, \bar{\mathcal{T}}_0) = (0, 0, 0)$  the result is true for  $(\mathcal{P}_0)$ . We suppose now  $(\mathcal{P}_n)$  true and we are going to show  $(\mathcal{P}_{n+1})$ .

To begin with we are going to show that  $1 + q^n$  is positive. In using the fact that  $B^{\frac{N}{2}} \hookrightarrow L^\infty$  and that we can take  $\varepsilon$  enough small, we have for  $t \in [0, T]$ :

$$\begin{aligned} \|q^n - q_0\|_{L^\infty((0,T) \times \mathbb{R}^N)} &\leq C_1 (\|\operatorname{div}(\bar{u}^n)\|_{L_T^1(B^{\frac{N}{2}})} + \|\operatorname{div}(q^{n-1}u^{n-1})\|_{L_T^1(B^{\frac{N}{2}})} \\ &\quad + \|\operatorname{div}(u^0)\|_{L_T^1(B^{\frac{N}{2}})}), \\ &\leq C_1 (2\sqrt{T}\varepsilon + \|q^{n-1}u^{n-1}\|_{L_T^1(B^{\frac{N}{2}+1})}), \end{aligned}$$

and by induction hypothesis  $(\mathcal{P}_{n-1})$ :

$$\begin{aligned} \|q^{n-1}u^{n-1}\|_{L_T^1(B^{\frac{N}{2}+1})} &\leq \sqrt{T} \|q^{n-1}\|_{L_T^\infty(B^{\frac{N}{2}})} \|u^{n-1}\|_{L_T^2(B^{\frac{N}{2}+1})} \\ &\quad + \sqrt{T} \|q^{n-1}\|_{L_T^2(B^{\frac{N}{2}+1})} \|u^{n-1}\|_{L_T^\infty(B^{\frac{N}{2}})}, \end{aligned}$$

thus:

$$\|q^n - q_0\|_{L^\infty((0,T) \times \mathbb{R}^N)} \leq C_1 \sqrt{T} (2\varepsilon + (A_0 + \varepsilon)\varepsilon).$$

Finally we have:

$$\begin{aligned} \|1 + q_0\|_{L^\infty((0,T) \times \mathbb{R}^N)} - \|q^n - q_0\|_{L^\infty((0,T) \times \mathbb{R}^N)} &\leq 1 + q^n \leq \|1 + q_0\|_{L^\infty((0,T) \times \mathbb{R}^N)} \\ &\quad + \|q^n - q_0\|_{L^\infty((0,T) \times \mathbb{R}^N)}, \end{aligned}$$

$$\frac{c}{2\bar{\rho}} \leq 1 + q^n \leq 1 + \frac{\|\rho_0\|_{L^\infty}}{\bar{\rho}}.$$

So we have shown that:

$$(*) \quad \|q^n\|_{L^\infty} \leq 2A_0$$

and that  $\rho^n$  is bounded away from 0.

To verify the uniform bound we use the propositions 5.11 and 5.12. For that we have to verify the different hypotheses of these propositions, so that we study the following terms:

$$a^n = \frac{\mu(\rho^n, \theta^n)}{1 + q^n}, \quad b^n = \frac{\zeta(\rho^n, \theta^n)}{1 + q^n}, \quad c^n = K(\rho_n), \quad d^n = \frac{\chi(\rho^n, \theta^n)}{1 + q^n}.$$

In using  $(\mathcal{P}_n)$  and by continuity of  $\mu$  and the fact that  $\mu$  is positive on  $[\bar{\rho}(1 + \min(q^0)) - \alpha, \bar{\rho}(1 + \max(q^0)) + \alpha] \times [\bar{\theta}(1 + \min(\mathcal{T}^0)) - \alpha, \bar{\rho}(1 + \max(\mathcal{T}^0)) + \alpha]$ , we have:

$$0 < c_1 \leq a^n = \frac{\mu(\rho^n, \theta^n)}{1 + q^n} \leq M_1.$$

We proceed similarly to verify the bounds of the other terms.

After we use the proposition 3.6 and the fact that  $q^n$  is bounded. We get:

$$\begin{aligned}\|\nabla a^n\|_{\tilde{L}_T^2(B^{\frac{N}{2}})} &\lesssim \|q^n\|_{\tilde{L}_T^2(B^{\frac{N}{2}+1, \frac{N}{2}+2})} + \|\mathcal{T}^n\|_{\tilde{L}_T^2(B^{\frac{N}{2}+1})}, \\ \|\nabla b^n\|_{\tilde{L}_T^2(B^{\frac{N}{2}})} &\lesssim \|q^n\|_{\tilde{L}_T^2(B^{\frac{N}{2}+1, \frac{N}{2}+2})} + \|\mathcal{T}^n\|_{\tilde{L}_T^2(B^{\frac{N}{2}+1})}, \\ \|\nabla c^n\|_{\tilde{L}_T^2(B^{\frac{N}{2}})} &\lesssim \|q^n\|_{\tilde{L}_T^2(\tilde{B}^{\frac{N}{2}+1, \frac{N}{2}+2})}.\end{aligned}$$

Next we want to estimate  $\partial_t c^n$  in  $L_T^1(B^{\frac{N}{2}})$ . For that, we use the fact that:

$$\partial_t c^n = K'(\rho^n)\partial_t q^n = K'(\rho^n)(\operatorname{div}(q^{n-1}u^{n-1}) - \operatorname{div}(u^n))$$

And we have:

$$\begin{aligned}\|K'(\rho^n)(\operatorname{div}(q^{n-1}u^{n-1}) - \operatorname{div}(u^n))\|_{L_T^1(B^{\frac{N}{2}})} &\leq \|K'(\rho^n)\operatorname{div}(q^{n-1}u^{n-1})\|_{L_T^1(B^{\frac{N}{2}})} \\ &\quad + \|K'(\rho^n)\operatorname{div}(u^n)\|_{L_T^1(B^{\frac{N}{2}})}, \\ &\lesssim (1 + \|q^n\|_{L_T^\infty(B^{\frac{N}{2}})})(\|u^n\|_{L_T^1(B^{\frac{N}{2}+1})} + \|q^{n-1}u^{n-1}\|_{L_T^1(B^{\frac{N}{2}+1})}), \\ &\lesssim (1 + \|q^n\|_{L_T^\infty(B^{\frac{N}{2}})})(\|u^n\|_{L_T^1(B^{\frac{N}{2}+1})} + \|q^{n-1}\|_{L_T^\infty(B^{\frac{N}{2}})}\|u^{n-1}\|_{L_T^1(B^{\frac{N}{2}+1})} \\ &\quad + \|q^{n-1}\|_{L_T^2(B^{\frac{N}{2}+1})}\|u^{n-1}\|_{L_T^2(B^{\frac{N}{2}})}).\end{aligned}$$

Now we want to show  $(\mathcal{P}_{n+1})$  by induction and in this goal we will apply the estimates of proposition 5.11 and proposition 5.12. This is possible as we have verified above the validity of the hypotheses. We obtain:

$$\begin{aligned}\|(\bar{q}^{n+1}, \bar{u}^{n+1}, \bar{\mathcal{T}}^{n+1})\|_{F_T} (1 - C(\|a^n\|_{L_T^2(B^{\frac{N}{2}+1})} + \|b^n\|_{L_T^2(B^{\frac{N}{2}+1})} + \|c^n\|_{L_T^2(B^{\frac{N}{2}+1})} \\ + \|d^n\|_{L_T^2(B^{\frac{N}{2}})} + \|\partial_t c^n\|_{L_T^1(B^{\frac{N}{2}})})) &\lesssim \|(\nabla F_n, G_n)\|_{L_T^1(B^{\frac{N}{2}})} + \|H_n\|_{L_T^1(B^{\frac{N}{2}})}.\end{aligned}\tag{5.45}$$

We want to control now the part on the right-hand side of (5.45), for this we do like previously in using proposition 3.6. We have:

$$\|F_n\|_{L_T^1(B^{\frac{N}{2}+1})} \leq \|\operatorname{div}(q^n u^n)\|_{L_T^1(B^{\frac{N}{2}+1})} + \|\operatorname{div}(u^0)\|_{L_T^1(B^{\frac{N}{2}+1})} + \|\Delta q^0\|_{L_T^1(B^{\frac{N}{2}+1})},$$

with:

$$\begin{aligned}\|u^n q^n\|_{L_T^1(B^{\frac{N}{2}+2})} &\leq \|q^n\|_{L_T^\infty(L^\infty)}\|u^n\|_{L_T^1(B^{\frac{N}{2}+2})} + \|q^n\|_{L_T^2(B^{\frac{N}{2}+2})}\|u^n\|_{L_T^2(B^{\frac{N}{2}})} \\ &\leq \|q^n\|_{L_T^\infty(\tilde{B}^{\frac{N}{2}, \frac{N}{2}+1})}\|u^n\|_{L_T^1(B^{\frac{N}{2}+2})} + \sqrt{T}\|q^n\|_{\tilde{L}_T^2(\tilde{B}^{\frac{N}{2}+1, \frac{N}{2}+2})}\|u^n\|_{L_T^\infty(B^{\frac{N}{2}})}.\end{aligned}$$

One ends up with:

$$\|F_n\|_{L_T^1(B^{\frac{N}{2}+1})} \leq C(A_0 + \varepsilon + \sqrt{\varepsilon})^2.$$

Next we want to control the different terms of  $G_n$ . We have:

$$\begin{aligned} \|(u^n)^* \cdot \nabla u^n\|_{L_T^1(B^{\frac{N}{2}})} &\leq \sqrt{T} \|u^n\|_{L_T^\infty(B^{\frac{N}{2}})} \|u^n\|_{L_T^2(B^{\frac{N}{2}+1})}, \\ \|\nabla \left( \frac{K_{\rho^n}}{2} |\nabla \rho^n|^2 \right)\|_{L_T^1(B^{\frac{N}{2}})} &\lesssim \sqrt{T} \|q^n\|_{L_T^\infty(B^{\frac{N}{2}+1})} \|q^n\|_{L_T^2(B^{\frac{N}{2}+2})}. \end{aligned}$$

After we have:

$$\|\nabla \left( \frac{\mu(\rho^n, \theta^n)}{\rho^n} \right) \operatorname{div} u^n\|_{L_T^1(B^{\frac{N}{2}})} \leq \|u^n\|_{L_T^2(B^{\frac{N}{2}+1})} (\sqrt{T} \|q^n\|_{L_T^\infty(B^{\frac{N}{2}+1})} + \|\mathcal{T}^n\|_{L_T^\infty(B^{\frac{N}{2}+1})}).$$

We treat similarly the term:

$$\nabla \left( \frac{\zeta(\rho^n, \theta^n)}{\rho^n} \right) \operatorname{div} u^n.$$

Next we study the term:

$$\left\| \frac{\lambda'(\rho^n, \theta^n) \nabla \rho^n \operatorname{div}(u^n)}{1 + q^n} \right\|_{L_T^1(B^{\frac{N}{2}})} \lesssim (1 + \|q^n\|_{L_T^\infty(B^{\frac{N}{2}})}) \sqrt{T} \|u^n\|_{L_T^2(B^{\frac{N}{2}+1})} \|q^n\|_{L_T^\infty(B^{\frac{N}{2}+1})}.$$

We proceed similarly for the following term:

$$\frac{(du^n + \nabla u^n) \mu'(\rho^n, \theta^n) \nabla \rho^n}{1 + q^n}.$$

Next we have:

$$\begin{aligned} \left\| \frac{[P'_0(\rho^n) + \tilde{T}^n P'_1(\rho^n)] \nabla q^n}{1 + q^n} \right\|_{L_T^1(B^{\frac{N}{2}})} &\lesssim T \|q^n\|_{L_T^\infty(B^{\frac{N}{2}})} \|q^n\|_{L_T^\infty(B^{\frac{N}{2}+1})} \\ &\quad + T \|\mathcal{T}^n\|_{L_T^\infty(B^{\frac{N}{2}})} \|q^n\|_{L_T^\infty(B^{\frac{N}{2}+1})} (1 + \|q^n\|_{L_T^\infty(B^{\frac{N}{2}})}), \\ \left\| \frac{P_1(\rho^n)}{\rho^n \Psi'(\tilde{T}^n)} \nabla \theta^n \right\|_{L_T^1(B^{\frac{N}{2}})} &\lesssim (\sqrt{T} (\|q^n\|_{L_T^\infty(B^{\frac{N}{2}})} + \|\mathcal{T}^n\|_{L_T^\infty(B^{\frac{N}{2}})})) \\ &\quad + \sqrt{T} \|\mathcal{T}^n\|_{L_T^\infty(B^{\frac{N}{2}})} \|q^n\|_{L_T^\infty(B^{\frac{N}{2}})} \|\mathcal{T}^n\|_{L_T^2(B^{\frac{N}{2}})}, \\ \left\| \operatorname{div} \left( \frac{\mu(\rho^n, \theta^n)}{1 + q^n} \nabla u^0 \right) \right\|_{L_T^1(B^{\frac{N}{2}})} &\lesssim (\|q^n\|_{L_T^\infty(B^{\frac{N}{2}})} + \|\mathcal{T}^n\|_{L_T^\infty(B^{\frac{N}{2}})}) \|u^0\|_{L_T^1(B^{\frac{N}{2}+2})} \\ &\quad + \|u^0\|_{L_T^2(B^{\frac{N}{2}+1})} (\sqrt{T} \|q^n\|_{L_T^\infty(B^{\frac{N}{2}+1})} + \|\mathcal{T}^n\|_{L_T^2(B^{\frac{N}{2}+1})}). \end{aligned}$$

We proceed similarly with the other terms:

$$-\Delta u^0, \quad \nabla \left( \frac{\zeta(\rho^n, \theta^n)}{1 + q^n} \operatorname{div}(u^0) \right), \quad \nabla(\kappa(\rho^n) \Delta q^0).$$

After we want to estimate the term  $\|H^n\|_{L_T^1(B^{\frac{N}{2}})}$ . So we have:

$$\begin{aligned} \left\| \nabla \left( \frac{1}{1 + q^n} \right) \cdot \nabla \theta^n \chi(\rho^n, \theta^n) \right\|_{L_T^1(B^{\frac{N}{2}})} &\lesssim (1 + \|q^n\|_{L_T^\infty(B^{\frac{N}{2}})} + \|\mathcal{T}^n\|_{L_T^\infty(B^{\frac{N}{2}})}) \\ &\quad \times \sqrt{T} \|q^n\|_{L_T^\infty(B^{\frac{N}{2}+1})} \|\mathcal{T}^n\|_{L_T^2(B^{\frac{N}{2}+1})}. \end{aligned}$$

We have after these last terms:

$$\begin{aligned} \left\| \frac{\tilde{T}^n P_1(\rho^n)}{\rho^n} \operatorname{div} u^n \right\|_{L_T^1(B^{\frac{N}{2}})} &\lesssim \sqrt{T} \left( \|q^n\|_{L_T^\infty(B^{\frac{N}{2}})} + \|\mathcal{T}^n\|_{L_T^\infty(B^{\frac{N}{2}})} (1 + \|q^n\|_{L_T^\infty(B^{\frac{N}{2}})}) \right) \\ &\quad \times \|u^n\|_{L_T^2(B^{\frac{N}{2}+1})}. \end{aligned}$$

and:

$$\begin{aligned} \|u^n \cdot \nabla \theta^n\|_{L_T^1(B^{\frac{N}{2}})} &\leq \sqrt{T} \|u^n\|_{L_T^\infty(B^{\frac{N}{2}})} \|\mathcal{T}^n\|_{L_T^2(B^{\frac{N}{2}+1})} \\ \|\nabla u^n : \nabla u^n\|_{L_T^1(B^{\frac{N}{2}})} &\leq \|u^n\|_{L_T^2(B^{\frac{N}{2}})}^2 \end{aligned}$$

we obtain in using (5.45), the hypothesis of recurrence to the state  $n$  and the previous inequalities:

$$\|(\bar{q}_{n+1}, \bar{u}_{n+1}, \bar{\mathcal{T}}_{n+1})\|_{F_T} (1 - C2\sqrt{\varepsilon}(A_0 + \sqrt{\varepsilon})) \leq C_1(\varepsilon(A_0 + \sqrt{\varepsilon})^2 + T(A_0 + \sqrt{\varepsilon})).$$

In taking  $T$  and  $\varepsilon$  small enough we have  $(\mathcal{P}_{n+1})$ , so  $(q^n, u^n, \mathcal{T}^n)$  is bounded in  $F_T$ . To conclude we proceed like in the proof of theorem 2.4 and we show in the same way that  $(\bar{q}^n, \bar{u}^n, \bar{\mathcal{T}}^n)$  is a Cauchy sequence in  $F_T$ , hence converges to some  $(q, u, \mathcal{T})$  in  $F_T$ . We verify after that  $(\rho, u, \theta)$  is a solution of the system.

### Uniqueness:

We compare the difference between two solutions with the same initial data and we use essentially the same type of estimates than in the part on contraction. The details are left to the reader.

## 6 Appendix

This part consists in one commutator lemma which enables us to conclude in proposition 5.12. Moreover we give the proof of proposition 3.7 on the composition of function in hybrid spaces adapted from Bahouri-Chemin in [2].

**Lemma 1** *Let  $0 \leq s \leq 1$ . Suppose that  $A \in \tilde{L}_T^2(B^{\frac{N}{2}+1})$  and  $B \in \tilde{L}_T^2(B^{\frac{N}{2}-1+s})$ . Then we have the following result :*

$$\|\partial_k[A, \Delta_l]B\|_{L_T^1(L^2)} \leq C c_l 2^{-l(\frac{N}{2}-1+s)} \|A\|_{\tilde{L}_T^2(B^{\frac{N}{2}+1})} \|B\|_{\tilde{L}_T^2(B^{\frac{N}{2}-1+s})}$$

with  $\sum_{l \in \mathbb{Z}} c_l = 1$ .

### Proof:

We have the following decomposition:

$$uv = T_u v + T_v' u$$

where:  $T_u v = \sum_{l \in \mathbb{Z}} S_{l-1} u \Delta_l v$  and:  $T'_v u = \sum_{l \in \mathbb{Z}} S_{l+2} v \Delta_l u$ .

We then have:

$$\partial_k [A, \Delta_l] B = \partial_k T'_{\Delta_l B} A - \partial_k \Delta_l T'_B A + [T_A, \Delta_l] \partial_k B + T_{\partial_k A} \Delta_l B - \Delta_l T_{\partial_k A} B. \quad (6.46)$$

From now on, we will denote by  $(c_l)_{l \in \mathbb{Z}}$  a sequence such that:

$$\sum_{l \in \mathbb{Z}} c_l \leq 1.$$

Now we are going to treat each term of (6.46). According to the properties of quasi-orthogonality and the definition of  $T'$  we have:

$$\partial_k T'_{\Delta_l B} A = \sum_{m \geq l-2} \partial_k (S_{m+2} \Delta_l B \Delta_m A).$$

Next, in using Bernstein inequalities, we have:

$$\begin{aligned} \|\partial_k T'_{\Delta_l B} A\|_{L_T^1(L^2)} &\lesssim \sum_{m \geq l-2} 2^m \|\Delta_l B\|_{L_T^2(L^\infty)} \|\Delta_m A\|_{L_T^2(L^2)} \\ &\lesssim 2^{l \frac{N}{2}} \|\Delta_l B\|_{L_T^2(L^2)} \sum_{m \geq l-2} 2^{-m \frac{N}{2}} (2^{m(\frac{N}{2}+1)}) \|\Delta_m A\|_{L_T^2(L^2)} \\ &\lesssim 2^{-l(N/2-1+s)} (2^{l(\frac{N}{2}-1+s)}) \|\Delta_l B\|_{L_T^2(L^2)} \sum_{m \geq l-2} (2^{m(\frac{N}{2}+1)}) \|\Delta_m A\|_{L_T^2(L^2)} \\ &\lesssim c_l 2^{-l(N/2-1+s)} \|B\|_{\tilde{L}_T^2(B^{\frac{N}{2}-1+s})} \|A\|_{\tilde{L}_T^2(B^{\frac{N}{2}+1})}. \end{aligned}$$

Next, we will use the classic estimates on the paraproduct to bound the second term of the right-hand side of (6.46). We obtain then:

$$\|T'_B A\|_{L_T^1(B^{\frac{N}{2}+s})} \lesssim \|B\|_{L_T^2(B^{\frac{N}{2}-1+s})} \|A\|_{L_T^2(B^{\frac{N}{2}+1})}.$$

After in using the spectral localization we have:

$$\begin{aligned} \|\partial_k \Delta_l T'_B A\|_{L_T^1(L^2)} &\lesssim 2^l \|\Delta_l T'_B A\|_{L_T^1(L^2)} \\ &\lesssim c_l 2^{-l(\frac{N}{2}-1+s)} \|B\|_{\tilde{L}_T^2(B^{\frac{N}{2}-1+s})} \|A\|_{\tilde{L}_T^2(B^{\frac{N}{2}+1})}. \end{aligned}$$

According to the properties of orthogonality of Littlewood-Payley decomposition we have:

$$[T_A, \Delta_l] \partial_k B = \sum_{|m-l| \leq 4} [S_{m-1} A, \Delta_l] \Delta_m \partial_k B.$$

In applying Taylor formula, we obtain for  $x \in \mathbb{R}^N$ :

$$[S_{m-1} A, \Delta_l] \Delta_m \partial_k B(x) = 2^{-l} \int_{\mathbb{R}^N} \int_0^1 h(y) (y \cdot S_{m-1} \nabla A(x - 2^{-l} \tau y)) \Delta_m \partial_k B(x - 2^{-l} y) d\tau dy.$$



By an inequality of convolution we have:

$$\|[S_{m-1}A, \Delta_l]\Delta_m\partial_k B\|_{L^2} \lesssim 2^{-l}\|\nabla A\|_{L^\infty}\|\Delta_m\partial_k B\|_{L^2}.$$

So we get:

$$\|[T_A, \Delta_l]\partial_k B\|_{L_T^1(L^2)} \lesssim c_l 2^{-l(\frac{N}{2}-1+s)}\|\nabla A\|_{L_T^2(L^\infty)}\|B\|_{\tilde{L}_T^2(B^{\frac{N}{2}-1+s})}.$$

Finally we have:

$$T_{\partial_k A}\Delta_l B = \sum_{|l-m|\leq 4} S_{m-1}\partial_k A\Delta_l\Delta_m B.$$

Hence:

$$\|T_{\partial_k A}\Delta_l B\|_{L_T^1(L^2)} \leq \|\partial_k A\|_{L_T^2(L^\infty)}\|\Delta_l B\|_{L_T^2(L^2)}.$$

And the classic estimates on the paraproduct give:

$$\|T_{\partial_k A}\Delta_l B\|_{L_T^1(B^{\frac{N}{2}-1})} \lesssim c_l 2^{-l(\frac{N}{2}-1+s)}\|\partial_k A\|_{\tilde{L}_T^2(B^{\frac{N}{2}})}\|B\|_{\tilde{L}_T^2(B^{\frac{N}{2}-1+s})}.$$

The proof is complete.  $\square$

### Proof of proposition 3.7:

To show (i) we use “first linéarisation” method introduced by Y.Meyer in [24], which amounts to write that:

$$F(u_1, u_2, \dots, u_d) = \sum_{p \in \mathbb{Z}} (F(S_{p+1}u_1, \dots, S_{p+1}u_d) - F(S_p u_1, \dots, S_p u_d)).$$

According to Taylor formula, we have:

$$F(S_{p+1}u_1, \dots, S_{p+1}u_d) - F(S_p u_1, \dots, S_p u_d) = m_p^1 u_1^p + \dots + m_p^d u_d^p$$

with  $u_i^p = \Delta_p u_i$  and

$$m_p^i = \int_0^1 \partial_i F(S_p u_1 + s u_1^p, \dots, S_p u_i + s u_i^p, \dots, S_p u_d + s u_d^p) ds.$$

Observe that:

$$\|m_p^i\|_{L^\infty} \leq \|\nabla F\|_{L^\infty}.$$

We have:

$$\Delta_p F(u_1, u_2, \dots, u_d) = \Delta_p^1 + \Delta_p^2$$

where we have decomposed the sum into two parts:

$$\Delta_p^{(1)} = \sum_{q \geq p} \left( \Delta_p(u_1^q m_q^1) + \dots + \Delta_p(u_d^q m_q^1) \right),$$

$$\Delta_p^{(2)} = \sum_{q \leq p-1} \left( \Delta_p(u_1^q m_q^1) + \cdots + \Delta_p(u_d^q m_q^1) \right).$$

Now we bound  $\|\Delta_p^{(1)}\|_{L_T^\rho(L^p)}$  in this way:

$$\begin{aligned} \|\Delta_p^{(1)}\|_{L_T^\rho(L^p)} &\leq \sum_{q \geq p} (\|u_1^q\|_{L_T^\rho(L^p)} \|m_q^1\|_{L_T^\infty(L^\infty)} + \cdots + \|u_d^q\|_{L_T^\rho(L^p)} \|m_q^1\|_{L_T^\infty(L^\infty)}) \\ &\leq \sum_{q \geq p} \|\nabla F\|_{L^\infty} 2^{-qs} c_q (\|u_1\|_{\tilde{L}_T^\rho(B_p^s)} + \cdots + \|u_d\|_{\tilde{L}_T^\rho(B_p^s)}) \end{aligned}$$

with  $(c_q) \in l^1(\mathbb{Z})$ .

Therefore, since  $s > 0$ :

$$\sum_{p \in \mathbb{Z}} 2^{ps} \|\Delta_p^{(1)}\|_{L_T^\rho(L^p)} \leq C \|\nabla F\|_{L^\infty} (\|u_1\|_{\tilde{L}_T^\rho(B_p^s)} + \cdots + \|u_d\|_{\tilde{L}_T^\rho(B_p^s)}).$$

To bound  $\|\Delta_p^{(2)}\|_{L_T^\rho(L^p)}$  we use the fact that the support of the Fourier transform of  $\Delta_p^{(2)}$  is included in the shell  $2^p \mathcal{C}$ , so that according to Bernstein inequality:

$$\begin{aligned} \|\Delta_p^{(2)}\|_{L_T^\rho(L^p)} &\leq \sum_{q \leq p-1} (\|\Delta_p(u_1^q m_q^1)\|_{L_T^\rho(L^p)} + \cdots + \|\Delta_p(u_d^q m_q^1)\|_{L_T^\rho(L^p)}), \\ &\leq C 2^{-p([s]+1)} \sum_{q \leq p-1} (\|\partial^{[s]+1}(u_1^q m_q^1)\|_{L_T^\rho(L^p)} + \cdots + \|\partial^{[s]+1}(u_d^q m_q^1)\|_{L_T^\rho(L^p)}). \end{aligned}$$

Moreover we have according to Faà-di-Bruno formula:

$$\partial^k m_q^i = \int_0^1 \sum_{l_1 + \cdots + l_m = k, l_m \neq 0} A_{l_1 \dots l_m}^k F^{m+1}(S_q(u) + su^q) \prod_{n=1}^m \partial_{l_n}(S_q(u) + su^q) ds.$$

Hence we get for all  $k \in \mathbb{N}$ :

$$\|\partial^k m_q^i\|_{L_T^\infty(L^\infty)} \leq C_{u_i, k} 2^{qk}$$

with:  $C_{u_i, k} = C(1 + \|u_i\|_{L_T^\infty(L^\infty)})$ .

We have then:

$$\|\Delta_p^{(2)}\|_{L_T^\rho(L^p)} \leq C 2^{-p([s]+1)} \sum_{q \leq p-1} c_q 2^{q(-s+[s]+1)} C_{u_1, \dots, u_d} (\|u_1\|_{\tilde{L}_T^\rho(B_p^s)} + \cdots + \|u_d\|_{\tilde{L}_T^\rho(B_p^s)}).$$

Hence the result:

$$\sum_{p \in \mathbb{Z}} 2^{ps} \|\Delta_p^{(2)}\|_{L_T^\rho(L^p)} \leq C_{u_1, \dots, u_d} (\|u_1\|_{\tilde{L}_T^\rho(B_p^s)} + \cdots + \|u_d\|_{\tilde{L}_T^\rho(B_p^s)}).$$

So the first part of the proof is complete.

For proving (ii) we proceed in the same way as before. We get:

$$F(u) = \sum_{q \in \mathbb{Z}} m_q u_q.$$

And we have for  $p > 0$ :

$$\Delta_p F(u) = \Delta_p^1 + \Delta_p^2$$

so:

$$\|\Delta_p^1\|_{L_T^\rho(L^2)} \leq \sum_{q \geq p} \|F'\|_{L^\infty} 2^{-qs_2} c_q \|u\|_{\tilde{L}_T^\rho(\tilde{B}^{s_1, s_2})}.$$

Hence in using convolution inequality:

$$\sum_{p > 0} 2^{ps_2} \|\Delta_p^1\|_{L_T^\rho(L^2)} \leq C \|F'\|_{L^\infty} \|u\|_{\tilde{L}_T^\rho(\tilde{B}^{s_1, s_2})}.$$

After we get for all  $s > 0$ :

$$\begin{aligned} \|\Delta_p^2\|_{L_T^\rho(L^2)} &\leq C 2^{-p([s]+1)} \sum_{q \leq p-1} \|\partial^{[s]+1}(m_q u_q)\|_{L_T^\rho(L^2)}, \\ &\leq C 2^{-p([s]+1)} \sum_{q \leq p-1} 2^{q([s]+1-s(q))} c_q \|u_q\|_{\tilde{L}_T^\rho(\tilde{B}^{s_1, s_2})}. \end{aligned}$$

with  $s(q) = s_1$  or  $s_2$ .

So we obtain:

$$\begin{aligned} \sum_{p > 0} 2^{ps_2} \|\Delta_p^2\|_{L_T^\rho(L^2)} &\leq \sum_{p > 0} 2^{-p([s]+1-s_2)} \sum_{q \leq 0} c_q 2^{q([s]+1-s_1)} \|u\|_{\tilde{L}_T^\rho(\tilde{B}^{s_1, s_2})} \\ &\quad + \sum_{p > 0} 2^{-p([s]+1-s_2)} \sum_{0 < q \leq p-1} c_q 2^{q([s]+1-s_2)} \|u\|_{\tilde{L}_T^\rho(\tilde{B}^{s_1, s_2})}. \end{aligned} \tag{6.47}$$

We have to choose  $s$ , so for the first term of (6.47) we just need that:  $[s] + 1 - s_2 > 0$  and  $[s] + 1 - s_1 > 0$  and for the second term of (6.47) we just have a inequality of convolution. So we can take  $s = 1 + \max(s_1, s_2)$ .

We do the same for  $p < 0$  and we have:

$$\begin{aligned} \sum_{p \leq 0} 2^{ps_1} \|\Delta_p^1\|_{L_T^\rho(L^2)} &\leq \sum_{p \leq 0} 2^{ps_1} \sum_{q \geq p} \|F'\|_{L^\infty} 2^{-qs_1} c_q \|u\|_{\tilde{L}_T^\rho(\tilde{B}^{s_1, s_2})} \\ &\quad + \sum_{p \leq 0} 2^{ps_1} \sum_{p \leq q \leq 0} \|F'\|_{L^\infty} 2^{-qs_2} c_q \|u\|_{\tilde{L}_T^\rho(\tilde{B}^{s_1, s_2})}. \end{aligned}$$

We conclude by a inequality of convolution.

And for the term  $\Delta_p^2$  we get:

$$\sum_{p \leq 0} 2^{ps_1} \|\Delta_p^2\|_{\tilde{L}^\rho(L^2)} \leq \sum_{p \leq 0} 2^{-p([s]+1-s_1)} \sum_{q \leq p-1} c_q 2^{q([s]+1-s_1)} \|u\|_{\tilde{L}_T^\rho(\tilde{B}^{s_1, s_2})}.$$

For proving (iii) and (iv), one just has to use the following identity:

$$G(v) - G(u) = (v - u) \int_0^1 H(u + \tau(v - u)) d\tau + G'(0)(v - u)$$

where  $H(w) = G'(w) - G'(0)$ , and we conclude by using (i), (ii) and proposition 3.6.  $\square$

## 7 Annex: Notations of differential calculus

If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , we denote:

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i = \partial_i f dx_i$$

with the summation convention on repeated indices and the simplified notation:

$$\partial_i f = \frac{\partial f}{\partial x_i}.$$

The vector field associated to the differential  $df$  is noted  $\nabla f$ ,

$$\nabla f = \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}.$$

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ . Let denote  $f_i$  the  $i$  th component of  $f$ , and:

$$(df)_{i,j} = \partial_j f_i.$$

By analogy with the case of the scalar, we denote:

$$\nabla f = (df)^*, \text{ so } (\nabla f)_{i,j} = \partial_i f_j.$$

The curl of  $f$  is given by:

$$(\text{curl} f)_{i,j} = \partial_i f_j - \partial_j f_i.$$

The divergence of the vector field  $f$  is given by:

$$\text{div} f = \text{tr} df = \partial_i f_i.$$

If  $A : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ , with coefficients  $a_{i,j}$ , we set:

$$(\text{div} A)_j = \partial_i a_{i,j}, \text{ div} A = \text{div} \left( A \frac{\partial}{\partial x_j} \right) dx_j.$$

In particular, for  $f$  scalar, we have:

$$\text{div}(fI) = df.$$

And finally we set:

$$A : B = a_{i,j} b_{i,j}.$$

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