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► **To cite this version:**

Valeriu Munteanu, Daniela Tarniceriu, Gheorghe Zaharia. Entropy and Average Cost of AUH Codes. Applied Mathematics & Information Sciences, 2013, 1 (1), pp.1-7. hal-00776585

HAL Id: hal-00776585

<https://hal.science/hal-00776585>

Submitted on 15 Jan 2013

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Paper title: ENTROPY AND AVERAGE COST OF AUH CODES

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ENTROPY AND AVERAGE COST OF AUH CODES

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Abstract— In this paper we address the class of anti-uniform Huffman (AUH) codes, named also unary codes, for sources with finite and infinite alphabet, respectively. Geometric, quasi-geometric, Fibonacci, exponential, Poisson, and negative binomial distributions lead to anti – uniform sources for some ranges of their parameters. Huffman coding of these sources results in AUH codes. We prove that as result of this encoding, in general, sources with memory are obtained. For these sources we attach the graph and determine the transition matrix between states, the state probabilities and the entropy. If c_0 and c_1 denote the costs for storing or transmission of symbols “0” and “1”, respectively, we compute the average cost for these AUH codes.

Keywords: Huffman coding, average codeword length, average cost, entropy.

1. Introduction

Let (p_1, p_2, \dots, p_n) be the probability distribution of a n messages source $\xi_n = \{s_1, s_2, \dots, s_n\}$. It is well known that the Huffman encoding algorithm [1] produces an optimal binary prefix-free code for ξ_n . A binary Huffman code is usually represented by a binary tree, whose leaves correspond to the source messages. The two edges emanating from each intermediate tree node (father) are labeled either 0 or 1. For related literature on Huffman coding and Huffman

trees, we refer the reader to [2]–[6]. The length between the root and a leaf is the length of the binary codeword associated with the corresponding message.

Assuming that $v_i, i = 1, 2, \dots, n$, is the codeword representing the message s_i , we denote the length of v_i by l_i . The optimality of Huffman coding implies that $l_i \leq l_j$, if $p_i > p_j$.

Anti uniform Huffman (AUH) codes were firstly introduced in [7]. A Huffman code representing a finite source ξ_n satisfying $p_1 \geq p_2 \geq \dots \geq p_n > 0$ is an anti-uniform code, if $l_i = i, i = 1, 2, \dots, n-1$ and $l_n = l_{n-1}$. A source ξ_n having an anti-uniform Huffman code is called an anti-uniform source. These sources were extensively analysed, concerning bounds on average codeword length, entropy and redundancy for different types of probability distribution [7]-[9]. The AUH sources appear in a wide variety of situations in the real world, because this class of sources have the property of achieving minimum redundancy in different situations and minimal average cost in highly unbalanced cost regime [10]-[12]. These properties determine a wide range of applications and motivate us to study these sources from an information theoretic perspective. One example is the telegraph channel with the alphabet $\{., -\}$ in which dashes are twice as long as dots [13]. Another is the $\{a, b\}$ run – length – limited codes used in magnetic and optical storage, in which the binary codewords are constrained so that each 1 must be preceded by at least a , and at most b , 0's [14]. The binary Huffman codes, constrained so all codewords must end in a 1, are used for group testing and self-synchronizing codes [15, 16]. As another example, binary codes whose codewords contain at most a specified number of 1's are used for energy minimization of transmissions in mobile environments [17]. AUH sources can be generated by several probability distributions. It has been shown that geometric, quasi-geometric, Fibonacci, exponential, Poisson and negative binomial distributions lie in the class of AUH sources for some regimes of their parameters [7], [18], [19], [20]. Related topic was addressed in [21], where the authors studied weakly super increasing (WSI) and partial WSI sources in connection with Fibonacci numbers and golden mean, which appeared extensively in modern science and, in particular, have applications in coding and information theory.

The rest of the paper is organized as follows. In Section 2 we present the Huffman encoding of an anti-uniform source and the graph of the source with memory resulting by binary Huffman encoding of AUH sources. We show that, in general, employing Huffman coding, a source with memory results. The entropy and the average cost of the code are also derived. In Sections 3 we compute the code entropy, as well as the average cost for AUH codes corresponding to sources with geometric, quasi-geometric, Fibonacci, exponential, Poisson and negative binomial distributions, respectively. Finally, we conclude the paper in Section 4.

2. Entropy and average cost of AUH codes

Let us consider a discrete and memoryless source, characterized by the distribution:

$$\xi_n : \begin{pmatrix} s_1 & s_2 & \cdots & s_n \\ p_1 & p_2 & \cdots & p_n \end{pmatrix}, \quad (1)$$

$$p_1 \geq p_2 \geq \dots \geq p_n \quad (2)$$

$$\sum_{i=1}^n p_i = 1 \quad (3)$$

If [7]

$$\sum_{k \geq i+2}^n p_k \leq p_i, 1 \leq i \leq n-3, \quad (4)$$

the source becomes anti-uniform.

After a binary Huffman encoding of the source with the distribution in (1) that fulfils (4), the graph in Fig. 1 is obtained.

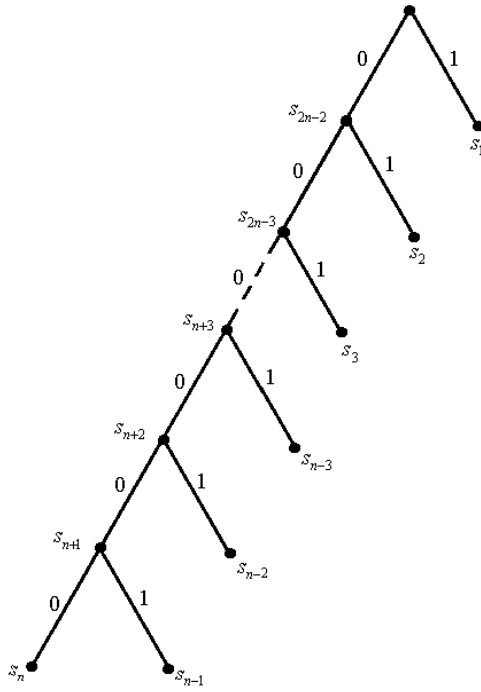


Fig. 1. The graph of binary Huffman encoding for the source ξ_n with distribution in (1)

The structure of codewords resulting from binary Huffman encoding is:

$$\begin{aligned}
 s_0 &\rightarrow v_0 \rightarrow 1 \\
 s_1 &\rightarrow v_1 \rightarrow 01 \\
 s_2 &\rightarrow v_2 \rightarrow 001 \\
 &\dots\dots\dots \\
 s_{n-1} &\rightarrow v_{n-1} \rightarrow \underbrace{00\dots01}_{n-2} \\
 s_n &\rightarrow v_n \rightarrow \underbrace{00\dots00}_{n-1}
 \end{aligned}$$

The length l_i of the codeword associated with the message s_i is the number of edges on the path between the root and the node s_i in the Huffman tree.

$$l_i = i, \quad i = 1, 2, \dots, n-1 \tag{5}$$

$$l_n = l_{n-1} \tag{6}$$

The average codeword length is determined with

$$\bar{l}_n = \sum_{i=1}^n p_i l_i = \sum_{i=1}^{n-1} i p_i + (n-1) p_n. \tag{7}$$

In Fig. 1 $s_{n+i}, i = 1, 2, \dots, n - 2$ denote the intermediate nodes in the graph, also called parents.

The probability of a parent is obtained as the sum of the two sibling probabilities. Denoting by p_{n+i} the probabilities of intermediate nodes, we have

$$p_{n+i} = \sum_{k=n-i}^n p_k; i = 1, 2, \dots, n - 2 \quad (8)$$

For a sequence of messages s_i the source delivers a string of binary symbols from the code alphabet $X = \{x_0 = 0, x_1 = 1\}$. As the probabilities of symbols in the binary string depend on the node from which they are generated, the set X , which is the output bitstream obtained as result of binary Huffman coding is a source with memory.

When a terminal node $s_i, i = 1, 2, \dots, n$ is reached, the source ξ_n will deliver another message and the source with memory X will generate another string of binary symbols.

The graph attached to the source with memory X can be obtained from the Huffman encoding graph in Fig. 1, as follows:

- a) We link the terminal nodes with the graph root;
- b) The branches between successive nodes have the probabilities equal to the ratio between the probability of the node in which the branch ends and the probability of the node from which it starts, excepting the branches linking the terminal nodes with graph root, whose probabilities are equal to unity;
- c) Each terminal node $s_i, i = 1, 2, \dots, n$, or intermediate ones $s_{n+i}, i = 1, 2, \dots, n - 2$, (excepting the root of the encoding graph) will represent the states $S_i, i = 1, 2, \dots, 2n - 2$, of the source with memory X . The set of these states is denoted by $S = \{S_1, S_2, \dots, S_{2n-2}\}$. The graph of the source X is shown in Fig. 2.

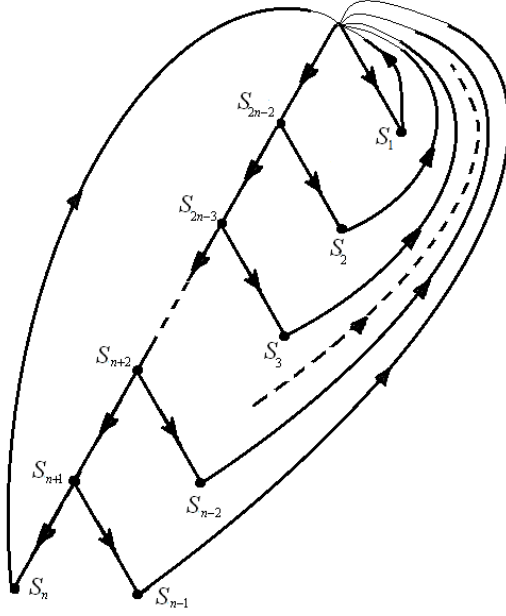


Fig. 2 The graph of the source with memory X

The transition probabilities from the state S_i in the state S_j , that is, $p(S_j | S_i)$, is equal to the probability of the branch between the node S_i and the node S_j . When the source enters the state S_j from the state S_i , it generates either the symbol $x_0 = 0$, or $x_1 = 1$. Therefore, the probability of delivering the symbol $x_j, j = 0, 1$, from the state S_i is the same as the probability to reach the state S_j , starting from the state S_i , that is:

$$p(x_1 | S_i) = p(S_1 | S_i) = p_1, i = 1, 2, \dots, n \quad (9)$$

$$p(x_0 | S_i) = p(S_{2n-2} | S_i) = \sum_{k=2}^n p_k, i = 1, 2, \dots, n \quad (10)$$

$$p(x_1 | S_{n+i}) = p(S_{n-i} | S_{n+i}) = \frac{p_{n-i}}{\sum_{k=n-i}^n p_k}, i = 1, 2, \dots, n-2 \quad (11)$$

$$p(x_0 | S_{n+i}) = p(S_{n+i-1} | S_{n+i}) = \frac{\sum_{k=n-i+1}^n p_k}{\sum_{k=n-i}^n p_k}, i = 1, 2, \dots, n-2 \quad (12)$$

Considering (9), (10), (11) and (12) as well as the graph in Fig. 2, the transition matrix between states is:

$$\mathbf{T} = \begin{array}{c} \begin{array}{cccccccccccc} S_1 & S_2 & \cdots & S_{n-2} & S_{n-1} & S_n & S_{n+1} & S_{n+2} & \cdots & S_{2n-3} & S_{2n-2} \\ \left[\begin{array}{cccccccccccc} p_1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \sum_{k=2}^n p_k \\ p_1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \sum_{k=2}^n p_k \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \sum_{k=2}^n p_k \\ 0 & 0 & \cdots & 0 & \frac{p_{n-1}}{\sum_{k=n-1}^n p_k} & \frac{\sum_{k=n}^n p_k}{\sum_{k=n-1}^n p_k} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \frac{p_{n-2}}{\sum_{k=n-2}^n p_k} & 0 & 0 & \frac{\sum_{k=n-1}^n p_k}{\sum_{k=n-2}^n p_k} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \frac{p_2}{\sum_{k=2}^n p_k} & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & \frac{\sum_{k=3}^n p_k}{\sum_{k=2}^n p_k} & 0 \end{array} \right] \begin{array}{l} S_1 \\ S_2 \\ \vdots \\ S_n \\ S_{n+1} \\ S_{n+2} \\ \vdots \\ S_{2n-2} \end{array} \end{array} \end{array} \quad (13)$$

where the entries of the matrix \mathbf{T} are $t_{ij}=p(S_j|S_i)$.

Let $\pi_i, i = 1, 2, \dots, 2n-2$, denote the stationary state probabilities of the source with memory. They can be determined by means of [22]

$$[\pi_1, \pi_2, \dots, \pi_{2n-2}] = [\pi_1, \pi_2, \dots, \pi_{2n-2}] \mathbf{T} \quad (14)$$

$$\sum_{i=1}^{2n-2} \pi_i = 1 \quad (15)$$

Considering (7) and (13), from (14) and (15) we obtain the state probabilities as:

$$\pi_i = \frac{p_i}{l_n}, i = 1, 2, \dots, n \quad (16)$$

$$\pi_{n+i} = \frac{1}{l_n} \sum_{k=n-i}^n p_k, i = 1, 2, \dots, n-2 \quad (17)$$

Generally, the entropy of the source with memory is computed by [23]

$$H(X) = - \sum_{i=1}^{2n-2} \sum_{j=1}^2 \pi_i p(x_j | S_i) \log p(x_j | S_i) \quad (18)$$

where π_i are given in (16) and (17) and $p(x_j | S_i)$ are given in (9), (10), (11) and (12).

Let c_0 and c_1 be the costs of storing or transmission of symbols “0” and “1”, respectively, resulted after the binary Huffman encoding of source ξ_n . The average cost is determined by [9]

$$\bar{C} = \sum_{i=1}^n p_i [n_0(i)c_0 + n_1(i)c_1], \quad (19)$$

where $n_0(i)$ and $n_1(i)$ denote the number of 0's and 1's in the codeword c_i .

Considering the structure of codewords for AUH sources, (5) and (6), the average cost is

$$\bar{C} = \sum_{i=1}^n p_i [(i-1)c_0 + c_1] + (n-1)c_0 p_n \quad (20)$$

3. Case studies for distributions leading to AUH sources

For all distributions we considered, we are interested to found out the condition imposed to the source parameter so that it is AUH, to derive the code entropy and the average cost.

3.1. AUH sources with geometric distribution

Let there be a discrete source characterized by the geometric distribution:

$$\xi_n : \left(\begin{array}{cccccc} s_1 & s_2 & s_3 & \cdots & s_n \\ p_1 = 1-p & p_2 = p(1-p) & p_3 = p^2(1-p) & \cdots & p_n = p^{n-1}(1-p) \end{array} \right), \quad (21)$$

In this case the source is not complete, because

$$\sum_{i=1}^n p_i = 1 - p^n. \quad (22)$$

To form a complete source, we normalize the probabilities p_i in (21), resulting the complete source with distribution:

$$\xi_n^{norm} : \left(\begin{array}{cccccc} s_1 & s_2 & s_3 & \cdots & s_n \\ p_1 = \frac{1-p}{1-p^n} & p_2 = \frac{p(1-p)}{1-p^n} & p_3 = \frac{p^2(1-p)}{1-p^n} & \cdots & p_n = \frac{p^{n-1}(1-p)}{1-p^n} \end{array} \right) \quad (23)$$

For this source to become AUH, relation (4) is required to be fulfilled. Replacing the probabilities from (23) in (4), we have [18]:

$$0 < p \leq \frac{\sqrt{5}-1}{2} \quad (24)$$

The average codeword length results by replacing the probabilities p_i from (23) into (7).

$$\overline{l}_n^{norm} = \frac{1}{(1-p)(1-p^n)} \left[1 - p^{n-1} - (n-1)p^n + (n-1)p^{n+1} \right] \quad (25)$$

Theorem

The entropy and the average cost of the source with memory resulted by binary encoding of the AUH source with Poisson distribution are determined by:

$$H_n(X) = -\frac{1}{\overline{l}_n^{norm}} \left[\log(1-p^n) - \log(1-p) - \frac{1-np^{n-1}+(n-1)p^n}{(1-p^n)(1-p)} p \log p \right] \quad (26)$$

$$\overline{C}_n = \frac{1}{1-p^n} \left[(1-p^{n-1})c_0 + \frac{1-np^{n-1}+(n-1)p^n}{1-p} pc_1 \right] \quad (27)$$

Proof

The stationary state probabilities are obtained by replacing the probabilities p_i from (23) into (16) and (17), and considering (25):

$$\left. \begin{aligned} \pi_i &= \frac{p^{i-1}(1-p)}{(1-p^n)\overline{l}_n^{norm}}, i=1,2,\dots,n \\ \pi_{n+i} &= \frac{p^{n-i-1}(1-p^{i+1})}{(1-p^n)\overline{l}_n^{norm}}, i=1,2,\dots,n-2 \end{aligned} \right\} \quad (28)$$

The probabilities to deliver the symbol $x_0 = 0$ and $x_1 = 1$, from the states $S_i, i=1,2,\dots,2n-2$, result by replacing the probabilities p_i from (23) into (9) – (12):

$$\left. \begin{aligned} p(x_1 | S_i) &= \frac{1-p}{1-p^n}, i=1,2,\dots,n \\ p(x_0 | S_i) &= \frac{p(1-p^{n-1})}{1-p^n}, i=1,2,\dots,n \\ p(x_1 | S_{n+i}) &= \frac{1-p}{1-p^{i+1}}, i=1,2,\dots,n-2 \\ p(x_0 | S_{n+i}) &= \frac{p(1-p^i)}{1-p^{i+1}}, i=1,2,\dots,n-2 \end{aligned} \right\} \quad (29)$$

Substituting (28) and (29) into (18), the relation (26) results.

Substituting (23) into (20), the relation (27) results.

When the number of messages, n , of the source ξ_n increases, at limit, when $n \rightarrow \infty$, we have

$$\bar{l}_\infty = \frac{1}{1-p} \quad (30)$$

$$H_\infty(X) = -p \log p - (1-p) \log(1-p) \quad (31)$$

$$\bar{C}_\infty = c_0 + \frac{p}{1-p} c_1 \quad (32)$$

3.2. AUH sources with quasi-geometric distribution

Let there be a discrete source characterized by the quasi-geometric distribution:

$$\xi_n : \left(\begin{array}{cccccc} s_1 & s_2 & s_3 & \cdots & s_{n-1} & s_n \\ p_1 = 1-p & p_2 = \frac{p}{2} & p_3 = \frac{p}{2^2} & \cdots & p_{n-1} = \frac{p}{2^{n-2}} & p_n = \frac{p}{2^{n-2}} \end{array} \right), \quad (33)$$

Following similar procedures as in the previous case, we get:

- The range for the source parameter for the source to be AUH

$$0 < p \leq \frac{2}{3}. \quad (34)$$

- The average codeword length

$$\bar{l}_n = 1 + \frac{2^{n-2} - 1}{2^{n-3}} p \quad (35)$$

- The entropy of the source with memory X

$$H_n(X) = -\frac{(1-p)^{n+1} \log(1-p) + [1 - np^{n-1} + (n-1)p^n] p \log p}{1 - p^{n-1} - (n-1)p^n + (n-1)p^{n+1}} \quad (36)$$

- The average cost of the source with memory X

$$\bar{C}_n = \frac{(2^{n-1} - 1)pc_1 + (2^{n-2} - p)c_0}{2^{n-2}} \quad (37)$$

When $n \rightarrow \infty$, we have

$$H_\infty(X) = -\frac{1}{1+2p} [(1-p) \log(1-p) + p \log p - 2p] \quad (38)$$

$$\bar{C}_\infty = 2pc_1 + c_0 \quad (39)$$

In the special case, when $p = \frac{1}{2}$, the source ξ_n becomes dyadic one. In this case X becomes

memoryless and then:

$$p(x_j | S_i) = \frac{1}{2}, \quad j = 0, 1; i = 1, 2, \dots, 2n - 2, \quad (40)$$

$$\bar{l}_{dn} = 2 - \frac{1}{2^{n-2}} \quad (41)$$

$$H_{dn}(X) = 1 \quad (42)$$

$$\bar{C}_{dn} = \frac{(2^{n-1} - 1)(C_1 + C_0)}{2^{n-1}} \quad (43)$$

Imposing $n \rightarrow \infty$ in (41) and (43), we have

$$\bar{l}_{d\infty} = 2 \quad (44)$$

$$\bar{C}_{d\infty} = c_0 + c_1 \quad (45)$$

3.3. AUH sources with Fibonacci distribution

Let there be a discrete and finite AUH source characterized by the Fibonacci distribution

$$\xi_n : \left(\begin{array}{cccccc} s_1 & s_2 & \cdots & s_{n-2} & s_{n-1} & s_n \\ p_1 = \frac{f_{n-1}}{f_{n+1}} & p_2 = \frac{f_{n-2}}{f_{n+1}} & \cdots & p_{n-2} = \frac{f_2}{f_{n+1}} & p_{n-1} = \frac{f_1}{f_{n+1}} & p_n = \frac{f_1}{f_{n+1}} \end{array} \right) \quad (46)$$

where f_n is the n^{th} Fibonacci number defined as

$$\left. \begin{array}{l} f_1 = f_2 = 1 \\ f_n = f_{n-1} + f_{n-2}, n \geq 3 \end{array} \right\} \quad (47)$$

The source ξ_n is also AUH, because relation (4) is fulfilled.

- The average codeword length

$$\bar{l}_n = \frac{f_{n+3} - 3}{f_{n+1}} \quad (48)$$

- The entropy of the source with memory X

$$H_n(X) = \frac{f_{n+1}}{f_{n+3}-3} \log f_{n+1} - \frac{1}{f_{n+3}-3} \sum_{i=1}^{n-1} f_i \log f_i \quad (49)$$

- The average cost of the source with memory X

$$\bar{C}_n = \frac{f_{n+1} - f_1}{f_{n+1}} c_0 + \frac{f_{n+2}}{f_{n+1}} c_1 \quad (50)$$

3.4. AUH sources with exponential distribution

The density probability function of exponential distribution is

$$f(x, \lambda) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases} \quad (51)$$

We define the cumulative density function

$$F(i) = 1 - e^{-\lambda i}, \quad (52)$$

and the probabilities

$$p_i = F(i) - F(i-1) = e^{-(i-1)\lambda} (1 - e^{-\lambda}). \quad (53)$$

Let us consider the source ξ_n with exponential distribution

$$\xi_n : \left(\begin{array}{ccccccc} s_1 & s_2 & \cdots & s_{n-1} & s_n & & \\ p_1 = 1 - e^{-\lambda} & p_2 = e^{-\lambda}(1 - e^{-\lambda}) & \cdots & p_{n-1} = e^{-(n-2)\lambda}(1 - e^{-\lambda}) & p_{n-1} = e^{-(n-1)\lambda}(1 - e^{-\lambda}) & & \end{array} \right) \quad (54)$$

- The range for the source parameter for the source to be AUH [8]

$$\lambda \geq \ln \left(\frac{\sqrt{5} + 1}{2} \right) \quad (55)$$

- The average codeword length

$$\bar{l}_n^{norm} = \frac{1}{(1 - e^{-\lambda})(1 - e^{-n\lambda})} \left[1 - e^{-(n-1)\lambda} - (n-1)e^{-n\lambda} + (n-1)e^{-(n+1)\lambda} \right] \quad (56)$$

- The entropy of the source with memory X

$$H_n(X) = -\frac{1}{\bar{l}_n^{norm}} \left[\log(1 - e^{-n\lambda}) - \log(1 - e^{-\lambda}) + \frac{1 - ne^{-(n-1)\lambda} + (n-1)e^{-n\lambda}}{(1 - e^{-n\lambda})(1 - e^{-\lambda})} \lambda e^{-\lambda} \log e \right] \quad (57)$$

- The average cost of the source with memory X

$$\bar{C}_n = \frac{1}{1 - e^{-n\lambda}} \left[(1 - e^{-(n-1)\lambda})C_0 + \frac{1 - ne^{-(n-1)\lambda} + (n-1)e^{-n\lambda}}{1 - e^{-\lambda}} e^{-\lambda} C_1 \right] \quad (58)$$

When $n \rightarrow \infty$, we have

$$H_\infty(X) = -(1 - e^{-\lambda}) \left(\log(1 - e^{-\lambda}) + \frac{e^{-\lambda} \log e^{-\lambda}}{1 - e^{-\lambda}} \right) \quad (59)$$

$$\bar{C}_\infty = c_0 \frac{e^{-\lambda}}{1 - e^{-\lambda}} + c_1 \quad (60)$$

$$\bar{l}_\infty = \sum_{k=0}^{\infty} (k+1)e^{-k\lambda}(1 - e^{-\lambda}) = \frac{1}{1 - e^{-\lambda}}. \quad (61)$$

3.5 AUH sources with Poisson distribution

Let there be a discrete source with infinite alphabet, characterized by Poisson distribution:

$$\xi_n : \left(\begin{array}{cccccc} s_0 & s_1 & s_2 & \cdots & s_n \\ p_0 = e^{-\lambda} & p_1 = e^{-\lambda} \frac{\lambda}{1!} & p_2 = e^{-\lambda} \frac{\lambda^2}{2!} & \cdots & p_n = e^{-\lambda} \frac{\lambda^n}{n!} \end{array} \right), \quad (62)$$

- The range of the parameter for the source to be AUH [18]

$$\lambda \leq 1 \quad (63)$$

- The average codeword length

$$\bar{l}_\infty = \lambda + 1. \quad (64)$$

The entropy of the source with memory X

$$H_\infty(X) = \frac{1}{\lambda + 1} \left(\lambda \log e - \lambda \log \lambda + e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \log(n!) \right) \quad (65)$$

The average cost of the source with memory X

$$\bar{C}_\infty = c_0 \lambda + c_1 \quad (66)$$

3.6. AUH sources with negative binomial distribution

Let there be a discrete source with infinite alphabet, characterized by the negative binomial distribution:

$$\xi_n : \left(\begin{array}{ccccccc} s_0 & s_1 & s_2 & \cdots & s_n & & \\ p_0 = C_{r-1}^{r-1} p^r & p_1 = C_r^{r-1} p^r q & p_2 = C_{r+1}^{r-1} p^r q^2 & \cdots & p_n = C_{r+n-1}^{r-1} p^r q^n & & \end{array} \right), \quad (67)$$

where $q=1-p$.

- The average codeword length

$$\bar{l}_\infty = \sum_{n=0}^{\infty} (n+1) C_{r+n-1}^{r-1} p^r q^n = \frac{rq+p}{p}. \quad (68)$$

The entropy of the source with memory X

$$H_\infty(X) = -\frac{p}{rq+p} \left(r \log p + \frac{qr}{p} \log q + p^r \sum_{n=1}^{\infty} C_{r+n-1}^{r-1} q^n \log C_{r+n-1}^{r-1} \right) \quad (69)$$

The average cost of the source with memory X

$$\bar{C}_\infty = c_0 \frac{qr}{p} + c_1 \quad (70)$$

7. Conclusions

In this paper we have considered the class of AUH sources with finite and infinite alphabets. Performing a binary Huffman encoding of these sources, we show that sources with memory result. For these sources we build the encoding graph and the graph of the source with memory X . The graph of the source with memory is obtained from the encoding graph by linking the terminal nodes with the graph root. The states of the source with memory correspond to the terminal or intermediate nodes in the encoding graph, excepting the root. We determined in the general case the state probabilities of the source with memory, as well as the transition probabilities between states. The entropy of this source with memory is computed. We assumed the costs c_0 and c_1 for the symbols “0” and “1”, respectively, and compute the average cost for these Huffman codes. Obviously, the Huffman encoding procedure assures minimum average length, but the average cost is not minimum. It can be easily verified that if the costs of symbols “0” and “1” are equal to unity,

the average cost becomes equal to the average length. We applied the results for several AUH sources with geometric, quasi-geometric, Fibonacci, exponential, Poisson and negative binomial distributions. We have also analyzed the limit cases, when the source alphabet increases unlimited, that is, $n \rightarrow \infty$.

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