Small time global null controllability for a viscous Burgers' equation despite the presence of a boundary layer

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Abstract

In this work, we are interested in the small time global null controllability for the viscous Burgers' equation $y_t - y_{xx} + yy_x = u(t)$ on the line segment $[0,1]$. The right-hand side is a scalar control playing a role similar to that of a pressure. We set $y(t,1) = 0$ and restrict ourselves to using only two controls (namely the interior one $u(t)$ and the boundary one $y(t,0)$). In this setting, we show that small time global null controllability still holds by taking advantage of both hyperbolic and parabolic behaviors of our system. We use the Cole-Hopf transform and Fourier series to derive precise estimates for the creation and the dissipation of a boundary layer.

1 Introduction

1.1 Description of the system and our main result

Let $T > 0$ be a positive time, possibly small. We consider the line segment $x \in [0,1]$ and the following one-dimensional viscous Burgers' controlled system:

$$\left\{\begin{array}{rcll} y_t + yy_x - y_{xx} & = u(t) & \text{in} & (0,T)\times(0,1), \\ y(t,0) & = v(t) & \text{in} & (0,T), \\ y(t,1) & = 0 & \text{in} & (0,T), \\ y(0,x) & = y_0(x) & \text{in} & (0,1). \end{array}\right. \tag{1}$$

The scalar controls are $u \in L^2(0,T)$ and $v \in H^{1/4}(0,T)$. The right-hand side control term $u(\cdot)$ plays a role somewhat similar to that of a pressure for multi-dimensional fluid systems. Unlike some other studies, our control term $u(\cdot)$ depends only on time and not on the space variable.

For any initial data $y_0 \in L^2(0,1)$ and any fixed controls in the appropriate spaces, it can be shown that system (1) has a unique solution in the space $X = L^2((0,T);H^1(0,1)) \cap \mathcal{C}^0([0,T];L^2(0,1))$. This type of existence result relies on standard *a priori* estimates and the use of a fixed point theorem. Such techniques are described in [35]. One can also use a semi-group method as in [37]. Our main result is the following small time global null controllability theorem for system (1):

Theorem 1. *Let $T > 0$ be any positive time and y_0 by any initial data in $L^2(0,1)$. Then there exists a control pair $u \in L^\infty(0,T)$ and $v \in H^{1/4}(0,T)$ such that the solution $y \in X$ to system (1) is null at time T: $y(T,\cdot) \equiv 0$.*

1.2 An open-problem for Navier-Stokes as a motivation

As a motivation for our study, let us introduce the following challenging open problem. Take some smooth connected bounded domain Ω in $\mathbb{R}^2$ or $\mathbb{R}^3$. Consider some open part Γ of its boundary $\partial\Omega$. This is the part of the boundary on which our control will act. We consider the following Navier-Stokes system:

$$\left\{\begin{array}{rcll} y_t - \Delta y + (y\cdot\nabla)y & = -\nabla p & \text{in} & (0,T)\times\Omega, \\ \operatorname{div} y & = 0 & \text{in} & (0,T)\times\Omega, \\ y & = 0 & \text{on} & (0,T)\times(\partial\Omega\setminus\Gamma), \\ y(0,\cdot) & = y_0(\cdot) & \text{in} & \Omega. \end{array}\right. \tag{2}$$

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We consider this system as an underdetermined system. Our control will be some appropriate trace of a solution on the controlled boundary Γ.

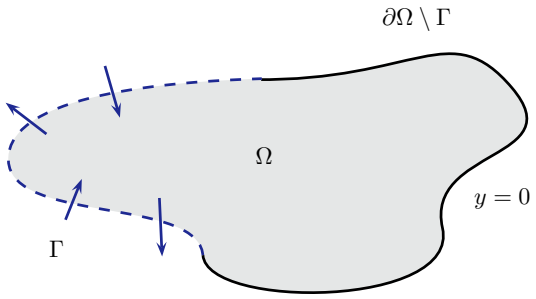

Figure 1: Setting of the Navier-Stokes control problem (2).

Open problem 1. *Is system (2) small time globally null controllable? That is to say, for any $T > 0$ and y_0 in some appropriate space, does there exist a trajectory of system (2) such that $y(T, \cdot) \equiv 0$?*

Many works have be done in this direction. Generally speaking, one can distinguish two approaches. First, one can think of the nonlinear term as a perturbation term and obtain the controllability by means of the Laplacian term. For instance, Fabre uses in [20] a truncation method for the Navier-Stokes equation. In [36], Lions and Zuazua use Galerkin approximations for various fluid systems. Of course, this approach is very efficient for local results. The most recent result concerning local controllability for system (2) is the one contained in [23] by Fernández-Cara, Guerrero, Imanuvilov and Puel. Their proof uses Carleman estimates.

The other approach goes the other way around. Indeed, in finite dimension, it is known that if $\dot{y} = F(y) + Bu$ where F is quadratic is controllable, then $\dot{y} = F(y) + Ay + Bu$ is controllable too (see [15, Theorem 3.8]). Likewise, for fluid systems, trying to get a small time controllability result implies to work at high Reynolds number (ie. with big fluid velocities, or low viscosity) inside the domain. Therefore, inertial forces prevail and the fluid system behaves like its null viscosity hyperbolic limit system. In our case, we expect to deduce results for Navier-Stokes from the Euler sytem. For Euler, global controllability has been shown in [10] by Coron for the 2D case (see also [12]) and by Glass for the 3D case in [27]. Their proofs rely on the return method introduced by Coron in [9] (see also [13, Chapter 6]). For Navier-Stokes, things get harder. In [16], Coron and Fursikov show a global controllability result in the case of a 2D manifold without boundary. In [24], Fursikov and Imanuvilov show a global exact controllability result for 3D Navier-Stokes with a control acting on the whole boundary (ie. $\Gamma = \partial\Omega$).

Other approaches exist. Let us mention for instance the work [2], where Agrachev and Sarychev control Navier-Stokes equations by means of low modes. They use methods of differential geometric / Lie algebraic control theory for finite dimensional control systems.

The main difficulty of Open problem 1 is the behavior of the system near $\partial\Omega \setminus \Gamma$. Indeed, although inertial forces prevail inside the domain, viscous forces play a crucial role near the uncontrolled boundary, and give rise to a boundary layer. An example of such a phenomenon can be found in [11] where Coron derives an approximate controllability result and highlights the creation of a boundary residue. Hence, the key question is whether one can handle such a boundary layer by means of the control.

Some authors have tried to study simplified geometries for Open problem 1. In [7], Chapouly studies a Navier-Stokes equation on a rectangle with Navier-slip boundary conditions on the uncontrolled part of the boundary. She obtains small time global null controllability. In [29] and [30], Guerrero, Imanuvilov and Puel prove approximate controllability for a Navier-Stokes system in a square (resp. in a cube) where one side (resp. one face) is not controlled and has zero Dirichlet boundary condition.

Burgers' equation has been extensively used as a toy model to investigate properties of more complex systems in a rather simple setting. This equation was introduced in the seminal paper [5] by Burgers. Both from a theoretical and a numerical point of view, it already exhibits some key behaviors (such as interaction between the non-linearity and the smoothing effect). Therefore, our Theorem 1 can be seen as an example for small time global null controllability despite the presence of a

Dirichlet boundary layer. Moreover, despite the simplicity of Burgers' equation, the analogy between systems (1) and (2) is quite striking. We can interpret our scalar control $u(t)$ as some one-dimensional counterpart of a pressure gradient for 2D or 3D.

1.3 Previous works concerning Burgers' controllability

Concerning the controllability of the inviscid Burgers' equation, some works have be carried out. In [3], Ancona and Marson describe the set of attainable states in a pointwise way for the Burgers' equation on the half-line $x \geq 0$ with only one boundary control at $x = 0$. In [32], Horsin describes the set of attainable states for a Burgers' equation on a line segment with two boundary controls. Thorough studies are also carried out in [1] by Adimurthi et al. In [38], Perrollaz studies the controllability of the inviscid Burgers' equation in the context of entropy solutions with the additional control $u(\cdot)$.

Let us recall known results concerning the controllability of the viscous Burgers' equation. We start with some positive results.

First, Fursikov and Imanuvilov have shown in [26] a small time local controllability result. It concerns local controllability in the vicinity of trajectories of system (1) and it only requires one boundary control (either $y(t,0)$ or $y(t,1)$). Their proof relies on Carleman estimates for the parabolic problem obtained by seeing the non-linear term yy_x as a small forcing term.

Global controllability towards steady states of system (1) is possible in large time both with one or two boundary controls. Such studies have be carried out by Fursikov and Imanuvilov in [25] for large time global controllability towards all steady states, and by Coron in [14] for global null-controllability in bounded time (ie. bounded with respect to the initial data).

When three scalar controls (namely $u(t)$, $y(t,0)$ and $y(t,1)$) are used, Chapouly has shown in [6] that the system is small time exactly controllable to the trajectories. Her proof relies on the return method and on the fact that the corresponding inviscid Burgers' system is small time exactly controllable (see [13, Chapter 6] for other examples of this method applied to Euler or Navier-Stokes).

Some studies have also been carried out in a two-dimensional setting. In [33], Imanuvilov and Puel study the global controllability of a 2D-Burgers system, where the control only acts on a part Γ_1 of the boundary. They derive geometric conditions on Γ_1 for small time controllability to hold.

Some negative results have also been obtained.

In the context of only one boundary control $y(t,1)$, first obstructions where obtained by Diaz in [18]. He gives a restriction for the set of attainable states. Indeed, they must lie under some limit state corresponding to an infinite boundary control $y(t,1) = +\infty$.

Still with only one boundary control, Fernández-Cara and Guerrero derived an asymptotic of the minimal null-controllability time $T(r)$ for initial states of H^1 norm lower than r (see [22]). This shows that the system is not small-time controllable.

Guerrero and Imanuvilov have shown negative results in [28] when two boundary controls $y(t,0)$ and $y(t,1)$ are used. They prove that neither small time null controllability nor bounded time global controllability hold. Hence, controlling the whole boundary does not provide better controllability.

1.4 Strategy for steering the system towards the null state

In view of these results, it seems that the pressure-like control $u(t)$ introduced by Chapouly is the key to obtaining small time global controllability results. In order to take advantage of both hyperbolic and parabolic behaviors of system (1), our strategy consists in splitting the motion in three stages:

Hyperbolic stage: Small time and approximate control towards the null state. During this very short stage $t \in [0, \varepsilon T]$ where $0 < \varepsilon \ll 1$, the systems behaves like the corresponding hyperbolic one, as the viscous term does not have enough time to act. This hyperbolic system is small time null controllable. During this first stage, we will use both $u(\cdot)$ and $v(\cdot)$ to try to get close to the null state, except for a boundary layer at $x = 1$.

Passive stage: Waiting. At the end of the first stage, we reach a state whose size is hard to estimate due to the presence of a boundary layer. During this stage, we use null controls $v(t) = u(t) = 0$. Regularization properties of the viscous Burgers equation dissipate the boundary layer and the size of $y(t,\cdot)$ decreases. We show that it tends to zero in $L^2(0,1)$ when $\varepsilon \to 0$. This is a crucial

stage as is enables us to get rid of the boundary residue. It seems to be a new idea and could also be applied for other boundary layers created when trying to get small time global controllability results.

Parabolic stage: Local exact controllability in the vicinity of zero. After the two first stages, we succeed in getting very close to the null state. The non-linear term becomes very small compared to the viscous one, and the system now behaves like a parabolic one. We use a small time local exact controllability result to steer the system exactly to zero. During this last stage, we only need the control $v(\cdot)$.

Most of the work to be done consists in deriving precise estimates for the creation and the dissipation of the boundary layer. We will use the Cole-Hopf transform and Fourier series to overcome this difficulty. The Cole-Hopf transform was introduced in [8] and [31] and was used for the first time in control theory by El Badia and Ain Seba in [19].

First, we will investigate the hyperbolic limit system (see Section 2). Then we will derive estimates for the creation of the boundary layer during our hyperbolic stage (see Section 3) and estimates for its dissipation during the passive stage (see Section 4). This will achieve the proof of a small time global approximate null controllability result for our system (1). In Section 5, we will explain the parabolic stage and the local exact controllability.

1.5 A comparison lemma for controlled Burgers' systems

Throughout our work, we will make an extensive use of the following comparison lemma for our Burgers' system, in order to derive precise estimates. When the viscosity is null, this comparison principle still holds for entropy solutions (as they are obtained as a limit of low viscosity solutions).

Lemma 1. *Let $T, \nu > 0$ and consider $y_0, \hat{y}_0 \in L^2(0,1)$, $u, \hat{u} \in L^2(0,T)$, $v_0, \hat{v}_0, v_1, \hat{v}_1, \in H^{1/4}(0,T)$. Assume these data satisfy the following conditions:*

$$y_0 \leq \hat{y}_0 \quad \textit{and} \quad u \leq \hat{u} \quad \textit{and} \quad v_0 \leq \hat{v}_0 \quad \textit{and} \quad v_1 \leq \hat{v}_1.$$

Consider the following system (which is a generalized version of system (1):

$$\left\{ \begin{array}{rll} y_t + yy_x - \nu y_{xx} &= u(t) & \textit{in} \quad (0,T)\times(0,1), \\ y(t,0) &= v_0(t) & \textit{in} \quad (0,T), \\ y(t,1) &= v_1(t) & \textit{in} \quad (0,T), \\ y(0,x) &= y_0(x) & \textit{in} \quad (0,1). \end{array} \right. \tag{3}$$

Then the associated solutions $y, \hat{y} \in X$ to system (3) are such that:

$$y \leq \hat{y} \quad \textit{on} \quad (0,T)\times(0,1).$$

One can find many comparison results in the literature (see for instance the book [39] and the references therein). However we give the proof of Lemma 1 both for the sake of completeness and because with have not found this precise version anywhere.

Proof. We introduce $w = \hat{y} - y$. Thus, $w \in X$ is a solution to the system:

$$\left\{ \begin{array}{rll} w_t - \nu w_{xx} &= (\hat{u}-u) - \frac{1}{2}(w\hat{y} + wy)_x & \text{in} \quad (0,T)\times(0,1), \\ w(t,0) &= \hat{v}_0(t) - v_0(t) & \text{in} \quad (0,T), \\ w(t,1) &= \hat{v}_1(t) - v_1(t) & \text{in} \quad (0,T), \\ w(0,x) &= \hat{y}_0(x) - y_0(x) & \text{in} \quad (0,1). \end{array} \right.$$

We want to study the negative part of w: $\delta = \min(w, 0)$. Hence, $\delta(t,0) = \delta(t,1) = 0$. Now we multiply the evolution equation by $\delta \leq 0$ and integrate by parts for $x \in [0,1]$ to get a L^2-energy estimate for δ:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^1 \delta^2 + \nu \int_0^1 \delta_x^2 &= (\hat{u}-u)\int_0^1 \delta + \frac{1}{2}\int_0^1 \delta(\hat{y}+y)\delta_x \\ &\leq \frac{\nu}{4}\int_0^1 \delta_x^2 + \frac{1}{4\nu}\int_0^1 \delta^2(\hat{y}+y)^2 \\ &\leq \frac{\nu}{4}\int_0^1 \delta_x^2 + \frac{1}{4\nu}\|\hat{y}(t,\cdot)+y(t,\cdot)\|_\infty^2 \cdot \int_0^1 \delta^2. \end{aligned}$$

Thus, we can incorporate the first term of the right-hand side in the left-hand side:

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \delta^2 \leq \frac{1}{4\nu}\|\hat{y}(t,\cdot)+y(t,\cdot)\|_\infty^2 \cdot \int_0^1 \delta^2.$$

Since $y, \hat{y} \in L^2\left((0,T);H^1(0,1)\right)$, we have that:

$$t \mapsto \|\hat{y}(t,\cdot)+y(t,\cdot)\|_\infty^2 \quad \text{belongs to} \quad L^1(0,T).$$

Hence we can use Grönwall's lemma. Since $\delta(0,\cdot)\equiv 0$, we deduce that $\delta \equiv 0$ and $y \leq \hat{y}$. □

2 Analysis of the hyperbolic limit system

2.1 Small time versus small viscosity scaling

Let us choose some $\varepsilon > 0$. We want to study what happens during the time interval $[0, \varepsilon T]$. To study this very short first stage, we perform the following change of scale. For $t \in [0,T]$ and $x \in [0,1]$, let:

$$\bar{y}(t,x) = \varepsilon y(\varepsilon t, x). \tag{4}$$

Hence, $\bar{y} \in X$ is now the solution to the small viscosity system:

$$\left\{\begin{array}{rll} \bar{y}_t + \bar{y}\bar{y}_x - \varepsilon \bar{y}_{xx} = \bar{u}(t) & \text{in} & (0,T)\times(0,1), \\ \bar{y}(t,0) = \bar{v}(t) & \text{in} & (0,T), \\ \bar{y}(t,1) = 0 & \text{in} & (0,T), \\ \bar{y}(0,x) = \bar{y}_0(x) & \text{in} & (0,1), \end{array}\right. \tag{5}$$

where we performed the following scalings: $\bar{u}(t) = \varepsilon^2 u(\varepsilon t)$, $\bar{v}(t) = \varepsilon v(\varepsilon t)$ and $\bar{y}_0(x) = \varepsilon y_0(x)$. This scaling is fruitful because it highlights the fact that, when small time scales are considered, the non-linear term is the key term. We want to understand the behavior of the limit system when $\varepsilon = 0$. Therefore, let us consider that $\bar{u}(\cdot)$, $\bar{v}(\cdot)$ and $\bar{y}_0(\cdot)$ are fixed data, and let ε go to zero.

2.2 Obtaining the entropy limit

When one considers the entropy limit $\varepsilon \to 0$ for system (5), it is not possible to keep on enforcing strong Dirichlet boundary conditions. A boundary layer appears and it is necessary to weaken the boundary conditions. Otherwise, the system would become over-constrained. The pioneer work concerning the derivation of such weak boundary conditions is the one by Bardos, Le Roux and Nédélec in [4]. In our particular setting, one gets the following system:

$$\left\{\begin{array}{rll} \bar{y}_t + \frac{1}{2}(\bar{y}^2)_x = \bar{u}(t) & \text{in} & (0,T)\times(0,1), \\ \bar{y}(t,0) \in E(\bar{v}(t)) & \text{in} & (0,T), \\ \bar{y}(t,1) \geq 0 & \text{in} & (0,T), \\ \bar{y}(0,x) = \bar{y}_0(x) & \text{in} & (0,1), \end{array}\right. \tag{6}$$

where

$$E(\alpha) = \left\{\begin{array}{ll}]-\infty;0] & \text{if} \quad \alpha \leq 0, \\ ]-\infty;-\alpha]\cup\{\alpha\} & \text{if} \quad \alpha > 0. \end{array}\right.$$

Let us explain the physical meaning of the set $E(\cdot)$. On the one hand, when one tries to enforce a negative boundary data on the left side, characteristics instantly flow out of the domain, and our actions are useless. On the other hand, if we set a positive boundary data, then: either it is satisfied, or a greater negative wave overwhelms it.

Without getting into the details of entropy solutions (for that subject, refer to the definition given in [4] or to the book [40]), we will use the following theorem that guarantees that system (6) is well-posed.

Theorem 2 (Bardos, Le Roux and Nédélec in [4]). *For any initial data $y_0 \in BV(0,1)$ and any pair of controls $u \in L^1(0,T)$, $v \in BV(0,T)$, system (6) has a unique entropy solution $\bar{y}$ in the space $BV((0,1)\times(0,T))$.*

2.3 Small time null controllability

We are going to show a small time null controllability result for the hyperbolic limit system. However, this will not imply small time global controllability since the system is not time reversible. Indeed, even though the PDE seems time-reversible, the definition of an entropy solution is not.

Theorem 3. *System (6) is small time globally exactly null controllable.*

Let us start by giving the intuition of the proof. In a first step, we enforce a constant left boundary data $H > 0$. It moves towards the right and overrides the initial data $\bar{y}_0(\cdot)$ provided that the shocks' propagation speed is sufficient. Therefore, H is chosen by using the Rankine-Hugoniot formula. Figure 2 shows a simulation of this first step for some smooth initial data $\bar{y}_0$. At the end of this step, we have $\bar{y}(\cdot) \equiv H$. During the second step, we use some constant negative $\bar{u}$ to get back down to the null state.

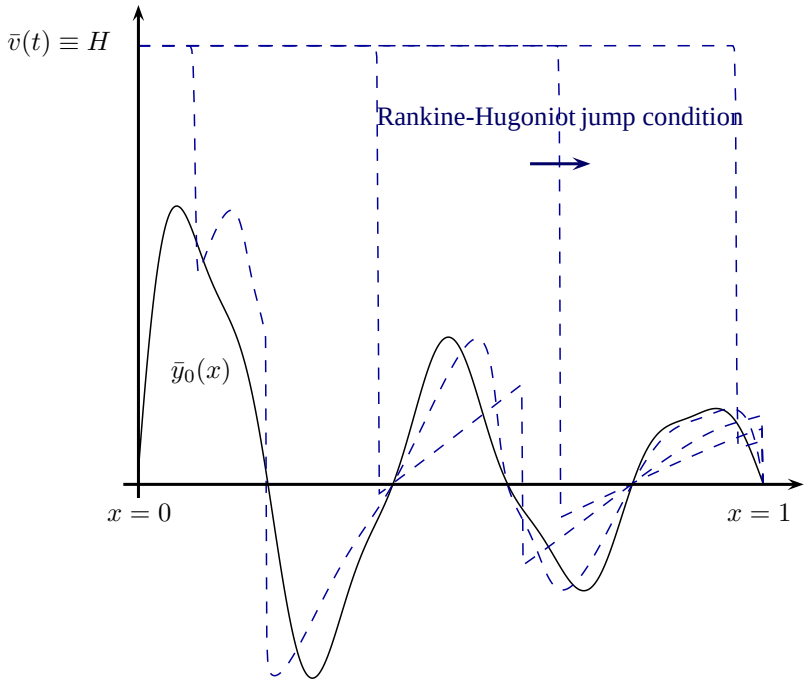

Figure 2: Overriding of an initial data $\bar{y}_0(x)$ by some constant state $\bar{y}(x) \equiv H$ for system (6).

Now let us give a rigorous proof using the comparison principle.

Proof. Let $\bar{y}_0(x) \in BV(0,1)$ and $T > 0$. Let us choose H such that:

$$\frac{1}{2}\left(H - \|\bar{y}_0\|_{L^\infty}\right) \geq \frac{2}{T}. \tag{7}$$

We enforce the following controls:

$$\begin{aligned} \bar{v}(t) &= \begin{cases} H & \text{for} \quad t \in [0, T/2], \\ 2H\left(1 - \frac{t}{T}\right) & \text{for} \quad t \in [T/2, T], \end{cases} \\ \bar{u}(t) &= \begin{cases} 0 & \text{for} \quad t \in [0, T/2], \\ -\frac{2H}{T} & \text{for} \quad t \in [T/2, T]. \end{cases} \end{aligned}$$

From Theorem 2, we know that there exists a unique entropy solution $\bar{y} \in BV((0,1) \times (0,T))$ for these data. Let us show that $\bar{y}(T/2, \cdot) \equiv H$. Therefore, we will easily deduce $\bar{y}(T, \cdot) \equiv 0$.

Let us extend our initial data from $[0,1]$ to $\mathbb{R}$. Since Theorem 2 guarantees the uniqueness of the solution, the restriction to $x \in [0,1]$ of our global solution will be the unique solution to (6). Therefore we consider $\hat{y}_0 \in BV(\mathbb{R})$:

$$\hat{y}_0(x) = \begin{cases} H & \text{for} \quad x < 0, \\ \bar{y}_0(x) & \text{for} \quad 0 < x < 1, \\ 0 & \text{for} \quad 1 < x. \end{cases} \tag{8}$$

Let us introduce $\hat{y}$ the weak entropy solution defined on $\mathbb{R} \times [0,T]$ associated to this initial data. Thanks to Rankine-Hugoniot formula and (7), we know that:

$$y(t,x) = H \quad \text{for} \quad x < t \cdot \frac{(H - \|y_0\|_\infty)}{2}.$$

Hence, $\hat{y}(T/2,x) = H$ for $x \in [0,1]$, and $y(t,0^+) \equiv H$. If we want the restriction of $\hat{y}$ to be a solution to (6), we need to check that $y(t,1^-) \geq 0$. Let us use the comparison principle for solutions to inviscid Burgers' equation. It can be obtained by taking the null viscosity limit in our Lemma 1. Hence $\hat{y}(t,x) \geq w(t,x)$ where w is the solution associated to the initial data:

$$w_0(x) = \begin{cases} H & \text{for} \quad x < 0, \\ -\|\bar{y}_0\|_\infty & \text{for} \quad 0 < x < 1, \\ 0 & \text{for} \quad 1 < x. \end{cases} \tag{9}$$

We have two Riemann problems. Near $x = 1$, we have a rarefaction wave. Hence $x \mapsto w(t,x)$ is continuous near $x = 1$ as long as the H shock wave has not reached $x = 1$. Hence $w(t,1^-) = 0$ before $T^* = 1/(2H - 2\|\bar{y}_0\|_\infty)$, then $w(t,1^-) = H$. This is why $w(t,1^-) \geq 0$. Thus $\hat{y}(t,1^-) \geq w(t,1^-) \geq 0$. The restriction $\hat{y}_{[0,1]}$ is the unique solution to (6) and it is equal to H at time $t = T/2$. □

This proof uses the comparison principle for Burgers' equation. Since we consider a 1-D system, this is not a problem. However, if we wanted to be able to handle multi-dimensional systems, we could use the generalized characteristics method from Dafermos (see [17]). This technique has been successfully used by Perrollaz in [38].

3 Hyperbolic stage and settling of the boundary layer

Thanks to the analysis of the hyperbolic limit system, we were able to exhibit controls steering the system towards the null state from any initial data. Now we want to apply the same strategy to the slightly viscous system (5) by using very similar controls. However, a boundary layer is going to appear. Our goal in this section is to derive bounds for the boundary layer at the end of this stage.

3.1 Steady states of system (5)

From now on, the viscosity is positive. Hence, since we have a zero Dirichlet boundary condition $\bar{y}(1) = 0$, we cannot hope to reach a constant state $\bar{y}(x) \equiv H > 0$. However, we expect that we can get very close to the corresponding steady state. Let us introduce the following steady state of system (5):

$$h^\varepsilon(x) = H \tanh\left(\frac{H}{2\varepsilon}(1-x)\right). \tag{10}$$

Lemma 2. *For any $H > 0$ and any $\varepsilon > 0$, h^ε defined by (10) is a stationary solution to system (5) with controls: $\bar{u}(t) = 0$ and $\bar{v}(t) = H \tanh\left(\frac{H}{2\varepsilon}\right)$.*

Proof. The proof is an easy computation. In fact, it is possible to compute explicitly all the steady states for system (5), at least when $\bar{u} = 0$. This is done in [25] with viscosity $\varepsilon = 1$. □

We have chosen a boundary data $\bar{v}(t) = H \tanh\left(\frac{H}{2\varepsilon}\right)$ for the definition of our steady state h^ε, but we will use a control $\bar{v}(t) = H$ for the motion. This technical trick will lighten some computations and is relevant since both terms are exponentially close as ε goes to zero. However, some proofs require the use of the exact steady state corresponding to a boundary data $\bar{v}(t) = H$. Therefore, we introduce:

$$k^\varepsilon(x) = K \tanh\left(\frac{K}{2\varepsilon}(1-x)\right), \tag{11}$$

where $K > 0$ is given by the implicit relation $K \tanh\left(K/(2\varepsilon)\right) = H$.

Lemma 3. *For any $H > 0$ and any $\varepsilon > 0$, k^ε defined by (11) is a stationary solution to system (5) with controls: $\bar{u}(t) = 0$ and $\bar{v}(t) = H$. Moreover, we have the estimate:*

$$\|k^\varepsilon - h^\varepsilon\|_{L^\infty(0,1)} \leq 2He^{-H/\varepsilon}. \tag{12}$$

Proof. Lemma 2 gives us that k^ε is a steady state. For the estimate, we write:

$$\begin{aligned}\|k^\varepsilon - h^\varepsilon\|_{L^\infty(0,1)} &\leq \left|K\tanh\left(\frac{K}{2\varepsilon}\right) - H\tanh\left(\frac{H}{2\varepsilon}\right)\right| \\ &\leq H\left|1-\tanh\left(\frac{H}{2\varepsilon}\right)\right| \\ &\leq 2He^{-H/\varepsilon}.\end{aligned}$$

□

3.2 First step: overriding the initial data

In order to get close to the steady state h^ε, it is necessary to choose H in such a way that a Rankine-Hugoniot type condition is satisfied. Once we get close enough to the steady state, the solution will very quickly converge to the steady state. Indeed, the eigenvalues of the linearized system around this steady state are real, negative, and of size at least $1/\varepsilon$. This guarantees very quick convergence to the steady state. Such a study of the linearized problem around a steady state for the Burgers' equation can be found in [34]. We give the following lemma describing the settling of the limit layer.

Lemma 4. *Let $T>0$, $H>0$ and $y_0 \in H_0^1(0,1)$ be given data. Then for $\varepsilon>0$ small enough, there exists a boundary control $\bar v \in H^{3/4}(0,T)$ such that $\bar v(\cdot) \leq H$ and such that the solution $\bar y \in X$ to system (5) with initial data $\bar y_0 = \varepsilon y_0$ and controls $\bar u = 0$ and $\bar v$ satisfies:*

$$\|\bar y(T,\cdot) - h_\varepsilon(\cdot)\|_{L^2(0,1)} = \mathcal{O}_{\varepsilon\to 0}\left(\varepsilon^{-1/2}e^{-\frac{H}{4\varepsilon}(HT-2)}\right). \tag{13}$$

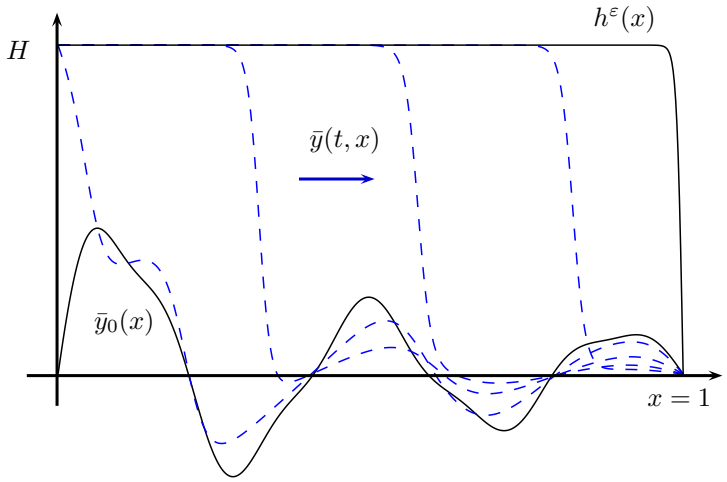

Figure 3: Example of evolution from an initial data towards a steady state.

Let us postpone the proof of Lemma 4 for the moment. We start by giving a few remarks concerning this statement and its proof. The intuition is to choose a boundary control $\bar v(t) \equiv H$, just like we have done for the hyperbolic case. Moreover, we want to use the Cole-Hopf transform and Fourier series to compute explicitly $\bar y(T,\cdot)$. Let us introduce the Cole-Hopf transform:

$$Z(t,x) = \exp\left(-\frac{1}{2\varepsilon}\int_0^x \bar y(t,s)ds\right).$$

This leads to the following heat system for the new unknown Z:

$$\begin{cases} Z_t - \varepsilon Z_{xx} &= -\left(\frac{1}{4\varepsilon}\bar y^2(t,0) - \frac{1}{2}\bar y_x(t,0)\right)Z \quad \text{on} \quad (0,T)\times(0,1), \\ Z(t,0) &= 1 \quad \text{on} \quad (0,T), \\ Z_x(t,1) &= 0 \quad \text{on} \quad (0,T), \\ Z(0,x) &= Z^0(x) \quad \text{on} \quad (0,1), \end{cases} \tag{14}$$

where the initial data Z^0 is computed from the initial data $\bar y_0 = \varepsilon y_0$:

$$Z^0(x) = \exp\left(-\frac{1}{2}\int_0^x y_0(s)ds\right). \tag{15}$$

Hence we see that it will not be possible to carry on explicit computations if we do choose $\bar{y}(t,0) \equiv H$. Indeed, in that case, we would not know explicitly $\bar{y}_x(t,0)$ (which is needed to compute the solution to system (14)). However, we are confident that this term is very small. Hence, we are going to go the other way around: we will choose our control explicitly in the Cole-Hopf domain and use it to compute our control $\bar{v}(\cdot)$. Therefore, we are interested in the following heat system:

$$\left\{\begin{array}{rcll} Z_t - \varepsilon Z_{xx} & = -\frac{H^2}{4\varepsilon} Z & \text{on} & (0,T)\times(0,1), \\ Z(t,0) & = 1 & \text{on} & (0,T), \\ Z_x(t,1) & = 0 & \text{on} & (0,T), \\ Z(0,x) & = Z^0(x) & \text{on} & (0,1). \end{array}\right. \tag{16}$$

If we go back to the Burgers' domain, this means that we somehow use the following boundary condition at $x = 0$:

$$\bar{y}_x(t,0) = \frac{1}{2\varepsilon}\left(\bar{y}^2(t,0) - H^2\right). \tag{17}$$

We expect that the solution Z will converge towards $H^\varepsilon(\cdot)$, where $H^\varepsilon(\cdot)$ is the Cole-Hopf transform of the steady state h^ε:

$$H^\varepsilon(x) = \frac{\cosh\left(\frac{H}{2\varepsilon}(1-x)\right)}{\cosh\frac{H}{2\varepsilon}}. \tag{18}$$

Indeed, we have the following lemma.

Lemma 5. *Let $T > 0$ and $Z^0 \in H^2(0,1)$ such that $Z^0(0) = 1$ and $Z^0_x(1) = 0$. Then system (16) has a unique solution Z in the space $L^2((0,T);H^3(0,1)) \cap H^1((0,T);H^1(0,1))$. Moreover, there exists a constant $C(Z^0) > 0$ depending only on $\|Z^0\|_{H^1}$ such that:*

$$\|Z(T,\cdot) - H^\varepsilon(\cdot)\|_{H^1(0,1)} \leq \varepsilon^{-1/2} C(Z^0) e^{-\frac{H^2 T}{4\varepsilon}}. \tag{19}$$

Proof. It is classical to show that system (16) has a unique solution in the space $L^2((0,T);H^3(0,1)) \cap H^1((0,T);H^1(0,1))$. One can even get more smoothness if needed. An efficient method is the semi-group method that one can find for instance in [37]. To compute the dynamics of system (16), we introduce the adequate Fourier basis of L^2:

$$f_n(x) = \sqrt{2}\sin\left(\left(n+\frac{1}{2}\right)\pi x\right) \quad \text{for} \quad n \geq 0.$$

Hence $f_n(0) = f_n'(1) = 0$. We will use the notation $\lambda_n = (n+\frac{1}{2})\pi$. Thus, $f_n'' = -\lambda_n^2 f_n$. Let us give the following scalar products, which can easily be computed using integration by parts:

$$\begin{aligned} \langle 1 | f_n \rangle &= \frac{\sqrt{2}}{\lambda_n}, \\ \langle H^\varepsilon | f_n \rangle &= \frac{\sqrt{2}\lambda_n}{\frac{H^2}{4\varepsilon^2} + \lambda_n^2}, \qquad (20) \\ \left|\langle Z^0 | f_n \rangle\right| &\leq \frac{\sqrt{2}}{\lambda_n}\left(1 + \frac{1}{2}\left\|Z^0\right\|_{H^1}\right). \qquad (21) \end{aligned}$$

In these equations $\langle\cdot|\cdot\rangle$ denotes the standard scalar product in $L^2(0,1)$. Let us write $Z = 1 + w$. Hence w will satisfy $w(t,0) = w_x(t,1) = 0$. Easy computations lead to the following ordinary differential equations for the components of w on our Fourier basis:

$$\dot{w}_n(t) = -\varepsilon\left(\lambda_n^2 + \frac{H^2}{4\varepsilon^2}\right) w_n(t) - \frac{H^2}{4\varepsilon}\langle 1 | f_n \rangle.$$

It is easy to see that the fixed points for these ODEs are the expected coefficients $\langle H^\varepsilon - 1 | f_n \rangle$. We can solve these ODEs with our initial condition:

$$w_n(t) = \alpha_n e^{-\varepsilon\left(\lambda_n^2 + \frac{H^2}{4\varepsilon^2}\right)t} + \langle H^\varepsilon - 1 | f_n \rangle,$$

where:

$$\alpha_n = \langle Z^0 | f_n \rangle - \langle H^\varepsilon | f_n \rangle.$$

Now we can estimate $Z(T,\cdot) - H^\varepsilon(\cdot)$:

$$\|Z(T,\cdot) - H^\varepsilon(\cdot)\|^2_{H^1(0,1)} = \sum_{n\geq 0} \lambda_n^2 \alpha_n^2 e^{-2\varepsilon\left(\lambda_n^2 + \frac{H^2}{4\varepsilon^2}\right)T}.$$

From the expression of α_n, (20) and (21) we get the easy bound:

$$\lambda_n^2 \alpha_n^2 \leq 16 + \|Z^0\|^2_{H^1(0,1)}, \quad \forall n \in \mathbb{N}.$$

Thus, we get

$$\|Z(T,\cdot) - H^\varepsilon(\cdot)\|^2_{H^1(0,1)} \leq \left(16 + \|Z^0\|^2_{H^1(0,1)}\right) e^{-\frac{H^2 T}{2\varepsilon}} \sum_{n\geq 0} e^{-2\varepsilon\lambda_n^2}.$$

Now we split the sum in two parts: $n \leq N = \lfloor 1/\varepsilon \rfloor$ and $n \geq N$. We get:

$$\begin{aligned}
\|Z(T,\cdot) - H^\varepsilon(\cdot)\|^2_{H^1(0,1)} &\leq \left(16 + \|Z^0\|^2_{H^1(0,1)}\right) e^{-\frac{H^2 T}{2\varepsilon}} \left(N + \sum_{k\geq 0} e^{-2\varepsilon(N+k+\frac{1}{2})^2\pi^2}\right) \\
&\leq \left(16 + \|Z^0\|^2_{H^1(0,1)}\right) e^{-\frac{H^2 T}{2\varepsilon}} \left(N + \frac{1}{1 - e^{-4\varepsilon N\pi^2}} e^{-2\varepsilon N^2\pi^2}\right).
\end{aligned}$$

Hence, for ε small enough, we have:

$$\|Z(T,\cdot) - H^\varepsilon(\cdot)\|^2_{H^1(0,1)} \leq \left(\frac{1}{\varepsilon} + 1\right)\left(16 + \|Z^0\|^2_{H^1(0,1)}\right) e^{-\frac{H^2 T}{2\varepsilon}}.$$

This concludes the proof of Lemma 5. □

Now we can prove Lemma 4.

Proof of Lemma 4. **Definition of the control:** Using Lemma 5, we start by considering the solution $Z \in L^2((0,T);H^3(0,1)) \cap H^1((0,T);H^1(0,1))$ to system (16) with the initial data (15). Since $Z_0(\cdot) > 0$, the usual strong maximum principle (see [40]) guarantees that $Z(t,x) > 0$. Thus, we can define:

$$\bar{y}(t,x) = -2\varepsilon \frac{Z_x(t,x)}{Z(t,x)}. \tag{22}$$

Hence $\bar{y} \in X$ is a solution to (5) with initial data εy_0 and boundary control $\bar{v}(t) = -2\varepsilon Z_x(t,0)$. Since $Z \in L^2((0,T);H^3(0,1)) \cap H^1((0,T);H^1(0,1))$, we can show that its boundary trace $Z_x(t,0)$ belongs to $H^{3/4}(0,T)$. Hence $\bar{v} \in H^{3/4}(0,T)$.

Proof of an L^∞ bound on the solution: If ε is small enough, then $\varepsilon\|y_0\|_\infty \leq H$. Moreover, we know that $\bar{v} \in H^{3/4}(0,T)$. Hence, $\bar{v} \in \mathcal{C}^0[0,T]$. Assume that $\sup_{[0,T]} \bar{v} > H$. Let T_0 be a time such that $\bar{v}(T_0) = \sup_{[0,T]} \bar{v} > H$. On the one hand, by the comparison principle from Lemma 1, we know that:

$$\bar{y} \leq \bar{v}(T_0) \quad \text{on} \quad (0,T)\times(0,1). \tag{23}$$

On the other hand, we recall relation (17):

$$\bar{y}_x(t,0) = \frac{1}{2\varepsilon}\left(\bar{y}^2(t,0) - H^2\right).$$

Hence, since $\bar{v}(T_0) > 0$, we get $\bar{y}_x(T_0,0) > 0$. Thus,there exists $x > 0$ such that $\bar{y}(T_0,x) > \bar{v}(T_0) = \sup_{[0,T]} \bar{v}$. This is in contradiction with assertion (23). Hence, if ε is small enough, $\bar{v}(\cdot) \leq H$ and $\bar{y}(T,\cdot) \leq H$.

Derivation of the L^2 estimate at time T: Now we want to prove estimate (13) from Lemma 4. We want to use estimate (19) from Lemma 5. We perform the following computation at time T and for any $x \in [0,1]$:

$$\begin{aligned}
|\bar{y} - h^\varepsilon| = 2\varepsilon\left|\frac{Z_x}{Z} - \frac{H^\varepsilon_x}{H^\varepsilon}\right| &= 2\varepsilon\left|\frac{Z\left(Z_x - H^\varepsilon_x\right) + Z_x\left(H^\varepsilon - Z\right)}{ZH^\varepsilon}\right| \\
&\leq 2\varepsilon\left|\frac{Z_x - H^\varepsilon_x}{H^\varepsilon}\right| + 2\varepsilon\left|\frac{Z_x}{Z}\right| \cdot \left|\frac{Z - H^\varepsilon}{H^\varepsilon}\right|.
\end{aligned}$$

Thus, we get:

$$\|\bar{y}(T,\cdot) - h^\varepsilon(\cdot)\|_{L^2(0,1)} \leq (2\varepsilon + \|\bar{y}(T,\cdot)\|_\infty) \times \sup_{[0,1]} \frac{1}{H^\varepsilon} \times \|Z(T,\cdot) - H^\varepsilon(\cdot)\|_{H^1(0,1)}\,.$$

Now we use that $\|\bar{y}(T,\cdot)\|_\infty \leq H$ and $\sup_{[0,1]} 1/H^\varepsilon \leq e^{+H/2\varepsilon}$. Hence , using also (19),

$$\|\bar{y}(T,\cdot) - h^\varepsilon(\cdot)\|_{L^2(0,1)} \leq \frac{1}{\sqrt{\varepsilon}}\,(2\varepsilon + H)\,C(Z^0)e^{-\frac{H}{4\varepsilon}(HT-2)}.$$

This estimate concludes the proof of Lemma 4. □

Remark 1. *In Lemma 4, we take an initial data $y_0 \in H_0^1(0,1)$. This is a technical assumption that enables us to use stronger solutions. We will get rid of it later on, by letting the Burgers' equation smooth our real initial data which is only in $L^2(0,1)$.*

3.3 Second step: going back to the null state

Once we have reached the steady state h^ε, we wish to go back to the null state. This is done by applying a suitable negative interior control $\bar{u}$. The control $\bar{v}$ will only be following the global movement. The intuitive idea is to apply some negative control $\bar{u}$ on $[0,T]$ such that $\int_0^T u(t)dt = -H$. Thus, we hope to reach some state that is below 0 and above a boundary residue $h^\varepsilon - H$. However, this last statement is only true up to some small L^2 function (small as $T \to 0$). The key will be to choose the duration T of this step small enough (with respect to ε).

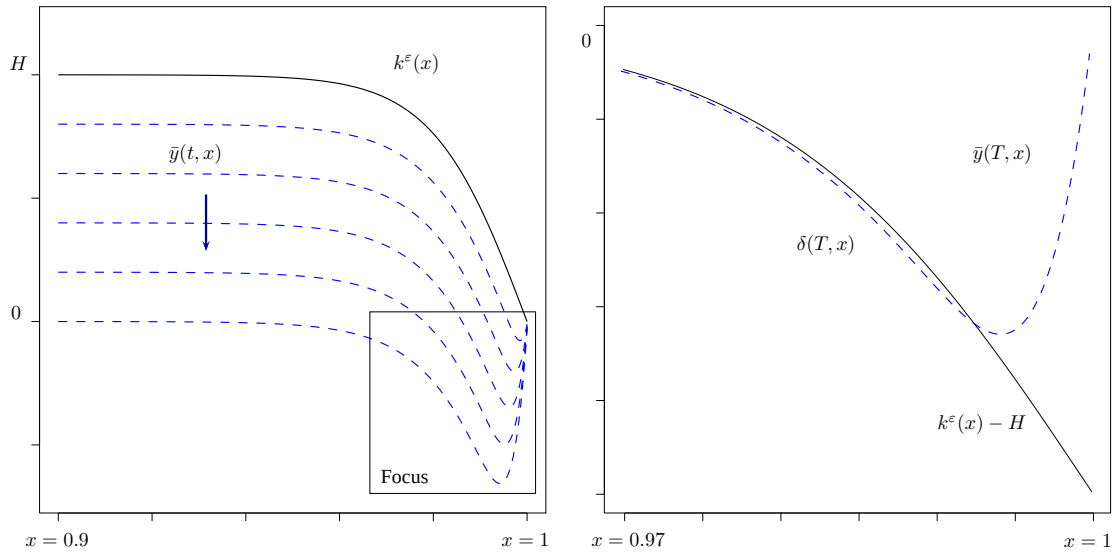

Figure 4: Numerical simulation of the push-down towards the null state and the creation of a boundary residue. The final state $\bar{y}(T,\cdot)$ is almost above the residue $k^\varepsilon(\cdot) - H$.

Lemma 6. *Let $\varepsilon > 0$ and $H > 0$ be given data. Assume that $2\varepsilon \leq H$. We consider the evolution of an initial data $\bar{y}_1 \in L^2(0,1)$. For any $T > 0$, we consider the following controls for $t \in [0,T]$:*

$$\bar{u}(t) = -\frac{H}{T}, \tag{24}$$

$$\bar{v}(t) = H + \int_0^t u(s)ds. \tag{25}$$

Then the associated solution $\bar{y} \in X$ to system (5) satisfies:

$$\bar{y}(T,\cdot) - k^\varepsilon(\cdot) + H \geq \delta(T,\cdot), \tag{26}$$

where $\delta \in X$ is the solution to some Burgers-like system given below and is such that:

$$\|\delta(T,\cdot)\|_{L^2} \leq e^{H^2T/4\varepsilon}\,\|\bar{y}_1 - k^\varepsilon\|_{L^2} + 2H\left(e^{H^2T/2\varepsilon} - 1\right). \tag{27}$$

Proof. Let $T > 0$ and consider the controls defined by (24) and (25). Let us consider the associated solution $\bar{y} \in X$ to (5). We compare $\bar{y}$ to the solution $z \in X$ to the following system:

$$\left\{\begin{array}{rlll} z_t + zz_x - \varepsilon z_{xx} &= \bar{u}(t) & \text{in} & (0,T)\times(0,1), \\ z(t,0) &= \bar{v}(t) & \text{in} & (0,T), \\ z(t,1) &= \bar{v}(t) - H & \text{in} & (0,T), \\ z(0,x) &= \bar{y}_0(x) & \text{in} & (0,1). \end{array}\right. \tag{28}$$

The comparison principle from Lemma 1 tells us that $y(T,\cdot) \geq z(T,\cdot)$. Now we want to derive precise estimates for the solution $z \in X$. We write:

$$z(t,x) = k^\varepsilon(x) + \int_0^t \bar{u}(s)ds + \delta(t,x), \tag{29}$$

where $\delta \in X$ is thus the solution to the following system:

$$\left\{\begin{array}{rlll} \delta_t - \varepsilon\delta_{xx} + k^\varepsilon\delta_x + \left(\delta + \int_0^t \bar{u}(s)ds\right)(k^\varepsilon+\delta)_x &= 0 & \text{in} & (0,T)\times(0,1), \\ \delta(t,0) &= 0 & \text{in} & (0,T), \\ \delta(t,1) &= 0 & \text{in} & (0,T), \\ \delta(0,x) &= \bar{y}_1(x) - k^\varepsilon(x) & \text{in} & (0,1). \end{array}\right. \tag{30}$$

Note that it is convenient in this proof to use k^ε in order to get exact zero boundary conditions $\delta(t,0) = \delta(t,1) = 0$. We multiply the evolution equation of (30) by δ and integrate by parts for $x \in [0,1]$ to get a L^2-energy estimate on δ:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^1 \delta^2 + \varepsilon\int_0^1 \delta_x^2 &= -\int_0^1 k^\varepsilon\delta\delta_x - \int_0^1 \left(\delta + \int_0^t \bar{u}(s)ds\right)(k^\varepsilon+\delta)_x\delta \\ &= \frac{1}{2}\int_0^1 \delta^2(k^\varepsilon)_x - \int_0^1 \left(\delta + \int_0^t \bar{u}(s)ds\right)\delta(k^\varepsilon)_x \\ &= -\frac{1}{2}\int_0^1 \delta^2(k^\varepsilon)_x - \int_0^t \bar{u}(s)ds\int_0^1 \delta(k^\varepsilon)_x. \end{aligned}$$

Now we use definition (11) and the assumption $2\varepsilon \leq H$:

$$\|k_x^\varepsilon\|_\infty \leq \frac{K^2}{2\varepsilon} \leq \frac{H^2}{2\varepsilon\tanh(1)^2} \leq \frac{H^2}{\varepsilon}.$$

Moreover, $\int_0^t \bar{u}(s)ds \leq H$. Hence,

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \delta^2 \leq \frac{H^2}{2\varepsilon}\int_0^1 \delta^2 + \frac{H^3}{\varepsilon}\left(\int_0^1 \delta^2\right)^{1/2}. \tag{31}$$

Let us denote $E(t) = \|\delta(t,\cdot)\|_{L^2}$. Hence, one has:

$$\dot{E}(t) \leq \frac{H^2}{2\varepsilon}E + \frac{H^3}{\varepsilon}. \tag{32}$$

From Grönwall's lemma, we get:

$$E(T) \leq (E(0) + 2H)\, e^{H^2T/2\varepsilon} - 2H. \tag{33}$$

This concludes the proof of Lemma 6. □

This is the end of the hyperbolic stage. We need to perform the reverse scaling of (4) to go back to y (and not $\bar{y}$). We have shown that we are above some boundary residue $h^\varepsilon - H$. Hence, we have to study the evolution of the following initial data:

$$\Phi^\varepsilon(x) = \frac{1}{\varepsilon}\left(h^\varepsilon(x) - H\right) = \frac{H}{\varepsilon}\left(\tanh\left(\frac{H}{2\varepsilon}(1-x)\right) - 1\right). \tag{34}$$

One should be scared by the size of this boundary residue that we are left with. Indeed, its L^2 size grows like $1/\sqrt{\varepsilon}$. However it has the important feature that its typical wavelength is ε. Hence, its spectral decomposition will mostly involve high frequencies that will decay rapidly during the passive stage thanks to smoothing effects of Burgers' equation.

4 Passive stage and dissipation of the boundary layer

The goal of this section is to prove the following estimate concerning the dissipation of the boundary residue Φ^ε created in the previous section. Indeed, although its L^2-norm increases as ε goes to zero, regularization effects of the Burgers equation will dissipate it in any positive time T.

Lemma 7. *Let $T > 0$ be a fixed positive time. For any $\varepsilon > 0$, let us consider $\phi \in X$ the solution to the following system:*

$$\left\{\begin{array}{rll} \phi_t + \phi\phi_x - \phi_{xx} &= 0 & \text{in } (0,T)\times(0,1), \\ \phi(t,0) &= 0 & \text{in } (0,T), \\ \phi(t,1) &= 0 & \text{in } (0,T), \\ \phi(0,x) &= \Phi^\varepsilon(x) & \text{in } (0,1), \end{array}\right.$$

where $\Phi^\varepsilon(x)$ is the boundary residue defined by (34). Then for any $\delta > 0$, we have the estimate:

$$\|\phi(T,\cdot)\|_{L^2(0,1)} = \mathcal{O}_{\varepsilon\to 0}\left(\varepsilon^{1-\delta}\right). \tag{35}$$

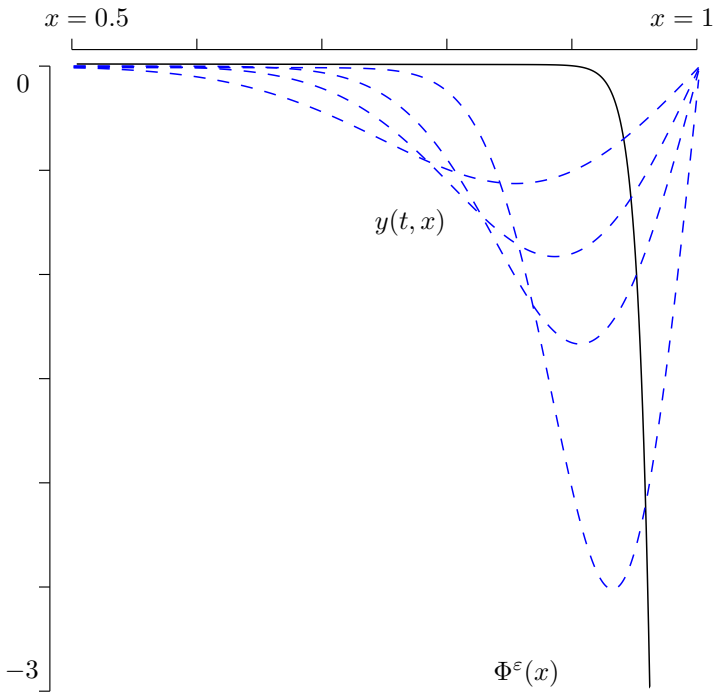

Figure 5: Numerical simulation of the dissipation of the boundary residue $\Phi^\varepsilon(\cdot)$. At time $t = 0$, the boundary residue was of size $\|\Phi^\varepsilon(\cdot)\|_\infty = 100$.

4.1 Cole-Hopf transform

Once again, we are going to use the Cole-Hopf transform to derive precise estimates. Therefore, let us introduce the following change of unknown for $x \in [0,1]$ and $t \in [0,T]$:

$$z(t,x) = \exp\left(-\frac{1}{2}\int_0^x \phi(t,s)ds\right).$$

This leads to the following heat system for the new unknown z:

$$\left\{\begin{array}{rll} z_t - z_{xx} &= 0 & \text{on } (0,T)\times(0,1), \\ z_x(t,0) &= 0 & \text{on } (0,T), \\ z_x(t,1) &= 0 & \text{on } (0,T), \\ z(0,x) &= Z^\varepsilon(x) & \text{on } (0,1), \end{array}\right. \tag{36}$$

where the initial data Z^ε is computed from the initial data Φ^ε:

$$\begin{aligned} Z^\varepsilon(x) &= \exp\left(-\frac{1}{2}\int_0^x \Phi^\varepsilon(s)ds\right) \\ &= \frac{1+e^{\frac{H}{\varepsilon}(x-1)}}{1+e^{-\frac{H}{\varepsilon}}}. \end{aligned} \tag{37}$$

An important remark is that $\Phi^\varepsilon \leq 0$. Thus, by the comparison principle from Lemma 1, $\phi \leq 0$ on $[0,T]\times[0,1]$ and $z \geq 1$ on $[0,T]\times[0,1]$. The backwards Cole-Hopf transform will give us:

$$\phi(T) = -2\frac{z_x(T)}{z(T)}.$$

Hence, using the fact that $z \geq 1$, we will have the following estimate:

$$|\phi(T,\cdot)| \leq 2\,|z_x(T,\cdot)|\,. \tag{38}$$

All we have to do is to study the L^2-norm of $z_x(T)$. To ease computations, let us introduce:

$$w = (1+e^{-\frac{H}{\varepsilon}})z_x, \tag{39}$$

such that w is the solution to:

$$\left\{\begin{array}{rll} w_t - w_{xx} &= 0 & \text{on} \quad (0,T)\times(0,1), \\ w(t,0) &= 0 & \text{on} \quad (0,T), \\ w(t,1) &= 0 & \text{on} \quad (0,T), \\ w(0,x) &= \frac{H}{\varepsilon}e^{\frac{H}{\varepsilon}(x-1)} & \text{on} \quad (0,1). \end{array}\right.$$

4.2 Fourier series decomposition

We use Fourier series to compute $w(T,\cdot)$. We will use the following Hilbert basis of L^2 made of the eigen-functions for the Laplace operator with Dirichlet boundary conditions on $[0,1]$:

$$e_n(x) = \sqrt{2}\sin(n\pi x) \quad \text{for} \quad n \geq 1.$$

Let us compute the decomposition of $w(0,\cdot)$ on this basis. We integrate by parts twice:

$$\begin{aligned} \langle w(0,\cdot)|e_n\rangle &= \sqrt{2}\frac{H}{\varepsilon}e^{-\frac{H}{\varepsilon}}\int_0^1 \sin(n\pi x)e^{\frac{H}{\varepsilon}x}dx \\ &= \sqrt{2}e^{-\frac{H}{\varepsilon}}\left[\sin(n\pi x)e^{\frac{H}{\varepsilon}x}\right]_0^1 - \sqrt{2}n\pi e^{-\frac{H}{\varepsilon}}\int_0^1 \cos(n\pi x)e^{\frac{H}{\varepsilon}x} \\ &= -\frac{\varepsilon\sqrt{2}}{H}n\pi e^{-\frac{H}{\varepsilon}}\left[\cos(n\pi x)e^{\frac{H}{\varepsilon}x}\right]_0^1 - \left(\frac{\varepsilon n\pi}{H}\right)^2\langle w(0,\cdot)|e_n\rangle \\ &= \frac{\sqrt{2}}{H}\frac{\varepsilon n\pi}{1+\frac{\varepsilon^2n^2\pi^2}{H^2}}\left((-1)^{n+1}+e^{-\frac{H}{\varepsilon}}\right). \end{aligned}$$

Now we can estimate the size of $w(T,\cdot)$ in $L^2(0,1)$:

$$\begin{aligned} \|w(T,\cdot)\|_{L^2}^2 &= \sum_{n\geq 1}\left(\langle w(0,\cdot)|e_n\rangle\cdot e^{-n^2\pi^2T}\right)^2 \\ &\leq 8\sum_{n\geq 1}\frac{\varepsilon^2n^2\pi^2H^{-2}}{\left(1+\varepsilon^2n^2\pi^2H^{-2}\right)^2}e^{-2n^2\pi^2T}. \end{aligned}$$

For $\alpha \in \mathbb{R}$, the following easy inequality holds:

$$\frac{\alpha^2}{(1+\alpha^2)^2} \leq \min\left(\alpha^2, \frac{1}{4}\right).$$

Hence we split the sum and cut at a level $N(\varepsilon)$:

$$\begin{aligned}\|w(T,\cdot)\|_{L^2}^2 &\leq 8\sum_{n=1}^{N-1}\frac{\varepsilon^2 n^2\pi^2}{H^2}+2\sum_{k\geq 0}e^{-2(N+k)^2\pi^2 T}\\ &\leq \frac{8\varepsilon^2N^3\pi^2}{3H^2}+2e^{-2N^2\pi^2T}\sum_{k\geq 0}e^{-4Nk\pi^2T}\\ &\leq \frac{8\varepsilon^2N^3\pi^2}{3H^2}+2\frac{e^{-2N^2\pi^2T}}{1-e^{-4N\pi^2T}}.\end{aligned}$$

We want to choose $N(\varepsilon)\to+\infty$ such that $\varepsilon^2N^3\to 0$. For instance, we can take $N=\lfloor\varepsilon^{-\eta}\rfloor$, where $\eta>0$ is small enough. For ε small enough, we have:

$$\|w(T,\cdot)\|_{L^2}^2\leq\frac{8\pi^2}{3H^2}\varepsilon^{2-3\eta}+4e^{-2\varepsilon^{-2\eta}\pi^2T}=\mathcal{O}\left(\varepsilon^{2-3\eta}\right). \tag{40}$$

Combining estimates (40) and (38), and the definition (39) we can easily deduce the estimate (35). This concludes the proof of Lemma 7. □

4.3 Approximate controllability towards the null state

First, let us prove the following technical lemma. Indeed, we have proven that the particular boundary layer Φ^ε dissipates, but all we also want to know what would happen if we were very close to it.

Lemma 8. *Let us change the initial data from Lemma 7 to $\Phi^\varepsilon(x)+\frac{1}{\varepsilon}\delta^\varepsilon$. We assume:*

$$\Phi^\varepsilon(x)+\frac{1}{\varepsilon}\delta^\varepsilon \leq 0, \tag{41}$$

$$\|\delta^\varepsilon(\cdot)\|_{L^2(0,1)} = \mathcal{O}_{\varepsilon\to 0}(\varepsilon^3). \tag{42}$$

Then, the conclusion of Lemma 7 still holds.

Proof. We follow the same scheme than for the proof of Lemma 7. Hence, we start by taking the Cole-Hopf transform of the new initial data $\Phi^\varepsilon(x)+\frac{1}{\varepsilon}\delta^\varepsilon$. Therefore, after the Cole-Hopf transform we have the following initial data:

$$Z^\varepsilon(x)+Z^\varepsilon(x)\cdot\left(\exp\left(-\frac{1}{2\varepsilon}\int_0^x\delta^\varepsilon\right)-1\right).$$

From our previous computation (37) of Z^ε, we know that $|(Z^\varepsilon)_x|=\mathcal{O}(1/\varepsilon)$. Hence, using condition (42), we have:

$$\left\|Z^\varepsilon(x)\cdot\left(\exp\left(-\frac{1}{2\varepsilon}\int_0^x\delta^\varepsilon\right)-1\right)\right\|_{H^1(0,1)}=\mathcal{O}_{\varepsilon\to 0}(\varepsilon).$$

Let us use the fact that our heat system (36) is linear. Therefore, using the conclusion of Lemma 7 we have:

$$\|z(T,\cdot)\|_{H^1(0,1)}=\mathcal{O}_{\varepsilon\to 0}(\varepsilon^{1-\delta})+\mathcal{O}_{\varepsilon\to 0}(\varepsilon).$$

Once again we apply the backwards Cole-Hopf transform. We use the fact that $z\geq 1$ (this comes from the comparison principle and the hypothesis (41)). Hence,

$$\|\phi(T,\cdot)\|_{L^2(0,1)}\leq 2\|z(T,\cdot)\|_{H^1(0,1)}.$$

Thus, the conclusion (35) of Lemma 7 still holds with this new initial data. □

Now everything is ready for us to show the following small time approximate controllability result for system (1). We have to combine the different estimates.

Theorem 4. *Let $T,r>0$ and $y_0\in L^2(0,1)$ be given data. Then there exists $u,v\in L^\infty(0,T)\times H^{1/4}(0,T)$ such that the associated solution $y\in X$ to system (1) on $[0,T]$ satisfies:*

$$\|y(T,\cdot)\|_{L^2(0,1)}\leq r.$$

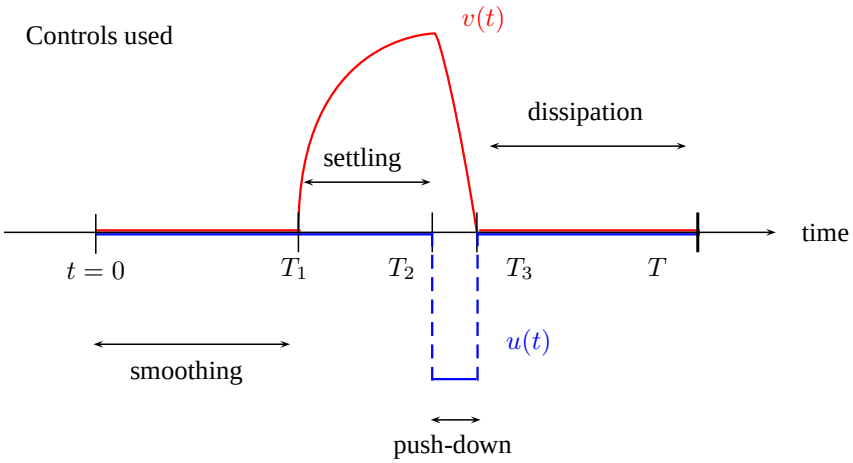

Figure 6: Approximate null-controllability strategy.

Proof. Take $T, r > 0$ and $y_0 \in L^2(0,1)$ given data. Let us take a small $\varepsilon > 0$ and break down our time interval into four parts. We introduce $T_1 = T/3$, $T_2 = T_1 + \varepsilon$ and $T_3 = T_2 + \varepsilon^4$. The first part $[0, T_1]$ of length $T/3$ is designed to smooth the initial data. The second part $[T_1, T_2]$ of length ε is the part where the settling of the boundary layer takes place. The third part $[T_2, T_3]$ of length ε^4 is the quick push down to zero. The fourth part $[T_3, T]$ of length at least $T/3$ (when ε is small enough) is the passive stage for the dissipation of the boundary layer. Let us give some details.

Smoothing of the initial data: First, for $t \in [0, T_1]$, we choose $u(t) = v(t) = 0$. The system evolves freely. Regularization effects of the Burgers' equation smooth our initial data $y_0 \in L^2(0,1)$. We have $y(T_1, \cdot) \in H_0^1(0,1)$. There are many ways to prove such a result. For instance, one can take the Cole-Hopf transform and use well-known regularization properties of the heat equation.

Settling of the boundary layer: Next, for $t \in [T_1, T_2]$, we perform the scaling (4). We want to apply Lemma 4 for a duration 1. Hence, let us choose some H such that $H - 2 > 0$. We take $\bar{v} \in H^{3/4}(0,T)$ the control from Lemma 4. For $t \in [T_1, T_2]$, we use:

$$\begin{aligned} u(t) &= 0, \\ v(t) &= \frac{1}{\varepsilon}\bar{v}\left(\frac{t - T_1}{\varepsilon}\right). \end{aligned}$$

From Lemma 4, we know that:

$$\left\| y(T_2, \cdot) - \frac{1}{\varepsilon} h_\varepsilon(\cdot) \right\|_{L^2(0,1)} = \mathcal{O}_{\varepsilon \to 0}\left(\varepsilon^{-3/2} e^{-\frac{H}{4\varepsilon}(H-2)}\right). \tag{43}$$

Push-down towards zero: Then, still in the context of scaling (4), we want to apply Lemma 6 during a very short duration ε^3. Hence, for $t \in [T_2, T_3]$, we choose the controls found in Lemma 6 (with a total time ε^3), and we scale them appropriately. That is to say:

$$\begin{aligned} u(t) &= \frac{1}{\varepsilon^2}\bar{u}\left(\frac{t - T_2}{\varepsilon}\right), \\ v(t) &= \frac{1}{\varepsilon}\bar{v}\left(\frac{t - T_2}{\varepsilon}\right). \end{aligned}$$

Combining (43) and Lemma 6, we get that, at the end of this hyperbolic stage:

$$0 \geq y(T_3, \cdot) \geq \Phi^\varepsilon + \frac{1}{\varepsilon}\delta(\varepsilon^3, \cdot) - \frac{1}{\varepsilon}\left\| h^\varepsilon - k^\varepsilon \right\|_\infty,$$

where (using estimate (12)):

$$\left\| \delta(\varepsilon^3, \cdot) \right\|_{L^2} + \left\| h^\varepsilon - k^\varepsilon \right\|_\infty = \mathcal{O}_{\varepsilon \to 0}(\varepsilon^3).$$

Dissipation of the boundary residue: Now we enter the passive stage. We choose $v(t) = u(t) = 0$ for $t \in [T_3, T]$. Since ε goes to zero, $T - T_3 \geq T/3$. Hence we can apply Lemma 8 on a time interval independent of ε. By using the comparison principle from Lemma 1 we can conclude that:

$$\|y(T,\cdot)\|_{L^2} = \mathcal{O}_{\varepsilon\to 0}(\varepsilon^{1-\eta}),$$

for any $\eta > 0$. For instance, one can choose $\eta = \frac{1}{2}$. Then we choose ε small enough to ensure that $\|y(T,\cdot)\|_{L^2} \leq r$. This concludes the proof of Theorem 4. □

Remark 2. *In the proof of Theorem 4, we concatenate different controls found in different parts. This could be a problem for smoothness because we did not check compatibility conditions at the jointures. However, the proof provides a control $v \in H^{1/4}(0,T)$ and this doesn't require compatibility conditions. If one wants smooth controls, it is also possible. One can choose a smooth control close to our control for the approximate controllability, then end with a smooth control for the exact controllability.*

5 Parabolic stage and exact local controllability

Theorem 4 takes care of the small time global approximate controllability towards the null state. To get Theorem 1, we need to combine it with a small time local exact controllability result in the vicinity of the null state. We give in this section two different approaches for this type of result.

5.1 Fursikov and Imanuvilov's theorem

The following theorem is due to Fursikov and Imanuvilov. Indeed, the techniques they expose in their book [26] can be applied to show the following result. However, the proof of this precise statement is not written, and one has to work to show that the control can be chosen to be smooth.

Theorem 5. *Let $T > 0$. There exists $r > 0$ such that, for any initial data $y_0 \in L^2(0,1)$ satisfying:*

$$\|y_0\|_{L^2(0,1)} \leq r, \tag{44}$$

there exists a control $v \in \mathcal{C}^1[0,T]$ such that the solution $y \in X$ to the system:

$$\begin{cases} y_t + yy_x - y_{xx} &= 0 & \text{in} \quad (0,T)\times(0,1), \\ y(t,0) &= v(t) & \text{in} \quad (0,T), \\ y(t,1) &= 0 & \text{in} \quad (0,T), \\ y(0,x) &= y_0(x) & \text{in} \quad (0,1), \end{cases} \tag{45}$$

satisfies $y(T,\cdot) \equiv 0$.

The full theorem is in fact more general since one obtains local exact controllability to the trajectories of system (45). The proof relies on Carleman estimates for parabolic equations. It is an extension of a previous result with two boundary controls whose proof can be read in [25].

5.2 Using Cole-Hopf and a moments method

In this section we give a proof of Theorem 5 (both for the sake of completeness and for avoiding Carleman estimates). It relies on the Cole-Hopf transform and a moments method introduced in [21] by Fattorini and Russell.

Proof. Let $T > 0$. First, we consider the following heat system:

$$\begin{cases} z_t - z_{xx} &= 0 & \text{in} \quad (0,T)\times(0,1), \\ z(t,0) &= \alpha(t) & \text{in} \quad (0,T), \\ z_x(t,1) &= 0 & \text{in} \quad (0,T), \\ z(0,x) &= z_0(x) & \text{in} \quad (0,1). \end{cases} \tag{46}$$

This is typically a setting for which we can apply the moments method of Fattorini and Russel exposed in [21]. They prove this system is null controllable for any positive time by means of very smooth controls. Let us use some control $\alpha \in \mathcal{C}^1[0,T]$. They also prove that there exists some constant C_T such that the size of the control is bounded from above by $C_T \times \|z_0\|_{L^2}$. Therefore, if z_0 is small enough in L^2, one can steer it to zero with a control $\alpha(\cdot)$ such that $\|\alpha(\cdot)\|_\infty < 1$.

Now we get back to our Burgers' system. For $y_0 \in L^2(0,1)$, let us choose:

$$z_0(x) = \exp\left(-\frac{1}{2}\int_0^1 y_0(s)ds\right) - 1.$$

Thus, if $y_0(\cdot)$ is small in $L^2(0,1)$ then $z_0(\cdot)$ too. If they are small enough, then we can steer z_0 to 0 with a control such that $\|\alpha(\cdot)\|_\infty < 1$. In that setting, we have $z(\cdot) > -1$ thanks to the maximum principle for the heat equation. Hence, if we let $y = -2z_x/(1+z)$, we get a solution $y \in X$ to (45) such that $y(T,\cdot) \equiv 0$ provided that condition (44) is satisfied for some $r > 0$ depending only on T. □

6 Conclusion and perspectives

In our work, we want to underline two important ideas. The first one is the rigorous analysis of the hyperbolic limit system and of the adequate weak boundary conditions. These weak boundary conditions somehow describe the behavior of the boundary layer and what it will be able to do or not. The second idea is the dissipation of the boundary layer by the fluid system itself during the passive stage. Once a boundary layer is created, will the system be able to dissipate it in short time or not?

These two ideas might be important for the analysis of more complex problems such as the Navier-Stokes Open problem 1. For instance, one could try to see if the boundary layer appearing in [11] when trying to control the 2D Navier-Stokes system with Navier slip boundary conditions can be dissipated in small time by the system itself. Another idea would be to investigate specific geometries (for example 2D rectangle domains). Indeed, one can then hope that the localization of the boundary layers can be analyzed in the same spirit.

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