Frontier estimation as a particular case of Conditional extreme value analysis

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Abstract

In this paper, we show that edge estimation, boundary estimation and frontier estimation can be seen as particular cases of Conditional Extreme Value Analysis.

Extreme value analysis (EVA) is a branch of statistics dealing with the extreme deviations from the mean of probability distributions. It seeks to assess the probability of events that are more extreme than any observed prior. Extreme value analysis is widely used in many disciplines, ranging from structural engineering, finance, earth sciences, traffic prediction, geological engineering, etc. For example, EVA might be used in the field of hydrology to estimate the value an unusually large flooding event, such as the 100-year flood. Similarly, for the design of a breakwater, a coastal engineer would seek to estimate the 50-year wave and design the structure accordingly.

Let X be a random variable and $p(x)$ its probability density function, $F(x) = P(X \leq x)$ its distribution function, and $\Psi(x) = 1 - F(x)$ its complementary distribution function. We can define a new random variable, Y_n, as the maximum of n copies of the random variable X: $Y_n = \max\{X_1, X_2, ..., X_n\}$. Y_n is the n-sample maximum of the random variable X. If the events generating the realizations of X are independent, the cumulative distribution of Y_n may be expressed as $[F(y)]^n$. Upon definition of a renormalized variable $S_n = (Y_n - b_n)/a_n$, the extreme value theorem establishes that

Theorem *If*

$$\lim_{n\to\infty} P(S_n < s) = \lim_{n\to\infty} F^n(a_n s + b_n) = H(s) \tag{1}$$

where $a_n > 0$ and b_n are normalization constants, then the function $H(s)$ in Eq. (1) must be one of the three following types:

- *EV1 or Gumbel:* $H(s) = \exp(-\exp(-s))$
- *EV2 or Fréchet:* $H(s) = \exp(-s^{-\alpha})$
- *EV3 or Weibull:* $H(s) = \exp(-|s|^{\alpha})$

The three asymptotic types, EV1-EV3, can be thought of as special cases of a single Generalized Extreme Value distribution (GEV) :

$$H_{GEV}(s) = \exp\left\{ -\left(1+\gamma\frac{s-\mu}{\sigma}\right)_+^{-1/\gamma} \right\} \quad (2)$$

where $(.)_+ = \max(.,0)$, μ is the location parameter, $\sigma > 0$ is the scale parameter, and γ is a shape parameter. The limit $\gamma = 0$ corresponds to the EV1 distribution, $\gamma > 0$ to the EV2 distribution (with $\alpha = 1/\gamma$) and $\gamma < 0$ to the EV3 distribution (with $\alpha = -1/\gamma$). The function $H_{GEV}(s)$ is usually fitted to the cumulative distribution of non-normalized maxima, so that the location parameter μ and the scale parameter σ are the renormalization parameters b_n and a_n respectively. However, it is important to note that the distribution describing the n-sample maximum will strictly be a GEV only for large values of n. How large the value of n needs to be should be determined by analyzing the convergence properties based on the observed realizations of X.

When the studied phenomena depends on a covariate, one has to deal with Conditional Extreme Value Analysis (CEVA). This branch of statistics has become very active these past ten years, the main contributions to this domain are listed below:

- Theoretical issues: [65, 62, 64, 26, 32, 20, 31, 33, 30, 5, 27, 13, 51, 25, 4, 1, 82, 81]
- Quantile regression: [56, 19, 66]
- Application to finance: [10, 3, 35, 11, 58, 80]

Edge estimation [53, 52], frontier estimation [63, 78, 75, 77, 72, 16, 14, 15, 9, 2, 39, 8, 61, 17, 70, 69, 21, 79, 60, 22, 48, 36, 59, 23, 24, 68, 6, 73, 12, 55, 57, 67, 54, 43, 45, 18, 29, 71, 74, 76, 38] and boundary estimation [44, 47, 46, 42, 41, 40, 49, 28, 50, 7, 34, 37] are particular cases of CEVA. They are embedded in the situation where the conditional extreme-value index is negative. In particular, the uniform distributed case corresponds to $\gamma = -1$.

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